PROPERTY (LR) AND AN EMBEDDING THEOREM FOR VIRTUALLY FREE GROUPS

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Abstract. We prove that every virtually free group G has property (LR) of Long and Reid: each finitely generated subgroup of G is a retract of a finite index subgroup. The main ingredient in the proof is a new embedding result stating that every countable virtually free group embeds in a double of a finite group.

As a corollary, we show that any group commensurable with the direct product of a free group and a finitely generated abelian group has (LR). This applies to generalized Baumslag-Solitar groups of arbitrary rank $n \in \mathbb{N}$ with finite monodromy, which, in particular, include all non-cyclic one-relator groups with center.

1. Introduction

A famous theorem of M. Hall [Hal49] implies that every finitely generated subgroup of a free group is a free factor of a finite index subgroup. Conversely, the work of Brunner and Burns [BB79], combined with the accessibility of finitely presented groups proved by Dunwoody [Dun85], tells us that every finitely presented group satisfying the latter *M. Hall's property* must be virtually free (i.e., it has a free subgroup of finite index). However, not all virtually free groups have this property, because finite order elements can have infinite centralizers and free factors are always malnormal (see [Min19] for a characterization of finitely generated virtually free groups with M. Hall's property).

The best analogue of M. Hall's theorem that one could hope to hold in all virtually free groups is *property (LR)*, defined by Long and Reid in [LR08], stating that every finitely generated subgroup is a retract of some subgroup of finite index. Recall that a subgroup H is a *retract* of a group K if there is a homomorphism $\rho : K \to H$ such that $\rho(h) = h$, for all $h \in H$ (equivalently, $K = N \rtimes H$, for some $N \lhd K$). A subgroup H is a *virtual retract* of a group G if H is a retract of a finite index subgroup of G. Thus G has (LR) if every finitely generated subgroup is a virtual retract.

The main result of this note is the following theorem, answering questions of the author from [Min19, Question 11.1] and [KM26, Question 20.59].

Theorem 1.1. *Virtually free groups have property (LR).*

Of course, M. Hall's theorem implies that free groups have (LR), but this property is quite sensitive and is not always inherited by finite index supergroups (see [Min19, Proposition 1.7 and Remark 5.4]). The only other known classes of groups with (LR) are finitely generated virtually abelian

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groups [Min19], fundamental groups of closed surfaces [Sco78], limit groups [Wil08] and some lamplighter groups [Min19]; additionally, property (LR) is preserved by free products [GMS03] (but not by direct products).

Every group with (LR) is *subgroup separable* (a.k.a. LERF), i.e., finitely generated subgroups are closed in the profinite topology, see [LR08; Min19], but property (LR) is much more restrictive. For example, every virtually polycyclic group is subgroup separable but it does not have (LR) unless it is virtually abelian, see [Min19, Proposition 9.1]. However, (LR) is preserved under passing to arbitrary subgroups, so in order to prove Theorem 1.1 for a finitely generated virtually free group, we first embed it in a virtually free group of simpler form.

Let C be a group with a subgroup $B \leqslant C$. Recall that a *double of* C *over* B is the amalgamated product

$$G = C *_B C' = \langle C, C' \mid b = \xi(b), \quad \text{for all } b \in B\rangle, \tag{1.1}$$

where C' is a copy of C and $\xi : C \to C'$ is an isomorphism. If the amalgamated subgroup B is not important, we will simply say that G is a *double of* C. The following, perhaps unexpected, embedding result is key to our proof of Theorem 1.1, and may also be of independent interest.

Theorem 1.2. *Every countable virtually free group embeds as a subgroup in a double of a finite group.*

We actually prove a stronger version of Theorem 1.2 in the case when the virtually free group is finitely generated, see Theorem 2.10. In particular, we obtain the following consequence.

Corollary 1.3. *If A is a finite p-group, for some prime $p \in \mathbb{N}$, (or a finite solvable group) then every finitely generated free-by-A group embeds in a double of a finite p-group (respectively, a finite solvable group).*

Since a double of a finite group is virtually free, we can combine Theorem 1.1 with Theorem 1.2 to deduce the following.

Corollary 1.4. *Every finitely generated virtually free group embeds as a virtual retract in a double of a finite group.*

In order to prove Theorem 1.2, we can assume that the virtually free group is finitely generated (by Scott's embedding theorem [Sco74]). By a theorem of Karrass, Pietrowski and Solitar [KPS73], building on Stallings' theory of groups with infinitely many ends [Sta70], each finitely generated virtually free group splits as the fundamental group of a finite graph of groups with finite vertex groups. Using this decomposition, in a series of lemmas we show that every finitely generated virtually free group embeds in a single special HNN-extension of a finite group A' (Proposition 2.9). Finally, in Theorem 2.10 we embed this special HNN-extension in a double of $A' \times C_p$, for any prime $p \in \mathbb{N}$.

We have thus reduced the proof of Theorem 1.1, in the case when the virtually free group is countable, to showing (LR) for a double G, given by (1.1) for some finite group C. We deal with this case using an idea of Scott from [Sco78] (and also Haglund's interpretation of this idea in [Hag08]), where property (LR) was proved for fundamental groups of compact surfaces. The group $G = C *_B C'$ admits a natural involution $\gamma \in \mathrm{Aut}(G)$ which interchanges the two factors C and C' and fixes B pointwise. We observe that the natural action of G on its Bass-Serre tree T extends to an action of $\widetilde{G} = G \rtimes \langle\gamma\rangle_2$ on T in such a way that every edge is inverted by a conjugate of γ. If $H \leqslant G$ is a finitely generated subgroup then there is a minimal invariant subtree U on which H acts cocompactly. The subgroup N, generated by the inversions in the edges that are on the boundary of U, will be normalized by H and will satisfy $|\widetilde{G} : HN| < \infty$ and $H \cap N = \{1\}$. The subgroup $K = HN \cap G = H(N \cap G)$ is thus a finite index subgroup of G retracting onto H.

