Theorem 7.12 Proof supplement

About renormalization.

The idea is to relate the $L^2(Q)$ norm between two functions in $\mathcal{F}$ to an L^2 norm between their subgraphs. fix $f, g \in \mathcal{F}$.

$$\Rightarrow \int |f-g|^2 dQ \le \int 2F(x) \, |f(x) - g(x)| \, dQ(x)$$

note that, for every 2 real number a and b

$$|a-b| = \int | I(t<a) - I(t<b) | \, dt$$

This gives

$$\begin{aligned}
\int |f-g|^2 dQ &\le \int 2F(x) \, |f(x) - g(x)| \, dQ(x) \\
&= \int 2F(x) \left(\int | I\{t < f(x)\} - I\{t < g(x)\} | \, dt \right) dQ(x) \\
&= \iint | I_{S(f)}(x,t) - I_{S(g)}(x,t) | \, 2F(x) \, dQ(x) \, dt \\
&= \iint_{(x,t):|t| \le F(x)} | I_{S(f)}(x,t) - I_{S(g)}(x,t) | \, 2F(x) \, dQ(x) \, dt \\
&= \left\{ \iint_{(x,t):|t| \le F(x)} 2F(x) \, dQ(x) \, dt \right\} \\
&\quad \times \iint_{(x,t):|t| \le F(x)} | I_{S(f)}(x,t) - I_{S(g)}(x,t) | \frac{2F(x) \, dQ(x) \, dt}{\iint_{(x,t):|t| \le F(x)} 2F(x) \, dQ(x) \, dt} \\
&= 4 \int F^2(x) \, dQ(x) \\
&\quad \times \iint_{(x,t):|t| \le F(x)} | I_{S(f)}(x,t) - I_{S(g)}(x,t) |^2 \frac{2F(x) \, dQ(x) \, dt}{\iint_{(x,t):|t| \le F(x)} 2F(x) \, dQ(x) \, dt}
\end{aligned}$$

thus → we proof that: $\| f - g \|_{L^2(Q)} = 2\|F\|_{L^2(Q)} \, \| I_{S(g)} - I_{S(f)} \|_{L^2(Q')}$ (1)

where Q' is a probability measure on $\mathcal{X} \times \mathbb{R}$ whose density with respect to $Q \times$ Lesbesgue is proportional to

$$2F(x) \, I\{(x,t) : |t| \le F(x)\}$$

It then follows from above (1)

$$M(\epsilon \|F\|_{L^2(Q)}, \mathcal{F}, L^2(Q)) \le M(\epsilon/2, \{I_{S(f)}, f \in \mathcal{F}\}, L^2(Q'))$$

to bound RHS, we can use the boolean classes as the space $\{ \cancel{I} \; I_{S(f)}, f \in \mathcal{F}\}$ is ~~indt~~ introductional space.

$$\Rightarrow \quad \sup_{Q} M(\epsilon \|F\|_{L^2(Q)}, \mathcal{F}, L^2(Q)) \le \left(\frac{c_1}{\epsilon}\right)^{c_2 D}, \quad 0 < \epsilon \le 1$$