We want to prove when $P[\|F_n - F\|_\infty \geq 8\sqrt{\frac{\log(n+1)}{n}} + t] \leq e^{-nt^2/2}$ for all $t > 0$, $\|F_n - F\| \xrightarrow{a.s.} 0$

We set $\varepsilon = 8\sqrt{\frac{\log(n+1)}{n}} + t$ $\left(\varepsilon > 8\sqrt{\frac{\log(n+1)}{n}}\right)$

By using Borel-Cantelli lemma, we want to prove

$$\sum_{n=1}^{\infty} P[\|F_n - F\|_\infty \geq \varepsilon] \leq \sum_{n=1}^{\infty} e^{-n\left(\varepsilon - 8\sqrt{\frac{\log(n+1)}{n}}\right)^2/2} < \infty$$

We define the right side of inequality as a series

$$U_n = \exp\left(-n\left(\varepsilon - 8\sqrt{\frac{\log(n+1)}{n}}\right)^2/2\right),$$

if we want to prove $\sum_{n=1}^{\infty} U_n < \infty$, we just need to prove U_n is convergence

Hence we need to prove $\lim_{n\to\infty} \frac{U_{n+1}}{U_n} < 1$

$$\frac{U_{n+1}}{U_n} = \exp\left[-(n+1)\cdot\left(\varepsilon - 8\sqrt{\frac{\log(n+2)}{n+1}}\right)^2/2 + n\cdot\left(\varepsilon - 8\sqrt{\frac{\log(n+1)}{n}}\right)^2/2\right]$$

$$= \exp\left[-\left(\varepsilon - 8\sqrt{\frac{\log(n+1)}{n}}\right)^2/2 - \frac{n}{2}\cdot\left[\left(\varepsilon - 8\sqrt{\frac{\log(n+2)}{n+1}}\right)^2 - \left(\varepsilon - 8\sqrt{\frac{\log(n+1)}{n}}\right)^2\right]\right]$$

because $\varepsilon > 8\sqrt{\frac{\log(n+1)}{n}}$ and $\sqrt{\frac{\log(n+1)}{n}}$ is a decreasing series

we can get $\left[\left(\varepsilon - 8\sqrt{\frac{\log(n+2)}{n+1}}\right)^2 - \left(\varepsilon - 8\sqrt{\frac{\log(n+1)}{n}}\right)^2\right]$ is positive

Hence $\lim_{n\to\infty} \frac{U_{n+1}}{U_n} < 1$, and U_n is convergence.

$\|F_n - F\| \xrightarrow{a.s.} 0$ can be proved.