# PROPERTY (VRC) AND VIRTUAL FIBERING FOR AMALGAMATED FREE PRODUCTS

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Abstract. This paper focuses on studying properties of amalgamated free products $G = G_1 *_{G_0} G_2$, where the amalgamated subgroup $G_0$ is virtually cyclic.

First, we prove that if the factors $G_1$ and $G_2$ are finitely generated virtually abelian groups then $G$ can be mapped to another virtually abelian group so that this homomorphism is injective on each factor.

We then present several applications of this result. In particular, we show that if $G_1$ and $G_2$ have property (VRC) (that is, every cyclic subgroup is a virtual retract), then the same is true for $G$. We also prove that $G$ inherits some residual properties (such as residual finiteness or virtual residual solvability) from the factors $G_i$, provided $G_0$ is a virtual retract of $G_i$, for $i = 1, 2$.

Finally, we give necessary and sufficient conditions for $G$ to be (virtually) $F_m$-fibered. In particular, we fully characterize when an amalgamated product of two (finitely generated free)-by-cyclic groups over a cyclic subgroup is free-by-cyclic or virtually free-by-cyclic.

## 1. Introduction

In this paper we study free products of groups amalgamating virtually cyclic subgroups. Our main technical result is the following statement.

**Theorem 1.1.** *Let $G = G_1 *_{G_0} G_2$ be an amalgamated free product of two finitely generated virtually abelian groups $G_1, G_2$ over a common virtually cyclic subgroup $G_0$. Then there exists a finitely generated virtually abelian group $E$ and a homomorphism $\nu : G \to E$ such that the restriction of $\nu$ to $G_i$ is injective, for $i = 1, 2$.*

While the statement of Theorem 1.1 is elementary, its proof requires significant work. As the reader will see below, this theorem has plenty of applications, and we believe that it will become a part of the standard toolkit for working with amalgamated free products over virtually cyclic subgroups.

The idea is that Theorem 1.1 can be applied to amalgamated free products

$$G = G_1 *_{G_0} G_2, \quad \text{where} \tag{1.1}$$

$G_0$ is virtually cyclic and is a virtual retract of $G_i$ (which tends to happen quite often: see the discussion of property (VRC) below), for $i = 1, 2$, but $G_1$ and $G_2$ are not necessarily virtually abelian. Indeed, in this case $G_i$ will have many homomorphisms to virtually abelian groups $G_i'$ that are injective on $G_0$ (see, [MM25a, Corollary 3.8] or Lemma 2.10 below), $i = 1, 2$. By the

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universal property of amalgamated free products, these homomorphisms can then be combined to produce homomorphisms $G \to G_1' *_{G_0} G_2'$, which, in view of Theorem 1.1, give rise to homomorphisms from $G$ to virtually abelian groups $E$ that are injective on $G_0$ and "remember" much information about $G$, see Proposition 6.1. Theorem 1.1 also has structural consequences for any amalgamated product $G = G_1 *_{G_0} G_2$ as in (1.1). For example, when $|G_0| = \infty$ it implies that $G$ has a finite index normal subgroup $K$ splitting as an extension of a finite free product by $\mathbb{Z}$: see Corollary 6.3.

In the case when the factors $G_1$ and $G_2$ are abelian, the claim of Theorem 1.1 holds without requiring $G_0$ to be virtually cyclic (this follows from the construction of central products, see [MM25a, Lemma 7.4]). However, the *virtually* abelian case is more subtle, as the following example shows.

*Example* 1.2. Let $n \geq 2$, $G_0 \coloneqq \mathbb{Z}^n$ and let $x, y \in \mathrm{GL}(n, \mathbb{Z})$ be two finite order matrices such that their product has infinite order. Set $G_1 \coloneqq G_0 \rtimes \langle x \rangle$, $G_2 \coloneqq G_0 \rtimes \langle y \rangle$ and $G \coloneqq G_1 *_{G_0} G_2$. If $E$ is a group and $\varphi : G \to E$ is a homomorphism that is injective on $G_0$ then the image of $\varphi(xy)$ will act on $\varphi(G_0) \cong \mathbb{Z}^n$ as an infinite order matrix. In particular, no positive power of $\varphi(xy)$ will commute with a finite index subgroup of $\varphi(G_0)$, showing that $E$ cannot be virtually abelian.

In fact, by [MM25a, Example 1.4 and Corollary 6.7], there is a homomorphism from $G$ to a virtually abelian group that is injective on each factor if and only if $\langle x, y \rangle$ is a finite subgroup of $\mathrm{GL}(n, \mathbb{Z})$.

We will now present several applications of Theorem 1.1, starting from more basic ones and ending with our main application to virtual fibering in Subsection 1.2.

1.1. **Applications to (VRC) and residual properties.** A subgroup $H$ of a group $G$ is a *virtual retract* (denoted $H \leqslant_{vr} G$) if $H$ is a retract of a finite index subgroup $K \leqslant_f G$ (that is, there is a homomorphism $\rho : K \to H$ such that $\rho|_H = \mathrm{Id}_H$). A group $G$ has property *(VRC)* if every cyclic subgroup is a virtual retract; equivalently, $G$ has (VRC) if and only if for every $g \in G$ there is a finitely generated virtually abelian group $P$ and a homomorphism $\varphi : G \to P$ such that $\varphi$ is injective on $\langle g \rangle$ (see [MM25a, Corollary 1.3]). In a finitely generated group $G$ with property (VRC), every finitely generated virtually abelian subgroup $H$ is a virtual retract; in particular, $H$ is closed in the profinite topology and undistorted (i.e., quasi-isometrically embedded) in $G$ [Min19]. In [MM25a] it was observed that property (VRC) naturally generalizes property RFRS, introduced by Agol [Ago08].

Basic examples of groups with (VRC) are *virtually special* groups in the sense of Haglund and Wise [HW08] (that is, groups that virtually embed in right angled Artin groups), see [Min19, Corollary 1.6]. Property (VRC) is known to be preserved under commensurability, direct products, amalgamated free products and HNN-extensions over finite subgroups, graph products, and wreath products with abelian base (see [Min19]). For fundamental groups of finite graphs of groups, (VRC) was studied in [MM25a]. In particular, it is shown in [MM25a] that when the vertex groups are finitely generated and virtually abelian, property (VRC) implies that the fundamental group of the graph of groups admits a geometric action on a complete CAT(0) space.

We use Theorem 1.1 to obtain the following result, which positively answers [Min19, Question 11.5].

**Theorem 1.3.** *Property (VRC) is stable under taking amalgamated free products over virtually cyclic subgroups.*

Let us mention one application of Theorem 1.3 to tubular groups, introduced by Wise in [Wis14]. Recall that a group $G$ is said to be *tubular* if it is isomorphic to the fundamental group of a finite graph of groups $(\mathcal{G}, \Gamma)$, where all vertex groups are free abelian of rank 2 and all edge groups are infinite cyclic. Button [But17] proved that when the underlying graph $\Gamma$ is a tree, the corresponding tubular group is virtually special (and hence it has (VRC)), and in [MM25a] the authors extended this statement to graphs of non-negative Euler characteristic, provided the tubular group is *balanced* (that is, for any $g \in G$ of infinite order, if $g^k$ is conjugate to $g^l$ then $l = \pm k$).

We will say that a non-empty finite connected graph $\Gamma$ is *cycle-disjoint* if every vertex of $\Gamma$ is contained in at most one non-trivial simple cycle (note that we allow loops and multiple edges).

**Corollary 1.4.** *Let $G$ be a tubular group such that the underlying graph $\Gamma$ is cycle-disjoint. If $G$ is balanced then all of the following hold:*

*(i) $G$ has (VRC);*
*(ii) $G$ is virtually special and CAT(0);*
*(iii) $G$ is virtually (finitely generated free)-by-$\mathbb{Z}$.*

*Proof.* Any cycle-disjoint graph $\Gamma$ can be constructed starting from a single vertex by inductively joining leaf edges or cycles, with the condition that a cycle can only be attached to a vertex of degree at most 1. Thus, if $G$ is a tubular group, with the underlying graph $\Gamma$, then $G$ can be obtained by iteratively amalgamating free abelian groups of rank 2 and tubular groups corresponding to cycle graphs along infinite cyclic subgroups. According to [MM25a, Corollary 1.8], any balanced tubular group corresponding to a cycle has (VRC), hence $G$ has (VRC) by Theorem 1.3. Therefore, $G$ is virtually special, CAT(0), and virtually (finitely generated free)-by-$\mathbb{Z}$, by [MM25a, Propositions 1.9, 1.10 and 12.3]. □

A group $G$ satisfying one of the conditions (i)–(iii) from Corollary 1.4 is always balanced (cf. [MM25a, Remark 9.2]). The requirement on the structure of the graph $\Gamma$ is necessary: [MM25a, Corollary 12.4] provides many examples of balanced tubular groups $G_k$, where the underlying graph consists of one vertex and two loops at that vertex, such that $G_k$ does not satisfy any of the properties (i)–(iii).

Theorem 1.1 is also useful for studying residual properties.

**Corollary 1.5.** *Suppose that $G = G_1 *_{G_0} G_2$, where $G_0$ is virtually cyclic and $G_0 \leqslant_{vr} G_i$, for $i = 1, 2$.*

*(i) If $G_1$, $G_2$ are residually finite then so is $G$.*
*(ii) If $G_1$, $G_2$ are virtually residually solvable then so is $G$.*

Claim (i) of Corollary 1.5 probably can also be established using the methods of Evans [Eva74] and Burillo–Martino [BM06], but we are not aware of ways to prove (ii) avoiding Theorem 1.1. The assumption that $G_0$ is a virtual retract in $G_i$ is important: in [KN17] Kar and Nikolov constructed an example of a double of $\mathrm{SL}(3, \mathbb{Z}[1/2])$ over a cyclic subgroup that is not residually amenable (hence, it is neither residually finite nor virtually residually solvable [Ber16, Lemma 2.3]). Note that $\mathrm{SL}(3, \mathbb{Z}[1/2])$ is both residually finite and virtually residually solvable (in fact, it is virtually residually $p$-finite, for all but finitely many primes $p$ by Platonov's theorem [Pla68]).

Since right angled Artin groups are residually nilpotent [Dro83, Chapter 3], Corollary 1.5.(ii) implies that amalgamated free products of virtually special groups over virtually cyclic subgroups are virtually residually solvable. Note that even if both factors are finite and solvable, their amalgamated free product may not be residually solvable: see Example 13.3 below.

1.2. **Applications to virtual $F_m$-fibering.** Our main application of Theorem 1.1 is to virtual fibering results for amalgamated free products. This application is motivated by the following basic question.

**Question 1.6.** When is an amalgamated free product of two free-by-cyclic groups over a cyclic subgroup free-by-cyclic? When is it virtually free-by-cyclic?

*Convention* 1.7. In this paper by a *free-by-cyclic group* we will mean a group isomorphic to a semidirect product of a *finite rank* free group with $\mathbb{Z}$. On the other hand, groups splitting as semidirect products $F \rtimes \mathbb{Z}$, for an arbitrary free group $F$, will be called *$F$-by-$\mathbb{Z}$*,

Showing that various groups are virtually $F$-by-$\mathbb{Z}$ has received significant attention recently: see the papers of Hagen–Wise [HW10], Kielak–Linton [KL24] and Fisher [Fis25b]. However, all of these results assume that the group is virtually special or virtually RFRS, which is not sufficient to answer Question 1.6. Indeed, in [WY25] Wu and Ye produced examples of cyclic amalgamations of free-by-cyclic groups that are not virtually $F$-by-$\mathbb{Z}$. Moreover, in Example 1.14 below, we use our results to construct an amalgam of two free-by-cyclic groups over a cyclic subgroup that is virtually free-by-cyclic but is not virtually RFRS.

We give complete answers to both parts of Question 1.6 in Corollaries 1.11 and 1.16 below, and our answer to the second part (about being virtually free-by-cyclic) uses Theorem 1.1 in a crucial way. We also give new sufficient criteria in the more general case when the factors $G_1$, $G_2$ are $F$-by-$\mathbb{Z}$, see Corollary 1.17.

An investigation of Question 1.6 naturally leads to studying (virtual) fibering of amalgamated free products.

**Definition 1.8.** Given $m \in \mathbb{N}$ and a group $G$, we will say that this group *$F_m$-fibers* (or that it is *$F_m$-fibered*) if there is an non-zero homomorphism $\chi : G \to \mathbb{Z}$ such that $\ker \chi$ is of type $F_m$. In the case when $m = 1$, we will simply say that $G$ *fibers* (or that $G$ is *fibered*).

Similarly, the group $G$ *virtually $F_m$-fibers* (*virtually fibers*) if there is a finite index subgroup $H \leqslant_f G$ such that $H$ $F_m$-fibers (respectively, fibers).

*Notation* 1.9. Further in this subsection we assume that $G = G_1 *_{G_0} G_2$, where $G_0$ is virtually cyclic and $G_0 \lneqq G_i$, $i = 1, 2$.

**Proposition 1.10.** *Suppose that for some $m \in \mathbb{N}$, $G_1$ and $G_2$ are of type $F_m$. Then the amalgamated product $G$ $F_m$-fibers if and only if the following two conditions hold for each $i = 1, 2$:*

*(i) $G_i$ $F_m$-fibers;*

*(ii) the natural image of $G_0$ in the abelianization $G_i/[G_i, G_i]$ is infinite.*

In fact condition (ii) of Proposition 1.10 can be strengthened to say that $G_0$ must have non-zero image under every epimorphism $G \to \mathbb{Z}$ with finitely generated kernel (in particular, the image of $G_0$ in the abelianization of $G$ is infinite): see Proposition 10.2 below, which describes the higher discrete Bieri–Neumann–Strebel–Renz (BNSR) invariants for fundamental groups of finite graphs of groups with virtually polycyclic edge groups, extending the work of Cashen–Levitt [CL16], who did this in the case $m = 1$. The other direction (sufficiency) in Proposition 1.10, is an elementary consequence of the openness of the higher BNSR invariants.

We can now obtain an answer to the first part of Question 1.6.

**Corollary 1.11.** *Suppose $G_1$, $G_2$ are free-by-cyclic and $G_0$ is cyclic. Then $G = G_1 *_{G_0} G_2$ is free-by-cyclic if and only if $|G_0| = \infty$ and $G_0 \cap [G_i, G_i] = \{1\}$ in $G_i$, for each $i = 1, 2$.*

*Proof.* The necessity is given by Proposition 1.10, applied to the case when $m = 1$. For the sufficiency, since finite rank free groups are finitely presented, we can use the same proposition for $m = 2$, to conclude that $G$ has a finitely presented normal subgroup $N \lhd G$ such that $G/N \cong \mathbb{Z}$. Since $G_i$ are free-by-cyclic and $G_0 \cong \mathbb{Z}$, the standard properties of cohomological dimension (see [Bie81, Chapter II]) imply that $\mathrm{cd}(G) \leq 2$, so $N$ is free by a result of Bieri [Bie81, Corollary 8.6 in Chapter II]. Thus $G$ is free-by-cyclic. □

For the second part of part of Question 1.6, we need an analogue of Proposition 1.10 for virtual fibering. We say that a subgroup $A$ of a group $G$ is an *almost virtual retract*, denoted $A \leqslant_{avr} G$, if there is a finite index subgroup $H \leqslant_f A$ such that $H \leqslant_{vr} G$. The next theorem uses Notation 1.9.

**Theorem 1.12.** *Suppose that for some $m \in \mathbb{N}$, $G_1$ and $G_2$ are of type $F_m$.*

*(a) If $G$ virtually $F_m$-fibers and is not virtually abelian then all of the following must be true:*

- *$G_i$ virtually $F_m$-fibers, for $i = 1, 2$;*
- *$|G_0| = \infty$;*
- *$G_0 \leqslant_{avr} G_i$, for $i = 1, 2$ (in fact, $G_0 \leqslant_{avr} G$).*

*(b) The group $G$ virtually $F_m$-fibers provided all of the following hold:*

- *$G_i$ is $F_m$-fibered, for $i = 1, 2$;*
- *$|G_0| = \infty$;*
- *$G_0 \leqslant_{vr} G_i$, for $i = 1, 2$.*

*Remark* 1.13. The Normal Form Theorem for amalgamated free products [LS77, Theorem IV.2.6] easily implies that the group $G$ from Notation 1.9 contains a free subgroup of rank 2, provided $\max\{|G_1 : G_0|, |G_2 : G_0|\} \geq 3$. Thus, $G$ is virtually abelian if and only if $|G_i : G_0| = 2$, for $i = 1, 2$, in which case it has a finite index subgroup splitting as a direct product $G_0 \times \mathbb{Z}$. Of course such a direct product $F_m$-fibers even when $|G_0| < \infty$, so the assumption that $G$ is not virtually abelian in Theorem 1.12.(a) cannot be dropped.

Claim (a) of Theorem 1.12 is established using a "virtual" version of Proposition 10.2 (see Proposition 11.1), while claim (b) uses a new criterion for fibering of fundamental groups of graphs of groups (Theorem 10.6) and Proposition 6.1. The latter proposition is a consequence of Theorem 1.1 and implies that, under the assumptions of claim (b), $G_0$ is "visible" in the character sphere of a finite index subgroup of $G$.

Part (b) of Theorem 1.12 does not seem to have any analogues in the literature. The most powerful known virtual fibering results, due to Kielak [Kie20] and Fisher [Fis24], require that the group $G$ virtually satisfies Agol's condition RFRS from [Ago08]. But Theorem 1.12.(b) applies in many cases where $G$ is not virtually RFRS.

*Example* 1.14. Let $G_1 = H$ be Gersten's free-by-cyclic group from [Ger94]

$$H \coloneqq \langle a, b, s, t \mid [a, b] = 1,\ sbs^{-1} = ab,\ tbt^{-1} = a^2b\rangle. \tag{1.2}$$

In [WY25, Lemma 5.18] Wu and Ye showed that $\langle a\rangle \not\leqslant_{vr} H$, in particular, $G_1$ does not have (VRC) and it is not virtually RFRS (see [MM25a]).

Let $G_2 \coloneqq \langle x, y \mid x^p = y^q\rangle$ be the fundamental group of the complement of a torus knot/link, for $|p|, |q| \geq 2$. Then $G_2$ is an amalgamated free product of two infinite cyclic groups $\langle x\rangle$ and $\langle y\rangle$, so it is fibered and has (VRC) (e.g., by Proposition 1.10 and Theorem 1.3; of course, in this case both statements were known previously). Finally, consider the amalgamated free product

$$G \coloneqq G_1 *_{\langle b\rangle = \langle [x,y]\rangle} G_2.$$

Clearly, $\langle b\rangle$ is a retract of $G_1$ (under the map sending $a, s, t$ to 1) and $\langle [x, y]\rangle \leqslant_{vr} G_2$, as $G_2$ has (VRC). Therefore, $G$ is virtually fibered by Theorem 1.12.(b) (in fact, it is virtually free-by-cyclic, by Corollary 1.16 below). However, $G$ is not fibered by Proposition 1.10 (because $[x, y] \in [G_2, G_2]$) and it is not virtually RFRS (because $G_1$ is not).

**Corollary 1.15.** *Suppose that for some $m \in \mathbb{N}$, $G_1$ and $G_2$ are $F_m$-fibered and have (VRC). Then the amalgamated free product $G$, from Notation 1.9, is virtually $F_m$-fibered and has (VRC).*

The reader will notice that there is a gap between the necessary conditions for virtual fibering of $G$, provided by part (a) of Theorem 1.12, and the sufficient conditions, given by part (b). One difference is that part (a) only concludes that $G_i$ is *virtually* $F_m$-fibered, for $i = 1, 2$, while part (b) requires both factors to be $F_m$-fibered on the nose. This disparity is the subject of Question 13.6. The second difference is that part (a) implies that $G_0$ is an *almost* virtual retract of $G_i$ (and of $G$), $i = 1, 2$, while part (b) requires it to be a virtual retract. Examples 13.8 and 13.9 show that this second difference is essential; however, it can only occur if at least one of the factors $G_i$ is not residually finite and $G_0$ contains non-trivial finite normal subgroups (see Lemma 2.12).

The next corollary answers the second part of Question 1.6.

**Corollary 1.16.** *Using Notation 1.9, assume that $G_1$, $G_2$ are free-by-cyclic groups and $G_0$ is cyclic. Then $G$ is virtually free-by-cyclic if and only if $G_0$ is infinite and $G_0 \leqslant_{vr} G_i$, for $i = 1, 2$.*

*Proof.* If $G$ is virtually free-by-cyclic then it virtually fibers. Hence, Theorem 1.12.(a) tells us that $|G_0| = \infty$ and $G_0 \leqslant_{avr} G_i$, for $i = 1, 2$. It follows that $G_0 \cong \mathbb{Z}$, so $G_0 \leqslant_{vr} G_i$, $i = 1, 2$, by Lemma 2.12.

Conversely, if $|G_0| = \infty$ and $G_0 \leqslant_{vr} G_i$, for $i = 1, 2$, then according to Theorem 1.12.(b), there is $K \leqslant_f G$ and a finitely presented normal subgroup $N \lhd K$ with $K/N \cong \mathbb{Z}$. The argument from the proof of Corollary 1.11 (using cohomological dimension) shows that $N$ must be free, so $G$ is virtually free-by-cyclic. □

A non-Euclidean Baumslag–Solitar group

$$BS(k,l) = \langle a, t \mid ta^k t^{-1} = a^l \rangle, \ k, l \in \mathbb{Z} \setminus \{0\}, \ |k| \neq |l|,$$

is never virtually free-by-cyclic. This shows that neither Corollary 1.16 nor part (b) of Theorem 1.12 can be easily extended to HNN-extensions (see also Example 13.10 for a more sophisticated construction, where the edge groups are all retracts of the resulting group).

Although so far we have restricted ourselves to considering free-by-cyclic groups, where the free normal subgroup is finitely generated, by combining our results with recent work of Fisher [Fis25a], we can extend the sufficiency implications in Corollaries 1.11, 1.16 to finitely generated $F$-by-$\mathbb{Z}$ groups.

**Corollary 1.17.** *Suppose that $G = G_1 *_{G_0} G_2$, where $G_1, G_2$ are finitely generated $F$-by-$\mathbb{Z}$ groups and $G_0$ is infinite cyclic.*

*(i) If $G_0 \cap [G_i, G_i] = \{1\}$, for each $i = 1, 2$, then $G$ is $F$-by-$\mathbb{Z}$.*
*(ii) If $G_0 \leqslant_{vr} G_i$, for each $i = 1, 2$, then $G$ is virtually $F$-by-$\mathbb{Z}$.*

Unlike Corollaries 1.11 and 1.16, the conditions listed in parts (i) and (ii) of Corollary 1.17 are not necessary. For example, the converse of (i) fails for the fundamental group $G$ of a closed orientable surface of genus 2, and the converse of (ii) fails for $G \coloneqq H *_{\langle a \rangle = \langle x \rangle} F_2$, where $H$ is Gersten's group (1.2) and $F_2$ is the free group with free basis $\{x, y\}$.

