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ε-ENTROPY OF CONVEX SETS AND FUNCTIONS

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Definition 1 [1]. Suppose that a minimal ε-net of a precompact metric set K contains $N_K(\varepsilon)$ points. The ε-entropy of the set K is defined by the quantity $\mathscr{H}_K(\varepsilon)=\log_2 N_K(\varepsilon)$.

Definition 2. Let M and N be closed subsets of the Euclidean space E^n. The Hausdorff distance between M and N is defined by the formula $\rho(M,N)=\max\{\sup_{m\in M} d(m,N),\ \sup_{n\in N} d(n,M)\}$, where $d(m,N)=\inf_{n\in N}|n-m|$; $|\cdot|$ is the Euclidean norm.

According to a well-known theorem of Hausdorff, the set of all closed subsets of a compact set in E^n is compact in a Hausdorff metric.

In §1 we shall prove the auxiliary Theorem 1.

In §2 we shall prove that the ε-entropy of a compact set of convex closed subsets of the unit sphere in Euclidean space E^n increases as $\varepsilon^{(1-n)/2}$. The same problem has been considered by Dudley [2], but he obtained a somewhat weaker result.

In §3 we shall prove that the ε-entropy of a compact set of uniformly bounded and uniformly Lipschitzian convex functions with a metric C defined on a cube in E^n increases as $\varepsilon^{-n/2}$.

§1

Let us denote by $\mathfrak{M}_n$ the class of convex closed subsets of the unit ball $T_1\subset E^n$. Let $M\in\mathfrak{M}_n$; $\varepsilon>0$. Let us introduce the notation

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$$\mathfrak{N}_\varepsilon(M) = \{N \in \mathfrak{M}_n : \rho(M, N) \leqslant \varepsilon\},$$
$$\tilde{\mathfrak{N}}_\varepsilon(M) = \{N \in \mathfrak{M}_n : \rho(M, N) \leqslant \varepsilon;\ M \subset N\}.$$

By M_r we shall denote an exterior set that is parallel to M at a distance r. It is evident that $\rho(M, N) = \rho(M_r, N_r)$. If $N \in \mathfrak{N}_\varepsilon(M)$, then $M_1 \subset N_{1+\varepsilon}$; $\rho(M_1, N_{1+\varepsilon}) \le \rho(M_1, M_{1+\varepsilon}) + \rho(M_{1+\varepsilon}, N_{1+\varepsilon}) \le 2\varepsilon$. Thus, the mapping $\varphi_{1+\varepsilon}: N \to N_{1+\varepsilon}$ will be an isometry of the space $\mathfrak{N}_\varepsilon(M)$ onto $\tilde{\mathfrak{N}}_{2\varepsilon}(M_1)$, i.e., in the space $\mathfrak{N}_\varepsilon(M)$ the number of points in a minimal δ-net will not be larger than in the space $\tilde{\mathfrak{N}}_{2\varepsilon}(M_1)$.

By S(y, r) [or T(y, r)] we shall denote a sphere (or ball) of the space E^n of radius r centered at the point $y \in E^n$.

Let us take a point $x \in M$. Since $M \subset T_1$, it follows that $M \subset T(x, 2)$. Thus, $T(x, 1) \subset M_1 \subset T(x, 3)$. Moreover, we shall require that for $\forall y \in \partial M_1$ there exists a point $\alpha \in E^n$ such that $T(\alpha, 1) \subset M_1$, $y \in S(\alpha, 1)$. Without loss of generality it can be assumed that x = 0.