Since property (LR) is stable under taking direct products with finitely generated virtually abelian groups [Min19, Proposition 5.6], Theorem 1.1 can be used to find even more groups with (LR). Recall that two groups are said to be *commensurable* if they have isomorphic subgroups of finite index.

Corollary 1.5. *Suppose that G is a group commensurable with the direct product of a free group and a finitely generated abelian group. Then G embeds as a finite index subgroup in the direct product of a virtually free group and a finitely generated virtually abelian group. Therefore,*

(i) *G has property (LR) and is subgroup separable;*
(ii) *G is hereditarily conjugacy separable;*
(iii) *G is subgroup conjugacy distinguished.*

Recall that a group G is *conjugacy separable* if for any two non-conjugate elements $x, y \in G$ there is a finite group M and a homomorphism $\varphi : G \to M$ such that $\varphi(x)$ is not conjugate to $\varphi(y)$ in M. A group is hereditarily conjugacy separable if every finite index subgroup is conjugacy separable. A subgroup H of a group G is *conjugacy distinguished* if for every element $g \in G$, not conjugate to an element of H, there is a homomorphism $\varphi : G \to M$, where $|M| < \infty$, such that $\varphi(g)$ is not conjugate to an element of $\varphi(H)$ in M. A group G is *subgroup conjugacy distinguished* if every finitely generated subgroup $H \leqslant G$ is conjugacy distinguished. In [RZ16] Ribes and Zalesskii proved that a hereditarily conjugacy separable group with (LR) is subgroup conjugacy distinguished.

Corollary 1.5 applies to many groups studied in Geometric Group Theory. This includes unimodular generalized Baumslag-Solitar groups [Lev07, Proposition 2.6] and, more broadly, generalized Baumslag-Solitar groups of arbitrary rank $n \in \mathbb{N}$ with finite monodromy (see [But25, Theorem 3.4]). In particular, one-relator groups with center have (LR) and are subgroup conjugacy distinguished, because they are either cyclic or unimodular generalized Baumslag-Solitar groups by [Pie74, Theorems 1, 3]. Finally, Corollary 1.5 also applies to any extension of a virtually non-abelian free group A by a finitely generated virtually abelian group B, as long as the natural image of B in $\mathrm{Out}(A)$ is finite (e.g., this is the case for a free-by-cyclic group where some power of the acting automorphism is inner), see Corollary 4.5.

In view of Corollary 1.5, it is natural to ask for which other classes of groups property (LR) is invariant under commensurability.

Conjecture 1.6. If a hyperbolic group G has (LR) then so does any finite index supergroup of G.

By a combination of results of Haglund–Wise [HW08], Wise [Wis21] and Linton [Lin25], every one-relator group with torsion has a finite index subgroup with (LR). So, a positive answer to Conjecture 1.6 would imply property (LR) for all one-relator groups with torsion.

In view of Wilton's theorem [Wil08] that limit groups have (LR), it seems sensible to propose the following.

Conjecture 1.7. Finite index supergroups of limit groups have (LR).

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2. Embedding virtually free groups

In this section we will prove our main embedding result, Theorem 1.2. We will make use of the following fundamental theorem, which was proved by Karrass, Pietrowski and Solitar [KPS73] in the case when the group is finitely generated, and by Dicks and Dunwoody [DD89] in the general case.

Theorem 2.1 ([KPS73, Theorem 1] and [DD89, Theorem 1.6 in Chapter IV])**.**

(a) *A finitely generated group G is virtually free if and only if G decomposes as the fundamental group of a finite graph of groups with finite vertex groups.*

(b) *An arbitrary group G is virtually free if and only if G is the fundamental group of a graph of groups such that all the vertex groups are finite of uniformly bounded orders.*

Lemma 2.2. *Suppose that G is the fundamental group of a graph of groups $(\mathcal{G}, \Gamma)$ and $\rho : G \to A$ is a group homomorphism. If ρ is injective on each vertex group G_v, $v \in V\Gamma$, then $\ker \rho$ is free.*

Proof. This is a well-known consequence of Bass-Serre theory. The group G acts on the Bass-Serre tree T corresponding to its given splitting and vertex stabilizers for this action are conjugates of the vertex groups G_v, $v \in V\Gamma$. Since $\ker \rho$ is a normal subgroup intersecting each G_v trivially, we can conclude that $\ker \rho$ acts on T freely, hence $\ker \rho$ is a free group (see [Ser03, Theorem 4 in Section I.3.3]). $\square$

2.1. Embedding general virtually free groups in multiple special HNN-extensions.

We start by showing that every virtually free group embeds in a multiple special HNN-extension of a finite group, see Lemma 2.4. An analogue of this statement for virtually free pro-p groups was proved by Zalesskii in [Zal19].

Given a group A, we will say that a group G is *free-by-A* if G has a free normal subgroup $F \lhd G$ with $G/F \cong A$.

Lemma 2.3. *If A is a finite group then every free-by-A group G embeds in the fundamental group H of a graph of groups $(\mathcal{H}, \Theta)$ such that the graph*

Θ has exactly one vertex with the corresponding vertex group A. Moreover, there is a retraction $\rho : H \to A$, and if G is finitely generated then Θ is finite.