## 1.3. Outline of the paper.

In Section 2 we give the necessary background on the profinite topology, virtual retractions, and graphs of groups. Sections 3–5 concern the proof of Theorem 1.1 (see Subsection 1.4 for the idea behind this proof). In Section 6 we deduce from Theorem 1.1 the key Proposition 6.1, which then allows us to establish Theorem 1.3. In Section 7 we use Proposition 6.1 to prove Corollary 1.5. Section 8 discusses necessary and sufficient criteria for a normal subgroup of the fundamental group of a graph of groups to be finitely generated and to satisfy higher finiteness properties. In Section 9 we collect the necessary background on the Bieri–Neumann–Strebel–Renz invariants that play a central role in our fibering and virtual fibering results. In Section 10 we prove Proposition 10.2, characterizing when the kernel of a discrete character $\chi : G \to \mathbb{R}$ is of type $F_m$, where $G$ is the fundamental group of a finite graph of groups with virtually polycyclic edge groups. We then use it to prove Proposition 1.10 and to give a new fibering criterion (Theorem 10.6). This criterion allows us to establish the results about virtual fibering in Section 11, where it is combined with Proposition 6.1 to prove Theorem 1.12. In Section 12, we show that our argument for part (b) of Theorem 1.12 actually works for more general "rationally open" invariants and deduce Corollary 1.17. Finally, in Section 13 we discuss natural open problems and give several examples demonstrating the sharpness of our results.

1.4. **Idea of the proof of Theorem 1.1.** Let us briefly describe the main steps in the proof of Theorem 1.1 in the harder case when $|G_0| = \infty$. First, for $i = 0, 1, 2$, we embed each group $G_i$ as a finite index subgroup in a larger virtually abelian group $P_i$, in such a way that

(1) the original embeddings $G_0 \hookrightarrow G_i$ extend to embeddings $P_0 \hookrightarrow P_i$, $i = 1, 2$;
(2) $P_0$ splits as a semidirect product $B_0 \rtimes Q$, where $B_0 \cong \mathbb{Z}$ and $|Q| < \infty$;
(3) $P_i$ has a finite index free abelian subgroup $B_i \lhd P_i$ such that $P_0 \cap B_i = B_0$ in $P_i$, for $i = 1, 2$.

To achieve this, in Section 3 we introduce a category $\mathcal{C}$, of group pairs, and embedding functors $\mathcal{F}_n : \mathcal{C} \to \mathcal{C}$, $n \in \mathbb{N}$. An object $(G, A) \in \mathcal{C}$ consists of a group $G$ and a torsion-free normal abelian subgroup $A \lhd G$. For each $n \in \mathbb{N}$, $\mathcal{F}_n((G, A)) = (P, B)$, where $B \cong A$, $P/B \cong G/A$ and there is a monomorphism $G \to P$, sending $A$ inside $B$. And if $n$ is divisible by $|G/A| = |P/B|$ then $P$ splits as a semidirect product $B \rtimes (P/B)$.

Property (1) allows us to embed the amalgamated free product $G_1 *_{G_0} G_2$ into the amalgamated free product $P_1 *_{P_0} P_2$, which has an easier structure. In Section 4 we simplify this structure further by embedding $P_1$ and $P_2$ into permutational wreath products $E_1 := \mathbb{Z} \wr_{\Omega_k} S_k$ and $E_2 := \mathbb{Z} \wr_{\Omega_l} S_l$, respectively, where $k, l \in \mathbb{N}$ and $S_k$ is the symmetric group of $\Omega_k := \{1, \dots, k\}$. We ensure that the images of $B_0$ (see (2)) under these embeddings are in the base groups $\mathbb{Z}^{\Omega_k}$ and $\mathbb{Z}^{\Omega_l}$, and the images of $Q$ are contained in the symmetric groups $S_k$ and $S_l$ and act freely on $\Omega_k$ and $\Omega_l$, respectively.

In Section 5, we first construct monomorphisms from $E_1$ and $E_2$ to $E := \mathbb{Z} \wr_{\Omega_{kl}} S_{kl}$, which agree on the subgroup $B_0 \lhd_f P_0$, and then modify the first of these monomorphisms (by composing it with an inner automorphism of $S_k$) to ensure that they also agree on $Q$. Since $P_0 = B_0 \rtimes Q$, this gives rise to a homomorphism $E_1 *_{P_0} E_2 \to E$, which we then use to produce the desired homomorphism $\nu : G_1 *_{G_0} G_2 \to E$ that is injective on $G_i$, for $i = 1, 2$.

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## 2. Preliminaries

2.1. **Profinite topology and separability.** Recall that the *profinite topology* on a group $G$ is the topology whose basic open sets are cosets of finite index subgroups in $G$. A subset of $G$ is said to be *separable* if it is closed in the profinite topology.

**Definition 2.1.** A group $G$ is said to be

- *residually finite* if $\{1\}$ is separable in $G$;
- *cyclic subgroup separable* if every cyclic subgroup is separable in $G$;
- *LERF* if every finitely generated subgroup is separable in $G$.

Clearly, LERF-ness implies cyclic subgroup separability and cyclic subgroup separability implies residual finiteness. The following facts are well-known.

*Remark* 2.2. Let $G$ be a group.

- A subgroup $H \leqslant G$ is separable if and only if $H$ is the intersection of a collection of finite index subgroups in $G$.
- A normal subgroup $N \lhd G$ is separable if and only if $G/N$ is residually finite.
- If $G$ is finitely generated and virtually abelian then every subgroup of $G$ is separable.

**Definition 2.3.** The *finite residual* $R(G)$, of a group $G$, is the intersection of all finite index subgroups of $G$.

The following basic fact about the finite residual is easy to verify.

*Remark* 2.4. If $G$ is a group then $R(G)$ is the smallest normal subgroup of $G$ such that $G/R(G)$ is residually finite. In other words, $G/R(G)$ is residually finite and if $G/N$ is residually finite, for $N \lhd G$, then $R(G) \subseteq N$.

**Lemma 2.5.** *If $G$ is a group and $L \leqslant G$ then $R(L) \subseteq R(G)$. Moreover, if $|G : L| < \infty$ then $R(L) = R(G)$.*

*Proof.* The first claim follows from the fact that $|L : (L \cap K)| < \infty$, for every $K \leqslant_f G$. And if $|G : L| < \infty$ then every finite index subgroup of $L$ has finite index in $G$, so we have $R(G) \subseteq R(L)$, and the second claim holds. □

2.2. **Virtual retractions.** Recall that a subgroup $H$ is a virtual retract of a group $G$ ($H \leqslant_{vr} G$) if there is $K \leqslant_f G$ such that $H \subseteq K$ and there is a retraction $\rho : K \to H$. Equivalently, $H \leqslant_{vr} G$ if there is a subgroup $N \leqslant G$ such that $N$ is normalized by $H$, $H \cap N = \{1\}$ and $|G : HN| < \infty$.

**Lemma 2.6** ([Min19, Lemma 3.2])**.** *Let $G$, $A$ be groups, let $H \leqslant K \leqslant G$ be subgroups of $G$, and let $\psi : G \to A$ be a group homomorphism.*

*(a) If $\psi(H) \leqslant_{vr} A$ and $\psi$ is injective on $H$ then $H \leqslant_{vr} G$.*
*(b) If $H \leqslant_{vr} K$ and $K \leqslant_{vr} G$ then $H \leqslant_{vr} G$. In particular, if $K \leqslant_{vr} G$ and $H \leqslant_f K$ then $H \leqslant_{vr} G$.*
*(c) If $H \leqslant_{vr} G$ then $H \leqslant_{vr} K$.*

**Lemma 2.7** ([Min19, Proposition 1.5 and Lemma 5.1.(ii)])**.** *If $G$ has (VRC) then every finitely generated virtually abelian subgroup is a virtual retract and is separable in $G$. In particular, $G$ is cyclic subgroup separable.*

**Lemma 2.8** ([Min19, Corollary 4.3])**.** *Every subgroup of a finitely generated virtually abelian group is a virtual retract.*

The following statement is similar to [Min19, Lemma 4.1].

**Lemma 2.9.** *Let $G$ be a group with a finite index subgroup $K \leqslant_f G$. Suppose that $N \lhd K$ is such that $K/N$ is finitely generated and virtually abelian. Then there exists a finitely generated virtually abelian group $A$ and a homomorphism $\varphi : G \to A$ such that $\ker\varphi \subseteq N$.*

*Proof.* Let $L \lhd_f G$ be the intersection of all conjugates of $K$ in $G$ and set $M := N \cap L \lhd L$. Then there are only finitely many different conjugates of $M$ in $G$: $M_1 = M, M_2, \dots, M_k \lhd L$, and

$$L/M_i \cong L/M \leqslant_f K/N$$

is finitely generated and virtually abelian, for all $i = 1, \dots, k$. Observe that the subgroup $O := \bigcap_{i=1}^k M_i$ is normal in $G$ and we have a diagonal embedding $L/O \hookrightarrow \bigtimes_{i=1}^k L/M_i$. Therefore, $L/O$ is finitely generated and virtually abelian. By construction, $L/O \lhd_f G/O$ so $A := G/O$ is finitely generated and virtually abelian. It remains to define $\varphi : G \to G/O$ as the quotient map, and to note that $\ker\varphi = O \subseteq M \subseteq N$. □

**Lemma 2.10.** *Suppose that $G$ is a group, $K_i \leqslant_f G$, $N_i \lhd K_i$, $i = 1, \dots, m$, and $H_j \leqslant_{vr} G$, $j = 1, \dots, n$. If the groups $K_i/N_i$, $i = 1, \dots, m$, and $H_j$, $j = 1, \dots, n$, are finitely generated and virtually abelian then there exist a finitely generated virtually abelian group $P$ and a homomorphism $\varphi : G \to P$ such that $\ker\varphi \subseteq N_i$, for $i = 1, \dots, m$, and $\varphi$ is injective on $H_j$, for each $j = 1, \dots, n$.*

*Proof.* By the assumptions, for each $j = 1, \dots, n$ there exist $L_j \leqslant_f G$ and $M_j \lhd L_j$ such that $H_j \subseteq L_j$, $H_j \cap M_j = \{1\}$ and $L_j = H_j M_j$. Now, Lemma 2.9 gives finitely generated virtually abelian groups $A_1, \dots, A_m$ and $B_1, \dots, B_n$, together with homomorphisms $\varphi_i : G \to A_i$ and $\psi_j : G \to B_j$, such that $\ker\varphi_i \subseteq N_i$ and $\ker\psi_j \subseteq M_j$, for all $i = 1, \dots, m$ and $j = 1, \dots, n$.

Evidently, the group

$$P := A_1 \times \dots \times A_m \times B_1 \times \dots \times B_n$$

is finitely generated and virtually abelian, and the diagonal homomorphism

$$\varphi := \varphi_1 \times \dots \times \varphi_m \times \psi_1 \times \dots \times \psi_n : G \to P$$

satisfies $\ker\varphi \subseteq \bigcap_{i=1}^m N_i \cap \bigcap_{j=1}^n M_j$. It follows that $\varphi$ enjoys the desired properties. □

**Definition 2.11.** A subgroup $A$ of a group $G$ will be called an *almost virtual retract* if there is a finite index subgroup $H \leqslant_f A$ such that $H \leqslant_{vr} G$. In this case we shall write $A \leqslant_{avr} G$.

For example, every finite subgroup of a group $G$ is always an almost virtual retract of $G$.

An almost virtual retract $A$ of a group $G$ need not be a virtual retract of $G$ in general (see [Min19, Examples 3.5 and 3.6]), but this is true in the case when $G$ is residually finite and $A$ is finitely generated virtually abelian [Min19, Theorem 1.4]. This statement can be generalized as follows.

**Lemma 2.12.** *Let $G$ be a group with a finitely generated virtually abelian subgroup $A \leqslant_{avr} G$, and let $R \lhd G$ be the finite residual of $G$. Suppose that $H \leqslant_f A$ satisfies $H \leqslant_{vr} G$. Then $H \cap R = \{1\}$ and $|A \cap R| < \infty$. Moreover, if $A \cap R = \{1\}$ then $A \leqslant_{vr} G$. In particular, $A \leqslant_{vr} G$ provided $G$ is residually finite or $A$ contains no non-trivial finite normal subgroups.*

*Proof.* By the assumptions, there exist $K \leqslant_f G$ and $N \lhd K$, such that $H \subseteq K$, $K = NH$ and $N \cap H = \{1\}$.

Since $K/N \cong H$ is residually finite, being a finitely generated virtually abelian group, $R(K) \subseteq N$ by Remark 2.4, so $R = R(G) \subseteq N$ by Lemma 2.5. It follows that $H \cap R = \{1\}$, and so $|A \cap R| \le |A : H| < \infty$.

Now, assume that $A \cap R = \{1\}$. Then the quotient map $\psi : G \to G/R$ is injective on $A$. Moreover, since $R \subseteq N$ and $N \cap H = \{1\}$ in $G$, we have $\psi(N) \cap \psi(H) = \{1\}$ in $G/R$, which implies that $\psi(H)$ is still a retract of $\psi(K) \leqslant_f G/R$, thus $\psi(A) \leqslant_{avr} G/R$.

According to Remark 2.4, $G/R$ is residually finite. Since $\psi(A) \cong A$ is finitely generated and virtually abelian, we can apply [Min19, Theorem 1.4] to deduce that $\psi(A) \leqslant_{vr} G/R$, hence $A \leqslant_{vr} G$ by Lemma 2.6.(a). □

**Lemma 2.13.** *Let $A$ be an infinite virtually cyclic subgroup of a finitely generated group $G$. Then the following are equivalent:*

*(i)* $A \leqslant_{avr} G$;

*(ii) for some (equivalently, for every) infinite cyclic subgroup $C \leqslant A$ one has $C \leqslant_{vr} G$;*

*(iii) there is a homomorphism $\varphi : G \to P$, where $P$ is finitely generated and virtually abelian, such that $|\ker \varphi \cap A| < \infty$ (equivalently, $\varphi(A)$ is infinite);*

*(iv) there exist a finite index subgroup $K \leqslant_f G$ and a homomorphism $\psi : K \to \mathbb{R}$ such that $\psi(K \cap A) \neq \{0\}$.*

*Proof.* If (i) holds, then there is a finite index subgroup $A' \leqslant_f A$ such that $A' \leqslant_{vr} G$. Since $A$ is infinite virtually cyclic, there is an infinite cyclic subgroup $C \leqslant_f A'$. Therefore, $C \leqslant_{vr} G$ by Lemma 2.6.(b). Now, for any other infinite cyclic subgroup $C' \leqslant A$ we have $C' \cap C \leqslant_f C$, so $C' \cap C \leqslant_{vr} G$, again by Lemma 2.6.(b). On the other hand, $C' \cap C$ has finite index in $C'$ and $C'$ is torsion-free, hence $C' \leqslant_{vr} G$, by Lemma 2.12. Thus, (ii) holds.

Now, suppose that (ii) is satisfied. Then, according to Lemma 2.10, there exist a finitely generated virtually abelian group $P$ and a homomorphism $\varphi : G \to P$ that is injective on $C$. Since $|A : C| < \infty$, we see that the intersection $\ker \varphi \cap A$ must be finite and $|\varphi(A)| = \infty$, i.e., (iii) holds.

If (iii) is true then choose a finitely generated free abelian finite index subgroup $L \leqslant_f P$ and set $K \coloneqq \varphi^{-1}(L) \leqslant_f G$. Since every finitely generated free abelian group embeds in $\mathbb{R}$, the restriction of $\varphi$ to $K$ gives rise to a homomorphism $\psi : K \to \mathbb{R}$ such that $|\ker \psi \cap A| < \infty$. Since $K \cap A$ is infinite, we can conclude that $\psi(K \cap A) \neq \{0\}$, thus (iv) is satisfied.

Finally, assume that (iv) holds. Since $G$ is finitely generated, so is its image in $\mathbb{R}$, thus $\psi(G) \cong \mathbb{Z}^n$, for some $n \in \mathbb{N}_0$. It follows that every subgroup of $\psi(G)$ is a virtual retract (see Lemma 2.8). Choose any element $c \in K \cap A$ with $\psi(c) \neq 0$. Then $c$ has infinite order in $G$, $\psi(\langle c \rangle) \leqslant_{vr} \psi(G)$ and $\psi$ is injective on $\langle c \rangle$. Therefore, $\langle c \rangle \leqslant_{vr} G$ by Lemma 2.6.(a). Since $A$

is virtually cyclic, we know that $|A : \langle c \rangle| < \infty$, thus $A \leqslant_{avr} G$. This shows that (iv) implies (i), so the proof is complete. □

2.3. **Graphs of groups.** In this paper we adopt Serre's notation for graphs. Namely, a graph $\Gamma$ is a 5-tuple $(V\Gamma, E\Gamma, \alpha, \omega, ^{-})$, where $V\Gamma$ and $E\Gamma$ are sets of *vertices* and *edges* of $\Gamma$, respectively, $\alpha, \omega : E\Gamma \to V\Gamma$ are the *incidence maps*, and $^{-} : E\Gamma \to E\Gamma$ is an involution sending each edge to its inverse.

We also use Serre's notation for graphs of groups $(\mathcal{G}, \Gamma)$, where $\Gamma$ is a connected graph as above and $\mathcal{G}$ is the data consisting of *vertex groups* $\{G_v\}_{v \in V\Gamma}$, *edge groups* $\{G_e\}_{e \in E\Gamma}$ and monomorphisms $\{\alpha_e\}_{e \in E\Gamma}$, such that $G_e = G_{\overline{e}}$ and $\alpha_e \colon G_e \to G_{\alpha(e)}$, for all $e \in E\Gamma$ (see [MM25a, Subsection 2.2] for more details). The Structure Theorem of Bass-Serre Theory [Ser80, Theorem 13 in Section I.5.4] gives the following statement.

**Theorem 2.14.** *Let $G$ be the fundamental group of a graph of groups $(\mathcal{G}, \Gamma)$ and let $H$ be a subgroup of $G$. Then $H$ is isomorphic to the fundamental group of a new graph of groups $(\mathcal{H}, \Delta)$, with vertex/edge groups equal to the intersections of $H$ with $G$-conjugates of the vertex/edge groups of $(\mathcal{G}, \Gamma)$. Moreover, if $\Gamma$ is finite and $|G : H| < \infty$ then $\Delta$ is also finite.*

More precisely, fix a maximal tree $T$ and an orientation on the edges $E\Gamma = E\Gamma^+ \sqcup E\Gamma^-$ in $\Gamma$. As in [Ser80, Section I.5.3], using this data we can construct the *Bass-Serre tree* $\mathcal{T}$ such that the fundamental group $G \coloneqq \pi_1(\mathcal{G}, \Gamma, T, E\Gamma^+)$ acts on $\mathcal{T}$ without inverting any edges, with the quotient $G \setminus \mathcal{T} \cong \Gamma$ and with the stabilizers of vertices and edges being precisely the conjugates of $G_v$ and $\alpha_e(G_e)$ in $G$, for $v \in V\Gamma$ and $e \in E\Gamma$.

This induces an action of $H$ on $\mathcal{T}$ with quotient graph $\Delta$, which, following [Ser80, Section I.5.4], gives rise to a splitting of $H$ as the fundamental group of a graphs of groups $(\mathcal{H}, \Delta)$, whose vertex/edge groups are chosen as $H$-conjugacy class representatives of the intersections of $H$ with the $G$-stabilizers of vertices/edges in $\mathcal{T}$. Moreover, [Ser80, Theorem 13 in Section I.5.4] tells us that we can choose a maximal tree in $\Delta$ and an orientation on the edges of $\Delta$ in such a way that the Bass-Serre tree associated to $(\mathcal{H}, \Delta)$ is naturally isomorphic to $\mathcal{T}$.

## 3. Splitting virtually abelian groups in finite index supergroups

This section develops an auxiliary tool allowing us to embed a virtually abelian group into a semidirect product of a torsion-free abelian group with a finite group, in a functorial manner. The embedding itself (see Definition 3.3) and the fact that the target group always splits (Proposition 3.6) are folklore; the authors are indebted to P. Kropholler for pointing these out to them.

**Definition 3.1.** Consider a category $\mathcal{C}$, whose objects are pairs $(G, A)$, where $G$ is a group and $A \lhd G$ is a torsion-free abelian normal subgroup of $G$. A morphism $\varphi : (G_1, A_1) \to (G_2, A_2)$ in this category is a group homomorphism $\varphi : G_1 \to G_2$ such that $A_1 = \varphi^{-1}(A_2)$. We say that this morphism is *injective* (*surjective*) if the group homomorphism $\varphi : G_1 \to G_2$ is injective (respectively, surjective).

*Remark* 3.2. The condition that $\varphi^{-1}(A_2) = A_1$ from Definition 3.1 can be restated by saying that we have a commutative diagram of the form

$$\begin{array}{ccccccccc} \{1\} & \longrightarrow & A_1 & \hookrightarrow & G_1 & \twoheadrightarrow & Q_1 & \longrightarrow & \{1\} \\ & & \downarrow & & \downarrow{\scriptstyle\varphi} & & \downarrow & & \\ \{1\} & \longrightarrow & A_2 & \hookrightarrow & G_2 & \twoheadrightarrow & Q_2 & \longrightarrow & \{1\} \end{array},$$

where the rows in this diagram are exact sequences and the right-most vertical map is an injective homomorphism from $Q_1 \coloneqq G_1/A_1$ to $Q_2 \coloneqq G_2/A_2$.

If we fix a group $Q$ and consider only objects $(G, A) \in \mathcal{C}$ such that $G/A \cong Q$ then we get a full subcategory of $\mathcal{C}$ which appears as $\left(\frac{Q}{}\right)$ in Gruenberg's notes [Gru70, Section 9.1]

If $(G, A) \in \mathcal{C}$ then $G$ acts on $A$ by conjugation, which allows us to consider the semidirect product $A \rtimes G = \{(a, g) \mid a \in A,\ g \in G\}$, where multiplication is defined by $(a, g)(b, h) = (a(gbg^{-1}), gh)$, for all $(a, g), (b, h) \in A \rtimes G$.