In E^n ($n \ge 2$) let us construct a cube S centered at the point 0 and having a side $2/\sqrt{n}$. Each of the 2n faces of the cube will be partitioned into small cubes with a side $c \cdot \sqrt{\varepsilon/n}$ [$c = 10^{-4}/\sqrt{n}(n-1)$]. The number of vertices thus obtained on each face will not exceed $2\varepsilon^{(1-n)/2}/c$. After that each small cube will be divided in a simplicial manner in such a way that no new vertices are adjoined. Thus the boundary ∂S of the cube S will be divided into simplexes with a diameter not exceeding $c\sqrt{\varepsilon}$, and a number of vertices not larger than

$$a = 4n\varepsilon^{(1-n)/2}/c. \tag{1}$$

Let us denote the vertices of the simplexes by $z_1, \ldots, z_N$, and the ray originating at the point 0 and passing through a point $z \in E^n$ by $\overrightarrow{0z}$.

The aim of §1 is to prove the following theorem.

THEOREM 1. Let $N' \in \tilde{\mathfrak{N}}_{4\varepsilon}(M_1)$, N being a polyhedron with vertices $\partial N' \cap \overrightarrow{0z_i}$. For $\varepsilon \le 10^{-12}/(n-1)$ we hence obtain

$$\rho(N', N) \leqslant \varepsilon/2 \quad (n \geqslant 2).$$

The proof is preceded by several lemmas.

LEMMA 1. If $d(z, M_1) \le 4\varepsilon$; $z \notin M_1$, then $|z - x(z)| \le 12\varepsilon$. Here $x(z) = \overrightarrow{0z} \cap \partial M_1$.

Proof. Let us denote by $\nu(x)$ ($x \in \partial M$) a unit vector of the outer normal to the convex set M at the point x. Through the ray $\overrightarrow{0z}$ let us draw a two-dimensional plane parallel to the vector $\nu[x(z)]$. Since $M_1 \supset T(0, 1)$, it follows that any reference plane to the set M_1 does not intersect the ball T(0, 1). Thus (Fig. 1) we have $|l| \ge 1$, $|x(z)| \le 3$. Evidently, $|z - z_1| \le 4\varepsilon$. Hence follows that $|x(z) - z| = |x(z)| \times |z - z_1|/|l| \le 12\varepsilon$.

In just as elementary a manner we can prove

LEMMA 2. Let $x \in \partial M_1$, $T(\alpha, 1) \subset M_1$, $x \in S(\alpha, 1)$. Then any two-dimensional plane that passes through the points 0 and x will intersect $T(\alpha, 1)$ along a circle of radius not smaller than 1/3.

Let us partition the space E^n into simplicial cones with a common vertex 0 and bases constructed by the simplexes of partition of the boundary ∂S of the cube S. These cones will be denoted by K_i.

LEMMA 3. Suppose that the points z_1 and z_2 lie in the same cone K_i of partition of the space E^n; $\rho(z_i, M_1) \le 4\varepsilon$, $z_i \notin M_1$ (i = 1, 2). Then $|z_1 - z_2| \le 19\pi c\sqrt{n}\sqrt{\varepsilon}$ for $\varepsilon \le 10^{-12}/(n-1)$ [$c = 10^{-4}/\sqrt{n}(n-1)$].

Proof. Let us project the points z_1 and z_2 onto a sphere $S(0, 1/\sqrt{n})$ that lies in the cube S. Let $z_i' = S(0, 1/\sqrt{n}) \cap \overrightarrow{0z_i}$, $z_i'' = \partial S \cap \overrightarrow{0z_i}$ (i = 1, 2). By virtue of our condition we have $|z_1'' - z_2''| \le c\sqrt{\varepsilon}$. Since z_i' is the projection of the point z_i'' onto the sphere $S(0, 1/\sqrt{n})$, and since projection onto a convex set does not increase the distances in an internal and (all the more so) an external metric ([3], p. 91), it follows that $|z_1' - z_2'| \le c\sqrt{\varepsilon}$. Thus the angle θ between the rays $\overrightarrow{0z_1}$ and $\overrightarrow{0z_2}$ will satisfy the inequality

$$\theta \leqslant \pi c\sqrt{n}\sqrt{\varepsilon}. \tag{2}$$

Now let us construct the points $x_i = \overrightarrow{0z_i} \cap \partial M_1$ (i = 1, 2). By virtue of Lemma 1,

$$|z_1 - z_2| \leqslant |z_1 - x_1| + |z_2 - x_2| + |x_1 - x_2| \leqslant 24\varepsilon + |x_1 - x_2|. \tag{3}$$

