Path Integrals. I&Z, Dirac QM, Feynman & Hibbs.

Revision QM (1D)

$$[p, q] = -i$$

Schrödinger rep^n: time dependence carried by wavef^n through $H|\psi\rangle = i\frac{\partial}{\partial t}|\psi\rangle$ (20.1)

(Schrödinger eq^n)

Can solve (20.1) for time indept Hamiltonian:

$$|\psi(t)\rangle = e^{-iHt}|\psi(0)\rangle \qquad (20.2)$$

Operators $\mathcal{O}$ do not carry time evolution

ie. $\frac{\partial}{\partial t}\mathcal{O} = 0$ but expectation values

$$\langle \mathcal{O}\rangle(t) = \langle\psi(t)|\mathcal{O}|\psi(t)\rangle$$

carries time dependence & using (20.1) ⇒

$$\frac{\partial}{\partial t}\langle\mathcal{O}\rangle = i\langle[H, \mathcal{O}]\rangle \qquad (20.5)$$

Heisenberg rep^n All physics results depend on expectation values or amplitudes $\langle\psi|\psi'\rangle$.

Get same physics ∴ if we use unitary transf^{ns}: $\mathcal{U}^{-1} = \mathcal{U}^{+}$

$$|\psi\rangle \mapsto \mathcal{U}|\psi\rangle \qquad \mathcal{O} \mapsto \mathcal{U}\mathcal{O}\mathcal{U}^{-1}$$

on all wavef^{ns} & operators (Dirac's transf^n theory)

Choose $\mathcal{U} = e^{iHt}$. This maps from Schrödinger repⁿ to Heisenberg repⁿ.

See from (20.2) or (20.1) that in Heisenberg repⁿ $\frac{\partial}{\partial t}|\psi\rangle = 0$ Wavefⁿˢ are fixed & time invariant.

But operators satisfy

$$\frac{\partial}{\partial t}\mathcal{O}(t)_{\text{Heisenberg}} = \frac{\partial}{\partial t}\left(e^{iHt}\,\mathcal{O}_{\text{Schrö}}\,e^{-iHt}\right)$$

$$= i\,[H, \underbrace{\mathcal{O}}_{\text{Heisenberg}}]$$

So now (20.5) holds as an <u>operator statement</u>

(Heisenberg eqⁿ motion related via Poisson brackets $\Leftrightarrow i[\,,\,]$ to Hamiltons eqⁿˢ.)

<u>N.B.</u> Like moving to a rotating coordinate system from an inertial system ...

Work in <u>Heisenberg</u> repⁿ from now on:

$p \equiv p(t)$, $q \equiv q(t)$ and (of course)

$$[p(t), q(t)] = -i \quad \text{still.} \qquad \boxed{\text{Ex!}} \qquad (22\cdot 2)$$

Define position and momentum eigenstates for the operators at different times i.e.

real # !

$$p(t)\,|p', t\rangle = p'\,|p', t\rangle$$

$$q(t)\,|q', t\rangle = q'\,|q', t\rangle$$

<u>N.B.</u> the 't' here is a <u>label</u> <u>not</u> how the ket evolves !!! (ket's <u>don't</u> evolve in Heisenberg repⁿ.)

It will be helpful in particular to look at the eigenstates for $q(0)$ & $p(0)$ i.e. for q & p at $t=0$. Use <u>shorthand</u> for these $|q'\rangle \equiv |q', 0\rangle$ & $|p'\rangle \equiv |p', 0\rangle$

i.e.

$$q(0)\,|q'\rangle = q'\,|q'\rangle$$

$$p(0)\,|p'\rangle = p'\,|p'\rangle$$

Expanding any ket $|\psi\rangle$ in $|q'\rangle$ basis gives pos^n rep^n:

Span & ortho normal

$$|\psi\rangle = \int_{-\infty}^{\infty} dq'\, |q'\rangle\, \psi(q') \qquad (23\cdot 2)$$

so by orthonormality: $\langle q'|q''\rangle = \delta(q'-q'')$,

$$\psi(q') = \langle q'|\psi\rangle \qquad (23\cdot 3)\ (23\cdot 4)$$

By (22·2) $p(0)$ has rep^n $-i\frac{\partial}{\partial q'}$ acting on $\psi(q')$. (Proof: $q(0)$ is represented by q' since $q(0)|\psi\rangle = \int_{-\infty}^{\infty} dq'\, |q'\rangle\, q'\psi(q')$

Clearly rep^n $p(0)|\psi\rangle = \int_{-\infty}^{\infty} dq'\, |q'\rangle\, -i\frac{\partial}{\partial q'}\psi(q')$ then satisfies (22·2). Uniqueness follows by theorem due to Neumann.) Considering eigenstate $|p'\rangle$ we then have by (23·3) & (23·4):

$$-i\frac{\partial}{\partial q'} \underbrace{\langle q'|p'\rangle}_{\psi_{p'}(q')} = p' \langle q'|p'\rangle$$

A result we will need

$$\Rightarrow \quad \langle q'|p'\rangle = \frac{1}{\sqrt{2\pi}}\, e^{ip'q'} \qquad (23\cdot 8)$$

Normalisation

$\psi_{p'}(q')$ is $|p'\rangle$ in pos^n basis.