Proof. By Theorem 2.1, G is the fundamental group of a graph of groups $(\mathcal{G}, \Gamma)$, where Γ is a connected graph and all vertex groups $\{G_v\}_{v\in V\Gamma}$ are finite; moreover if G is finitely generated then we can choose Γ to be finite. Let $\varphi : G \to A$ be a homomorphism with free kernel. Free groups are torsion-free, hence φ is injective on each G_v, $v \in V\Gamma$. Let Δ be a new graph obtained from Γ by adding a single vertex u and an edge e_v from u to v, for each $v \in V\Gamma$ (in other words, Δ is a cone over Γ with apex u). We set $G_u = A$, and for each of the newly added edges e_v, we let G_{e_v} be G_v, whose embedding into the vertex group G_v is the identity map and the embedding into G_u is given by $\varphi|_{G_v}$. Obviously, G embeds in the fundamental group H of this new graph of groups. After fixing any vertex $v_0 \in V\Gamma$ and collapsing all of the graph Γ to it, we see (cf. [Ser03, Lemma 6 in Section I.5.2 and Section I.5.1]) that H has the relative presentation

$$H = \left\langle G, A, \{t_v\}_{v\in V\Gamma} \;\middle|\; \bigcup_{v\in V\Gamma} \{t_v g t_v^{-1} = \varphi(g)\}_{g\in G_v},\ t_{v_0} = 1 \right\rangle.$$

This presentation immediately shows that there is a retraction $\rho : H \to A$ such that $\rho|_G = \varphi$, $\rho|_A = \mathrm{Id}_A$ and $\rho(t_v) = 1$, for all $v \in V\Gamma$.

Clearly, the union of the edges e_v, $v \in V\Gamma$, is a maximal tree in Δ and the subgroup of H, generated by the vertex groups for vertices in this maximal tree, is A, by construction. Contracting this maximal tree to the vertex u will result in a graph of groups $(\mathcal{H}, \Theta)$, where the underlying graph Θ has a single vertex u, with $H_u = A$, and all edges are loops at u. As before, the fundamental group of $(\mathcal{H}, \Theta)$ is isomorphic to H. Moreover, if G if finitely generated then Θ is a finite graph, by construction. $\square$

Recall that a *multiple special HNN-extension* of a group A is a group G with presentation

$$G = \langle A, \{t_i\}_{i\in I} \mid \{t_i b t_i^{-1} = b, \quad \text{for all } b \in B_i\}_{i\in I}\rangle, \tag{2.1}$$

where I is an index set, $\{B_i\}_{i\in I}$ is a family of subgroups of A, and $\{t_i\}_{i\in I}$ are the *stable letters* of this HNN-extension. If $|I| = 1$ then we will say that G is a *single special HNN-extension* of A.

Lemma 2.4. *If A is a finite group then every free-by-A group G embeds in a multiple special HNN-extension H of A. Moreover, if G is finitely generated then H has finitely many stable letters.*

Proof. In view of Lemma 2.3, we know that G embeds in the group H with presentation

$$H = \langle A, \{t_i\}_{i\in I} \mid \{t_i b t_i^{-1} = \beta_i(b), \quad \text{for all } b \in B_i\}_{i\in I}\rangle, \tag{2.2}$$

where I is an index set, $\{B_i\}_{i\in I}$ is a family of subgroups of A and $\beta_i : B_i \to A$ are monomorphisms, $i \in I$. Moreover, there is a retraction $\rho : H \to A$. Denote $s_i = \rho(t_i) \in A$, for $i \in I$. Then for each $i \in I$ and all $b \in B_i$ we have

$\beta_i(b) = \rho(\beta_i(b)) = s_i b s_i^{-1}$ in A. Hence, after setting $r_i = s_i^{-1} t_i$, $i \in I$, we can use Tietze transformations to re-write the presentation (2.2) as

$$H = \langle A, \{r_i\}_{i\in I} \mid \{r_i b r_i^{-1} = b, \quad \text{for all } b \in B_i\}_{i \in I}\rangle.$$

Thus, H is a multiple special HNN-extension of the finite group A. Moreover, if G is finitely generated then $|I| < \infty$, by Lemma 2.3. $\square$

2.2. **Embedding finitely generated virtually free groups in doubles of finite groups.** We now aim to improve the statement of Lemma 2.4 in the case when the virtually free group is finitely generated, see Proposition 2.9. To this end, the following observation will be important.

Lemma 2.5. *Let G be the multiple special HNN-extension (2.1), and let N be the normal closure of the stable letters $\{t_i\}_{i\in I}$ in G. Then N is free and $G \cong N \rtimes A$; in particular, $G/N \cong A$. Moreover, N is generated by the set*

$$U = \bigcup_{i \in I} \{d t_i d^{-1} \mid d \in D_i\}, \tag{2.3}$$

where D_i is a left transversal for B_i in A, $i \in I$.

Proof. As evident from (2.1), there is a retraction $\rho : G \to A$ such that $\rho(t_i) = 1$, for all $i \in I$. Thus $N = \ker \rho$ and $G \cong N \rtimes A$. From Lemma 2.2, we also know that N is a free group.

Finally, since N is the normal closure of the elements $\{t_i\}_{i\in I}$ in G and A is a subgroup, it is easy to see that N is generated by the subset $\bigcup_{i\in I}\{a t_i a^{-1} \mid a \in A\}$, which is the set U from the statement of the lemma because for each $i \in I$, t_i centralizes B_i in G. $\square$

Remark 2.6. The subset U in Lemma 2.5 is in fact a free generating set of N. In the case when $|I| < \infty$, this can be seen by observing that the rank of N is $\sum_{i\in I} |A|/|B_i| = \sum_{i\in I} |D_i|$, by [KPS73, Theorem 2] (see also the Euler characteristic [Ser03, Section II.2.9]). When $|I| = \infty$, one can show this by adopting the argument from the proof of Proposition 2.9 below.