Let l be a straight line that passes through the points x_1 and x_2. It is easy to show that for $\varepsilon \le 10^{-12}/(n-1) < 1/9nc^2$ the straight line l does not intersect the ball T(0, 1/2). Let us denote by d the base of the perpendicular dropped from the point 0 onto the straight line l. There can be two possibilities:

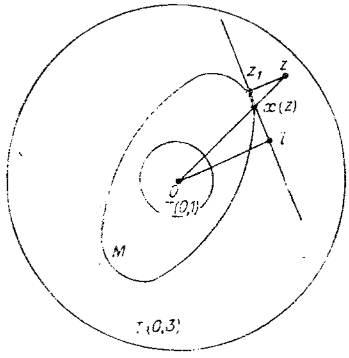

Fig. 1

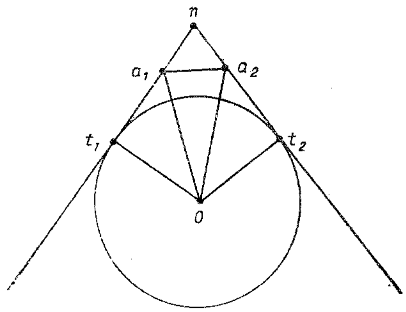

Fig. 2

a) $x_1 \in [d, x_2]$ (z_1 and z_2 can be exchanged);

b) $d \in [x_1, x_2]$.

Here $[a, b]$ is a segment with ends a and b.

Case (a). $\sin < 0x_2d = |d|/|x_2| \geq 1/6$, since $|d| \geq 1/2$; $|x_2| \leq 3$. The length

$$|x_1 - x_2| = |x_2| \cdot \sin\angle x_1 0 x_2 / \sin\angle 0 x_2 x_1 \leqslant 18 \sin\theta.$$

From (2) we find that $|x_1 - x_2| \leq 18\pi c\sqrt{n}\sqrt{\varepsilon}$.

Case (b). $|x_1 - x_2| = |x_1 - d| + |x_2 - d|$. From (2) we obtain

$$|x_i - d| = |x_i| \cdot \sin\angle x_i 0 d \leqslant 3\sin\theta \leqslant 3\pi c\sqrt{n}\sqrt{\varepsilon}.$$

Hence $|x_1 - x_2| \leq 6\pi c\sqrt{n}\sqrt{\varepsilon}$.

Finally we find from (3) that $|z_1 - z_2| \leq 24\varepsilon + 18\pi c\sqrt{n}\sqrt{\varepsilon} \leq 19\pi c\sqrt{n}\sqrt{\varepsilon}$ for $\varepsilon \leq 10^{-12}/(n-1)$. This completes the proof of Lemma 3.

The following two lemmas refer to the case of the plane E^2; in them, $n \geq 2$ serves as a parameter.

LEMMA 4. In the Euclidean plane E^2 let the set M and the points a and b be such that $T(0, 1) \subset M \subset T(0, 3)$; $0 \in M \in \mathfrak{M}_2$; $d(a, M) \leq 4\varepsilon$; $a \notin M$; $b \in \partial M$; $\Delta\beta = \angle a0b \leq \pi c\sqrt{n}\sqrt{\varepsilon}$. Suppose also that there exists a point $\alpha \in M$ such that $T(\alpha, 1/3) \subset M$; $b \in S(\alpha, 1/3)$. Then $d[a, T(\alpha, 1/3)] \leq 3(5 + 450\pi^2c^2n)\varepsilon$; $\angle a\alpha b \leq 57\pi^2 c\sqrt{n}\sqrt{\varepsilon}$ for $\varepsilon \leq 10^{-8}/(n-1)$ $[c = 10^{-4}/\sqrt{n}(n-1)]$.

