$$\boxed{\triangle\ E(e^{\lambda x}) \le e^{\frac{\lambda^2(b-a)^2}{8}}}$$

3.8

$\Rightarrow$ convex $\Rightarrow f(x) \le f(a) + \frac{f(b)-f(a)}{b-a}(x-a)$

$\Rightarrow$ let $f(x) = e^{\lambda x} \Rightarrow$

$$e^{\lambda x} \le e^{\lambda a} + \frac{e^{\lambda b}-e^{\lambda a}}{b-a}\cdot(x-a) = \frac{b-x}{b-a}\cdot e^{\lambda a} + \frac{x-a}{b-a}e^{\lambda b} \quad \forall x \in [a,b]$$

regarding x as the random variables $\in [a,b] \to EX = 0$.

$$\Rightarrow E(e^{\lambda x}) \le \frac{b-EX}{b-a}\cdot e^{\lambda a} + \frac{EX-a}{b-a}e^{\lambda b}$$

$$= \frac{b}{b-a}e^{\lambda a} - \frac{a}{b-a}e^{\lambda b}$$

$$= \frac{-a}{b-a}\cdot e^{\lambda a}\left(-\frac{b}{a} + e^{\lambda(b-a)}\right)$$

let $q = -\frac{a}{b-a}$, $h = \lambda(b-a)$

$\Rightarrow$ right-hand-~~right~~ side $\Rightarrow q\cdot e^{-qh}\left(\frac{1}{q} - 1 + e^{h}\right)$

$$= e^{-qh + \ln(1-q+qe^{h})}$$

let $L(h) = -qh + \ln(1-q+qe^{h})$

$\Leftrightarrow$ goal: $E(e^{\lambda x}) \le e^{L(h)}$

$$\Rightarrow L'(h) = -q + \frac{qe^{h}}{1-q+qe^{h}}$$

$$L''(h) = \frac{qe^{h}(1-q+qe^{h}) - (qe^{h})^2}{(1-q+qe^{h})^2}$$

$\Rightarrow$ ~~Taylor~~ $\Rightarrow$ (power series) $\Rightarrow$

$$L(h) = \frac{L(0)}{0!}h^0 + \frac{L'(0)}{1!}h^1 + \frac{L''(0)}{2!}h^2 + o(h^2)$$

$\Rightarrow L'(0) = 0 \quad L''(0) = q(1-q) \le \frac{1}{4}$

$L(h) = \frac{L''(0)}{2!}h^2 \le \frac{\lambda^2(b-a)^2}{8}$

$\therefore E(e^{\lambda x}) \le e^{\frac{\lambda^2 (b-a)^2}{8}}$

lemma 3.16

(i) $\triangleright$ $E[e^{\xi_i^2/(6\|a^{(i)}\|^2)}] \leq 2$

let $A = \frac{\xi^2}{6\|a\|^2} = \left(\frac{\sum a_k \varepsilon_k}{\sqrt{6}\|a^{(i)}\|}\right)^2$

$= \sum_{k=1}^{n} \frac{a_k^2}{6\|a\|^2} + \sum_k \sum_j \frac{a_j a_k}{6\|a\|^2} \underline{\varepsilon_k \varepsilon_j}$ new random variable

$\Rightarrow E[e^{\xi_i^2/6\|a^{(i)}\|^2}] = E[e^A]$

$= e^{\frac{1}{6}} E[e^{\sum\sum \frac{a_j a_k}{6\|a\|^2} \cdot \varepsilon_{kj}}]$

$= e^{\frac{1}{6}} \prod_k \prod_j E\, e^{\frac{a_j a_k}{6\|a\|^2} \varepsilon_{kj}}$

$E e^{a\varepsilon} \leq e^{\lambda^2/2} \Rightarrow$ $\leq e^{\frac{1}{6}} \cdot e^{\frac{\sum a_j^2 a_k^2}{72\|a\|^4}}$

$= e^{\frac{1}{4}} \approx 2$

(ii) let $z = \frac{\xi_i}{\sqrt{6}\|a\|}$ $\Rightarrow \varphi(z) = e^{z^2}$

$\Rightarrow$ lemma 3.15

$E\max|\xi_i| \leq \varphi^{-1}(LN) \max c_i$

let $L=2$, $c_i = \sqrt{6}\|a^{(i)}\|$, $\varphi^{-1} = \sqrt{\log \varphi(\cdot)}$

$\Rightarrow E\max|\xi_i| \leq \sqrt{\log 2N} \cdot \sqrt{6} \max\|a^{(i)}\|$

$\exists$ constant C

$E\max|\xi_i| \leq C\sqrt{\log N} \cdot \max\|a^{(i)}\|$

prove done.