**Definition 3.3.** Given any pair $(G, A) \in \mathcal{C}$ and any $n \in \mathbb{N}$ we define the group $P_n = P_n(G, A)$ as the quotient of $A \rtimes G$ by the normal subgroup

$$N_n = \{(a^n, a^{-1}) \mid a \in A\}.$$

We let $\psi_n : A \rtimes G \to P_n$ denote the natural quotient map and define $B_n = B_n(G, A)$ by $B_n \coloneqq \psi_n((A, 1)) \lhd P_n$, where $(A, 1) = \{(a, 1) \mid a \in A\} \lhd A \rtimes G$. We will also denote by $\xi_n : G \to P_n$ the induced map $\xi_n(g) \coloneqq \psi_n((1, g))$, $g \in G$.

One readily verifies that $N_n$ is indeed a normal subgroup of $A \rtimes G$ because $A \lhd G$ is abelian. Moreover, since $A$ is torsion-free, the homomorphism $\psi_n$ is injective on the copies $(A, 1)$ and $(1, G) = \{(1, g) \mid g \in G\}$ of $A$ and $G$ in $A \rtimes G$. Intuitively, $P_n$ is obtained from $G$ by adding $n$-th roots to all elements of $A$, because $(a, 1)^n = (a^n, 1)$ is identified with $(1, a)$ in $P_n$, so that $\xi_n(A) = (B_n)^n$. In particular, when $A \cong \mathbb{Z}^m$, for some $m \in \mathbb{N} \cup \{0\}$, then $\xi_n(G) \cong G$ has index $n^m$ in $P_n$.

**Lemma 3.4.** *For each $n \in \mathbb{N}$ and every $(G, A) \in \mathcal{C}$, the pair $(P_n, B_n)$ belongs to $\mathcal{C}$ and $\xi_n : G \to P_n$ defines an injective morphism between $(G, A)$ and $(P_n, B_n)$. Moreover, $B_n \cong A$ and $\xi_n$ induces a natural isomorphism between the quotients $G/A$ and $P_n/B_n$.*

*Proof.* We have already noted that $\xi_n$ is injective on $G$. Since $(A, 1) \cap N_n = \{(1, 1)\}$ in $A \rtimes G$, we have

$$B_n \cong (A, 1)/((A, 1) \cap N_n) = (A, 1) \cong A,$$

so $B_n$ is a torsion-free abelian normal subgroup of $P_n$, thus $(P_n, B_n) \in \mathcal{C}$.

Observe that $\xi_n(g) \in B_n$ in $P_n$ if and only if $(1, g) \in (A, 1)N_n = (A, A)$ in $A \rtimes G$, which is equivalent to $g \in A$. Thus $\xi_n^{-1}(B_n) = A$ in $G$, so $\xi_n$ gives rise to a morphism between $(G, A)$ and $(P_n, B_n)$ in $\mathcal{C}$.

The final claim of the lemma follows from the second isomorphism theorem:

$$\begin{aligned} P_n/B_n &\cong (A \rtimes G)/\big((A, 1)N_n\big) = \big((A, A)(1, G)\big)/(A, A) \\ &\cong (1, G)/(1, A) \cong G/A. \end{aligned}$$

$\square$

For every $n \in \mathbb{N}$, Lemma 3.4 allows us to define a map $\mathcal{F}_n : \mathcal{C} \to \mathcal{C}$, by $\mathcal{F}_n((G,A)) \coloneqq (P_n, B_n)$, given in Definition 3.3.

*Remark* 3.5. In view of Lemma 3.7 below, it is not difficult to see that for every $n \in \mathbb{N}$ the map $\mathcal{F}_n : \mathcal{C} \to \mathcal{C}$ defines a covariant functor from the category $\mathcal{C}$ to itself. We will not need this fact here, so we leave its verification to the reader.

If $Q = G/A$, the previous lemma implies that $P_n$ fits into the commutative diagram (3.1), where $\theta_n : A \to B_n$ is given by

$$\theta_n(a) \coloneqq \psi_n((1,a)) = \psi_n((a^n,1)), \quad \text{for all } a \in A.$$

$$\begin{array}{ccccccccc} \{1\} & \longrightarrow & A & \hookrightarrow & G & \twoheadrightarrow & Q & \longrightarrow & \{1\} \\ & & \Big\downarrow{\scriptstyle \theta_n} & & \Big\downarrow{\scriptstyle \xi_n} & & \Big\| & & \\ \{1\} & \longrightarrow & B_n & \hookrightarrow & P_n & \twoheadrightarrow & Q & \longrightarrow & \{1\} \end{array} \tag{3.1}$$

Since $A$ is abelian, the conjugation action of $G$ on $A$ gives rise to an action of $Q$ on $A$. The action of $Q$ on $B_n$ is inherited from it, i.e.,

$$q.\psi_n((a,1)) \coloneqq \psi_n((q.a,1)), \quad \text{for all } q \in Q \text{ and } a \in A.$$

In particular, $\theta_n$ is a monomorphism of $Q$-modules.

The main reason why we need the construction from Definition 3.3 is the following fact.

**Proposition 3.6.** *Suppose that $(G,A) \in \mathcal{C}$, $Q = G/A$ is finite and $n \in \mathbb{N}$ is divisible by the order $|Q|$. If $(P_n, B_n) = \mathcal{F}_n((G,A)) \in \mathcal{C}$, then $P_n$ splits as the semidirect product $B_n \rtimes Q$, where the action of $Q$ on $B_n$ is inherited from the action of $Q$ on $A$ in $G$.*

*Proof.* Let $f : Q \times Q \to A$ be a 2-cocycle determining the extension

$$A \to G \to Q.$$

In view of Lemma 3.4 and diagram (3.1), the composition $\overline{f} = \theta_n \circ f : Q \times Q \to B_n$ is a 2-cocycle, determining the extension

$$B_n \to P_n \to Q.$$

By construction, $\frac{1}{n}\theta_n(A) \subseteq B_n$ (we are using additive notation on $A$ and $B_n$ as we treat them as $Q$-modules here), so $h = \frac{1}{n}\bar{f} : Q \times Q \to B_n$ is also a 2-cocycle. But multiplication by $n$ annihilates the cohomology group $H^2(Q, B_n)$ because $|Q|$ divides $n$ (see [Bro82, Corollary III.10.2]), so $\bar{f} = nh$ is a coboundary, hence the extension $B_n \to P_n \to Q$ splits and $P_n \cong B_n \rtimes Q$. $\square$

**Lemma 3.7.** *Given $(G_1, A_1), (G_2, A_2) \in \mathcal{C}$ and $n \in \mathbb{N}$, any morphism $\varphi : (G_1, A_1) \to (G_2, A_2)$ defines a morphism $\tilde{\varphi} : (P_{n,1}, B_{n,1}) \to (P_{n,2}, B_{n,2})$ such that the corresponding group homomorphisms fit into the following commutative diagram (here $P_{n,i} = P_n(G_i, A_i)$, $B_{n,i} = B_n(G_i, A_i)$ and $\xi_{n,i} : G_i \to P_{n,i}$, $i = 1, 2$, are given by Definition 3.3):*

$$\begin{array}{ccc} G_1 & \xrightarrow{\varphi} & G_2 \\ \Big\downarrow{\scriptstyle \xi_{n,1}} & & \Big\downarrow{\scriptstyle \xi_{n,2}} \\ P_{n,1} & \xrightarrow{\tilde{\varphi}} & P_{n,2} \end{array} . \tag{3.2}$$

*Moreover, if $\varphi$ is injective (respectively, surjective), then so is $\tilde{\varphi}$.*

*Proof.* Since $\varphi(A_1) \subseteq A_2$, we can define the map

$$\hat{\varphi} : A_1 \rtimes G_1 \to A_2 \rtimes G_2,\ (a, g) \overset{\hat{\varphi}}{\mapsto} (\varphi(a), \varphi(g)), \tag{3.3}$$

that fits into the commutative diagram:

$$\begin{array}{ccc} G_1 & \xrightarrow{\varphi} & G_2 \\ \downarrow & & \downarrow \\ A_1 \rtimes G_1 & \xrightarrow{\hat{\varphi}} & A_2 \rtimes G_2 \end{array}, \tag{3.4}$$

where the vertical maps are the natural embeddings $g \mapsto (1, g)$, of $G_i$ into $A_i \rtimes G_i$, $i = 1, 2$.

Let $N_{n,i} \lhd A_i \rtimes G_i$ denote the kernel of the epimorphism $\psi_{n,i} : A_i \rtimes G_i \to P_{n,i}$, $i = 1, 2$, given by Definition 3.3. Note that $\hat{\varphi}(N_{n,1}) \subseteq N_{n,2}$ by (3.3), hence $\hat{\varphi}$ factors through the quotients, giving the commutative diagram

$$\begin{array}{ccc} A_1 \rtimes G_1 & \xrightarrow{\hat{\varphi}} & A_2 \rtimes G_2 \\ \downarrow{\scriptstyle \psi_{n,1}} & & \downarrow{\scriptstyle \psi_{n,2}} \\ P_{n,1} & \xrightarrow{\tilde{\varphi}} & P_{n,2} \end{array}, \tag{3.5}$$

where $\tilde{\varphi} : P_{n,1} \to P_{n,2}$ is the induced homomorphism. Thus, (3.2) follows by combining (3.4) with (3.5).

To check that $\tilde{\varphi}$ defines a morphism between the objects $(P_{n,1}, B_{n,1})$ and $(P_{n,2}, B_{n,2})$ in $\mathcal{C}$, we need to verify that $B_{n,1} = \tilde{\varphi}^{-1}(B_{n,2})$. The inclusion $B_{n,1} \subseteq \tilde{\varphi}^{-1}(B_{n,2})$ is clear from (3.5) because $B_{n,i} = \psi_{n,i}((A_i, 1))$, $i = 1, 2$, and $\hat{\varphi}((A_1, 1)) \subseteq (A_2, 1)$, as $\varphi$ is a morphism between $(G_1, A_1)$ and $(G_2, A_2)$. For the opposite inclusion, suppose that $p \in P_{n,1}$ satisfies $\tilde{\varphi}(p) \in B_{n,2}$. Since $\psi_{n,1}$ is surjective, there is $(a, g) \in A_1 \rtimes G_1$ such that $p = \psi_{n,1}((a, g))$, and (3.5) implies that

$$(\varphi(a), \varphi(g)) = \hat{\varphi}((a, g)) \in \psi_{n,2}^{-1}(B_{n,2}) = (A_2, 1)N_{n,2} = (A_2, A_2).$$

Thus $\varphi(g) \in A_2$, so $g \in \varphi^{-1}(A_2) = A_1$. It follows that

$$p = \psi_{n,1}((a, g)) \in \psi_{n,1}((A_1, A_1)) = \psi_{n,1}((A_1, 1)N_{n,1}) = B_{n,1},$$

i.e., $\tilde{\varphi}^{-1}(B_{n,2}) \subseteq B_{n,1}$, as required.

If $\varphi$ is surjective then $\varphi(A_1) = A_2$, so, in view of (3.3), $\hat{\varphi}$ is also surjective. Therefore, $\tilde{\varphi}$ is surjective by (3.5). Assume, now, that $\varphi$ is injective. In view of (3.5), to show injectivity of $\tilde{\varphi}$ it is enough to check that $\hat{\varphi}^{-1}(N_{n,2}) \subseteq N_{n,1}$. Indeed, suppose that $\hat{\varphi}((a, g)) \in N_{n,2}$, for some $(a, g) \in A_1 \rtimes G_1$. Then

$$\hat{\varphi}((a, g)) = (\varphi(a), \varphi(g)) = (c^n, c^{-1}),\ \text{for some } c \in A_2,$$

which implies that $\varphi(g) = c^{-1} \in A_2$, so $g \in \varphi^{-1}(A_2) = A_1$ (because $\varphi$ defines a morphism between $(G_1, A_1)$ and $(G_2, A_2)$ in $\mathcal{C}$). Consequently, $\varphi(a) = c^n = \varphi(g^{-n})$, hence $a = g^{-n}$ in $G_1$, as $\varphi$ is injective. The latter shows that $(a, g) = (g^{-n}, g) \in N_{n,1}$, so $\hat{\varphi}^{-1}(N_{n,2}) \subseteq N_{n,1}$ and $\tilde{\varphi}$ is injective. □

## 4. Permutational wreath products

Let $S$ be a group acting on a set $\Omega$ on the left. For any group $A$ we can define the *permutational wreath product* $A \wr_\Omega S$ as the semidirect product $A^\Omega \rtimes S$, where $A^\Omega$ consists of all functions $f : \Omega \to A$ with finite support and $S$ acts on $A^\Omega$ according to the formula

$$(g.f)(x) \coloneqq f(g^{-1}.x), \quad \text{for all } g \in S \text{ and } x \in \Omega.$$

The subgroup $A^\Omega$ is said to be the *base* of the wreath product $A \wr_\Omega S$.

*Remark* 4.1. Suppose that $A$, $S$ are groups and $S$ acts on a set $\Omega$. If $T \leqslant S$ and $\Sigma \subseteq \Omega$ is a $T$-invariant subset then $A \wr_\Sigma T$ naturally embeds as a subgroup of $A \wr_\Omega S$.

In this embedding, any function $f : \Sigma \to A$ is sent to a function $\hat{f} : \Omega \to A$ such that $\hat{f}(x) = f(x)$, for every $x \in \Sigma$, and $\hat{f}(y) = 1$, for every $y \in \Omega \setminus \Sigma$.

If $\Omega = S$ and the action of $S$ on itself is by left translations, then the permutational wreath product $A \wr_S S$ is called the *standard wreath product*, denoted by $A \wr S$.

For any set $\Omega$, we will denote by $\mathrm{Sym}(\Omega)$ the group of all permutations of this set. We will always assume that the action of $\mathrm{Sym}(\Omega)$ on $\Omega$ is the standard left action, unless specified otherwise.

If $k \in \mathbb{N}$ then we will write $\Omega_k \coloneqq \{1, \dots, k\}$ and $S_k \coloneqq \mathrm{Sym}(\Omega_k)$, the symmetric group on $k$ elements. Given any group $A$, the group $A \wr_{\Omega_k} S_k$ is called the *complete monomial group of $A$ of degree $k$* in some literature. It was introduced and studied by Ore [Ore42] and can be represented as a group of $k \times k$ matrices, where each row and column contain exactly one non-zero entry from $A$. Such *monomial matrices* are multiplied in a natural way, establishing a group embedding $A \wr_{\Omega_k} S_k \hookrightarrow \mathrm{GL}(k, \mathbb{Z}A)$. In this section, we will produce embeddings of finitely generated virtually abelian groups into complete monomial groups satisfying additional properties (see Proposition 4.6).

If $L$ is a normal subgroup of a group $G$, any subgroup $Q \leqslant G$ satisfying $G = LQ$ and $Q \cap L = \{1\}$ will be called a *complement to $L$* in $G$. If $S$ is a group acting on a set $\Omega$ and $x \in \Omega$, we will write $S_x$ to denote the *stabilizer* of $x$ in $S$. The next proposition is a simplification of a result of Parker and Quick [PQ03], which generalized a classical result of Ore for complete monomial groups [Ore42, Theorem I.11].

**Proposition 4.2** ([PQ03, Theorem 2.6])**.** *Let $A$ be any group, let $S$ be a group acting transitively on a finite set $\Omega$, and let $G = A \wr_\Omega S$. Suppose that for some $x \in \Omega$ the stabilizer $S_x$ does not admit non-trivial homomorphisms to $A$. Then there is exactly one conjugacy class of complements to $A^\Omega$ in $G$.*

We will extend this result to non-transitive actions, which will help us describe conjugacy classes of subgroups that are disjoint from base groups in wreath products.

**Proposition 4.3.** *Suppose that $A, S$ are groups, $S$ acts on a non-empty finite set $\Omega$, $G = A \wr_\Omega S$ and $Q \leqslant G$ is a subgroup which has trivial intersection with the base $L \coloneqq A^\Omega$ of the wreath product $G$. Let $R \leqslant S$ denote the image of $Q$ under the natural projection $G \to S$. If $R_x$ does not admit* *non-trivial homomorphisms to $A$, for every $x \in \Omega$, then there is $h \in L$ such that $Q = hRh^{-1}$ in $G$.*

Before proving this proposition we need an auxiliary lemma.

**Lemma 4.4.** *Suppose that $H$ is a group, $N_1, N_2 \lhd H$ are normal subgroups with trivial intersection, and $Q, R \leqslant H$ are two complements to $L \coloneqq N_1N_2$ in $H$. Let $\alpha_i : H \to H/N_i$ be the natural homomorphisms, $i = 1, 2$. If $\alpha_i(Q)$ is conjugate to $\alpha_i(R)$ in $H/N_i$, for each $i = 1, 2$, then there is $h \in L$ such that $Q = hRh^{-1}$ in $H$.*

*Proof.* By the assumptions, for each $i = 1, 2$ there exists $h_i \in H$ such that $\alpha_i(Q) = \alpha_i(h_iRh_i^{-1})$. Since $H = N_1N_2R$ and $N_i = \ker \alpha_i$ for $i = 1, 2$, we can assume that $h_1 \in N_2$ and $h_2 \in N_1$.

Thus, given any $q \in Q$ there is $r_i = r_i(q) \in R$ such that

$$\alpha_i(q) = \alpha_i(h_ir_ih_i^{-1}) \text{ in } H/N_i, \text{ for } i = 1, 2. \tag{4.1}$$

Let $\rho : H \to R$ denote the natural retraction of $H$ onto $R$ with kernel $L = N_1N_2$. Then $\rho$ factors through $\alpha_i$, for each $i = 1, 2$, so (4.1) implies that

$$\rho(q) = \rho(h_ir_ih_i^{-1}) = \rho(r_i) = r_i, \text{ for } i = 1, 2,$$

i.e., $r_1 = r_2 = \rho(q)$ in $R$. Recall that $\alpha_1(h_2) = 1$, so (4.1) yields

$$\alpha_1(h_1h_2\rho(q)h_2^{-1}h_1^{-1}) = \alpha_1(h_1r_1h_1^{-1}) = \alpha_1(q), \text{ for every } q \in Q.$$

Similarly, $\alpha_2(h_1h_2\rho(q)h_2^{-1}h_1^{-1}) = \alpha_2(q)$, for all $q \in Q$. Since $\ker \alpha_1 \cap \ker \alpha_2 = N_1 \cap N_2 = \{1\}$ by the assumptions, we can conclude that

$$q = h_1h_2\rho(q)h_2^{-1}h_1^{-1} \text{ in } H, \text{ for every } q \in Q. \tag{4.2}$$

Since $h \coloneqq h_1h_2 \in L = \ker \rho$ and the restriction of $\rho$ to $Q$ is an isomorphism between $Q$ and $R$, (4.2) shows that $Q = hRh^{-1}$ in $H$, as claimed. □

*Proof of Proposition 4.3.* Set $H = LR \leqslant G$ and observe that $Q \leqslant H$ and $H \cong A \wr_\Omega R$ is itself a wreath product, where the action of $R$ on $\Omega$ is induced by the action of $S$ on $\Omega$ (see Remark 4.1). Moreover, $Q$ and $R$ are complements to $L = A^\Omega$ in $H$.

Let $x_1, \dots, x_k \in \Omega$ be a list of orbit representatives for the action of $R$ on $\Omega$. We will argue by induction on $k$. If $k = 1$ then the desired statement follows from Proposition 4.2, so assume that $k \geq 2$.

Let $\Sigma_1 = R.x_1$ be the $R$-orbit of $x_1$ and $\Sigma_2 = \Omega \setminus \Sigma_1$ be the union of the remaining orbits. Observe that the subgroups $N_1 = A^{\Sigma_1}$ and $N_2 = A^{\Sigma_2}$ are normal in $H$, $N_1 \cap N_2 = \{1\}$, $L = N_1N_2$, and we have natural isomorphisms $H/N_1 \cong A \wr_{\Sigma_2} R$, $H/N_2 \cong A \wr_{\Sigma_1} R$. Since the image of $Q$ in $A \wr_{\Sigma_2} R$ is a complement to $A^{\Sigma_2}$ and $R$ acts with $k-1 < k$ orbits on $\Sigma_2$, we can use the induction hypothesis to deduce that the images of $Q$ and $R$ are conjugate in $H/N_1$. Similarly, the images of $Q$ and $R$ are conjugate in $H/N_2$. Therefore, we can apply Lemma 4.4 to find $h \in L$ such that $Q = hRh^{-1}$ in $H$. This completes the proof of the proposition by induction. □

The following immediate corollary of Proposition 4.3 will be useful.

**Corollary 4.5.** *Suppose that $A$ is a torsion-free group and $S$ is a group acting on a finite set $\Omega$. If $G = A \wr_\Omega S$ and $Q \leqslant G$ is any finite subgroup then there exists $h \in A^\Omega$ such that $Q \subseteq hSh^{-1}$ in $G$.*

If $S$ is a group acting on a set $\Omega$, then an element $\tau \in S$ is said to act *freely* if it has no fixed points. A subgroup $T \leqslant S$ acts *freely* on $\Omega$ if every non-trivial element $\tau \in T$ acts freely.

**Proposition 4.6.** *Let $P$ be a group with a finite index normal subgroup $B \cong \mathbb{Z}^j$, for some $j \in \mathbb{N}$, and let $Q \leqslant P$ be a finite subgroup. Then there exist $k \in \mathbb{N}$ and a monomorphism $\eta : P \to \mathbb{Z} \wr_{\Omega_k} S_k$, where $\Omega_k = \{1, \dots, k\}$, such that all of the following hold:*

*(i)* $\eta(B) \subseteq \mathbb{Z}^{\Omega_k}$;
*(ii)* $\eta(Q) \subseteq S_k$ *and* $\eta(Q)$ *acts freely on* $\Omega_k$.

*Proof.* Set $T := P/B$, then, by the Universal Embedding Theorem of Krasner and Kaloujnine (see [DM96, Theorem 2.6.A]), there is a monomorphism $\psi : P \to B \wr T$, where $B \wr T$ is the standard wreath product and $\psi(B) \subseteq B^T$.

By the assumptions, $B \cong \mathbb{Z}^j$ and we denote $k := j|T| \in \mathbb{N} \cup \{0\}$. It is easy to see that there is an isomorphism $\xi : B \wr T \to \mathbb{Z} \wr_{\Omega_k} T$, where $\Omega_k = \{1, \dots, k\}$ and $T$ acts on $\Omega_k$ freely with $j$ orbits, such that $\xi(B^T) = \mathbb{Z}^{\Omega_k}$ and the restriction of $\xi$ to $T$ is the identity map. It follows that $\xi \circ \psi : P \to \mathbb{Z} \wr_{\Omega_k} T$ is an embedding satisfying $(\xi \circ \psi)(B) \subseteq \mathbb{Z}^{\Omega_k}$.