Proof. Let us select a coordinate system in E^2 in such a way that the origin coincides with the point 0 and the x axis is parallel to the reference line l_b to the set M at the point b. Suppose that the point α has in this system the coordinates (x_α, y_α). Let us denote by β the angle formed by the ray $\vec{0b}$ with the x axis. Similarly to the proof of Lemma 1 we find that $|\sin\beta| \geq 1/3$, i.e., for $0 \leq \Delta\beta < \arcsin 1/3$ we have $0 < \beta \pm \Delta\beta < \pi$. It is easy to see that for such values of ε this constraint on $\Delta\beta$ is satisfied. Let us denote by e the point $\vec{0a} \cap l_b'$, where l_b' denotes a straight line parallel to l_b at a distance 4ε from the latter and located outside the set M. Since $d(a, M) \leq 4\varepsilon$, it follows that $d[a, T(\alpha, 1/3)] \leq d[e, T(\alpha, 1/3)]$. The point e has the coordinates $x_e = (y_\alpha + 1/3 + 4\varepsilon)\cot(\beta \pm \Delta\beta)$; $y_e = y_\alpha + 1/3 + 4\varepsilon$.

Let us transform the formula for the coordinates x_e. $x_e = (y_\alpha + 1/3 + 4\varepsilon)(1 \mp \tan\beta\tan\Delta\beta)/(\tan\beta \pm \tan\Delta\beta)$. Since $\tan\beta = (y_\alpha + 1/3)/x_\alpha$, we obtain $x_e = (y_\alpha + 1/3 + 4\varepsilon)(x_\alpha \mp y_\alpha \tan\Delta\beta \mp {}^1\!/_3 \tan\Delta\beta)/(y_\alpha + 1/3 \pm x_\alpha \tan\Delta\beta)$. Now let us estimate $|x_e - x_\alpha|$:

$$|x_e - x_\alpha| \leqslant \left(\operatorname{tg}\Delta\beta\left[(y_\alpha + {}^1\!/_3 + 4\varepsilon)(y_\alpha + {}^1\!/_3) + x_\alpha^2\right] + 4\varepsilon|x_\alpha|\right) / (y_\alpha + {}^1\!/_3 - |x_\alpha|\operatorname{tg}\Delta\beta).$$

Since $|x_\alpha| \leq 3$, $|y_\alpha| \leq 3$ and for $\varepsilon \leq (20c\sqrt{n})^{-2}$ we have $0 \leq \Delta\beta \leq \pi/20$, so that $0 \leq \tan\Delta\beta < 2\Delta\beta \leq 1/18$, we obtain

$$|x_e - x_\alpha| \leqslant 30\pi c\sqrt{n}\sqrt{\varepsilon}. \tag{4}$$

Now let us estimate $d[e, T(\alpha, 1/3)] = |e - \alpha| - 1/3$. By virtue of (4) we obtain $|e - \alpha| = [(x_e - x_\alpha)^2 + (y_e - y_\alpha)^2]^{1/2} \leq 1/3 + 3(4 + 450\pi^2c^2n)\varepsilon + 24\varepsilon^2 \leq 1/3 + 3(5 + 450\pi^2c^2n)\varepsilon$. Hence follows that $d[a, T(\alpha, 1/3)] \leq d[e, T(\alpha, 1/3)] \leq 3(5 + 450\pi^2c^2n)\varepsilon$. Since $|a - b| \leq 19\pi c\sqrt{n}\sqrt{\varepsilon}$ (Lemma 3), we obtain by projecting the points a and b onto $S(\alpha, 1/3)$ the formula