Given a prime $p \in \mathbb{N}$ and a finite group A, we will say that a finite group C is *p-by-A* if there is a normal p-subgroup $N \lhd C$ such that $C/N \cong A$.

Lemma 2.7. *Let G be a free-by-A group, for some finite group A. Then for every prime $p \in \mathbb{N}$ and any finite subset $S \subseteq G$ there is a finite p-by-A group C and a homomorphism $\varphi : G \to C$ such that φ is injective on S.*

Proof. By the assumptions, there is a normal free subgroup $F \lhd G$ such that $G/F \cong A$. Since free groups are residually p-finite (see [Tak51, Theorem 6]), there exists a normal subgroup $K \lhd F$ such that F/K is a finite p-group and $K \cap S^{-1}S = \{1\}$ in G (without loss of generality, we assume that $S \neq \emptyset$).

Choose any transversal $g_1, \dots, g_n \in G$ for F in G, where $n = |A|$. Then $N = \bigcap_{i=1}^n g_i K g_i^{-1}$ is a normal subgroup of G, and since F/N embeds in the direct power $(F/K)^n$, it is a finite p-group. Clearly, $N \cap S^{-1}S = \{1\}$, so the quotient map $\varphi : G \to G/N$ is injective on S. Moreover, $C = G/N$ is a p-by-A group, as it is an extension of F/N by $G/F \cong A$. $\square$

The next statement is an easy consequence of the ping-pong lemma. It will be very useful for simplifying the technical arguments when dealing with normal forms.

Lemma 2.8 ([MV25, Lemma 2.8]). *Let F be a free group and let $f_1, \ldots, f_l \in F \setminus \{1\}$. If $\langle f_i \rangle \cap \langle f_j \rangle = \{1\}$, for all $i \neq j$, then there is $n \in \mathbb{N}$ such that the elements $f_1^n, \ldots, f_l^n$ freely generate a free subgroup of F of rank l.*

Proposition 2.9. *Let A be a finite group and let $p \in \mathbb{N}$ be any prime. Every finitely generated free-by-A group G embeds in a single special HNN-extension of a finite p-by-A group.*

Proof. According to Lemma 2.4 and Theorem 2.1.(a), we can assume that

$$G = \langle A, t_1, \ldots, t_k \mid \{t_i b t_i^{-1} = b, \quad \text{for all } b \in B_i\}_{i=1}^k \rangle, \tag{2.4}$$

where $B_i \leqslant A$, $i = 1, \ldots, k$, are some subgroups. Define a finite subset $S \subseteq G$ by

$$S = \{t_i^{-1} a t_j \mid i, j \in \{1, \ldots, k\}, \ a \in A\} \cup A.$$

By Lemma 2.7, there exist a finite p-by-A group C and a homomorphism $\varphi : G \to C$ such that φ is injective on S. Abusing the notation, we will identify A and $B_1, \ldots, B_k$ with their φ-images in C. We will show that G embeds in the single special HNN-extension

$$H = \langle C, t \mid tat^{-1} = a, \quad \text{for all } a \in A \rangle.$$

For each $i = 1, \ldots, k$, denote $s_i = \varphi(t_i) \in C$. Consider the elements $h_i = s_i t s_i^{-1} \in H$, $i = 1, \ldots, k$, and let $\rho : H \to C$ be the natural retraction whose kernel F is the normal closure of t in H. Then $h_1, \ldots, h_k \in F$ and F is a free subgroup of H (see Lemma 2.5). Let D_i be a left transversal for B_i in A, $i = 1, \ldots, k$, and consider the finite subset V of F, defined by

$$V = \bigcup_{i=1}^{k} \{d h_i d^{-1} \mid d \in D_i\}.$$

We will aim to apply Lemma 2.8 to the subset V to show that there is $n \in \mathbb{N}$ such that the subset $W = \{f^n \mid f \in V\} \subseteq F$ freely generates a free subgroup of F. To this end, consider two elements $f_1 = d_1 h_i d_1^{-1}$ and $f_2 = d_2 h_j d_2^{-1}$ in V such that $d_1 \in D_i$, $d_2 \in D_j$ and either $i \neq j$ or $d_1 \neq d_2$. Note that both of these elements have infinite order, being conjugates of t in H. Suppose that $f_1^m = f_2^q$, for some $m \in \mathbb{N}$ and $q \in \mathbb{Z} \setminus \{0\}$. Then in H we have the equation

$$1 = f_1^m f_2^{-q} = d_1 h_i^m d_1^{-1} d_2 h_j^{-q} d_2^{-1} = d_1 s_i t^m s_i^{-1} d_1^{-1} d_2 s_j t^{-q} s_j^{-1} d_2^{-1}. \tag{2.5}$$

Since H retracts onto its infinite cyclic subgroup $\langle t \rangle$ (with the retraction sending all of C to 1), we must have $m - q = 0$, so $q = m \in \mathbb{N}$ and (2.5) becomes

$$d_1 s_i t^m s_i^{-1} d_1^{-1} d_2 s_j t^{-m} s_j^{-1} d_2^{-1} = 1 \ \text{ in } H. \tag{2.6}$$

Now, by Britton's Lemma for HNN-extensions [LS77, Section IV.2], we have

$$s_i^{-1} d_1^{-1} d_2 s_j \in A \ \text{ in } C. \tag{2.7}$$

Recall that we have identified A, d_1 and d_2 with their φ-images in C. Thus $s_i^{-1} d_1^{-1} d_2 s_j = \varphi(t_i^{-1} d_1^{-1} d_2 t_j)$, and $t_i^{-1} d_1^{-1} d_2 t_j \in S$ because $d_1^{-1} d_2 \in A$. Since φ is injective on S and $A \subseteq S$, (2.7) implies that $t_i^{-1} d_1^{-1} d_2 t_j \in A$ in G, i.e.,

$$t_i^{-1} d_1^{-1} d_2 t_j a^{-1} = 1 \ \text{ in } G, \ \text{for some } a \in A. \tag{2.8}$$

