QFT2
Gaussian Integration and functional derivatives
Problem Sheet 1

For Sussex students: Solutions should be handed in for marking at the school office, before noon, Thurs 21st February 2019. For Southampton students: please put your solutions in my pigeon hole. Please present your work in a well structured and concise manner.

Note: The two questions carry a maximum of 10 marks each.

1. Gaussian Integration

Prove the following results:

(i)

$$\int_{-\infty}^{\infty} dx \exp\left[-\frac{1}{2}\alpha x^2\right] = \left(\frac{2\pi}{\alpha}\right)^{1/2}$$

3 marks

(ii)

$$\int_{-\infty}^{\infty} dx \exp\left[-\frac{1}{2}\alpha x^2 + \beta x\right] = \left(\frac{2\pi}{\alpha}\right)^{1/2} \exp\left[\frac{\beta^2}{2\alpha}\right]$$

3 marks

(iii)

$$\int_{-\infty}^{\infty} \prod_{i=1}^{n} dx_i \exp\left[-\frac{1}{2}\sum_{i,j=1}^{n} x_i \alpha_{ij} x_j + \sum_{k=1}^{n} \beta_k x_k\right]$$
$$= \left(\frac{(2\pi)^n}{\det\alpha}\right)^{1/2} \exp\left[\frac{1}{2}\sum_{i,j=1}^{n} \beta_i (\alpha^{-1})_{ij} \beta_j\right]$$

4 marks

[Hint: Regarding α as a real symmetric matrix, diagonalise it with a real orthogonal matrix, then change to the diagonal variables x'_i, assuming $\prod_{i=1}^{n} dx_i = \prod_{i=1}^{n} dx'_i$, and use the result from (ii), with $\det\alpha$ being a product of the eigenvalues of α.]

2. Functional derivatives [10]

Using the definition of a functional derivative:

$$\frac{\delta F[f(x)]}{\delta f(y)} = \lim_{\varepsilon \to 0} \frac{F[f(x) + \varepsilon\delta(x-y)] - F[f(x)]}{\varepsilon}$$

(i) Compute the functional derivative w.r.t. $f(y)$ of the functional:

$$F[f(x)] = \int_{-\infty}^{\infty} f(x)dx$$

1 mark

(ii) Compute the functional derivative w.r.t. $q(t')$ of the functional:

$$S[q(t)] = \int_{-\infty}^{\infty} L(q(t))dt$$

3 marks

(iii) Compute the functional derivative w.r.t. $q(t')$ of the functional:

$$S[q(t)] = \int_{-\infty}^{\infty} L(\dot{q}(t))dt, \qquad \text{where} \quad \dot{q}(t) = \frac{dq(t)}{dt}$$

3 marks

[Hint: use definition of functional derivative followed by integration by parts.]

(iv) Compute the functional derivative w.r.t. $J(t')$ of the functional:

$$Z[J(t)] = \int \mathcal{D}q(t) \exp\left[i \int (L(q(t)) + J(t)q(t))dt\right]$$

3 marks