$$\angle a\alpha b \leqslant 3\cdot 19\pi c\sqrt{n}\sqrt{\varepsilon}.$$

LEMMA 5. Let $T(0, 1/3)$ be a circle on the Euclidean plane E^2; $d[a_i, T(0,1/3)] \le 3(5 + 450\pi^2c^2n)\varepsilon$; $a_1 \notin T(0, 1/3)$ $(i = 1, 2)$; $\angle a_1 0 a_2 \le 57\pi^2 c\sqrt{n}\sqrt{\varepsilon}$; $|a_1 - a_2| \le 19\pi c\sqrt{n}\sqrt{\varepsilon}$. Let also $S \supset T(0, 1/3)$ be a convex set such that $a_1 \in \partial S$ $(i = 1, 2)$. For $\varepsilon \le 1/30$, we then have $\rho(S \cap K, S_1) \le \varepsilon/2(n-1)$. (Here S_1 is a triangle with vertices 0, a_1, and a_2; K is an acute angle with vertex 0 whose sides contain the points a_1 and a_2 $[c = 10^{-4}/\sqrt{n}(n-1)]$.)

Proof. It follows from Definition 2 that since $S_1 \subset S \cap K$, it suffices to show that if $z_0 \in \partial S \cap K$, then $d(z_0, [a, b]) \le \varepsilon/2(n-1)$.

It is evident that the point z_0 lies in the triangle $a_1 a_2 m$ (Fig. 2). $0a_1 \le 1/3 + 3(5 + 450\pi^2c^2n)\varepsilon$, whence follows that $\angle t_i 0 a_i \le \pi\cdot 3\sqrt{3(5+450\pi^2c^2n)}\cdot\sqrt{\varepsilon}/2$ $(i = 1, 2)$, $\angle t_1 0 t_2 \le (3\pi\sqrt{3(5+450\pi^2c^2n)} + 57\pi^2c\sqrt{n})\sqrt{\varepsilon}$. From elementary considerations we can see that $\angle m a_1 a_2 + \angle m a_2 a_1 = \angle t_1 0 t_2$. For $\varepsilon \le 1/30$ we have $\angle m a_1 a_2 + \angle m a_2 a_1 \le \pi/2$, i.e., $d(z_0, [a_1, a_2]) \le d(m, [a_1, a_2])$.

Let $\angle m a_1 a_2 \le \angle m a_2 a_1$. Then $d(m, [a_1, a_2]) \le |a_1 - a_2| \sin \angle m a_1 a_2 \le 19\pi c\sqrt{n}\cdot(3\pi\sqrt{3(450\pi^2c^2n+5)} + 57\pi^2c\sqrt{n})\varepsilon/2 < \varepsilon/2(n-1)$, since $c = 10^{-4}/\sqrt{n}(n-1)$.

Proof of Theorem 1. It suffices to show that for any cone K_i of partition of the space E^n we have $\rho(N' \cap K_i, N \cap K_i) \le \varepsilon/2$, since it is evident that if for any i we have $\rho(A_i, B_i) \le a$, then $\rho(\bigcup_i A_i, \bigcup_i B_i) \le a$. Let us fix a cone K. Let us denote by $K^{(s)}$ $(s = 1, 2, \ldots, n)$ an s-dimensional hull of a simplicial cone K. Let us prove by induction on s that if $z \in K^{(s)} \cap N'$, then

$$d(z, N) \leqslant \varepsilon(s-1)/2(n-1). \tag{5}$$

Since $N \subset N'$, we obtain for $s = n$ the assertion of the theorem.