Observe, from (2.4), that by taking the quotient of G by the normal closure of A we obtain the free group of rank k, freely generated by $t_1, \dots, t_k$. Therefore, in view of (2.8), we must have $i = j$. Once again, we can apply Britton's lemma (but now in the multiple HNN-extension G), to conclude that $d_1^{-1} d_2 \in B_i$. Since $d_1, d_2 \in D_i$, the latter implies that $d_1 = d_2$, contradicting our assumption. Thus, we have shown that $\langle f_1 \rangle \cap \langle f_2 \rangle = \{1\}$, for any two distinct elements $f_1, f_2 \in V$. Therefore, we can apply Lemma 2.8 to find $n \in \mathbb{N}$ such that the subset $W = \{f^n \mid f \in V\}$ freely generates a free subgroup M of F, of rank $|W| = \sum_{i=1}^{k} |D_i| = \sum_{i=1}^{k} |A : B_i|$.

Observe that for each $i = 1, \dots, k$, the element h_i centralizes B_i in H because t centralizes all of A and s_i is the φ-image of t_i which centralizes B_i in G. Therefore, there is a group homomorphism $\psi : G \to H$, given by

$$\psi(a) = \varphi(a), \text{ for all } a \in A, \quad \text{and} \quad \psi(t_i) = h_i^n, \text{ for } i = 1, \dots, k.$$

Let $N \lhd G$ be the normal closure of $\{t_1, \dots, t_k\}$. According to Lemma 2.5, N is generated by the set U, given by (2.3). By construction, the homomorphism $\psi : G \to H$ sends U bijectively onto the set W, which is a free generating set of $M = \langle W \rangle$. It follows that N is free on U and the restriction of ψ to N is an isomorphism between N and M. Thus, the restrictions of ψ to N and to A are both injective. Since $G = NA$ and $\psi(A) \cap \psi(N) \subseteq C \cap F = \{1\}$, we can conclude that ψ is injective. Hence, we have shown that G embeds in H, and the proof is complete. □

In [Sco74, Embedding Theorem] Scott proved that every countable virtually free group embeds in a finitely generated virtually free group. Therefore, Theorem 1.2, stated in the Introduction, follows from the following result.

Theorem 2.10. *Suppose that A is a finite group. Then for every finitely generated free-by-A group G and each prime $p \in \mathbb{N}$, G embeds in a double of a finite p-by-A group.*

Proof. In view of Proposition 2.9, we can assume that

$$G = \langle A', t \mid tbt^{-1} = b, \quad \text{for all } b \in B \rangle,$$

for some finite p-by-A group A' and a subgroup $B \leqslant A'$. Let Y be a cyclic group of order p, generated by $y \in Y$. Then $C = A' \times Y$ is again a finite p-by-A group, and we denote by C' a copy of C, with a fixed isomorphism $\xi : C \to C'$. We will show that G embeds in the double $H = C *_B C'$, of C over B, given by (1.1). To this end, we will use use an argument similar to the one in the proof of Proposition 2.9, even though in this case one could also argue by using normal forms more directly.

Note that there is a canonical retraction $\rho : H \to C$ which restricts to the identity map on C and to ξ^{-1} on C'. Let $y' = \xi(y) \in C'$ and set $h = y^{-1} y' \in H$, so that $h \in \ker \rho$. Since $\ker \rho \cap C = \{1\} = \ker \rho \cap C'$, we know that $F = \ker \rho$ is free, by Lemma 2.2. Choose a left transversal D for B in A' and define

$$U = \{dtd^{-1} \mid d \in D\} \subseteq N \quad \text{and} \quad V = \{dhd^{-1} \mid d \in D\} \subseteq F,$$

where $N \lhd G$ is the normal closure of t.

Using the Normal Form Theorem for amalgamated free products [LS77, Theorem 2.6 in Section IV.2], we see that h has infinite order in H. Now, suppose that there are distinct $d_1, d_2 \in D$ such that $d_1 h^m d_1^{-1} = d_2 h^q d_2^{-1}$ in H, for some $m \in \mathbb{N}$ and $q \in \mathbb{Z} \setminus \{0\}$. Then $d_1^{-1} d_2 \in A' \setminus B$ and we have

$$(2.9) \qquad 1 = d_2^{-1} d_1 h^m d_1^{-1} d_2 h^{-q} = d_2^{-1} d_1 (y^{-1} y')^m d_1^{-1} d_2 (y^{-1} y')^{-q} \text{ in } H.$$

If $q > 0$ then (2.9) amounts to

$$(2.10) \qquad (d_2^{-1} d_1 y^{-1})(y') \dots (y^{-1})(y')(d_1^{-1} d_2)(y'^{-1})(y) \dots (y'^{-1})(y) = 1,$$

where we placed the brackets in such a way that each term either belongs to $C \setminus B$ or to $C' \setminus \xi(B)$ and consecutive terms are from different factors. This means that the left-hand side of (2.10) is a non-empty reduced sequence in the amalgamated free product H (see [LS77, Section IV.2]), so equation (2.10) contradicts the Normal Form Theorem for amalgamated free products.