Ex: $\int_{-\infty}^{\infty} dq'\, \langle p'|q'\rangle \langle q'|p''\rangle = \delta(p'-p'')$

Solve for eigenstates $|q', t\rangle$ in terms of eigenstates of $q(0)$, i.e. $|q'\rangle$:

$$\frac{\partial}{\partial t} q(t) = i[H, q(t)]$$

$$\Rightarrow \quad q(t) = e^{iHt} q(0) e^{-iHt}$$

$$\text{or} \quad q(0) = e^{-iHt} q(t) e^{iHt}$$

Then $q(0)|q'\rangle = q'|q'\rangle \Rightarrow$

$$q(t)\{e^{iHt}|q'\rangle\} = q'\{e^{iHt}|q'\rangle\}$$

Since for given q' the eigenvector of $q(t)$ is unique we have

$$\underline{\underline{|q', t\rangle = e^{iHt}|q'\rangle}} \qquad (24.6)$$

Ex: A question of interpretation. Compare (20.2). Theres a sign difference in the exponential, but does not indicate an error ....

Ex: Substitute (23.3) into (23.2) to get

<u>Completeness Rel^n</u> : $1 = \int_{-\infty}^{\infty} dq' \, |q'\rangle\langle q'|$

Amplitude for particle initially known to be at pos^n q_i at time t_i to be found later at pos^n q_f at time t_f is:

$$\langle q_f, t_f \mid q_i, t_i \rangle$$

= component of $|q_f, t_f\rangle$ in expansion of wavef^n $|q_i, t_i\rangle$ in $q(t_f)$ eigenstate basis.

N.B. Use (24.6) to write:

$$= \langle q_f | e^{-iH(t_f - t_i)} | q_i \rangle \qquad (25.5)$$

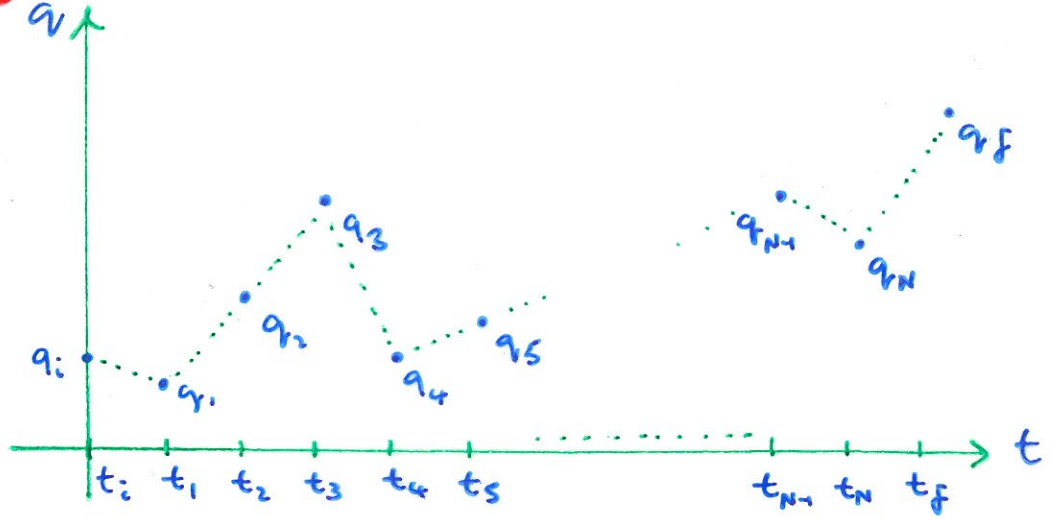
→ by (20.2) this is the amplitude for the process in the Schrödinger rep^n.
→ Same physics from both rep^{ns} as req^d.