For $s = 1$, the assertion (5) is evident, since we selected in this way the vertices of the polyhedron N. Suppose that it holds for s. Let us take a point $z \in N' \cap K^{(s+1)}$. Let $x = \overline{0z} \cap \partial N'$. There exists a point $\alpha \in E^n$ such that $T(\alpha, 1) \subset M_1$, $x \in S(\alpha, 1)$. Let us denote by π_{s+1} the (s+1)-dimensional face of the cone K containing the point z. Through the points 0 and z let us construct a two-dimensional plane π_2 such that $\pi_2 \subset \pi_{s+1}$ is an angle whose outer rays lie in $K^{(s)}$. Suppose that these rays intersect $\partial N'$ at the points a and b. The radius of the circle $\pi_2 \cap T(\alpha, 1)$ is not smaller than 1/3. With the aid of Lemmas 4 and 5 we obtain $d(z, [a, b]) \le \varepsilon/2\cdot(n-1)$. But $d(a, N)$ and $d(b, N)$ do not exceed $\varepsilon(s-2)/2(n-1)$ by virtue of the induction hypothesis. Since N is a convex set, it follows that $d(z, N) \le \varepsilon(s-1)/2(n-1)$. Thus we have proved the inequality (5), and hence also Theorem 1.

§2

At first let us prove a theorem on the number of points in an ε-net of a 2ε-neighborhood of a set $M \in \mathfrak{M}_n$.

THEOREM 2. Let $M \in \mathfrak{M}_n$. Then $\forall \varepsilon \le 10^{-12}/(n-1)$ in the space $\mathfrak{N}_{2\varepsilon}(M)$ there exists an ε-net containing not more than $12^{\gamma(\sqrt{\varepsilon})^{1-n}}$ points, $\gamma \le 4\cdot 10^4\cdot n^{5/2}$.

Proof. The analysis presented at the beginning of § 1 shows that it suffices to obtain an upper bound for the number of points in an ε-net of the space $\tilde{\mathfrak{N}}_{4\varepsilon}(M_1)$. As before, let the z_i be vertices of simplexes in the case of a simplicial partition of the boundary ∂S of the cube S. Let us construct the rays $\overrightarrow{0z_i}$. Let $N \in \tilde{\mathfrak{N}}_{4\varepsilon}(M_1)$, $y_i = \partial N \cap \overrightarrow{0z_i}$. It then follows from Theorem 1 that $\rho(N, \overline{\bigcup_i y_i}) \le \varepsilon/2$ ($\bar{A}$ being the convex hull of the set A). If the points y_i' are such that $|y_i - y_i'| \le \varepsilon/2$, then $\rho(\overline{\bigcup_i y_i}, \overline{\bigcup_i y_i'}) \le \varepsilon/2$. Hence we can see that an ε-net of the space $\tilde{\mathfrak{N}}_{4\varepsilon}(M_1)$ can be constructed as follows: We divide the sections of the rays $\overrightarrow{0z_i}$ of length 12ε, beginning with the points of intersection with the boundary ∂M_1 (Lemma 1), into segments of length ε; after that we form all possible collections of division points, one from each ray $\overrightarrow{0z_i}$, and then we construct the convex hull of each collection. The total number of collections of division points does not exceed 12^p, where p is the number of rays $\overrightarrow{0z_i}$. It follows from (1) that p can be taken equal to $4\cdot 10^4\cdot n^{5/2}(\sqrt{\varepsilon})^{1-n}$. This completes the proof of Theorem 2.

THEOREM 3. For any positive $\varepsilon \le 10^{-12}/(n-1) = \varepsilon_0$ there exists in $\mathfrak{M}_n$ $(n \ge 2)$ an ε-net containing not more than $\beta(n)\cdot 12^{4\gamma(\sqrt{\varepsilon})^{1-n}}$ points ($\gamma \le 4\cdot 10^4\cdot n^{5/2}$).

Proof. Let us express the number set $(0, \varepsilon_0]$ in the form $(0, \varepsilon_0] = \bigcup_{k=1}^{\infty} A_k$, $A_k = (\varepsilon_0/2^k, \varepsilon_0/2^{k-1}]$. The theorem will be proved by induction on k, i.e., for all $\varepsilon \in A_k$.