By Corollary 4.5, there is $h \in \mathbb{Z}^{\Omega_k}$ such that $(\xi \circ \psi)(Q) \subseteq hTh^{-1}$. Therefore, after applying an inner automorphism of $\mathbb{Z} \wr_{\Omega_k} T$ (which preserves the base $\mathbb{Z}^{\Omega_k}$), we can assume that $(\xi \circ \psi)(Q) \subseteq T$ in $\mathbb{Z} \wr_{\Omega_k} T$.

Finally, since $T$ acts on $\Omega_k$ freely, hence faithfully, we have an embedding $T \hookrightarrow S_k$ which extends to a monomorphism $\alpha : \mathbb{Z} \wr_{\Omega_k} T \to \mathbb{Z} \wr_{\Omega_k} S_k$ by Remark 4.1. We now see that the composition $\eta = \alpha \circ \xi \circ \psi : P \to \mathbb{Z} \wr_{\Omega_k} S_k$ satisfies both conditions from the statement of the proposition. □

## 5. Proof of Theorem 1.1

**Lemma 5.1.** *Given any $k, l \in \mathbb{N}$, consider the groups $E_1 := \mathbb{Z} \wr_{\Omega_k} S_k$ and $E_2 := \mathbb{Z} \wr_{\Omega_l} S_l$. Suppose $c_1 \in \mathbb{Z}^{\Omega_k}$ and $c_2 \in \mathbb{Z}^{\Omega_l}$ are non-trivial elements in $E_1$ and $E_2$ respectively. Then for $m := kl \in \mathbb{N}$ there exist monomorphisms $\beta_i : E_i \to \mathbb{Z} \wr_{\Omega_m} S_m$, $i = 1, 2$, satisfying the following conditions:*

- $\beta_1(\mathbb{Z}^{\Omega_k}), \beta_2(\mathbb{Z}^{\Omega_l}) \subseteq \mathbb{Z}^{\Omega_m}$;
- $\beta_1(S_k) \subseteq S_m$ *and if an element* $\sigma \in S_k$ *acts freely on* $\Omega_k$ *then* $\beta_1(\sigma)$ *acts freely on* $\Omega_m$;
- $\beta_2(S_l) \subseteq S_m$ *and if an element* $\tau \in S_l$ *acts freely on* $\Omega_l$ *then* $\beta_2(\tau)$ *acts freely on* $\Omega_m$;
- $\beta_1(c_1) = \beta_2(c_2)$.

*Proof.* Recall that in our notation, $\Omega_k = \{1, \dots, k\}$ and $\Omega_l = \{1, \dots, l\}$. We set $\Omega = \Omega_k \times \Omega_l$ and $S = \mathrm{Sym}(\Omega)$. Evidently, for $m = kl$ there is an isomorphism between wreath products $E := \mathbb{Z} \wr_\Omega S$ and $\mathbb{Z} \wr_{\Omega_m} S_m$, taking $\mathbb{Z}^\Omega$ to $\mathbb{Z}^{\Omega_m}$ and $S$ to $S_m$. Therefore, we can work with $E$ instead of $\mathbb{Z} \wr_{\Omega_m} S_m$.

In this proof we will think of the base group $\mathbb{Z}^{\Omega_k}$ in $E_1$ ($\mathbb{Z}^{\Omega_l}$ in $E_2$) as the set of all vectors in $\mathbb{Z}^k$ (respectively, $\mathbb{Z}^l$). It will therefore be natural to think of $\mathbb{Z}^\Omega \leqslant E$ as the set of all $k \times l$ matrices with integer entries, under addition.

Let $e_1 = (1, 0, \dots, 0)$, $\dots$, $e_k = (0, 0, \dots, 1)$ be the standard basis of $\mathbb{Z}^k$. Similarly, let $f_1, \dots, f_l$ be the standard basis of $\mathbb{Z}^l$. By the definition of the action of $S_k$ on $\mathbb{Z}^k \cong \mathbb{Z}^{\Omega_k}$, we have

$$\sigma.e_i = e_{\sigma(i)}, \text{ for all } \sigma \in S_k \text{ and } i = 1, \dots, k. \tag{5.1}$$

By the assumptions, $c_1 = (u_1, \dots, u_k)$ is a non-zero vector in $\mathbb{Z}^k \leqslant E_1$, and $c_2 = (v_1, \dots, v_l)$ is a non-zero vector in $\mathbb{Z}^l \leqslant E_2$. Define a homomorphism $\beta_1 : E_1 \to E$ as follows. For each $i \in \{1, \dots, k\}$, we let $\beta_1(e_i)$ be the $k \times l$ matrix $L_i \in \mathbb{Z}^\Omega$ such that the $i$-th row of $L_i$ is the vector $(v_1, \dots, v_l)$ and all the other entries are 0. For every permutation $\sigma \in S_k$, we let $\beta_1(\sigma) \in S$ be the corresponding permutation of rows of matrices in $\mathbb{Z}^\Omega$, i.e.,

$$\big(\beta_1(\sigma)\big)(i, j) = (\sigma(i), j), \text{ for all } (i, j) \in \Omega.$$

Observe that if $\sigma \in S_k$ acts freely on $\Omega_k$ then $\beta_1(\sigma)$ acts freely on $\Omega$. We also note that

$$\beta_1(\sigma).L_i = L_{\sigma(i)}, \text{ for all } \sigma \in S_k \text{ and } i = 1, \dots, k. \tag{5.2}$$

Since $(v_1, \dots, v_l) \neq (0, \dots, 0)$, the matrices $L_1, \dots, L_k$ are linearly independent in $\mathbb{Z}^\Omega$. Combined with (5.1) and (5.2), this easily implies that $\beta_1$ extends to an injective homomorphism from $E_1$ to $E$. Clearly, $\beta_1(\mathbb{Z}^{\Omega_k}) \subseteq \mathbb{Z}^\Omega$ and $\beta_1(S_k) \subseteq S$.

The homomorphism $\beta_2 : E_2 \to E$ is defined similarly, but now using columns. For every $j \in \{1, \dots, l\}$ we let $\beta_2(f_j) \in \mathbb{Z}^\Omega$ be the $k \times l$ matrix $M_j$ whose $j$-th column vector is $(u_1, \dots, u_k)^T$ and all the other entries are 0. And for each permutation $\tau \in S_l$, we let $\beta_2(\tau) \in S$ be the corresponding permutation of columns of $k \times l$ matrices. Again, we observe that elements of $S_l$ acting freely on $\Omega_l$ are sent to elements of $S$ acting freely on $\Omega$. As before, one can check that this gives an injective homomorphism $\beta_2$ from $E_2$ to $E$, satisfying $\beta_2(\mathbb{Z}^{\Omega_l}) \subseteq \mathbb{Z}^\Omega$ and $\beta_2(S_l) \subseteq S$.

Finally, since $c_1 = \sum_{i=1}^k u_i e_i$ in $\mathbb{Z}^k$ and $c_2 = \sum_{j=1}^l v_j f_j$ in $\mathbb{Z}^l$, we see that

$$\beta_1(c_1) = \sum_{i=1}^k u_i L_i = \begin{pmatrix} u_1 v_1 & \dots & u_1 v_l \\ \vdots & & \vdots \\ u_k v_1 & \dots & u_k v_l \end{pmatrix} = \sum_{j=1}^l v_j M_j = \beta_2(c_2).$$

This completes the proof of the lemma. □

**Lemma 5.2.** *Let $H = C \rtimes Q$, where $C = \langle c \rangle$ is an infinite cyclic group and $Q$ is a finite group. Suppose that $E = A \wr_\Omega S_m$, where $A$ is any group, $m \in \mathbb{N}$, $\Omega \coloneqq \{1, \dots, m\}$ and $\gamma_i : H \to E$, $i = 1, 2$, are two monomorphisms such that $\gamma_1(c) = \gamma_2(c) \in A^\Omega \setminus \{1\}$ and $\gamma_1(Q), \gamma_2(Q) \subseteq S_m$ in $E$. If $\gamma_i(Q)$ acts freely on $\Omega$, for $i = 1, 2$, then there exists $\sigma \in S_m$ such that $\sigma \in \mathrm{C}_E(\gamma_1(c))$ and $\sigma\gamma_1(h)\sigma^{-1} = \gamma_2(h)$ in $E$, for all $h \in H$.*

*Proof.* Denote $f \coloneqq \gamma_1(c) = \gamma_2(c) \in A^\Omega$ and let $A_f \coloneqq f(\Omega)$ be the set of values in $A$ attained by $f$. For each $a \in A_f$ we let $\Omega_a \coloneqq \{x \in \Omega \mid f(x) = a\}$, so that $\Omega = \bigsqcup_{a \in A_f} \Omega_a$. Throughout this proof we will think of $\mathrm{Sym}(\Omega_a)$ as the subgroup of $S_m = \mathrm{Sym}(\Omega)$ consisting of permutations supported on $\Omega_a$.

Denote by $\mathrm{St}(f)$ the stabilizer of $f$ in $S_m \leqslant E$. By the definition of the action of $S_m$ on $A^\Omega$, a permutation $\sigma \in S_m$ belongs to $\mathrm{St}(f)$ if and only if $\sigma^{-1}.\Omega_a = \Omega_a$ (equivalently, $\sigma.\Omega_a = \Omega_a$), for every $a \in A_f$.

By the assumptions, for each $i = 1, 2$ the group $Q_i := \gamma_i(Q) \leqslant S_m$ is isomorphic to $Q$ and acts freely on $\Omega$. Since $Q$ normalizes the infinite cyclic subgroup $C$ in $H$, for every $q \in Q$ either $qcq^{-1} = c$ or $qcq^{-1} = c^{-1}$.

*Case 1:* $c$ is central in $H$. Then $Q_1, Q_2 \leqslant \mathrm{St}(f)$ in $S_m$, so each of these subgroups preserves $\Omega_a$, for every $a \in A_f$. Therefore, for all $a \in A$ and $i = 1, 2$, $Q_i = \gamma_i(Q)$ acts freely on $\Omega_a$. Since any two free actions of the group $Q$ on the same set $\Omega_a$ are conjugate in $\mathrm{Sym}(\Omega_a)$ (cf. [Ser80, Lemma 9 in Section II.2.6]), there exists a permutation $\sigma_a \in \mathrm{Sym}(\Omega_a)$ such that

$$\left(\sigma_a^{-1}\gamma_2(q)\sigma_a\right).x = \gamma_1(q).x, \text{ for all } q \in Q \text{ and } x \in \Omega_a. \tag{5.3}$$

Note that $\sigma_a \in \mathrm{St}(f)$, for each $a \in A_f$, hence the product

$$\sigma := \prod_{a \in A_f} \sigma_a \in S_m$$

commutes with $f = \gamma_1(c) = \gamma_2(c)$ in $E$. In view of (5.3), we have

$$\sigma^{-1}\gamma_2(q)\sigma = \gamma_1(q), \text{ for all } q \in Q. \tag{5.4}$$

Since $H = \langle c, Q\rangle$, it follows that $\sigma^{-1}\gamma_2(h)\sigma = \gamma_1(h)$, for all $h \in H$.

*Case 2:* there is $r \in Q$ such that $r^{-1}cr = c^{-1}$ in $H$. Then the centralizer $Q^+ := Q \cap \mathrm{C}_H(c)$ has index 2 in $Q$ and $Q = Q^+ \sqcup rQ^+$. Set

$$Q_i^+ := \gamma_i(Q^+) \leqslant \mathrm{C}_{S_m}(f) \text{ and } r_i := \gamma_i(r) \in S_m, \text{ for } i = 1, 2.$$

As before, the action of $Q_i^+$ preserves $\Omega_a$ setwise, for each $a \in A_f$. On the other hand, since $r_i^{-1}fr_i = f^{-1}$ in $E$, for each $s \in r_iQ_i^+ = Q_i^+r_i$ we have

$$f(s.x) = (s^{-1}fs)(x) = (r_i^{-1}fr_i)(x) = (f(x))^{-1}, \text{ for each } x \in \Omega.$$

Therefore, $A_f^{-1} = A_f$ and $s.\Omega_a = \Omega_{a^{-1}}$, for all $a \in A_f$, $i = 1, 2$ and $s \in r_iQ_i^+$.

Let $I$ denote the set of all involutions in $A_f$ (i.e., elements satisfying $a = a^{-1}$). Choose a single element from each pair $\{a, a^{-1}\}$, where $a \in A_f \setminus I$, and let $J$ denote the set of all such representatives. Thus $A_f = I \sqcup J \sqcup J^{-1}$ and

$$\Omega = \bigsqcup_{a \in I} \Omega_a \sqcup \bigsqcup_{a \in J} \left(\Omega_a \sqcup \Omega_{a^{-1}}\right). \tag{5.5}$$

If $a \in I$ is an involution, then the action of $Q_i$ preserves $\Omega_a$ setwise, for $i = 1, 2$, and we can argue as in Case 1 to find a permutation $\sigma_a \in \mathrm{Sym}(\Omega_a)$ such that (5.3) holds.

Now, consider any $a \in J$, so that $a \neq a^{-1}$. Then the action of $Q_i$ preserves the subset $\Omega_a \sqcup \Omega_{a^{-1}}$ setwise, so it decomposes in the union of $Q_i$-orbits:

$$\Omega_a \sqcup \Omega_{a^{-1}} = \bigsqcup_{k=1}^{l} O_{i,k}, \text{ for } i = 1, 2.$$

Note that, since $Q_i$ acts freely on $\Omega$, $|O_{i,k}| = |Q|$, for all $i = 1, 2$ and $k = 1, \ldots, l$, in particular, $l = |\Omega_a \sqcup \Omega_{a^{-1}}|/|Q|$ is independent of $i$. For each $k \in \{1, \ldots, l\}$ the $Q_i$-orbit $O_{i,k}$ splits into two $Q_i^+$-orbits $O_{i,k} \cap \Omega_a$ and $O_{i,k} \cap \Omega_{a^{-1}}$, that are interchanged by the action of $r_i$, $i = 1, 2$.

Choose arbitrary basepoints $p_{i,k} \in O_{i,k} \cap \Omega_a$, for all $k = 1, \dots, l$ and $i = 1, 2$. Let $\sigma_a \in \mathrm{Sym}(\Omega_a \sqcup \Omega_{a^{-1}})$ be the permutation defined as follows:

$$\sigma_a.(\gamma_1(t).p_{1,k}) \coloneqq \gamma_2(t).p_{2,k}\,, \text{ for all } t \in Q \text{ and } k = 1, \dots, l. \tag{5.6}$$

We shall now check that

$$\left(\sigma_a^{-1}\gamma_2(q)\sigma_a\right).x = \gamma_1(q).x, \text{ for all } q \in Q \text{ and } x \in \Omega_a \sqcup \Omega_{a^{-1}}. \tag{5.7}$$

Indeed, given any $x \in \Omega_a \sqcup \Omega_{a^{-1}}$, there exist unique $k \in \{1, \dots, l\}$ and $t \in Q$ such that $x = \gamma_1(t).p_{1,k}$. Then, in view of (5.6), for any $q \in Q$ we have

$$\begin{aligned}\left(\sigma_a^{-1}\gamma_2(q)\sigma_a\right).x &= \left(\sigma_a^{-1}\gamma_2(q)\sigma_a\right).(\gamma_1(t).p_{1,k}) = \left(\sigma_a^{-1}\gamma_2(q)\right).(\gamma_2(t).p_{2,k})\\ &= \sigma_a^{-1}.(\gamma_2(qt).p_{2,k}) = \gamma_1(qt).p_{1,k} = \gamma_1(q).(\gamma_1(t).p_{1,k})\\ &= \gamma_1(q).x.\end{aligned}$$

Thus, (5.7) has been verified.

For any $x \in \Omega_a$ there are unique $k \in \{1, \dots, l\}$ and $t \in Q$ such that $x = \gamma_1(t).p_{1,k}$. Note that $t \in Q^+$ because both $x$ and $p_{1,k}$ are in $\Omega_a$ and any element from $\gamma_1(rQ^+)$ interchanges $\Omega_a$ with $\Omega_{a^{-1}}$. Applying (5.6), we see that $\sigma_a.x \in Q_2^+.p_{2,k}$ is also in $\Omega_a$, thus $\sigma_a.\Omega_a \subseteq \Omega_a$. Since $\Omega_a \sqcup \Omega_{a^{-1}}$ is a finite set and $\sigma_a \in \mathrm{Sym}(\Omega_a \sqcup \Omega_{a^{-1}})$, we can conclude that

$$\sigma_a.\Omega_a = \Omega_a \text{ and } \sigma_a.\Omega_{a^{-1}} = \Omega_{a^{-1}}.$$

It follows that $\sigma_a \in \mathrm{St}(f) \leqslant \mathrm{C}_E(f)$ (again, we are treating $\mathrm{Sym}(\Omega_a \sqcup \Omega_{a^{-1}})$ as a subgroup of $S_m$).

We can now define the permutation $\sigma \in S_m$ by the formula

$$\sigma \coloneqq \prod_{a \in I \sqcup J} \sigma_a,$$

and conclude that it belongs to the centralizer $\mathrm{C}_E(f)$ in $E$. Moreover, in view of (5.5), (5.3) (when $a \in I$) and (5.7) (when $a \in J$), we see that (5.4) is satisfied. Therefore, $\sigma^{-1}\gamma_2(h)\sigma = \gamma_1(h)$, for all $h \in H$, and the lemma is proved. □

The following lemma takes care of the easy case in Theorem 1.1, when the amalgamated subgroup $G_0$ is finite.

**Lemma 5.3.** *Suppose that $G = G_1 *_{G_0} G_2$ is an amalgamated free product of finitely generated virtually abelian subgroups $G_1, G_2$ over a common finite subgroup $G_0$. Then there is a finitely generated virtually abelian group $E$ and a homomorphism $\nu : G \to E$ that is injective on each $G_i$, $i = 1, 2$.*

*Proof.* The groups $G_1, G_2$ have property (VRC) by Lemma 2.8, and since $|G_0| < \infty$, the amalgamated free product $G$ has (VRC) by [Min19, Corollary 6.5]. According to [Min19, Proposition 1.5], $G_1, G_2 \leqslant_{vr} G$, therefore, the existence of $E$ and $\nu : G \to E$ follows from Lemma 2.10. □

We are finally ready to prove the main technical result for amalgamated products of virtually abelian groups over a virtually cyclic subgroup.

*Proof of Theorem 1.1.* Lemma 5.3 takes care of the case when $|G_0| < \infty$, so we can further assume that $G_0$ is infinite.

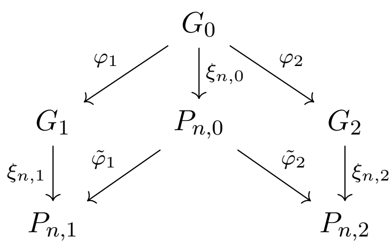
Let $\varphi_i : G_0 \to G_i$ be the subgroup inclusions, $i = 1, 2$, so that the amalgamated product $G$ has the presentation

$$G = \langle G_1, G_2 \mid \varphi_1(g) = \varphi_2(g), \text{ for all } g \in G_0 \rangle. \tag{5.8}$$

Choose finite index free abelian subgroups $N_i \lhd_f G_i$, $i = 1, 2$. Since $G_1, G_2$ are finitely generated and virtually abelian, they are both quasipotent and cyclic subgroup separable (for example, by [BM06, Theorem 5.5]), hence we can apply [BM06, Lemma 3.3] to find finite index normal subgroups $A_i \lhd_f G_i$, such that $A_i \subseteq N_i$, $i = 1, 2$, and $A_1 \cap G_0 = A_2 \cap G_0$. In other words, if we let $A_0 \coloneqq A_1 \cap G_0 = A_2 \cap G_0$, then $\varphi_i$ becomes an injective morphism in the category $\mathcal{C}$ (see Definition 3.1) between the objects $(G_0, A_0)$ and $(G_i, A_i)$, for each $i = 1, 2$. Note that $A_0 \cong \mathbb{Z}$ as it is free abelian and has finite index in the infinite virtually cyclic group $G_0$.

Set $Q \coloneqq G_0/A_0$ and $n \coloneqq |Q| \in \mathbb{N}$. Let $P_{n,i} = P_n(G_i, A_i)$ and $B_{n,i} = B_n(G_i, A_i)$, $i = 0, 1, 2$, be the groups together with their normal abelian subgroups given by Definition 3.3, and let $\xi_{n,i} : (G_i, A_i) \to (P_{n,i}, B_{n,i})$ be the corresponding injective morphisms (see Lemma 3.4), $i = 0, 1, 2$. Recall that $B_{n,i} \cong A_i$ and $P_{n,i}/B_{n,i} \cong G_i/A_i$, for $i = 0, 1, 2$, by Lemma 3.4.

Now, by Lemma 3.7, the injective morphisms $\varphi_i : (G_0, A_0) \to (G_i, A_i)$ extend to injective morphisms $\tilde{\varphi}_i : (P_{n,0}, B_{n,0}) \to (P_{n,i}, B_{n,i})$, $i = 1, 2$. Therefore, we have the following commutative diagram, where all of the maps are injective:




This allows us to consider the amalgamated free product

$$P \coloneqq P_{n,1} *_{P_{n,0}} P_{n,2} \coloneqq \langle P_{n,1}, P_{n,2} \mid \tilde{\varphi}_1(g) = \tilde{\varphi}_2(g), \text{ for all } g \in P_{n,0} \rangle.$$

Since $\tilde{\varphi}_i$ extends $\varphi_i$, for $i = 1, 2$, we have a homomorphism

$$\varkappa : G = G_1 *_{G_0} G_2 \to P = P_{n,1} *_{P_{n,0}} P_{n,2}, \tag{5.9}$$

whose restriction to $G_i$ is $\xi_{n,i}$, for $i = 1, 2$. In particular, $\varkappa$ is injective on $G_1$ and $G_2$ (however, it need not be injective on all of $G$).