Thus we must have $q < 0$, so (2.9) becomes

$$(2.11) \qquad (d_2^{-1} d_1 y^{-1})(y') \dots (y^{-1})(y')(d_1^{-1} d_2 y^{-1})(y') \dots (y^{-1})(y') = 1.$$

As before this yields a contradiction, hence $\langle d_1 h d_1^{-1} \rangle \cap \langle d_2 h d_2^{-1} \rangle = \{1\}$ in H. Therefore, we can apply Lemma 2.8 to find $n \in \mathbb{N}$ such that the set $W = \{f^n \mid f \in V\}$ freely generates a free subgroup $M \leqslant F$ of rank $|D| = |A' : B|$. Since $h = y^{-1} y'$ centralizes $B = \xi(B)$ in H, we can define a homomorphism $\psi : G \to H$ by

$$\psi(a) = a, \text{ for all } a \in A', \quad \text{and } \psi(t) = h^n.$$

Just like in Proposition 2.9, one can show that ψ restricts to a bijection between N and M and use this to verify that ψ is injective. Thus, G embeds in H. □

Corollary 1.3 from the Introduction is an immediate consequence of Theorem 2.10 and the fact that finite p-groups are solvable.

3. Property (LR) for virtually free groups

In this section we prove the main Theorem 1.1. If G is a group, we will write $H \leqslant_{vr} G$ to say that H is a virtual retract of G.

Lemma 3.1 ([Min19, Lemma 3.2])**.** *Suppose that G is a group and there are subgroups $H \leqslant K \leqslant G$.*

(i) *If $H \leqslant_{vr} G$ then $H \leqslant_{vr} K$. In particular, every subgroup of a group with (LR) has (LR).*
(ii) *If $H \leqslant_{vr} K$ and $K \leqslant_{vr} G$ then $H \leqslant_{vr} G$.*

Let C be a finite group with a subgroup $B \leqslant C$, and let G be the double of C over B, as defined in (1.1). Let us recall the construction of the Bass-Serre tree T for G (see [Ser03, Theorem 7 in Section I.4]). The set vertices of T consists of the left cosets $\{gC \mid g \in G\} \sqcup \{gC' \mid g \in G\}$, and two vertices xC and yC' are adjacent if and only if there exists $z \in G$ such that $xC = zC$ and $yC' = zC'$. The group G acts on T naturally, on the left, without edge inversions. We denote by $\mathrm{Stab}_G(v)$ the *G-stabiliser* of a vertex v of T.

Note that the group G admits an involution $\gamma \in \mathrm{Aut}(G)$ defined by

$$\gamma(c) = \xi(c), \text{ for all } c \in C, \quad \text{and } \gamma(c') = \xi^{-1}(c'), \text{ for all } c' \in C'.$$

We let $\widetilde{G}$ denote the semidirect product $\widetilde{G} = G \rtimes \langle \gamma \rangle_2$, so that $|\widetilde{G} : G| = 2$. The next lemma is straightforward to verify using the observation that the action of γ interchanges C with C' in G.

Lemma 3.2. *The natural action of G on the Bass-Serre T extends to an action of $\widetilde{G}$ on T, where the element $\gamma \in \widetilde{G}$ acts as an inversion in the edge e_0 joining the vertices C and C' in T. In other words,*

$$\gamma.(xC) = \gamma(x)C' \quad \text{and} \quad \gamma.(xC') = \gamma(x)C, \quad \text{for all } x \in G.$$

Moreover, for every vertex v of T we have $\mathrm{Stab}_{\widetilde{G}}(v) = \mathrm{Stab}_G(v)$.

Let E denote the set of unoriented edges of T (any such edge is just a 2-element set $\{gC, gC'\}$, for some $g \in G$). Let $e_0 \in E$ be the edge of T between the vertices C and C'. Then G acts on E transitively, so we can use γ to define the *inversion in an edge* $e \in E$ as $i_e = g\gamma g^{-1} \in \widetilde{G}$, where $g \in G$ is any element such that $e = g.e_0$. Note that i_e is well-defined because γ commutes with $B = \mathrm{Stab}_G(e_0)$ in $\widetilde{G}$. We also observe that

$$i_e^2 = 1 \quad \text{and} \quad h i_e h^{-1} = i_{h.e}, \quad \text{for all } h \in G \text{ and } e \in E. \tag{3.1}$$

We are now ready to prove the main result of this section.

Theorem 3.3. *Let G be a double of a finite group C over a subgroup $B \leqslant C$. Then G has property (LR).*

Proof. We will assume that G is given by (1.1). Let T, E, $\gamma \in \mathrm{Aut}(G)$ and $\widetilde{G} = G \rtimes \langle \gamma \rangle_2$ be as described above. Suppose that $H \leqslant G$ is a finitely generated subgroup. Then, by [DD89, Theorem I.4.12 and Proposition I.4.13], there is a non-empty H-invariant subtree U in T on which H acts with finitely many orbits of vertices and edges.

Let ∂U denote the set of all edges $e \in E$ such that exactly one of the endpoints of e belongs to U. Since U is H-invariant, we see that the same is true for ∂U, so, in view of (3.1), we conclude that the subgroup

$$N = \langle i_e \mid e \in \partial U \rangle$$

is normalized by H in $\widetilde{G}$. We will show that $|\widetilde{G} : HN| < \infty$ and $H \cap N = \{1\}$, i.e., H is a virtual retract of $\widetilde{G}$.

Fix any vertex $u \in U$. Since H acts on U with finitely many orbits of vertices, there exist elements $g_1, \ldots, g_k \in G$ such that $G.u \cap U = \bigsqcup_{j=1}^k Hg_j.u$. Consider any $g \in \widetilde{G}$. We will use the induction on the edge-path distance $\mathrm{d}_T(g.u, U)$ to show that

$$g.u \in (\bigcup_{j=1}^k NHg_j).u. \tag{3.2}$$

If $g.u \in U$ then $g.u \in \bigcup_{j=1}^k Hg_j.u$ by the definition of g_j. Hence, we can assume that $\mathrm{d}_T(g.u, U) > 0$. Let $e \in \partial U$ be the first edge of the geodesic path in T joining U to $g.u$. Evidently, $\mathrm{d}_T(i_e.(g.u)), U) < \mathrm{d}_T(g.u, U)$, so $i_e.(g.u) \in (\bigcup_{j=1}^k NHg_j).u$, by the induction hypothesis. Since $i_e \in N$, we can conclude that (3.2) holds.