For k = 1 it suffices to select the value of $\beta(n)$. Suppose that the assertion of the theorem is true for any $\varepsilon \in A_k$. Let us prove that it is true also for any $\varepsilon \in A_{k+1}$. If $\varepsilon \in A_{k+1}$, then $2\varepsilon \in A_k$. By the induction hypothesis, there exists in $\mathfrak{M}_n$ a 2ε-net with a number of points not exceeding $\beta(n) \cdot 12^{4\gamma(\sqrt{2\varepsilon})^{1-n}}$ It follows from Theorem 2 that $\forall M \in \mathfrak{M}_n$ there exists in the space $\mathfrak{N}_{2\varepsilon}(M)$ an ε-net containing not more than $12^{\gamma(\sqrt{\varepsilon})^{1-n}}$ points. An ε-net of the space $\mathfrak{M}_n$ can be obtained from any of its 2ε-nets by replacing each of its elements by an ε-net of its 2ε-neighborhood. By taking the above-mentioned 2ε-net of the space $\mathfrak{M}_n$, we obtain an ε-net containing not more than

$$\beta(n) \cdot 12^{4\gamma(\sqrt{2\varepsilon})^{1-n}+\gamma(\sqrt{\varepsilon})^{1-n}} \leqslant \beta(n) \cdot 12^{4\gamma(\sqrt{\varepsilon})^{1-n}}$$

points. This completes the proof of Theorem 3.

Now let us obtain a lower bound for the number of points of a minimal ε-net.

Definition 3 [1]. Let K be a compact metric set. We shall say that the points $a_1, \ldots, a_N \in K$ form an ε-distinguishable set in K if $\rho(a_i, a_j) \geq \varepsilon$ $(i \neq j)$.

It was proved in [1] that the number of points in any ε-distinguishable compact set does not exceed the number of points in any of its ε-nets.

THEOREM 4. In the space $\mathfrak{M}_n$ there exists for any positive $\varepsilon \leq 1/64$ an ε-distinguishable set containing not less than $\frac{\varkappa_{n-1}}{2^{8^{n-1}(n-1)}}(\sqrt{\varepsilon})^{1-n}$ points, $\varkappa_{n-1}$ being the measure of the (n−1)-dimensional unit ball.

Proof. Let us consider the unit sphere $S = \{x \in E^n : |x| = 1\}$ and let $K \subset S$ be a $2\sqrt{\varepsilon}$-distinguishable subset of the latter in an external Euclidean metric. Let $x \in K$. Let us denote by $(S)_x$ the tangent plane of the sphere at the point x. Now we construct a plane that is parallel to $(S)_x$ at a distance $\varepsilon \leq 1/64$ from the latter and that intersects the sphere S. It is easy to see that this plane separates the point x and the set $K \setminus \{x\}$, i.e., the polyhedra with vertices belonging to K form an ε-distinguishable set in $\mathfrak{M}_n$. It contains 2^k points, where k is the number of points in the finite set K. Now let us estimate the number of points k.

Let us project the set K onto an (n−1)-dimensional plane. The projection operator will be denoted by P. Since the operator P does not increase distances if the set P[K] is ε-distinguishable, it follows that a subset K of the sphere will do likewise. Thus it suffices to find a lower bound for the number of points in a $2\sqrt{\varepsilon}$-distinguishable set of the unit ball in E^{n-1}. From Mikhlin's result ([4], p.300) it follows that for $\varepsilon \leq 1/64$ there exists a set with the required properties and with a number of points not smaller than $\varkappa_{n-1}(\sqrt{\varepsilon})^{1-n}/8^{n-1} \cdot$ (n−1). This completes the proof of Theorem 4.

By comparing Theorems 3 and 4, we obtain the principal result of this paper.