Insert a complete set of states $|q_j, t_j\rangle$ at time intervals t_j $j = 1, 2, \ldots, N$ s.t.
$t_i < t_1 < t_2 \cdots < t_N < t_f$

$$\langle q_f, t_f \mid q_i, t_i \rangle =$$

$$\int_{-\infty}^{\infty} dq_1 \, dq_2 \cdots dq_N \, \langle q_f, t_f | q_N, t_N \rangle \langle q_N, t_N | q_{N-1}, t_{N-1} \rangle \cdots \cdots \langle q_2, t_2 | q_1, t_1 \rangle \langle q_1, t_1 | q_i, t_i \rangle \qquad (25.10)$$

Physical interpretation:



For given values q_j we know the particle passed through these points.

The full amplitude is a 'sum' over all such paths (cf. Young's slits). weighted by the corresponding amplitudes (cf. [25.10]).

Evaluating amplitude elements [cf (24.6)]:

$$\langle q_{\alpha+1}, t_{\alpha+1} | q_\alpha, t_\alpha \rangle = \langle q_{\alpha+1} | e^{-iH\underbrace{(t_{\alpha+1} - t_\alpha)}_{\delta t_\alpha}} | q_\alpha \rangle \quad (27.1)$$

Take $H = \frac{p^2}{2m} + V(q)$, then

$$e^{-iH\delta t_\alpha} = e^{-i\frac{p^2}{2m}\delta t_\alpha} \, e^{-iV(q)\delta t_\alpha} + O(\delta t_\alpha^2) \quad (27.3)$$

[N.B. Baker Campbell Hausdorff formula:

$$e^A e^B = \exp\left\{ A + B + \tfrac{1}{2}[A, B] + \text{double commutators} + \cdots \right\}$$]

Inserting a complete set of states $|p_\alpha\rangle$ & using (27.1, 3) $\Rightarrow$

$$\langle q_{\alpha+1}, t_{\alpha+1} | q_\alpha, t_\alpha \rangle = \int_{-\infty}^{\infty} dp_\alpha \, \langle q_{\alpha+1} | e^{i\frac{p^2}{2m}\delta t_\alpha} | p_\alpha \rangle \langle p_\alpha | e^{-iV(q)\delta t_\alpha} | q_\alpha \rangle$$

$$= \int_{-\infty}^{\infty} dp_\alpha \, e^{-i\frac{(p_\alpha)^2}{2m}\delta t_\alpha - iV(q_\alpha)\delta t_\alpha} \underbrace{\langle q_{\alpha+1} | p_\alpha \rangle \langle p_\alpha | q_\alpha \rangle}_{\frac{1}{2\pi} e^{i(q_{\alpha+1} - q_\alpha) p_\alpha}} \quad (27.6)$$

cf. [23.8]

A useful intermediate result:

(27.6) $\Rightarrow$

$$\langle q_{\alpha+1}, t_{\alpha+1} | q_\alpha, t_\alpha \rangle = \int_{-\infty}^{\infty} dp_\alpha \; e^{i L_\alpha \delta t_\alpha}$$

where $$\mathcal{L}_\alpha = \frac{q_{\alpha+1} - q_\alpha}{\delta t_\alpha} p_\alpha - H(q_\alpha, p_\alpha)$$

or $$H(q_\alpha, p_\alpha) = \frac{q_{\alpha+1} - q_\alpha}{\delta t_\alpha} p_\alpha - L_\alpha$$

Discretized version of $H = \dot{q} p - L$

$\Rightarrow L_\alpha =$ Lagrangian $+ O(t_\alpha^2)$

Integrate over p_α $\boxed{\text{Ex.!}}$ to get:

$$\langle q_{\alpha+1}, t_{\alpha+1} | q_\alpha, t_\alpha \rangle \propto e^{i L_\alpha \delta t_\alpha}$$

large const $\sim \frac{1}{\sqrt{\delta t_\alpha}}$

$$\mathcal{L}_\alpha = \frac{m}{2} \left(\frac{q_{\alpha+1} - q_\alpha}{\delta t_\alpha} \right)^2 - V(q_\alpha)$$

All together (25.10) reads:

$$\langle q_f, t_f | q_i, t_i \rangle \propto \int_{-\infty}^{\infty} dq_1 \cdots dq_N \, \exp\left\{ i L_{(i)} \delta t_{(i)} + i \sum_{\alpha=1}^{N} \delta t_\alpha L_\alpha \right\}$$

$\alpha = i$

Take "limit" in which $N \to \infty$ ($\delta t_\alpha \to 0$)

then

$$\langle q_f, t_f | q_i, t_i \rangle = \frac{1}{Z_0} \int \mathcal{D} q(t) \; e^{iS[q]} \qquad (29.1)$$

"Path integral" or "Functional Integral"

$$S[q] = \int_{t_i}^{t_f} dt \; L(q(t), \dot{q}(t))$$

= Integral over ∞ dim^{nl} space!

Comments.