Now, recall that $D = \mathrm{Stab}_{\widetilde{G}}(u)$ is a finite group isomorphic to C, by Lemma 3.2. Combined with (3.2), this shows that $\widetilde{G} = \bigcup_{j=1}^{k} NHg_jD$, thus $|\widetilde{G} : NH| < \infty$.

It remains to show that $N \cap H = \{1\}$. For each edge $e \in \partial U$, removing the interior of e divides T into the union of two disjoint subtrees; let T_e denote the subtree that does not contain U. Note that $U \cap T_e = \emptyset$ and

$$i_e.(U \cup T_f) \subseteq T_e, \quad \text{whenever } e, f \in \partial U,\ e \neq f. \tag{3.3}$$

Now, if $g \in N \setminus \{1\}$ then $g = i_{e_1} \cdots i_{e_n}$, for $n \in \mathbb{N}$ and some $e_1, \ldots, e_n \in E$ satisfying $e_j \neq e_{j+1}$, for $j = 1, \ldots, n-1$. Then (3.3) implies that $g.u \in T_{e_1}$, hence $g.u \notin U$ and so $g \notin H$ (as $u \in U$ and U is H-invariant). Thus $N \cap H = \{1\}$. (Moreover, this also shows that $N \cong *_{e\in\partial U} \langle i_e \rangle$, i.e., N is isomorphic to the free product of cyclic groups of order 2.)

Therefore, we have proved that $H \leqslant_{vr} \widetilde{G}$, and since $H \leqslant G \leqslant \widetilde{G}$, we also have $H \leqslant_{vr} G$, by Lemma 3.1.(i). Hence G has (LR), as required. □

Remark 3.4. Note that in Theorem 3.3 the kernel N, of a virtual retraction of $\widetilde{G}$ onto H, only depends on the choice of an H-invariant subtree U in T, such that H acts on U cocompactly. If H is infinite then there is a natural choice for U as the unique minimal H-invariant subtree of T (U will be the union of all the axes of hyperbolic elements in H, see [DD89, Proposition I.4.13]). Thus, in the case when $|H| = \infty$ there is a canonical choice for N. Moreover, as shown in the proof of Theorem 3.3, N is isomorphic to a free product of cyclic groups of order 2 and H acts on this free product by permuting the free factors.

We are now ready to prove Theorem 1.1 from the Introduction.

Proof of Theorem 1.1. By Lemma 2.4, every virtually free group can be embedded in a multiple special HNN-extension G of a finite group A, given by (2.1). So, in view of Lemma 3.1.(i), it suffices to prove that G has (LR). Let H be a finitely generated subgroup of G. Then there is a finite subset $J \subseteq I$ such that H is contained in the subgroup $K = \langle A, \{t_j\}_{j\in J} \rangle$ in G. Evidently, there is a retraction $\rho : G \to K$ such that $\rho(t_i) = 1$, for all $i \in I \setminus J$.

Now, by Theorem 2.10, K embeds in a double L of some finite group, and L has (LR) by Theorem 3.3. Therefore, K has (LR) by Lemma 3.1.(i), so $H \leqslant_{vr} K$. Since K is a retract of G, it remains to apply Lemma 3.1.(ii) to conclude that $H \leqslant_{vr} G$. Thus G has (LR), and the proof is complete. □

4. More groups with (LR)

Property (LR) is not generally preserved by direct products (e.g., the direct square of the free group of rank 2 is not even subgroup separable [AG73]), but the situation is much better when one of the factors is finitely generated and virtually abelian.

Lemma 4.1 ([Min19, Proposition 5.6])**.** *If A is a finitely generated virtually abelian group and B is a group with (LR) then $A \times B$ has (LR).*

This lemma, combined with Lemma 3.1.(i) and Theorem 1.1, allows us to show that a group has property (LR) by embedding it in a direct product of a virtually free group and a virtually abelian group. In this section we give several examples when such embeddings arise.

Lemma 4.2. *Let G be a group commensurable with the direct product of a free group and a free abelian group of finite rank. Then G embeds as a finite index subgroup in the direct product of a virtually free group and a finitely generated virtually abelian group.*

Proof. The assumptions imply that G has a finite index subgroup K isomorphic to the direct product $F \times A$, where F is free and $A \cong \mathbb{Z}^n$, for some $n \in \mathbb{N}_0$. Let $N \lhd G$ be the normal core of K, so that $|G : N| < \infty$. Without loss of generality, we can assume that N projects onto the direct factors F and A of K.

If $\mathrm{rank}(F) \le 1$ then K is abelian, so G is virtually abelian. Thus we can further assume that $\mathrm{rank}(F) \ge 2$. Then F has trivial centre, so the center of N is $Z = N \cap A \lhd G$ and $|A : Z| < \infty$. Thus $FZ \cong F \times Z$ has finite index in K, hence $Z \cong \mathbb{Z}^n$ is a normal virtual retract of G. Now, by [MM25, Lemma 4.3], there is a normal subgroup $R \lhd G$ such that

$$Z \cap R = \{1\} \quad \text{and} \quad |G : ZR| < \infty. \tag{4.1}$$

Note that $N/Z \cong F$ has finite index in G/Z, hence G/Z is virtually free. Moreover, the image of Z has finite index in G/R, so G/R is finitely generated and virtually abelian. Now, (4.1) implies that the homomorphism $G \to G/Z \times G/R$, $g \mapsto (gZ, gR)$, is injective and its image has finite index. □

We can now prove Corollary 1.5 from the Introduction.