Recall that by the choice of $n = |Q|$ and Proposition 3.6, the group $P_{n,0}$ splits as a semidirect product $B_{n,0} \rtimes Q$, where $B_{n,0} \cong A_0 \cong \mathbb{Z}$. Set $C \coloneqq B_{n,0}$ and $H \coloneqq P_{n,0}$, so that $\tilde{\varphi}_i(C) \subseteq B_{n,i}$ is infinite cyclic and $\tilde{\varphi}_i(Q)$ is a finite subgroup of $P_{n,i}$, for $i = 1, 2$. We can now apply Proposition 4.6 to find $k, l \in \mathbb{N}$ and group embeddings

$$\eta_1 : P_{n,1} \to E_1 \coloneqq \mathbb{Z} \wr_{\Omega_k} S_k, \quad \eta_2 : P_{n,2} \to E_2 \coloneqq \mathbb{Z} \wr_{\Omega_l} S_l,$$

such that $\Omega_k = \{1, \ldots, k\}$, $\Omega_l = \{1, \ldots, l\}$, and for $\alpha_i \coloneqq \eta_i \circ \tilde{\varphi}_i : H \to E_i$, $i = 1, 2$, we have

$$\alpha_1(C) \subseteq \mathbb{Z}^{\Omega_k}, \ \alpha_2(C) \subseteq \mathbb{Z}^{\Omega_l}, \ \alpha_1(Q) \subseteq S_k, \ \alpha_2(Q) \subseteq S_l,$$

and $\alpha_i(Q)$ acts freely on $\Omega_i$, $i = 1, 2$.

Observe that we have a natural monomorphism

$$\lambda : P = P_{n,1} *_{P_{n,0}} P_{n,2} \to E_1 *_H E_2, \tag{5.10}$$

where the latter amalgamated product is defined by the presentation

$$E_1 *_H E_2 \coloneqq \langle E_1, E_2 \mid \alpha_1(h) = \alpha_2(h), \text{ for all } h \in H\rangle. \tag{5.11}$$

Let $c$ be a generator of $C$ and set $c_1 \coloneqq \alpha_1(c) \in E_1$, $c_2 \coloneqq \alpha_2(c) \in E_2$. By Lemma 5.1, for $m \coloneqq kl \in \mathbb{N}$ and $E \coloneqq \mathbb{Z} \wr_{\Omega_m} S_m$, there exist monomorphisms $\beta_i : E_i \to E$ enjoying all the properties from its claim. In particular, if we define $\gamma_i \coloneqq \beta_i \circ \alpha_i : H \to E$ then all the conditions of Lemma 5.2 will be satisfied, so we can find $\sigma \in S_m$ such that

$$\sigma\gamma_1(h)\sigma^{-1} = \gamma_2(h) \text{ in } E, \text{ for all } h \in H. \tag{5.12}$$

Let $\delta : E \to E$ be the inner automorphism corresponding to conjugation by $\sigma$. Equation (5.12) shows that after we replace $\beta_1$ by $\delta \circ \beta_1$, we will have

$$\beta_1(\alpha_1(h)) = \beta_2(\alpha_2(h)), \text{ for all } h \in H.$$

Since $E_1 *_H E_2$ has presentation (5.11), it follows that we have a homomorphism $\mu : E_1 *_H E_2 \to E$ such that the restriction of $\mu$ to $E_i$ is $\beta_i$, $i = 1, 2$. In particular, $\mu$ is injective on $E_1$ and $E_2$. Therefore, the homomorphism

$$\nu \coloneqq \mu \circ \lambda \circ \varkappa : G \to E,$$

where $\varkappa$ and $\lambda$ are given by (5.9) and (5.10) respectively, is injective on $G_i$, for $i = 1, 2$. Thus the theorem is proved. □

## 6. Amalgamated products of (VRC) groups over virtually cyclic subgroups have (VRC)

In this section we give first applications of Theorem 1.1; in particular, we prove Theorem 1.3 from the Introduction. The following important consequence of Theorem 1.1 will be used throughout the rest of the paper.

**Proposition 6.1.** *Let $G_1, G_2$ be groups with a common virtually cyclic subgroup $G_0$, and let $G \coloneqq G_1 *_{G_0} G_2$. Suppose that we are given normal subgroups $N_i \lhd G_i$, $i = 1, 2$, and $N \lhd G$, such that $G_i/N_i$, $i = 1, 2$, and $G/N$ are finitely generated and virtually abelian. If $G_0 \leqslant_{vr} G_i$, for $i = 1, 2$, then for arbitrary finitely generated virtually abelian virtual retracts $H_i \leqslant_{vr} G_i$, $i = 1, 2$, there exists a finitely generated virtually abelian group $P$ and a homomorphism $\psi : G \to P$ such that*

- *$\ker\psi \cap G_i \subseteq N_i$, for each $i = 1, 2$;*
- *$\ker\psi \subseteq N$;*
- *$\psi$ is injective on $G_0$ and on each subgroup $H_i$, $i = 1, 2$.*

*Proof.* According to Lemma 2.10, for each $i = 1, 2$ there exist a finitely generated virtually abelian group $P_i$ and a homomorphisms $\varphi_i \colon G_i \to P_i$, such that $\ker\varphi_i \subseteq N_i$ and $\varphi_i$ is injective on $H_i$ and on $G_0$. The latter allows us to view $G_0$ as a subgroup of each $P_i$, $i = 1, 2$. Then, using the universal property of amalgamated free products, we obtain a homomorphism

$$\varphi : G \to P_1 *_{G_0} P_2,$$

such that $\varphi|_{G_i} = \varphi_i$, for $i = 1, 2$. By Theorem 1.1, there exists a homomorphism $\nu \colon P_1 *_{G_0} P_2 \to E$ such that $E$ is a finitely generated virtually abelian group and $\nu$ is injective on $P_1$ and $P_2$. We then have that the group $P := E \times G/N$ and the homomorphism

$$\psi := (\nu \circ \varphi) \times \pi \colon G \to P,$$

where $\pi : G \to G/N$ is the quotient map, satisfies all conditions of the statement, by construction. □

*Proof of Theorem 1.3.* Let $G = G_1 *_{G_0} G_2$ be the free amalgamated product of two groups $G_1$ and $G_2$ over a virtually cyclic subgroup $G_0$. Suppose that $G_1$ and $G_2$ satisfy (VRC); in particular, $G_0 \leqslant_{vr} G_i$, for $i = 1, 2$, by Lemma 2.7. Now, according to [Min19, Corollary 8.5] (or by Lemma 2.7 and Corollary 7.6 below), $G$ is cyclic subgroup separable, hence the virtually cyclic subgroup $G_0$ is separable in $G$ (as a finite union of left cosets to its finite index cyclic subgroup). This allows us to apply [MM25a, Corollary 6.6] to deduce that for each *hyperbolic element* $g \in G$ (that is, $g$ is not conjugate to an element of $G_i$, for $i = 1, 2$), we have $\langle g \rangle \leqslant_{vr} G$.

Now, suppose that $g \in G$ is an *elliptic element*, i.e., $g = xhx^{-1}$, where $h \in G_i$, for some $i \in \{1, 2\}$, and $x \in G$. Since $G_i$ has (VRC), we know that $\langle h \rangle \leqslant_{vr} G_i$, therefore, by Proposition 6.1, there exist a finitely generated virtually abelian group $P$ and a homomorphism $\psi : G \to P$ such that $\psi$ is injective on $\langle h \rangle$ (and, consequently, on $\langle g \rangle$). By Lemma 2.8, $\psi(\langle g \rangle) \leqslant_{vr} P$, hence $\langle g \rangle \leqslant_{vr} G$ by Lemma 2.6.(a). Thus $G$ has (VRC). □

**Corollary 6.2.** *Let $G$ be the fundamental group of a finite tree of groups, where all the vertex groups have (VRC) and all the edge groups are virtually cyclic. Then $G$ has (VRC).*

The next corollary is useful for describing the structure of an amalgamated product of two groups over a common virtually cyclic virtual retract.

**Corollary 6.3.** *Suppose that $G = G_1 *_{G_0} G_2$, where $G_0$ is virtually cyclic and $G_0 \leqslant_{vr} G_i$, for $i = 1, 2$. Then*

*(i) every finitely generated virtually abelian virtual retract $H \leqslant_{vr} G_i$, $i = 1, 2$, is a virtual retract of $G$; in particular, $G_0 \leqslant_{vr} G$.*

*(ii) If $|G_0| < \infty$ then there is a finite index subgroup $F \lhd_f G$ such that $F = F_0 * F_1 * \cdots * F_k$, where $k \in \mathbb{N}_0$, $F_0$ is a finitely generated free group (possibly trivial) and each $F_j$ is isomorphic to a finite index subgroup of some $G_i$, for $i = i(j) \in \{1, 2\}$.*

*(iii) If $G_0$ is infinite then there is a finite index normal subgroup $K \lhd_f G$ isomorphic to a semidirect product*

$$K \cong (N_0 * N_1 * \cdots * N_k) \rtimes \mathbb{Z},$$

*where $k \in \mathbb{N}_0$, $N_0$ is a finitely generated free group (possibly trivial), and for each $j = 1, \dots, k$ there is $i = i(j) \in \{1, 2\}$ such that $N_j$ is isomorphic to some $M_j \lhd (G_i \cap K)$ with $(G_i \cap K)/M_j \cong \mathbb{Z}$.*

*Proof.* (i) According to Proposition 6.1, there exists a group homomorphism $\psi : G \to P$, where $P$ is a finitely generated virtually abelian group, such that $\psi$ is injective on $H$. Lemma 2.8 tells us that $\psi(H) \leqslant_{vr} P$, so, by Lemma 2.6.(a), $H \leqslant_{vr} G$.

(ii) By part (i), $G_0 \leqslant_{vr} G$, so there is a subgroup $G' \leqslant G$ such that $G_0$ normalizes $G'$, $G' \cap G_0 = \{1\}$ and $|G : G_0G'| < \infty$. Since $|G_0| < \infty$, we can conclude that $|G : G'| < \infty$, so there exists $F \lhd_f G$, with $F \subseteq G'$. In particular, $F \cap G_0 = \{1\}$, hence, by Theorem 2.14, $F$ has a free product decomposition with the desired properties.

(iii) Again, since $G_0 \leqslant_{vr} G$, we can apply [MM25a, Lemma 12.2] to find $K \lhd_f G$ admitting a homomorphism $\xi : K \to \mathbb{Z}$ such that $\xi$ is injective on $K \cap gG_0g^{-1}$, for all $g \in G$. By Theorem 2.14, $K$ decomposes as the fundamental group of a finite graph of groups $(\mathcal{K}, \Gamma)$, where each vertex group $K_v$, $v \in V\Gamma$, is isomorphic to $K \cap G_i$, for some $i \in \{1, 2\}$, and each edge group $K_e$, $e \in E\Gamma$, is isomorphic to $K \cap G_0$.

Since $|G_0| = \infty$, $K \cap G_0 \lhd_f G_0$ and $\xi$ is injective on $\alpha_e(K_e)$, we see that $\xi(K)$ is non-zero (so we can assume that $\xi$ is surjective) and $\xi(\alpha_e(K_e))$ has finite index in $\mathbb{Z}$, for all $e \in E\Gamma$. Hence, for $N := \ker \xi \lhd K$ we have

$$K \cong N \rtimes \mathbb{Z},\ N \cap \alpha_e(K_e) = \{1\} \text{ and } |K : N\alpha_e(K_e)| < \infty, \text{ for all } e \in E\Gamma.$$

Therefore, we can combine Corollary 8.5 below with Theorem 2.14 to conclude that $N$ splits as the free product $N_0 * N_1 * \cdots * N_k$, where $N_0$ is free of finite rank and for each $j = 1 \ldots, k$, $N_j$ is isomorphic to a normal subgroup $M_j$ of some $K_v$, $v \in V\Gamma$, with $K_v/M_j \cong \mathbb{Z}$. □

## 7. Residual properties of amalgamated free products

In this section we give applications of Theorem 1.1 to residual properties of amalgamated free products over virtually cyclic subgroups.

*Notation* 7.1. Throughout this section we assume that $G = G_1 *_{G_0} G_2$ is an amalgamated free product of groups $G_1, G_2$, where $G_0$ is virtually cyclic and $G_0 \leqslant_{vr} G_i$, for $i = 1, 2$.

The next statement establishes claim (ii) of Corollary 1.5.

**Corollary 7.2.** *If $G_1$ and $G_2$ are virtually residually solvable then so is $G$.*

*Proof.* Let $N_i \lhd_f G_i$ be a finite index normal residually solvable subgroup, $i = 1, 2$. By Proposition 6.1, there exist a finitely generated virtually abelian group $P$ and $\psi : G \to P$ such that $\psi$ is injective on $G_0$ and

$$\ker \psi \cap G_i \subseteq N_i, \quad \text{for } i = 1, 2. \tag{7.1}$$

Note that $N := \ker \psi \lhd G$ trivially intersects each conjugate of $G_0$ in $G$. Therefore, by a generalization of Kurosh's theorem [Ser80, Theorem 14 in Section I.5.5] (or by Theorem 2.14), $N$ is isomorphic to a free product $*_{j \in J} M_j * F$, where $F$ is a free group and for each $j \in J$ there is $g_j \in G$ and $i = i(j) \in \{1, 2\}$ such that $M_j = N \cap g_j G_i g_j^{-1}$.

Inclusion (7.1) tells us that for every $j \in J$, $M_j \leqslant g_j N_{i(j)} g_j^{-1}$, so $M_j$ is residually solvable. Since $F$ is also residually solvable, we can apply a result of Gruenberg [Gru57, Corollary after Theorem 4.1] to deduce that $N$ is residually solvable. Let $A \lhd_f \psi(G)$ be an abelian subgroup of finite index and let $H = \psi^{-1}(A) \lhd_f G$. Then $H$ is an extension of the residually solvable group $N$ by the abelian group $A$, hence $H$ is residually solvable (see [Gru57, Lemma 1.5]), and the proof is complete. □

We now focus on showing that if $G_1$ and $G_2$ are residually finite, then so is $G$. We will say that a subgroup $H$ of $G$ is *topologically embedded* if the profinite topology on $G$ induces the full profinite topology on $H$ (i.e., every closed subset of $H$ is an intersection of $H$ with a closed subset of $G$). This is equivalent to saying that for each $H' \leqslant_f H$ there is $G' \leqslant_f G$ such that $H \cap G' \subseteq H'$ (cf. [Lor08, p. 1707]).

**Lemma 7.3.** *For each $i = 0, 1, 2$, $G_i$ is topologically embedded in $G$.*

*Proof.* Recall that $G_0 \leqslant_{vr} G$, by Corollary 6.3.(i). Thus there exist $K \leqslant_f G$, with $G_0 \subseteq K$, and a retraction $\rho : K \to G_0$. So, for any $G_0' \leqslant_f G_0$, we have

$$G' := \rho^{-1}(G_0') \leqslant_f G \ \text{ and } \ G' \cap G_0 = G_0'.$$

Therefore, $G_0$ is topologically embedded in $G$.

Now, consider any $i \in \{1, 2\}$, and suppose that $N_i \leqslant_f G_i$. By Proposition 6.1, there is a finitely generated virtually abelian group $P$ and a homomorphism $\psi : G \to P$ such that $\ker\psi \cap G_i \subseteq N_i$. Since $\psi(N_i)$ is separable in $P$ (by Remark 2.2), there exists a finite index subgroup $L \leqslant_f P$ such that $L \cap \psi(G_i) = \psi(N_i)$ (cf. [MM25b, Lemma 4.17]). Then $K := \psi^{-1}(L) \leqslant_f G$ satisfies $K \cap G_i = N_i$. Thus $G_i$ is topologically embedded in $G$. □

**Lemma 7.4.** *Suppose that $G_0$ is separable in $G_i$, for $i = 1, 2$. Then $G$ is edge-approximated by the family of finitely generated virtually abelian groups, in the sense of [MM25a, Definition 6.1]. This means that for arbitrary finite subset $F_i \subseteq G_i$, $i = 1, 2$, there is a finitely generated virtually abelian group $P$ and a homomorphism $\psi : G \to P$ such that*

$$\psi(F_i \setminus G_0) \subseteq P \setminus \psi(G_0), \quad \textit{for } i = 1, 2, \tag{7.2}$$

*and $\psi(G_0)$ is separable in $P$.*

*Proof.* Let $F_i \subseteq G_i$ be an arbitrary finite subset, $i = 1, 2$. Since $G_0$ is separable in $G_i$, according to [MM25a, Lemma 2.2]there exists $N_i \lhd_f G_i$ such that

$$F_i \cap G_0 N_i = F_i \cap G_0, \quad \text{for} \quad i = 1, 2. \tag{7.3}$$

Now, by Proposition 6.1 there is a finitely generated virtually abelian group $P$ and a homomorphism $\psi : G \to P$ such that $\ker\psi \cap G_i \subseteq N_i$, for $i = 1, 2$. If $f \in F_i$, for some $i \in \{1, 2\}$, and $\psi(f) \in \psi(G_0)$ in $P$, then

$$f \in \psi^{-1}(\psi(G_0)) \cap G_i = G_0 \ker\psi \cap G_i = G_0(\ker\psi \cap G_i) \subseteq G_0 N_i,$$

hence $f \in G_0$ by (7.3). Thus (7.2) is satisfied. Moreover, $\psi(G_0)$ is separable in $P$, by Remark 2.2. □

**Proposition 7.5.** *Assume that $G_0$ is separable in $G_1$ and in $G_2$. Then*

(i) *$G_1$, $G_2$ and $G_0$ are all separable in $G$. It follows that every separable subset of $G_i$ is separable in $G$, for $i = 0, 1, 2$.*
(ii) *If $g \in G$ is not conjugate to an element of $G_1 \cup G_2$ then $\langle g \rangle \leqslant_{vr} G$.*

*Proof.* In view of Lemma 7.4, we can apply [MM25a, Proposition 6.4] to deduce that $G_1$ and $G_2$ and $G_0 = G_1 \cap G_2$ are separable in $G$. Since $G_i$ is topologically embedded in $G$ (see Lemma 7.3), for $i = 0, 1, 2$, this implies claim (i). Claim (ii) follows from Lemma 7.4 and [MM25a, Theorem 6.3]. □

In view of Definition 2.1 and the fact that virtual retracts of residually finite groups are always separable (see [Min19, Lemma 2.2]), Proposition 7.5 yields the following amplification of Corollary 1.5.(i).

**Corollary 7.6.** *Assume that $G = G_1 *_{G_0} G_2$, where $G_0$ is virtually cyclic and $G_0 \leqslant_{vr} G_i$, for $i = 1, 2$. If $G_1$ and $G_2$ are residually finite (respectively, cyclic subgroup separable) then $G$ is residually finite (respectively, cyclic subgroup separable).*

*Remark* 7.7. An amalgamated product of two residually finite groups over a common retract is well-known to be residually finite (see [BE73, Theorem 1]), but this may fail if the amalgamated subgroup is only assumed to be a *virtual* retract of the factors. The most prominent examples of this failure are the simple groups constructed by Burger and Mozes [BM00], which decompose as amalgamated free products of finite rank free groups over subgroups that have finite index in both factors. Another class of interesting examples was given by Leary and Minasyan in [LM21, Section 11], who constructed amalgamated products of virtually $\mathbb{Z}^2$ groups over finite index subgroups that are not residually finite (and can even be non-Hopfian).

## 8. Finiteness properties of normal subgroups in graphs of groups

In this section we study necessary and sufficient conditions for a normal subgroup of the fundamental group of a finite graph of groups to be finitely generated and to satisfy higher finiteness properties.

Recall that an action of a group on a tree is said to be *minimal* if there is no non-empty proper invariant subtree. If $(\mathcal{G}, \Gamma)$ is a graph of groups, following Bass [Bas93, Section 7] we will say that a vertex $v \in V\Gamma$ is *terminal* if there is a unique edge $e \in E\Gamma$ such that $\alpha(e) = v$ (in particular, $\alpha(e) \neq \omega(e)$) and $G_v = \alpha_e(G_e)$. Clearly, any terminal vertex can be removed from the graph of groups without changing its fundamental group.

**Lemma 8.1** ([Bas93, Proposition 7.12])**.** *Let $(\mathcal{G}, \Gamma)$ be a finite graph of groups. Fix a maximal tree $T$ and an orientation on $\Gamma$, and suppose that $G = \pi_1(\mathcal{G}, \Gamma, T, E\Gamma^+)$ and $\mathcal{T}$ is the corresponding Bass-Serre tree. Then the action of $G$ on $\mathcal{T}$ is minimal if and only if $(\mathcal{G}, \Gamma)$ contains no terminal vertices.*

**Lemma 8.2.** *If $G$ is a finitely generated group acting minimally on a tree $\mathcal{T}$ then every finite index subgroup $G' \leqslant_f G$ also acts minimally on $\mathcal{T}$.*

*Proof.* Without loss of generality we may assume that $\mathcal{T}$ has at least one edge (otherwise the statement is obvious).

Let $M \lhd_f G$ be a finite index normal subgroup contained in $G'$. If $M$ fixes a vertex of $\mathcal{T}$ then the finite group $G/M$ acts on the subtree $\operatorname{Fix}(M)$, consisting of $M$-fixed points, hence $G/M$ must fix a vertex $v \in \operatorname{Fix}(M)$ (see [Ser80, Example 6.3.1 in Section I.6]). Therefore, $v$ is a global fixed vertex for the action of $G$ on $\mathcal{T}$, contradicting the assumption that $\mathcal{T}$ has at least one edge and the action of $G$ on it is minimal.

Thus, the action of $M$ does not fix a vertex of $\mathcal{T}$. Since $M$ is finitely generated (because $G$ is), it must contain at least one hyperbolic element by [Bas93, Corollary 7.3]. Therefore, $\mathcal{T}$ contains a unique $M$-invariant subtree $\mathcal{S}$ such that the action of $M$ on $\mathcal{S}$ is minimal (see [Bas93, Proposition 7.5]). Since $M \lhd G$, $\mathcal{S}$ will be invariant under the action of $G$, hence $\mathcal{S} = \mathcal{T}$, by minimality. It follows that the action of $M$ on $\mathcal{T}$ is minimal, yielding the same statement for the action of $G'$ on $\mathcal{T}$. □

The following statement, observed by Bridson and Howie in [BH07], is the reason why minimality of the action is important for us.

**Lemma 8.3.** *Let $(\mathcal{G},\Gamma)$ be a finite graph of groups, whose fundamental group $G$ acts minimally on the corresponding Bass-Serre tree $\mathcal{T}$ (this is equivalent to the absence of terminal vertices, by Lemma 8.1). If $N \lhd G$ is a finitely generated normal subgroup then at least one of the following is true:*

$$N \subseteq \alpha_e(G_e), \quad \text{for each } e \in E\Gamma, \tag{8.1}$$

*or*

$$|G : N\alpha_e(G_e)| < \infty, \quad \text{for each } e \in E\Gamma. \tag{8.2}$$

*Proof.* If $N \subseteq \alpha_f(G_f)$, for some $f \in E\Gamma$, then $N$ fixes an edge of $\mathcal{T}$, and the set of fixed points of $N$ forms a non-empty subtree of $\mathcal{T}$. Since $N \lhd G$ this subtree is $G$-invariant, hence it must be the whole of $\mathcal{T}$, by minimality. Therefore, $N$ must fix every edge of $\mathcal{T}$, which shows that (8.1) is true.