THEOREM 5. The ε-entropy of the space $\mathfrak{M}_n$ satisfies for $\varepsilon \leq 10^{-12}/(n-1)$ the inequalities

$$\frac{\varkappa_{n-1}}{8^{n-1}(n-1)}(\sqrt{\varepsilon})^{1-n} \leqslant \mathscr{H}_{\mathfrak{M}_n}(\varepsilon) \leqslant 10^6 \cdot n^{5/2} \cdot \log 12 \cdot (\sqrt{\varepsilon})^{1-n}$$

or, in the notation of [1],

$$\mathscr{H}_{\mathfrak{M}_n}(\varepsilon) \underset{\cap}{\cup} \left(\frac{1}{\varepsilon}\right)^{\frac{n-1}{2}}.$$

Remark 1. Let us denote by $\mathfrak{M}_n(R)$ the totality of convex closed subsets of an n-dimensional ball of radius R. It then follows directly from Theorem 5 that

$$\mathscr{H}_{\mathfrak{M}_n(R)} \underset{\cap}{\cup} \left(\frac{R}{\varepsilon}\right)^{(n-1)/2}.$$

Remark 2. For n = 2, more exact estimates were obtained by Yu. G. Reshetnyak and A. P. Orlov (not published).

§ 3

Let us denote by $F_n(M, C)$ a compact set (endowed with a uniform metric) of all convex functions defined on a cube S with a side 2 in E^n such that $|f(x)| \leq M$; $|f(x) - f(y)| \leq C\,|x-y|$ $(M, C > 0)$.

THEOREM 6. $\mathscr{H}_{F_n(M,C)} \underset{\cap}{\cup} \left(\frac{1}{\varepsilon}\right)^{n/2}$.

Proof. Let $f(x) \in F_n(M, C)$. To the function $f(x)$ let us assign a convex closed set $V(f) \subset E^{n+1}$ according to the following rule: $V(f) = \{x_1, \ldots, x_{n+1}): (x_1, \ldots, x_n) \in S; f(x_1, \ldots, x_n) \leq x_{n+1} \leq M\}$. By virtue of the fact that through any point of the boundary of the set $V(f)$ of the form $[x_1, \ldots, x_n, f(x_1, \ldots, x_n)]$ it is possible to construct for the set $V(f)$ a reference plane that forms with the base vector e_{n+1} an angle not exceeding arctan C, it is easy to see that

$$\|f-g\|/\sqrt{1+C^2} \leqslant \rho(V(f), V(g)) \leqslant \|f-g\|. \tag{6}$$

Here $\|f\|$ is the uniform norm of the function f.

All the sets $V(f)$ lie in an (n+1)-dimensional ball of radius $\sqrt{n+M^2}$. From (6) we can see that the number of points in a minimal ε-net of the space $F_n(M, C)$ is not larger than the number of points in a minimal $\varepsilon/2\sqrt{1+C^2}$-net of the space $\mathfrak{M}_{n+1}(\sqrt{n+M^2})$. By using Theorem 5 and Remark 1, we hence obtain an upper bound

$\mathcal{H}_{F_n(M,C)}(\varepsilon) \leqslant \left(\frac{1}{\varepsilon}\right)^{n/2}$ (the notation has been adopted from [1]).

As we can see from (6), a lower bound can be obtained by estimating the number of points in an ε-distinguishable set of the space of all sets of the form $V(f)$, where $f \in F_n(M, C)$. For this purpose it is possible to use a construction like the one used in the proof of Theorem 4. In E^{n+1} let us construct a sphere of radius R centered at the point $(0, \ldots, 0, \delta)$. The numbers R and δ can be selected in such a way that in the cube $\{|x_i| \leq 1 \ (i \leq n);\ x_{n+1} = 0\}$ part of the sphere serves as a plot of the function $f \in F_n(M, C)$. By constructing a $2\sqrt{R\varepsilon}$-distinguishable set on this part of the sphere, we can obtain (similarly to the proof of Theorem 4) an ε-distinguishable set in a space of sets of the form $V(f)$ $[f \in F_n(M, C)]$. Hence follows that $\mathcal{H}_{F_n(M,C)} \geqslant (1/\varepsilon)^{n/2}$.

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