Proof of Corollary 1.5. According to Lemma 4.2, G embeds as a finite index subgroup in the direct product of a virtually free group with a finitely generated virtually abelian group. Therefore, we deduce that G has (LR) by combining Theorem 1.1, Lemma 4.1 and Lemma 3.1.(i). It then follows that G is subgroup separable, see [Min19, Lemma 5.1.(iii)].

Finitely generated virtually abelian groups are conjugacy separable by [Seg83, Proposition 1 in Section 4.C], and virtually free groups are conjugacy separable by Dyer's result [Dye79, Theorem 3]. Both of these classes of groups are closed under taking arbitrary subgroups, hence groups in these classes are hereditarily conjugacy separable. So, G embeds as a finite index subgroup in a direct product of hereditarily conjugacy separable groups, hence G is itself hereditarily conjugacy separable by [MM12, Lemma 7.3], and (ii) holds.

In [RZ16, Lemma 6] it is shown that a virtual retract of a hereditarily conjugacy separable group is always conjugacy distinguished. Therefore, claim (iii) follows from claims (i) and (ii). □

Given a natural number $n \in \mathbb{N}$, a *generalized Baumslag-Solitar group of rank n* (GBS_n for short) is a group decomposing as the fundamental group of a finite graph of groups, where all vertex and edge groups are isomorphic to $\mathbb{Z}^n$. Such a group G will necessarily commensurate any vertex group, which gives rise to a *modular homomorphism* $\mathcal{M} : G \to \mathrm{GL}(n, \mathbb{Q})$, from G to the abstract commensurator $\mathrm{GL}(n, \mathbb{Q})$ of $\mathbb{Z}^n$, see [But25]. If the image

$\mathcal{M}(G)$ is finite in $\mathrm{GL}(n,\mathbb{Q})$, we will say that the GBS_n group G has *finite monodromy*. When $n = 1$ we have $\mathrm{GL}(1,\mathbb{Q}) = \mathbb{Q}^*$, so a GBS_1 group G has finite monodromy if and only if $\mathcal{M}(G) \subseteq \{-1, 1\}$, i.e., G is *unimodular* in the sense of Levitt [Lev07].

With the help of Corollary 1.5, we can characterize property (LR) in the class of all GBS_n groups.

Corollary 4.3. *For any $n \in \mathbb{N}$, a GBS_n group G has (LR) if and only if it has finite monodromy.*

Proof. If G has (LR), then, in particular, it has (VRC), which means that every cyclic subgroup of G is a virtual retract. The latter implies that G has finite monodromy, by a recent result of Wang [Wan26, Theorem 1.2].

Conversely, suppose that G has finite monodromy. In this case Button [But25, Theorem 3.4] proved that G has a finite index subgroup splitting as a direct product of a finite rank free group with $\mathbb{Z}^n$, hence G has (LR) by Corollary 1.5. □

Another method to generate groups satisfying the assumptions of Corollary 1.5 is by using certain extensions. If A is a normal subgroup of a group G then G acts on A by conjugation, giving rise to a homomorphism $G \to \mathrm{Aut}(A)$. By taking a further quotient by $\mathrm{Inn}(A)$, we get natural homomorphism $G \to \mathrm{Out}(A)$. Since A is contained in the kernel of the latter homomorphism, it induces a *natural homomorphism* $G/A \to \mathrm{Out}(A)$.

Lemma 4.4. *Assume that G is an extension of a normal subgroup A by a group $B = G/A$, where A has finite center, B is virtually polycyclic and the natural homomorphism $B \to \mathrm{Out}(A)$ has finite image. Then G has a finite index subgroup isomorphic to the direct product $A \times B'$, where B' is a finite index subgroup of B.*

Proof. The assumption that the image of B in $\mathrm{Out}(A)$ is finite implies that $|G : A\,\mathrm{C}_G(A)| < \infty$ (cf. [MM25, Remark 4.6]). We also know that $A \cap \mathrm{C}_G(A) = \mathrm{Z}(A)$ is finite and $\mathrm{C}_G(A)/\mathrm{Z}(A)$ is a subgroup of the virtually polycyclic group B. Therefore, $\mathrm{C}_G(A)$ is itself virtually polycyclic (see [Seg83, Proposition 2 in Section 1.A]) and, in particular, it is residually finite ([Seg83, Theorem 1 in Section 1.C]). Thus, there is a finite index subgroup $B' \lhd_f \mathrm{C}_G(A)$ such that $B' \cap A = B' \cap \mathrm{Z}(A) = \{1\}$. Then $AB' \cong A \times B'$, $|G : AB'| < \infty$ and B' is isomorphic to a finite index subgroup of B because it maps injectively into B under the quotient homomorphism $G \to G/A = B$. □

Since virtually free groups that are not virtually cyclic have finite centers, we can combine Lemma 4.4 with Lemma 4.2 to obtain the following.

Corollary 4.5. *Suppose that a group G has a virtually free normal subgroup $A \lhd G$ such that $B = G/A$ is finitely generated and virtually abelian. If A is not virtually cyclic and the natural homomorphism $B \to \mathrm{Out}(A)$ has finite image then G embeds as a finite index subgroup in the direct product of a virtually free group and a finitely generated virtually abelian group. In particular, G satisfies claims (i)–(iii) of Corollary 1.5.*

In Corollary 4.5 it is indeed necessary to assume that A is not virtually cyclic. For example, the Heisenberg group $H_3(\mathbb{Z})$ is a central extension of $\mathbb{Z}$ by $\mathbb{Z}^2$ but it does not have (LR) (see [Min19, Example 9.2]).

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