Thus we can assume that $N \not\subseteq \alpha_e(G_e)$, for every $e \in E\Gamma$. Choose any edge $e \in E\Gamma$. By the definition of the Bass-Serre tree, $\alpha_e(G_e)$ is the $G$-stabilizer of some edge $a$ of $\mathcal{T}$. By contracting all edges outside the orbit $G.a$ to points, we obtain a new $G$-tree $\mathcal{S}$. Since the action of $G$ on $\mathcal{T}$ was minimal, so is the action of $G$ on $\mathcal{S}$. The group $G$ acts on $\mathcal{S}$ with one orbit of edges and $N$ does not fix any edge. We can now apply the argument from [BH07, Proposition 2.2] to deduce that $|G : N\alpha_e(G_e)| < \infty$, as required. □

**Lemma 8.4.** *Let $(\mathcal{G},\Gamma)$ be a finite graph of groups with fundamental group $G$, and let $\mathcal{T}$ be the corresponding Bass-Serre tree. Suppose that $N \lhd G$ is a normal subgroup satisfying (8.2). Then the induced action of $N$ on $\mathcal{T}$ is cocompact (i.e., $N\backslash\mathcal{T}$ is a finite graph).*

*Proof.* From Bass-Serre Theory [Ser80, Section I.5.3], we know that $G$ acts on $\mathcal{T}$ with finitely many orbits of vertices, finitely many orbits of edges and without edge inversion. Every vertex stabilizer in this action is a conjugate of $G_v \leqslant G$, for some $v \in V\Gamma$, and every edge stabilizer is a conjugate of $\alpha_e(G_e)$, for some $e \in E\Gamma$.

Let $e_1, \dots, e_k \in E\mathcal{T}$ be a finite set of representatives of the $G$-orbits of edges in $\mathcal{T}$, and let $E_i \leqslant G$ be the $G$-stabilizer of $e_i$, $i = 1, \dots, k$. For each $i = 1, \dots, k$, condition (8.2) implies that $|G : NE_i| < \infty$, hence there is $n_i \in \mathbb{N}$ and a collection of elements $g_{i,1}, \dots, g_{i,n_i} \in G$ such that

$$G = \bigsqcup_{j=1}^{n_i} g_{i,j} N E_i.$$

If $e \in E\mathcal{T}$ is an arbitrary edge, then there is a unique $i \in \{1, \dots, k\}$ such that $e \in G.e_i$. Since $N \lhd G$, it follows that for some $j \in \{1, \dots, n_i\}$ we have

$$e \in (g_{i,j} N E_i).e_i = (g_{i,j} N).e_i = N.(g_{i,j}.e_i).$$

Thus the induced action of $N$ on $\mathcal{T}$ has at most $\sum_{i=1}^{k} n_i$ orbits of edges. Consequently, there are also finitely many $N$-orbits of vertices in $\mathcal{T}$, so the quotient $N\backslash\mathcal{T}$ is a finite graph, as claimed. □

**Corollary 8.5.** *Let $G$ be the fundamental group of a finite graph of groups $(\mathcal{G}, \Gamma)$ and suppose that $N \lhd G$ satisfies condition (8.2). Then $N$ splits as the fundamental group of a finite graph of groups $(\mathcal{N}, \Delta)$, where for every $w \in V\Delta$ and each $f \in E\Delta$ there are $v \in V\Gamma$ and $e \in E\Gamma$ such that $N_w \cong N \cap G_v \leqslant G$ and $N_f \cong N \cap \alpha_e(G_e) \leqslant G$. And conversely, for each $v \in V\Gamma$ there is $w \in V\Delta$ such that $N \cap G_v \cong G_w$.*

*Proof.* The statement follows from Lemma 8.4 and the Structure Theorem of Bass-Serre Theory [Ser80, Theorem 13 in Section I.5.4]. □

In the remainder of this section $R$ will denote a non-zero commutative ring with unity. Let us briefly recall the finiteness properties $F_m$ and $FP_m(R)$ (see [Geo08] and [Bie81] for details). A group $G$ is said to be of type $F_m$, if it admits an Eilenberg-Maclane space $K(G, 1)$ with finite $m$-skeleton. In particular, $G$ is of type $F_1$ if and only if it is finitely generated, and of type $F_2$ if and only if it is finitely presented. A group $G$ is said to be of type $FP_m$ over $R$ if the trivial $RG$-module $R$ admits a resolution by projective $RG$-modules that are finitely generated up to degree $m$. Note that $F_1$ coincides with $FP_1$ over $R$ (for any $R$), but for $m \geq 2$, $F_m$ is generally stronger and not always equivalent to $FP_m$ over $R$.

**Lemma 8.6** ([Geo08, Corollary 7.2.4],[Bie81, Proposition 2.5 in Chapter I])**.** *If $G$ is a group and $H \leqslant_f G$ is a subgroup of finite index then for each $m \in \mathbb{N}$*

- *$G$ if of type $F_m$ if and only if $H$ is of type $F_m$;*
- *$G$ is of type $FP_m$ over $R$ if and only if $H$ is of type $FP_m$ over $R$.*

We will explore the following well-known connection between finiteness properties of the fundamental group of a finite graph of groups $(\mathcal{G}, \Gamma)$ and those of the vertex groups, assuming that the edge groups are sufficiently well-behaved. The latter will always be satisfied for virtually polycyclic edge groups, as such groups are of type $F_m$ (and $FP_m$ over $R$), for all $m \in \mathbb{N}$.

**Proposition 8.7.** *Suppose that $G$ is the fundamental group of a finite graph of groups with virtually polycyclic edge groups. Then, given any $m \in \mathbb{N}$, $G$ is of type $F_m$ ($FP_m$ over $R$) if and only if $G_v$ is of type $F_m$ (respectively, $FP_m$ over $R$), for all $v \in V\Gamma$.*

*Proof.* In the case of the homological finiteness conditions $FP_m$ over $R$, the statement follows from [Bie81, Proposition 2.13 in Chapter I], using induction on the number of edges in $\Gamma$.

In the case of the homotopical finiteness properties $F_m$, the sufficiency is given by [Geo08, Exercise 3 in Section 7.2] and the necessity was proved in [HW21, Theorem 1.2] (see also [GL17, Lemma 4.7 and Proposition 4.9], which give the necessity in the case $m \leq 2$ and can be combined with [Bie81, Proposition 2.13 in Chapter I] to deduce it for all $m \in \mathbb{N}$). □

By combining Corollary 8.5 with Proposition 8.7 we obtain the following.

**Corollary 8.8.** *Assume that $(\mathcal{G}, \Gamma)$ is a finite graph of groups with fundamental group $G$ and with virtually polycyclic edge groups. If $N \lhd G$ is a normal subgroup satisfying (8.2) then for each $m \in \mathbb{N}$*

- *$N$ is of type $F_m$ if and only if $N \cap G_v$ is of type $F_m$, for all $v \in V\Gamma$;*

- *$N$ is of type $FP_m$ over $R$ if and only if $N \cap G_v$ is of type $FP_m$ over $R$, for all $v \in V\Gamma$.*

*Remark* 8.9. In the case when $m = 1$ and $\Gamma$ has only one edge, Corollary 8.8 was proved by Ratcliffe in [Rat14, Theorems 1 and 2].

## 9. Background on BNSR invariants

In this section we will briefly summarize basic properties of the Bieri–Neumann–Strebel–Renz (BNSR) invariants that play a crucial role in the study of (virtual) fibering. Throughout this section we assume that $G$ is a finitely generated group and $R$ is a non-zero commutative ring with unity.

**Definition 9.1.** We say that a homomorphism $\chi : G \to \mathbb{R}$ is a *character* of $G$ and we use $\mathrm{Hom}(G,\mathbb{R})$ to denote the set of all characters. If the $\mathbb{Q}$-rank of $\chi(G)$ is one (i.e., $\chi(G) \cong \mathbb{Z}$), we say that the character $\chi$ is *discrete* (or *rational*). Equivalently, $\chi$ is discrete if and only if there exists a homomorphism $\chi' \colon G \to \mathbb{Z}$ and a positive real number $r$ such that $\chi = r\chi'$.

We say that two characters $\chi_1$, $\chi_2$ of $G$ are *equivalent* if $\chi_1 = r\chi_2$ for some positive real $r$. The set of equivalence classes

$$S(G) := \{[\chi] \mid \chi \in \mathrm{Hom}(G,\mathbb{R}) \setminus \{0\}\},$$

with the induced structure coming from the finite dimensional normed real vector space $\mathrm{Hom}(G,\mathbb{R})$, will be called the *character sphere* of $G$. For a subgroup $H \leqslant G$, we denote by $S(G,H)$ the *great subsphere* of $S(G)$ consisting of equivalence classes of characters of $G$ that vanish on $H$.

Note that the dimension of the sphere $S(G)$ is $n-1$ (by a sphere if dimension $-1$ we mean the empty set), where $n \in \mathbb{N}_0$ is the $\mathbb{Q}$-rank of the abelianization $G/[G,G]$ (i.e., $G/[G,G] \cong \mathbb{Z}^n \times B$, for some finite abelian group $B$). If $H \leqslant G$ is a subgroup whose image in $G/[G,G]$ is infinite, then the great subsphere $S(G,H)$ has strictly lower dimension than the sphere $S(G)$. This observation implies claim (i) of the following lemma; claim (ii) is given by [Str13, Lemma B3.24].

**Lemma 9.2.** *Let $H$ be a subgroup of a finitely generated group $G$.*

*(i) If the natural image of $H$ in $G/[G,G]$ is infinite (equivalently, if there is a homomorphism $\varphi : G \to A$, for a torsion-free abelian group $A$, such that $\varphi(H) \neq \{0\}$) then $S(G,H)$ is a closed nowhere dense subset of $S(G)$.*

*(ii) The set of equivalence classes of discrete characters is dense in $S(G)$.*

Within the character sphere $S(G)$, the *invariant* $\Sigma^1(G)$ was defined by Bieri, Neumann and Strebel [BNS87]. For the higher invariants ($m \geq 2$), one distinguishes between the *homotopical BNSR invariants* $\Sigma^m(G) \subseteq S(G)$, introduced by Renz in [Ren88], and the *homological BNSR invariants* $\Sigma^m(G,R) \subseteq S(G)$, defined by Bieri and Renz in [BR88]. We will not give formal definitions of these invariants here, since we will only need their basic properties stated in Corollary 9.4 below. A nice introduction to $\Sigma^1(G)$ is given in Strebel's notes [Str13], and main properties of the higher invariants are summarized in [BGK10, Sections 1.2 and 1.3].

BNSR invariants are important because they control finiteness properties of co-abelian normal subgroups.

**Theorem 9.3** ([BR88; Ren88; Ren89])**.** *Let $G$ be a group of type $F_m$ ($FP_m$ over $R$), for some $m \in \mathbb{N}$, with a normal subgroup $N$ such that $G/N$ is abelian. Then $N$ is of type $F_m$ (respectively, $FP_m$ over $R$) if and only if $S(G,N) \subseteq \Sigma^m(G)$ (respectively, $S(G,N) \subseteq \Sigma^m(G,R)$).*

The main fact about these invariants is that they form open subsets of the character sphere $S(G)$, see [Ren88, Theorem A (2.7) in Section IV.2] and [BR88, Theorem A].

Note that in the case when $N = \ker\chi$, for a discrete character $\chi : G \to \mathbb{R}$, we have $S(G,N) = \{[\chi], -[\chi]\}$. Since we are interested in studying finiteness properties of the kernels of discrete characters, it makes sense to consider the *symmetric BNSR invariants* in $S(G)$:

$$\Sigma^m_\pm(G) := \Sigma^m(G) \cap (-\Sigma^m(G)) \quad \text{and} \quad \Sigma^m_\pm(G,R) := \Sigma^m(G,R) \cap (-\Sigma^m(G,R))\,.$$

A combination of the above results gives the following.

**Corollary 9.4** ([BR88; Ren88; Ren89])**.** *Let $m \in \mathbb{N}$ and let $G$ be a group of type $F_m$ ($FP_m$ over $R$). Then*

*(i) the invariant $\Sigma^m_\pm(G)$ (respectively, $\Sigma^m_\pm(G,R)$) is open in $S(G)$;*

*(ii) if $\chi : G \to \mathbb{R}$ is a discrete character then $\ker\chi$ is of type $F_m$ (respectively, $FP_m$ over $R$) if and only if $[\chi] \in \Sigma^m_\pm(G)$ (respectively, $[\chi] \in \Sigma^m_\pm(G,R)$).*

**Lemma 9.5.** *Let $H$ and $L$ be finitely generated groups, and let $\psi\colon H \to L$ be a homomorphism. Then for any open subset $\Upsilon \subseteq S(H)$ the set*

$$A := \{[\xi] \mid \xi \in \mathrm{Hom}(L,\mathbb{R}) \ \textit{s.t.}\ [\xi \circ \psi] \in \Upsilon\}$$

*is open in $S(L)$. In particular, if $H$ is of type $F_m$, for some $m \in \mathbb{N}$, then*

$$\{[\xi] \mid \xi \in \mathrm{Hom}(L,\mathbb{R}) \ \textit{s.t.}\ [\xi \circ \psi] \in \Sigma^m_\pm(H)\}$$

*is an open subset of $S(L)$.*

*Proof.* Let $\Phi\colon \mathrm{Hom}(L,\mathbb{R}) \to \mathrm{Hom}(H,\mathbb{R})$ be the linear map defined by precomposition with $\psi$. Since these vector spaces are finite dimensional, the map $\Phi$ is necessarily continuous, so we have the following commutative diagram of continuous maps:

$$\begin{array}{ccccc} \mathrm{Hom}(L,\mathbb{R}) & & \xrightarrow{\quad\Phi\quad} & \mathrm{Hom}(H,\mathbb{R}) & \\ \uparrow & & & \uparrow & \\ B := \Phi^{-1}\left(\mathrm{Hom}(H,\mathbb{R}) \setminus \{0\}\right) & \xrightarrow{\Phi_B} & & \mathrm{Hom}(H,\mathbb{R}) \setminus \{0\} & \xrightarrow{q_H} \mathrm{S}(H) \end{array} \quad , \qquad (9.1)$$

where $B$ is an open subset of $\mathrm{Hom}(L,\mathbb{R}) \setminus \{0\}$ (as $\Phi$ is continuous), $\Phi_B$ is the restriction of $\Phi$ to $B$, and $q_H$ is the quotient map sending each character to its equivalence class.

The continuity of $\Phi_B \circ q_H$ implies that the full preimage

$$C := (q_H \circ \Phi_B)^{-1}(\Upsilon) = \{\xi \in \mathrm{Hom}(L,\mathbb{R}) \mid [\xi \circ \psi] \in \Upsilon\}$$

is open in $B$, hence it is also open in $\mathrm{Hom}(L,R) \setminus \{0\}$. Observe that the quotient map $q_L : \mathrm{Hom}(L,\mathbb{R}) \setminus \{0\} \to S(L)$ is an open map, by definition, therefore $A = q_L(C)$ is open in $S(L)$, and the first claim of the lemma is proved. The second claim now follows from Corollary 9.4.(i). □

*Remark* 9.6. The same argument can also be used to get a homological version of Lemma 9.5, where $F_m$ and $\Sigma^m_\pm(H)$ are replaced by $FP_m$ over $R$ and $\Sigma^m_\pm(H, R)$, respectively.

## 10. Fibering graphs of groups

In this section we establish necessary and sufficient criteria for $F_m$-fibering of fundamental groups of graphs of groups with virtually polycyclic edge groups, and we use it to prove Proposition 1.10 from the Introduction.

The following observation stems from the fact that every non-trivial subgroup of $\mathbb{Z}$ has finite index.

*Remark* 10.1. Let $G$ be the fundamental group of a finite graph of groups $(\mathcal{G}, \Gamma)$. If $\chi : G \to \mathbb{R}$ is a non-zero discrete character and $N \coloneqq \ker\chi \lhd G$ then condition (8.2) is equivalent to the condition

$$\chi(\alpha_e(G_e)) \neq 0, \quad \text{for all } e \in E\Gamma. \tag{10.1}$$

The following statement can be regarded as an analogue of [CL16, Theorem 1.2] for higher BNSR invariants. Instead of assuming that the graph of groups is reduced (as it is done in [CL16]) we suppose that it has no terminal vertices, because the latter behaves well under passing to finite index subgroups (see Lemmas 8.1 and 8.2), while the former does not.

**Proposition 10.2.** *Let $G$ be the fundamental group of a finite graph of groups $(\mathcal{G}, \Gamma)$ with virtually polycyclic edge groups and no terminal vertices. Suppose that $G$ is of type $F_m$, for some $m \in \mathbb{N}$, and $\chi : G \to \mathbb{R}$ is a non-zero discrete character.*

*Then $[\chi] \in \Sigma^m_\pm(G)$ provided the following two conditions hold:*

*(i) $\chi(\alpha_e(G_e)) \neq \{0\}$, for all $e \in E\Gamma$;*
*(ii) $[\chi|_{G_v}] \in \Sigma^m_\pm(G_v)$, for each $v \in V\Gamma$.*

*And conversely, if $[\chi] \in \Sigma^m_\pm(G)$ and $\ker\chi \neq \alpha_e(G_e)$, for every $e \in E\Gamma$, then both conditions (i) and (ii) are satisfied.*

*Proof.* Note that, by Proposition 8.7, each vertex group $G_v$ has type $F_m$, so it makes sense to talk about its BNSR invariant $\Sigma^m_\pm(G_v)$. Moreover, $G$ acts minimally on its Bass-Serre tree $\mathcal{T}$, by Lemma 8.1.

Assume, first, that conditions (i) and (ii) hold, and denote $N \coloneqq \ker\chi \lhd G$. Then $N \cap G_v$ has type $F_m$ by Corollary 9.4.(ii), for every $v \in V\Gamma$. Remark 10.1 allows us to apply Corollary 8.8 to conclude that $N = \ker\chi$ is of type $F_m$, thus $[\chi] \in \Sigma^m_\pm(G)$ by Corollary 9.4.(ii).

Conversely, suppose that $[\chi] \in \Sigma^m_\pm(G)$ and $\ker\chi \neq \alpha_e(G_e)$, for every $e \in E\Gamma$. In view of Corollary 9.4.(ii), this means that the normal subgroup $N = \ker\chi \lhd G$ is of type $F_m$, in particular, it is finitely generated. Therefore, we can apply Lemma 8.3 to deduce that either (8.1) or (8.2) is true. If the former is true then $N = \ker\chi \subseteq \alpha_e(G_e)$, for all $e \in E\Gamma$, which, combined with the assumption that $N \neq \alpha_e(G_e)$, implies condition (10.1), hence (8.2) is satisfied by Remark 10.1. Thus we can assume that (8.2) is true. Then (i) holds by Remark 10.1 and (ii) holds by Corollaries 8.8 and 9.4.(ii). □

*Remark* 10.3. The assumption that $\ker\chi \neq \alpha_e(G_e)$, for every $e \in E\Gamma$, in the converse direction of Proposition 10.2 is important. Indeed, any semidirect product $G \coloneqq H \rtimes \mathbb{Z}$ can be considered as an HNN-extension over $H$. If $H$ is of type $F_m$ then for the natural projection $\chi : G \to \mathbb{Z}$, with $\ker \chi = H$, we have $[\chi] \in \Sigma^m_{\pm}(G)$ and $\chi(H) = \{0\}$, in particular, $[\chi|_H] \notin \Sigma^m_{\pm}(H)$.

It is well-known that every infinite virtually cyclic group $G_0$ has a finite normal subgroup $K$ such that $G_0/K$ is either infinite cyclic or infinite dihedral (see, for example, [FJ95, Lemma 2.5]). Clearly, in the latter case $G_0$ cannot map onto $\mathbb{Z}$, therefore we can make the following observation.

*Remark* 10.4. Let $G_0$ be a virtually cyclic group admitting a non-zero homomorphism to $\mathbb{Z}$. Then there is a finite normal subgroup $M \lhd G_0$ and an infinite order element $c \in G_0$ such that $G_0 = M\langle c\rangle \cong M \rtimes \langle c\rangle$. Every homomorphism $\chi : G_0 \to \mathbb{Z}$ sends $M$ to $\{0\}$, thus $\chi$ is completely determined by the image $\chi(c) \in \mathbb{Z}$.

Let us now prove the criterion for $F_m$-fibering of amalgamated free products mentioned in the Introduction.

*Proof of Proposition 1.10.* We can treat $G$ as the fundamental group of a graph of groups $(\mathcal{G}, \Gamma)$, where $\Gamma$ consists of two vertices and two mutually inverse edges joining them, with vertex groups $G_1$ and $G_2$ and with the edge group $G_0$. Note that $(\mathcal{G}, \Gamma)$ has no terminal vertices because $G_0$ embeds as a proper subgroup of $G_1$ and $G_2$, and $G$ is of type $F_m$ by Proposition 8.7.

Observe that $G_0$ cannot be the kernel of a character $G \to \mathbb{R}$ because if $G_0 \lhd G$ then $G/G_0 \cong G_1/G_0 * G_2/G_0$ splits as a non-trivial free product, so $G/G_0$ is necessarily non-abelian. Therefore, if $G$ $F_m$-fibers then (i) and (ii) hold by Proposition 10.2 and Corollary 9.4.(ii).

Thus, it remains to prove the sufficiency, so suppose that (i) and (ii) are true. Take any $i \in \{1, 2\}$. Condition (i), together with Corollary 9.4, imply that $\Sigma^m_{\pm}(G_i)$ is a non-empty open subset of the sphere $S(G_i)$. In view of condition (ii), we can apply Lemma 9.2 to deduce that there exists a non-zero homomorphism $\chi_i : G_i \to \mathbb{Z}$ such that $[\chi_i] \in \Sigma^m_{\pm}(G_i)$ and $\chi_i(G_0) \neq \{0\}$. Let $c \in G_0$ be an infinite order element provided by Remark 10.4, and denote $n_i \coloneqq \chi_i(c) \in \mathbb{Z} \setminus \{0\}$. Since $\Sigma^m_{\pm}(G_i) = -\Sigma^m_{\pm}(G_i)$, we can replace $\chi_i$ by $-\chi_i$, if necessary, to assume that $n_i > 0$.

Observe that
$$n_2\chi_1(c) = n_2 n_1 = n_1\chi_2(c),$$
so, in view of Remark 10.4, we see that the homomorphisms
$$n_2\chi_1 : G_1 \to \mathbb{Z} \ \text{ and } \ n_1\chi_2 : G_2 \to \mathbb{Z}$$
agree on the subgroup $G_0$. Therefore, we can define a homomorphism $\chi : G \to \mathbb{Z}$ by $\chi|_{G_1} \coloneqq n_2\chi_1$ and $\chi|_{G_2} \coloneqq n_1\chi_2$.

By construction, $\chi(G_0) \neq \{0\}$ and $[\chi|_{G_i}] = [\chi_i] \in \Sigma^m_{\pm}(G_i)$, for $i = 1, 2$, hence $[\chi] \in \Sigma^m_{\pm}(G)$ by Proposition 10.2. Thus $G$ $F_m$-fibers, by Corollary 9.4.(ii). □

**Definition 10.5.** Let $H$ be a group with a subgroup $K \leqslant H$. We will say that $H$ *$F_m$-fibers relative to* $K$ if there is a non-zero homomorphism $\chi : H \to \mathbb{Z}$ such that $K \subseteq \ker \chi$ and $\ker \chi$ is of type $F_m$.

The next theorem gives a new fibering criterion for fundamental groups of graphs of groups and may be of independent interest.

**Theorem 10.6.** *Let $(\mathcal{H}, \Delta)$ be a finite graph of groups with fundamental group $H$, such that $H$ is of type $F_m$, for some $m \in \mathbb{N}$, and all edge groups are virtually polycyclic. Suppose that there exist a free abelian group $L$ of rank $n \in \mathbb{N}$ and homomorphisms $\psi : H \to L$ and $\varphi : H \to \mathbb{Z}$ such that all of the following conditions hold:*

*(a)* $\ker \psi \subseteq \ker \varphi$*;*
*(b)* $\psi(\alpha_e(H_e))$ *is non-trivial in $L$, for each $e \in E\Delta$;*
*(c) for every $v \in V\Delta$ either $[\varphi|_{H_v}] \in \Sigma^m_\pm(H_v)$ or $S(H_v, K_v) \subseteq \Sigma^m(H_v)$, where $K_v \coloneqq H_v \cap \ker \psi$.*

*Then the group $H$ $F_m$-fibers relative to* $\ker \psi$.

*Proof.* Without loss of generality, we can assume that $\Delta$ has no terminal vertices and $\psi$ is surjective. Let $U \subseteq V\Delta$ be the subset consisting of all vertices $u \in V\Delta$ such that $[\varphi|_{H_u}] \in \Sigma^m_\pm(H_u)$. Then Lemma 9.5 tells us that the subset

$$C \coloneqq \bigcap_{u \in U} \{[\xi] \in S(L) \mid \xi \in \operatorname{Hom}(L, \mathbb{R}) \text{ s.t. } [\xi \circ \psi|_{H_u}] \in \Sigma^m_\pm(H_u)\}$$

is open in $S(L)$ (if $U = \emptyset$ then we set $C \coloneqq S(L)$). According to (a), there is a homomorphism $\eta : L \to \mathbb{Z}$ such that $\varphi = \eta \circ \psi$. If $U \neq \emptyset$ then $[\eta] \in C$, by the definition of $C$, otherwise $C = S(L)$. In either case, we have verified that $C$ is a non-empty open subset of $S(L)$.

Now, since $L \cong \mathbb{Z}^n$, condition (b) together with Lemma 9.2.(i) tell us that for each $e \in E\Delta$, the subset

$$D_e \coloneqq S\big(L, \psi(\alpha_e(H_e))\big) = \{[\xi] \in S(L) \mid (\xi \circ \psi)(\alpha_e(H_e)) = \{0\}\}$$

is closed and nowhere dense in $S(L)$. Consequently, the finite union $\cup_{e \in E\Delta} D_e$ is also closed and nowhere dense in $S(L)$. On the other hand, the set of discrete characters is dense in $S(L)$ by Lemma 9.2.(ii), hence there exists a non-zero discrete character $\xi_0 : L \to \mathbb{Z}$ such that $[\xi_0] \in C$ and $[\xi_0] \notin D_e$, for all $e \in E\Delta$. It follows that the character $\chi \coloneqq \xi_0 \circ \psi : H \to \mathbb{Z}$ satisfies

- $[\chi|_{H_u}] \in \Sigma^m_\pm(H_u)$, for each $u \in U$, and
- $\chi(\alpha_e(H_e)) \neq \{0\}$, for all $e \in E\Delta$.

Since each vertex of $\Delta$ is incident to some edge, the latter condition implies that the restriction of $\chi$ to $H_v$ is non-zero, for all $v \in V\Delta$. To apply Proposition 10.2, it remains to check that $[\chi|_{H_v}] \in \Sigma^m_\pm(H_v)$, for each $v \in V\Delta \backslash U$. Note that $\ker \psi \subseteq \ker \chi$, by construction, hence $N_v \coloneqq H_v \cap \ker \chi$ contains $K_v = H_v \cap \ker \psi$, so

$$S(H_v, N_v) \subseteq S(H_v, K_v) \subseteq \Sigma^m(H_v), \quad \text{for every } v \in V\Delta \setminus U,$$

by the second part of condition (c). Thus, in view of Theorem 9.3 and Corollary 9.4.(ii), we see that $[\chi|_{H_v}] \in \Sigma^m_\pm(H_v)$, for all $v \in V\Delta$, so $[\chi] \in \Sigma^m_\pm(H)$ by Proposition 10.2. Corollary 9.4.(ii) now tells us that $\ker \chi$ is of type $F_m$, showing that $H$ $F_m$-fibers relative to $\ker \psi$. □

*Remark* 10.7. Essentially the same argument proves the homological versions of Proposition 10.2 and Theorem 10.6, where $F_m$ replaced by $FP_m$ over $R$ and $\Sigma^m_\pm(G)$ replaced by $\Sigma^m_\pm(G, R)$ (for some non-zero commutative ring $R$ with unity).

## 11. Virtual fibering

In this section we prove Theorem 1.12. The following statement, giving necessary conditions for virtual $F_m$-fibering of the fundamental group of a graph of groups with virtually cyclic edge groups, is a straightforward consequence of Bass-Serre Theory and Proposition 10.2.

**Proposition 11.1.** *Let $G$ be the fundamental group of a finite graph of groups $(\mathcal{G}, \Gamma)$ without terminal vertices and with virtually cyclic edge groups. Suppose that $G$ is of type $F_m$, for some $m \in \mathbb{N}$. If $G$ virtually $F_m$-fibers and is not virtually abelian then every vertex group $G_v$ virtually $F_m$-fibers, $v \in V\Gamma$, and for each $e \in E\Gamma$, the edge group $G_e$ is infinite and satisfies $\alpha_e(G_e) \leqslant_{avr} G$.*

*Proof.* Let $H \leqslant_f G$ be a finite index subgroup admitting a non-zero character $\chi : H \to \mathbb{Z}$ with $\ker \chi$ of type $F_m$, so that $[\chi] \in \Sigma^m_{\pm}(H)$, by Corollary 9.4.

The given splitting of $G$ induces a splitting of $H$ as the fundamental group of a finite graph of groups $(\mathcal{H}, \Delta)$, which will have virtually cyclic edge groups and will be without terminal vertices by Lemmas 8.1 and 8.2.

If $\ker \chi = \alpha_f(H_f)$, for some $f \in E\Delta$, then $H$ is virtually abelian (because it will be an extension of the virtually cyclic group $\alpha_f(H_f)$ by $\mathbb{Z}$). Hence, the group $G$ will also be virtually abelian, contradicting the assumptions.

Thus we must have $\ker \chi \neq \alpha_f(H_f)$, for all $f \in E\Delta$. The group $H$ is of type $F_m$, as a finite index subgroup of $G$, by Lemma 8.6, so Proposition 10.2 tells us that $\chi(\alpha_f(H_f)) \neq \{0\}$, for all $f \in E\Delta$, and $[\chi|_{H_u}] \in \Sigma^m_{\pm}(H_u)$, for all $u \in V\Delta$. The latter means that every vertex group $H_u$ is $F_m$-fibered (see Corollary 9.4.(ii)).

For each $v \in V\Gamma$, $G_v$ is the $G$-stabilizer of some vertex $\tilde{v}$ of the Bass-Serre tree $\mathcal{T}$ for $G$. The graph of groups $(\mathcal{H}, \Delta)$ is constructed from the induced action of $H$ on $\mathcal{T}$ (see the paragraphs below Theorem 2.14 in Subsection 2.3), so for all $v \in V\Gamma$ there exist $g = g(v) \in H$ and $u = u(v) \in V\Delta$ such that $G_v \cap H = \mathrm{St}_H(\tilde{v}) = gH_ug^{-1}$. Since $G_v \cap H \leqslant_f G_v$, we can conclude that $G_v$ virtually $F_m$-fibers.

Similarly, for each $e \in E\Gamma$ there are $h = h(e) \in H$ and $f = f(e) \in E\Delta$ such that $\alpha_e(G_e) \cap H = h\alpha_f(H_f)h^{-1} \leqslant_f \alpha_e(G_e)$ in $G$. Since

$$\chi(h\alpha_f(H_f)h^{-1}) = \chi(\alpha_f(H_f)) \neq \{0\},$$

we see that $|G_e| = \infty$ and, by Lemma 2.13, $\alpha_e(G_e) \leqslant_{avr} G$, as desired. □

In view of Proposition 8.7 and Lemma 2.6.(c), part (a) of Theorem 1.12 from the Introduction is a special case of Proposition 11.1. For the second part of this theorem we will need an auxiliary lemma.

**Lemma 11.2.** *Let $G = G_1 *_{G_0} G_2$, where $G_0$ is infinite virtually cyclic and $G_0 \leqslant_{vr} G_i$, for $i = 1, 2$. Suppose that we have homomorphisms $\chi_i : G_i \to \mathbb{Z}$, $i = 1, 2$, such that $\chi_1(G_0) = \{0\}$ and $\chi_2(G_0) \neq \{0\}$. Then there is a finitely generated virtually abelian group $P$ and homomorphisms $\psi : G \to P$ and $\varphi : G \to \mathbb{Z}$ such that all of the following conditions are true:*

- $\ker \psi \subseteq \ker \varphi$;
- $\varphi|_{G_1} = \chi_1$;
- $\ker \psi \cap G_2$ *has finite index in* $\ker \chi_2$;

- *$\psi$ is injective on $G_0$.*

*Proof.* Since $\chi_2(G_0) \neq \{0\}$, Remark 10.4 tells us that $G_0$ decomposes as a semidirect product $M \rtimes C$, where $M \lhd G_0$ is a finite normal subgroup and $C$ is infinite cyclic. As $G_0 \leqslant_{vr} G_2$, we know, by Lemma 2.12, that for each $x \in M \setminus \{1\}$ there is a finite index normal subgroup $N_x \lhd_f G_2$ such that $x \notin N_x$. Then $N := \bigcap_{x \in M \setminus \{1\}} N_x$ is a finite index normal subgroup of $G_2$ intersecting $M$ trivially.

The assumption $\chi_2(G_0) \neq \{0\}$ implies that $M = \ker \chi_2 \cap G_0$ (see Remark 10.4). It follows that

$$O := \ker \chi_2 \cap N \lhd G_2$$

is a normal subgroup in $G_2$, satisfying $O \cap G_0 = \{1\}$.

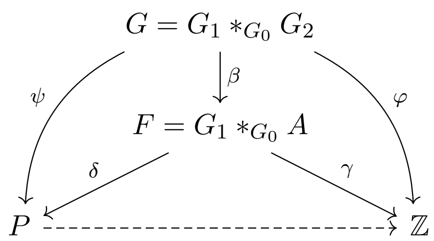
Observe that the group $A := G_2/O$ is virtually cyclic, as $O \lhd_f \ker \chi_2$ and $G_2/\ker \chi_2 \cong \mathbb{Z}$. Since $O \cap G_0 = \{1\}$ the natural homomorphism $\alpha : G_2 \to A$ is injective on $G_0$. Therefore, after identifying $G_0$ with its image in $A$ and using the universal property of amalgamated free products, we obtain a homomorphism

$$\beta : G = G_1 *_{G_0} G_2 \to F := G_1 *_{G_0} A, \quad \text{such that } \beta|_{G_1} = \mathrm{Id}_{G_1} \text{ and } \beta|_{G_2} = \alpha.$$

Since $\chi_1(G_0) = \{0\}$, we have a homomorphism $\gamma : F \to \mathbb{Z}$ whose restriction to $G_1$ is $\chi_1$ and whose restriction to $A$ is the zero map. Since $G_0 \leqslant_{vr} G_1$, by the assumptions, and $G_0 \leqslant_{vr} A$ (in fact, $|A : G_0| < \infty$ as $|G_0| = \infty$ and $A$ is virtually cyclic), we can use Proposition 6.1 to find a finitely generated virtually abelian group $P$ and an epimorphism $\delta : F \to P$ such that $\ker \delta \subseteq \ker \gamma$ and $\delta$ is injective on $A$. Denote

$$\psi := \delta \circ \beta : G \to P \quad \text{and} \quad \varphi := \gamma \circ \beta : G \to \mathbb{Z},$$

so that we have the commutative diagram (11.1).




(11.1)

By construction, we have $\ker \psi \subseteq \ker \varphi$ and $\varphi|_{G_1} = \chi_1$. We also have

$$\ker \psi \cap G_2 = \ker \beta \cap G_2 = \ker \alpha = O \lhd_f \ker \chi_2,$$

which implies that $\ker \psi \cap G_0 = O \cap G_0 = \{1\}$, as required. $\square$

Statement (b) of Theorem 1.12 is given by the following.

**Theorem 11.3.** *Suppose that $G = G_1 *_{G_0} G_2$, where $G_0$ is infinite virtually cyclic and $G_1, G_2$ are of type $F_m$, for some $m \in \mathbb{N}$. If the group $G_i$ $F_m$-fibers and $G_0 \leqslant_{vr} G_i$, for every $i = 1, 2$, then $G$ virtually $F_m$-fibers.*

*Proof.* By the assumptions and Proposition 8.7, the group $G$ is of type $F_m$. We also know that for each $i = 1, 2$ there exists a non-zero homomorphism $\chi_i : G_i \to \mathbb{Z}$ such that $\ker \chi_i$ is of type $F_m$. We will consider 3 different cases depending on the behaviors of these homomorphisms on $G_0$.

*Case 1.* $\chi_i(G_0) \neq \{0\}$, for $i = 1, 2$. Then $G$ $F_m$-fibers by Proposition 1.10.

*Case 2.* $\chi_i(G_0) = \{0\}$, for $i = 1, 2$. Then these homomorphisms agree on $G_0$, so there is a homomorphism $\varphi : G \to \mathbb{Z}$ such that $\varphi|_{G_i} = \chi_i$, for $i = 1, 2$.

Since $G_0 \leqslant_{vr} G_i$, for $i = 1, 2$, by Proposition 6.1, there is a finitely generated virtually abelian group $P$ and a homomorphism $\psi : G \to P$ such that $\ker \psi \subseteq \ker \varphi$ and $\psi$ is injective on $G_0$. Let $L \lhd_f P$ be a free abelian finite index normal subgroup of finite rank, and let $H \coloneqq \psi^{-1}(L) \lhd_f G$.

The decomposition of $G$ as the amalgamated free product of $G_1$ and $G_2$ over $G_0$ induces a splitting of $H$ as the fundamental group of a finite graph of groups $(\mathcal{H}, \Delta)$, as described in Theorem 2.14. We will now aim to apply Theorem 10.6 to show that $H$ $F_m$-fibers relative to $\ker \psi \cap H$.

Let $\psi' : H \to L$ and $\varphi' : H \to \mathbb{Z}$ denote the restrictions of $\psi$ and $\varphi$ to $H$, respectively. Clearly we still have $\ker \psi' \subseteq \ker \varphi'$. Since $H \lhd_f G$, for each edge $e \in E\Delta$ there exists $g \in G$ and such that

$$\alpha_e(H_e) = H \cap gG_0g^{-1} \lhd_f gG_0g^{-1} \text{ in } G.$$

Thus $H_e$ is infinite virtually cyclic and $\psi'$ is injective on $\alpha_e(H_e)$, as $\psi$ is injective on $G_0$. In particular, $\psi'(\alpha_e(G_e))$ is non-trivial in $L$, for all $e \in E\Delta$.

Similarly, for every $v \in V\Delta$ there are $g \in G$ and $i \in \{1, 2\}$ such that $H_v = H \cap gG_ig^{-1} \lhd_f gG_ig^{-1}$ in $G$. Hence,

$$\ker(\varphi'|_{H_v}) = \ker \varphi \cap gG_ig^{-1} \cap H \lhd_f g(\ker \varphi \cap G_i)g^{-1} = g(\ker \chi_i)g^{-1}. \quad (11.2)$$

Therefore, according to Lemma 8.6, $H_v$ and $\ker(\varphi'|_{H_v})$ are of type $F_m$. In view of Corollary 9.4.(ii), the latter shows that $[\varphi'|_{H_v}] \in \Sigma^m_{\pm}(H_v)$, for every $v \in V\Delta$. We can now apply Theorem 10.6 to conclude that $H$ $F_m$-fibers relative to $\ker \psi'$, so $G$ virtually $F_m$-fibers.

*Case 3.* $\chi_j(G_0) = \{0\}$ and $\chi_k(G_0) \neq \{0\}$, where $\{j, k\} = \{1, 2\}$. Without loss of generality, we can assume that $j = 1$ and $k = 2$. Then we can apply Lemma 11.2 to find a finitely generated virtually abelian group $P$ and homomorphisms $\psi : G \to P$ and $\varphi : G \to \mathbb{Z}$ from its claim.

Now, as in Case 2, we let $L \lhd_f P$ be a finitely generated free abelian normal subgroup of finite index, denote $H \coloneqq \psi^{-1}(L) \lhd_f G$, and let $\psi' : H \to L$ and $\varphi' : H \to \mathbb{Z}$ be the restrictions of the homomorphisms $\psi$ and $\varphi$ to $H$, respectively. Then $\ker \psi' \subseteq \ker \varphi'$, by construction. As before, $H$ decomposes as the fundamental group of a finite graph of groups $(\mathcal{H}, \Delta)$, and we can check that for each $e \in E\Delta$, $H_e$ is infinite cyclic and $\psi'$ is injective on $\alpha_e(H_e)$. In particular, $\psi'(\alpha_e(H_e))$ is non-trivial in $L$.

Consider any vertex $v \in V\Delta$. By Theorem 2.14, there are $i \in \{1, 2\}$ and $g \in G$ such that $H_v = H \cap gG_ig^{-1}$ (in particular, $H_v$ is of type $F_m$, by Lemma 8.6). If $i = 1$ then, as in (11.2), we see that

$$\ker(\varphi'|_{H_v}) \lhd_f g(\ker \chi_1)g^{-1},$$

which implies that $[\varphi'|_{H_v}] \in \Sigma^m_{\pm}(H_v)$.

Let us now suppose that $i = 2$, i.e., $H_v = H \cap gG_2g^{-1}$. Then

$$K_v \coloneqq \ker \psi' \cap H_v \lhd_f \ker \psi \cap gG_2g^{-1} = g(\ker \psi \cap G_2)g^{-1}. \quad (11.3)$$

But $\ker \psi \cap G_2$ has finite index in $\ker \chi_2$, by Lemma 11.2, so (11.3) implies that $K_v$ is isomorphic to a finite index subgroup of $\ker \chi_2$. In view of

Lemma 8.6, we deduce that $K_v$ is of type $F_m$. Hence, $S(H_v, K_v) \subseteq \Sigma^m(H_v)$, by Theorem 9.3.

Thus, we have verified that the maps $\psi'$ and $\varphi'$ satisfy all the conditions of Theorem 10.6. We can therefore conclude that $H$ $F_m$-fibers (relative to $\ker \psi'$), and so $G$ virtually $F_m$-fibers. □

## 12. An adaptation to general open invariants

Given a group $G$, one may be interested in studying properties of the kernels of homomorphisms $G \to \mathbb{Z}$ other than $F_m$ or $FP_m$. To this end, it makes sense to introduce invariants corresponding to any such property, in a similar spirit to the BNSR invariants $\Sigma^m(G)$. Here we will restrict ourselves to the symmetric rational (i.e., discrete) invariants.

For a finitely generated group $G$, we let

$$S_{\mathbb{Q}}(G) \coloneqq \{[\chi] \mid \chi \in \operatorname{Hom}(G, \mathbb{Q}) \setminus \{0\}\}$$

be the *rational character sphere of* $G$. The equivalence of two characters is defined as in Definition 9.1. Of course, $S_{\mathbb{Q}}(G)$ can be naturally identified with a dense subset of $S(G)$ (see Lemma 9.2.(ii)), and so it has a natural topology induced by the topology of $S(G)$. Given a property of groups $\mathcal{P}$ we define the invariant $\Sigma^{\mathcal{P}}_{\mathbb{Q}}(G) \subseteq S_{\mathbb{Q}}(G)$ by

$$\Sigma^{\mathcal{P}}_{\mathbb{Q}}(G) \coloneqq \{[\chi] \in S_{\mathbb{Q}}(G) \mid \ker \chi \text{ has } \mathcal{P}\}.$$

Equivalent characters have the same kernel, so this invariant is well-defined.

**Definition 12.1.** Let $\mathcal{P}$ be a property of groups and let $\mathfrak{G}$ be a class of finitely generated groups. We will say that $\mathcal{P}$ *is rationally open in* $\mathfrak{G}$ if $\Sigma^{\mathcal{P}}_{\mathbb{Q}}(G)$ is open in $S_{\mathbb{Q}}(G)$, for every $G \in \mathfrak{G}$.

Of course, having type $F_m$ (or $FP_m$ over a ring $R$) are examples of rationally open properties in the class of all finitely generated groups, by Corollary 9.4. However, by the recent work of Fisher [Fis25a], there are also other interesting rationally open properties in the class of groups of type $FP$, such as being a free group (possibly of infinite rank) or having cohomological dimension strictly smaller than that of the whole group.

We will look at group properties $\mathcal{P}$ satisfying the following conditions.

(C1) $\mathbb{Z}$ has $\mathcal{P}$;
(C2) if $A$ and $B$ have $\mathcal{P}$ then so does $A * B$;
(C3) if $G$ has $\mathcal{P}$ and $H \lhd_f G$ then $H$ has $\mathcal{P}$.

We will say that a finitely generated group $G$ is $\mathcal{P}$*-fibered* if $\Sigma^{\mathcal{P}}_{\mathbb{Q}}(G) \neq \emptyset$, i.e., if there is a non-zero homomorphism $\chi : G \to \mathbb{Z}$ such that $\ker \chi$ has $\mathcal{P}$. A group $G$ is *virtually* $\mathcal{P}$*-fibered* if it has a $\mathcal{P}$-fibered subgroup of finite index. We can now generalize one direction of Proposition 1.10 to rationally open properties (here we need to assume that $G_0 \cong \mathbb{Z}$).

**Proposition 12.2.** *Let $\mathcal{P}$ be property of groups satisfying (C1)–(C2). Suppose that $G = G_1 *_{G_0} G_2$, where $G_0$ is infinite cyclic and $G_i$ is finitely generated and $\mathcal{P}$-fibered, for $i = 1, 2$. If the image of $G_0$ in the abelianization $G_i/[G_i, G_i]$ is infinite, for $i = 1, 2$, and $\mathcal{P}$ is rationally open in the class $\{G_1, G_2\}$ then $G$ is $\mathcal{P}$-fibered.*

*Proof.* The "sufficiency" part of the argument from the proof of Proposition 1.10, given in Section 10, goes through verbatim to show that there is a character $\chi : G \to \mathbb{Z}$ such that $\chi(G_0) \neq \{0\}$ and $[\chi|_{G_i}] \in \Sigma^{\mathcal{P}}_{\mathbb{Q}}(G_i)$, for $i = 1, 2$. The latter condition shows that $\ker \chi \cap G_i$ has property $\mathcal{P}$, for $i = 1, 2$, and the former condition implies that $\ker \chi \cap G_0 = \{1\}$ (because $G_0$ is infinite cyclic).

In view of Remark 10.1, we can apply Corollary 8.5 to conclude that $\ker \chi$ splits as the fundamental group of a finite graph of groups, where all edge groups are trivial and all vertex groups have $\mathcal{P}$. Thus $\ker \chi$ is the free product of finitely many groups with $\mathcal{P}$ and a finitely generated free group, so $\ker \chi$ satisfies $\mathcal{P}$ by conditions (C1) and (C2). □

For finitely generated groups $G_1$, $G_2$ we denote by $\mathfrak{FI}(G_1, G_2)$ the class of groups consisting of finite index subgroups of $G_1$ and $G_2$. The following statement generalizes Theorem 11.3.

**Theorem 12.3.** *Let $\mathcal{P}$ be property of groups satisfying (C1)–(C3). Suppose that $G = G_1 *_{G_0} G_2$, where $G_0$ is infinite virtually cyclic and $G_i$ is finitely generated and $\mathcal{P}$-fibered, for $i = 1, 2$. If $G_0 \leqslant_{vr} G$, for $i = 1, 2$, and $\mathcal{P}$ is rationally open in the class $\mathfrak{FI}(G_1, G_2)$ then $G$ is virtually $\mathcal{P}$-fibered.*

*Proof.* We only sketch the argument, leaving the details to the reader.

By the assumptions, for each $i \in \{1, 2\}$ there is a non-zero character $\chi_i : G_i \to \mathbb{Z}$, where $\ker \chi_i$ satisfies property $\mathcal{P}$. We can argue as in each of the three cases of the proof of Theorem 11.3 (using Proposition 6.1), to produce a finitely generated virtually abelian group $P$ and homomorphisms $\varphi : G \to \mathbb{Z}$ and $\psi : G \to P$ such that

- $\ker \psi \subseteq \ker \varphi$;
- $\psi$ is injective on $G_0$;
- for each $i \in \{1, 2\}$, either $\varphi|_{G_i} = \chi_i$ or $\ker \psi \cap G_i$ has finite index in $\ker \chi_i$.

Next, we choose a free abelian subgroup $L \lhd_f P$, set $H := \psi^{-1}(N) \lhd_f G$, and denote by $\psi' : H \to L$ and $\varphi' : H \to \mathbb{Z}$ the restrictions of $\psi$ and $\varphi$ to $H$, respectively. As before, $H$ will have an induced decomposition as a fundamental group of a finite graph of groups $(\mathcal{H}, \Delta)$, and $\psi'$ will be injective on the edge groups. Since all the edge groups are infinite virtually cyclic and $L$ is torsion-free, we deduce that each $H_e$ is infinite cyclic.

For any vertex group $H_v$, there will be $g \in G$ and $i \in \{1, 2\}$ such that $H_v = g(H \cap G_i)g^{-1}$. If $i \in \{1, 2\}$ is such that $\varphi|_{G_i} = \chi_i$, then $\ker(\varphi'|_{H_v}) \lhd_f g(\ker \chi_i)g^{-1}$ has property $\mathcal{P}$ by (C3), thus $[\varphi'|_{H_v}] \in \Sigma^{\mathcal{P}}_{\mathbb{Q}}(H_v)$. In the other case, when $\ker \psi \cap G_i$ has finite index in $\ker \chi_i$, we have

$$K_v := H_v \cap \ker \psi' \lhd_f g(\ker \chi_i)g^{-1}, \tag{12.1}$$

thus $K_v$ has $\mathcal{P}$ by (C3). On the other hand, (12.1) also shows that $H_v/K_v$ is infinite virtually cyclic, as it is commensurable up to finite kernels with $G_i/\ker \chi_i \cong \mathbb{Z}$. But $H_v/K_v$ is a subgroup of the torsion-free group $L$, hence $H_v/K_v \cong \mathbb{Z}$. It follows that $S(H_v, K_v) \subseteq \Sigma^{\mathcal{P}}_{\mathbb{Q}}(H_v)$.

The argument from Theorem 10.6 now goes through with minimal adaptation (after "rationalizing" everything and using the assumption that $\Sigma^{\mathcal{P}}_{\mathbb{Q}}(H_v)$ is open in $S_{\mathbb{Q}}(H_v)$, for each $v \in V\Delta$), to show that there is a non-zero character $\chi : H \to \mathbb{Z}$ such that $[\chi|_{H_v}] \in \Sigma^{\mathcal{P}}_{\mathbb{Q}}(H_v)$, for all $v \in V\Delta$, and $\chi(\alpha_e(H_e)) \neq \{0\}$, for each $e \in E\Delta$. The latter condition implies that $\ker\chi \cap \alpha_e(H_e) = \{1\}$, for all $e \in E\Delta$ (because $H_e \cong \mathbb{Z}$). As in Proposition 12.2, we can now apply Corollary 8.5 to deduce that $\ker\chi$ splits as a free product of finitely many groups satisfying $\mathcal{P}$ with a free group of finite rank. Conditions (C1) and (C2) then imply that $\ker\chi$ has $\mathcal{P}$, thus $H$ $\mathcal{P}$-fibers and $G$ virtually $\mathcal{P}$-fibers. □

*Proof of Corollary 1.17.* In [Fis25a, Corollary C] Fisher proved that being free is a rationally open property in the class of groups of type $FP$. Finitely generated $F$-by-$\mathbb{Z}$ groups are of type $FP$ (and even of type $F$) by the work of Feighn and Handel [FH99, Theorem 1.2], and this class of groups is obviously closed under taking finite index subgroups. Therefore, statements (i) and (ii) of Corollary 1.17 follow from Proposition 12.2 and Theorem 12.3 respectively. □

## 13. Examples and open questions

If one analyzes the proof of Theorem 1.1, one will notice that we have a good control over the intersection of the images $(\beta_1 \circ \alpha_1)(G_1)$ and $(\beta_2 \circ \alpha_2)(G_2)$ in $E$, but we may lose this control after replacing $\beta_1$ by $\delta \circ \beta_1$. Therefore, the next question arises naturally, and the positive answer to it may be useful in future applications.

**Question 13.1.** Given an amalgamated free product $G = G_1 *_{G_0} G_2$ of two finitely generated virtually abelian groups $G_1$, $G_2$ over a common virtually cyclic subgroup $G_0$, is it always possible to find a finitely generated virtually abelian group $E$ and a homomorphism $\nu : G \to E$ such that $\nu$ is injective on each $G_i$, $i = 1, 2$, and the intersection $\nu(G_1) \cap \nu(G_2)$ is virtually cyclic?

*Example* 13.2. Theorem 1.3 and Lemma 2.8 imply that any amalgamated free product $G$ of two finitely generated virtually abelian groups over a common virtually cyclic subgroup has (VRC). However, such $G$ does not necessarily have the stronger property (LR) (i.e., some finitely generated subgroup may not be a virtual retract of $G$). Indeed, Long and Niblo [LN91, Theorem 4] gave an example of a double of the $(4, 4, 2)$ triangle group (which is virtually $\mathbb{Z}^2$), over an infinite cyclic subgroup, that is not LERF, so this double does not have (LR) by [Min19, Lemma 5.1.(iii)].

The next example shows that amalgamated free products of solvable groups over virtually cyclic virtual retracts need not be residually solvable, so one cannot drop the word "virtually" from claim (ii) of Corollary 1.5.

*Example* 13.3. Let $C_3$ denote the cyclic group of order 3, $G_0 \coloneqq (C_3)^5$, and let $G_1$ and $G_2$ be of the form $G_0 \rtimes C_3$, where each $C_3$ acts on $G_0$ by shuffling its factors using the permutations (123) and (345) respectively. Then $G_1$, $G_2$ are finite metabelian groups, and, of course, $G_0 \leqslant_{vr} G_i$, for $i = 1, 2$. However, the group $G = G_1 *_{G_0} G_2$ is not residually solvable. Indeed, if $G$ were residually solvable, there would exist a solvable group $H$ and an epimorphism $\varphi\colon G \to H$ that is injective on $G_0$ (because $G_0$ is finite and the direct product of a finite number of solvable groups is solvable). Then

$H$ acts on $\varphi(G_0) \lhd H$ by conjugation in the same way as $G$ acts on $G_0$, i.e., we have the following commutative diagram:

$$\begin{array}{ccc} G & \xrightarrow{\varphi} & H \\ \downarrow & & \downarrow \\ \mathrm{Aut}(G_0) & \xrightarrow{\phi} & \mathrm{Aut}(\varphi(G_0)) \end{array}, \tag{13.1}$$

where the vertical arrows come from the actions of $G$ and $H$ on $G_0$ and $\varphi(G_0)$ by conjugation, and $\phi$ is the isomorphism defined by $\psi \mapsto \varphi \circ \psi \circ \varphi^{-1}$, for all $\psi \in \mathrm{Aut}(G_0)$. By construction of $G$, the image of the left vertical map is isomorphic to the alternating group $A_5$, and since $\phi$ is an isomorphism, the same holds for the image of $H$ under the right vertical map. Since $H$ is solvable, the latter would mean that $A_5$ is solvable, giving a contradiction.

**Question 13.4.** Suppose that $G = G_1 *_{G_0} G_2$, where $G_1$, $G_2$ are residually amenable, $G_0$ is virtually cyclic and $G_0 \leqslant_{vr} G_i$, for $i = 1, 2$. Is $G$ residually amenable?

Berlai [Ber16, Proposition 1.4] showed that there exist (residually amenable group)-by-amenable groups that are not residually amenable (in other words, amenability is not a root property in the sense of Gruenberg [Gru57]). Hence, our argument for proving Corollary 1.5.(ii) (see Corollary 7.2) does not give an answer to Question 13.4.

Since virtually special groups are prime examples of groups with (VRC), Theorem 1.3 naturally prompts the following.

**Question 13.5.** Is the amalgam of two finitely generated virtually special groups over a virtually cyclic subgroup virtually special?

The next question addresses the gap between conditions in parts (a) and (b) of Theorem 1.12.

**Question 13.6.** Let $G = G_1 *_{G_0} G_2$, with $G_0$ infinite virtually cyclic. Suppose that $G_i$ is virtually fibered and $G_0 \leqslant_{vr} G_i$, for $i = 1, 2$. Must $G$ be virtually fibered?

We note that a positive answer to Question 13.5 implies a positive answer to Question 13.6 in the case when $G_1$ and $G_2$ are virtually special. Indeed, virtually special groups are virtually RFRS by [Ago08, Theorem 2.2], and $\beta_1^{(2)}(G_i) = 0$, for $i = 1, 2$, by [Kie20, Theorem 2.27]. Then $\beta_1^{(2)}(G) = 0$ by [FV17, Theorem 1], so we can apply Kielak's result [Kie20, Theorem 5.3] to see that $G = G_1 *_{G_0} G_2$ is virtually fibered.

We don't even know the answer to the following weaker version of Question 13.6 in the case when $\Gamma$ is the path of length 2.

**Question 13.7.** Let $G$ be the fundamental group of a finite graph of groups $(\mathcal{G}, \Gamma)$. Suppose that $\Gamma$ is a tree, each vertex group $G_v$ is fibered, each edge group $G_e$ is infinite virtually cyclic and $\alpha_e(G_e) \leqslant_{vr} G_{\alpha(e)}$, for all $e \in E\Gamma$. Does it follow that $G$ is virtually fibered?

The following example shows that the assumption that $G_0 \leqslant_{vr} G_i$ in part (b) of Theorem 1.12 cannot be weakened to $G_0 \leqslant_{avr} G_i$, for $i = 1, 2$.

*Example* 13.8. Let $D = \langle a, b \mid a^2 = b^2 = 1\rangle$ be the infinite dihedral group, let $C = \langle c \mid c^2 = 1\rangle$ be cyclic of order 2, and let

$$F \coloneqq \langle D \times C, t \mid \ tat^{-1} = c\rangle$$

be the HNN-extension of $D \times C$ with associated subgroups $\langle a\rangle$ and $\langle c\rangle$. We define $G_1 \coloneqq F \times E$, where $E = \langle e \mid \ \rangle$ is infinite cyclic. Evidently, $G_1$ is fibered and it has (VRC) by [Min19, Corollary 6.5 and Lemma 5.2], therefore the virtually cyclic subgroup $\langle [a, b], c\rangle \cong C_\infty \times C_2$ is a virtual retract of $G_1$.

Now, let $G_2 \coloneqq Z \times Q$, where $Z = \langle z\rangle$ is infinite cyclic and $Q$ is a finitely generated group such that the finite residual of $Q$ contains an involution $q \in R(Q)$. Thus $G_2$ is fibered and $Z$ is a retract, so $\langle z, q\rangle \leqslant_{avr} G_2$.

We define $G$ as the free product of $G_1$ and $G_2$ amalgamated along their infinite virtually cyclic subgroups $H_1 \coloneqq \langle [a, b], c\rangle$ and $H_2 \coloneqq \langle z, q\rangle$, namely

$$G \coloneqq \langle G_1, G_2 \mid [a, b] = z, \ c = q\rangle.$$

Let us show that $G$ is not virtually fibered. Note that, by Lemma 2.5, $c = q \in R(Q) \subseteq R(G)$, hence $a = t^{-1}ct \in R(G)$, so $[a, b] \in R(G)$, as $R(G) \lhd G$, hence $H_1 \subseteq R(G)$. Since every finitely generated virtually abelian group $P$ is residually finite, Remark 2.4 implies that for any homomorphism $\varphi : G \to P$, we must have $\varphi(H_1) = \{1\}$. Lemma 2.13 now tells us that $H_1$ cannot be an almost virtual retract of $G$, so $G$ does not virtually fiber by Proposition 11.1.

In the next example, we construct an amalgamated free product as in Notation 1.9 such that $G_1$ is virtually fibered, $G_2$ is fibered and $G_0 \leqslant_{avr} G$ has infinite image in the abelianization of $G$, but $G$ is not virtually fibered. This example is in contrast with Proposition 1.10 and Theorem 1.12.

*Example* 13.9. Let $G_1 \coloneqq D \times F$, where $D$ is the infinite dihedral group generated by involutions $a, b$ and $F$ is the free group with basis $\{x, y\}$. Let $G_2 = Z \times Q$ be the group from Example 13.8, with an involution $q \in R(Q)$. Clearly, $G_1$ is virtually fibered (it has an index 2 subgroup isomorphic to $C_\infty \times F$) and $G_2$ is fibered.

We define $G$ as the free product of $G_1$ and $G_2$ amalgamated along their infinite virtually cyclic subgroups $H_1 = \langle a, x\rangle \cong C_\infty \times C_2$ and $H_2 = \langle z, q\rangle \cong C_\infty \times C_2$. Thus

$$G \coloneqq \langle G_1, G_2 \mid a = q, \ x = z\rangle.$$

Evidently, $G$ retracts onto its infinite cyclic subgroup $\langle x\rangle = \langle z\rangle$, with the kernel of the retraction normally generated by $a, b, y$ and $Q$, hence the edge group $G_0 \coloneqq H_1 = H_2$ is an almost virtual retract of $G$, and it also has infinite images in the abelianizations of $G$, $G_1$ and $G_2$.

Arguing as in Example 13.8, one can see that $a = q \in R(G)$, so $D$ has finite image in every virtually abelian quotient of $G$, thus $D$ is not an almost virtual retract of $G$ by Lemma 2.13. However, it is easy to see that $G$ splits as the amalgamated free product $G = \langle D, x, G_2\rangle *_D \langle D, y\rangle$, so $G$ does not virtually fiber by Proposition 11.1, as $D \not\leqslant_{avr} G$.

The following example shows that there is no converse to Proposition 11.1 for HNN-extensions even in the case when all edge groups are retracts.

*Example* 13.10. Consider the tubular group $G$ given by the presentation

$$G \coloneqq \langle a, b, d, s, t \mid [a, b] = [a, d] = 1, \ sbs^{-1} = ab, \ tbt^{-1} = a^2b\rangle. \tag{13.2}$$

On one hand, $G = \langle K, s, t \mid sbs^{-1} = ab,\ tbt^{-1} = a^2b\rangle$ is a double HNN-extension of the fibered subgroup $K := \langle a, b, d\rangle \cong C_\infty \times F_2$. And presentation (13.2) shows that the associated cyclic subgroups $\langle b\rangle$, $\langle ab\rangle$ and $\langle a^2b\rangle$, of this double HNN-extension, are all retracts of $G$, with the kernel of the retraction normally generated by $a, d, s$ and $t$ (in particular, $\langle b\rangle$, $\langle ab\rangle$ and $\langle a^2b\rangle$ have infinite images in the abelianization of $G$).

Alternatively, $G$ is also an HNN-extension $\langle J, t \mid tbt^{-1} = a^2b\rangle$, of the tubular group

$$J := \langle a, b, d, s \mid [a, b] = [a, d] = 1,\ sbs^{-1} = ab\rangle.$$

Now, the group $J$ has (VRC) by [MM25a, Corollary 10.3 and Remark 10.4], because the images of the cyclic subgroups $\langle b\rangle$ and $\langle ab\rangle = \{1\}$ have trivial intersection in the abelianization of the subgroup $\langle a, b, d\rangle$. Therefore, according to [MM25a, Proposition 12.3], $J$ is virtually free-by-cyclic, in particular it virtually fibers.

On the other hand, $G$ can be viewed as an amalgamated free product of Gersten's free-by-cyclic group $H$, given by (1.2), with the free abelian group $\langle a, d\rangle \cong \mathbb{Z}^2$ over $\langle a\rangle$. In [WY25, Lemma 5.18] it is shown that $\langle a\rangle$ is not a virtual retract of $H$, hence $\langle a\rangle \not\leqslant_{vr} G$, so $\langle a\rangle \not\leqslant_{avr} G$, by Lemma 2.12 (as $a$ has infinite order in $G$). Therefore, $G$ does not virtually fiber by Proposition 11.1.

In our final example we construct an amalgam $G$ of two virtually abelian RFRS $\mathbb{Z}^2 \rtimes \mathbb{Z}$ groups over a common normal virtual retract such that $G$ does not virtually fiber. This serves as further evidence that Theorem 1.12.(b) may fail when $G_0$ is not virtually cyclic.

*Example* 13.11. Let $G_0 := \mathbb{Z}^2$, $G_1 := G_0 \rtimes_x \mathbb{Z}$ and $G_2 := G_0 \rtimes_y \mathbb{Z}$, where $x, y$ are two finite order matrices, such that $\langle x, y\rangle = \mathrm{SL}(2, \mathbb{Z})$. Then $G_1$ and $G_2$ are finitely generated, locally indicable and virtually abelian, hence both of them are RFRS by [OS24, Lemma 6.1]. Moreover, $G_0 \leqslant_{vr} G_i$, for $i = 1, 2$, as $x$ and $y$ have finite orders.

Let us prove that $G = G_1 *_{G_0} G_2$ is not virtually fibered. Indeed, suppose there exists $H \leqslant_f G$ and a nontrivial homomorphism $\varphi\colon H \to \mathbb{Z}$ such that $\ker\varphi$ is finitely generated (so $[\varphi] \in \Sigma^1_\pm(H)$ by Corollary 9.4.(ii)). By Theorem 2.14, $H$ is the fundamental group of a finite graph of groups $(\mathcal{H}, \Delta)$ without terminal vertices, and since $G_0 \lhd G$, we have

$$\alpha_e(H_e) = H \cap G_0 \lhd H, \text{ for all } e \in E\Delta.$$

Note that $G/G_0$ is the free group of rank 2, by construction, so $H/\alpha_e(H_e) \leqslant_f G/G_0$ is non-abelian, in particular $\ker\varphi \neq \alpha_e(H_e)$, for all $e \in E\Delta$. Therefore, we can apply the converse direction of Proposition 10.2 to deduce that $\varphi(\alpha_e(H_e)) \neq \{0\}$, for some $e \in E\Delta$. Since $\alpha_e(H_e) \cong \mathbb{Z}^2$, this implies that $\ker\varphi|_{\alpha_e(H_e)} \cong \mathbb{Z}$, so

$$\ker\varphi \cap \alpha_e(H_e) = \langle\vec{v}\rangle \lhd H, \tag{13.3}$$

for some non-zero vector $\vec{v} \in \mathbb{Z}^2$. Since $\langle x, y\rangle = \mathrm{SL}(2, \mathbb{Z})$, the natural homomorphism $\psi\colon G \to \mathrm{Aut}(G_0)$ has image $\mathrm{SL}(2, \mathbb{Z})$. Then (13.3) implies that all matrices from $\psi(H)$ must have $\vec{v}$ as an eigenvector, which is clearly impossible as $\psi(H) \leqslant_f \mathrm{SL}(2, \mathbb{Z})$. Thus $G$ does not virtually fiber.

Since the conditions given in Corollary 1.17 are not necessary, the following problem remains open.

**Problem 13.12.** Find a condition that is both necessary and sufficient for an amalgamated free product of two finitely generated $F$-by-$\mathbb{Z}$ groups over an infinite cyclic subgroup to be $F$-by-$\mathbb{Z}$ (or virtually $F$-by-$\mathbb{Z}$).

The discussion in Section 12 naturally suggests the following.

**Problem 13.13.** Find other interesting examples of rationally open properties in natural classes of groups.

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