① Could have left p_α integral

$$\langle q_f, t_f | q_i, t_i \rangle = \frac{1}{\tilde{Z}_0} \int \mathcal{D}[q(t), p(t)] \; e^{iS} \qquad (29.7)$$

Ex!

where $S = \int_{t_i}^{t_f} dt \left\{ \dot{q}(t) p(t) - H(q, p) \right\}$

② If we had not set $\hbar = 1$ we would have found:

$$\langle q_f, t_f | q_i, t_i \rangle = \frac{1}{Z_0} \int \mathcal{D}[q(t)] \; e^{\frac{i}{\hbar} S[q]}$$

Ex!
$[p, q] = -i\hbar$
or
by dim^{nl}!

③ Large fluctuations in $q \Rightarrow S \gg \hbar$

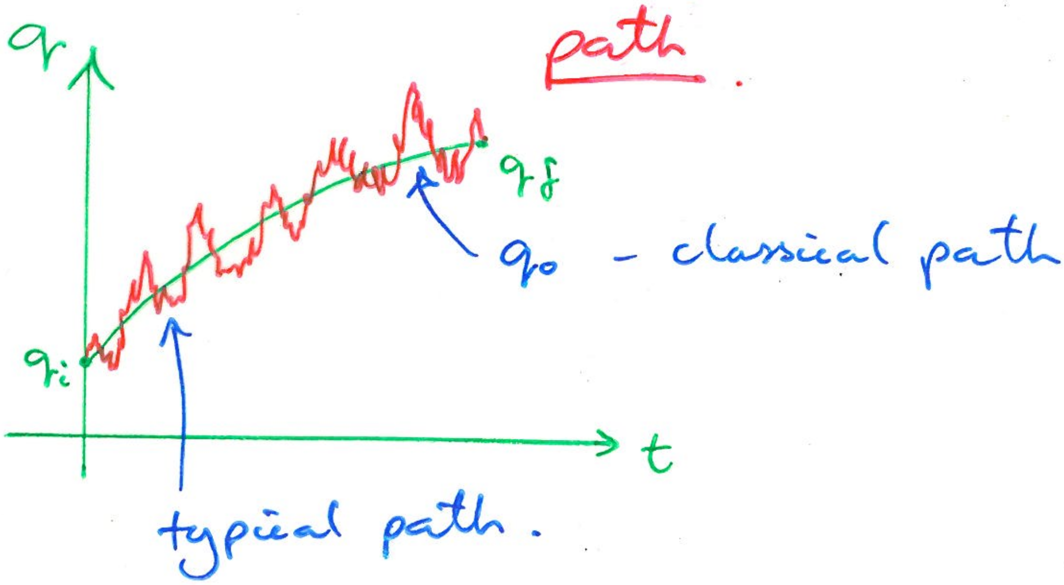
$\Rightarrow$ rapidly oscillating phases $e^{\frac{i}{\hbar}S}$ that cancel each other out.

Paths that contribute as $\hbar \to 0$ are those $q(t) = q_0(t) + \delta q(t)$ such that →

$$S[q_0+\delta q] = S[q_0] + \int_{t_i}^{t_f} \frac{\delta S}{\delta q(t)}\,\delta q(t)\,dt + O(\delta q^2)$$

→ 1st order fluctuations vanish i.e.

$$\frac{\delta S}{\delta q}[q_0] = 0$$ i.e. q_0 is classical path.



As $\hbar \to 0$

$$\langle q_f, t_f | q_i, t_i \rangle \propto e^{\frac{i}{\hbar}S[q_0]}$$

④ General wavef^{ns}.

Schrödinger picture: Suppose system starts at time $t = t_i$ in state $|\psi(t_i)\rangle$ then by (25·5) & (23·3): $\psi(q_i, t_i) = \langle q_i | \psi(t_i)\rangle$

& $\psi(q_f, t_f) = \langle q_f | \psi(t_f)\rangle$

$$= \int_{-\infty}^{\infty} dq_i \, \underbrace{\langle q_f, t_f | q_i, t_i\rangle}_{\text{Green's } f^n \text{ for time evolution.}} \langle q_i | \psi(t_i)\rangle$$

Heisenberg Picture: Amplitude to go from $\psi_i(q_i)$ @ t_i to $\psi_f(q_f)$ @ t_f is:

$$\langle \psi_f | \psi_i \rangle = \iint_{-\infty}^{\infty} dq_f \, dq_i \, \langle \psi_f | q_f, t_f\rangle \langle q_f, t_f | q_i, t_i\rangle \langle q_i, t_i | \psi_i\rangle$$

$$\overset{[23\cdot 3]}{=} \iint_{-\infty}^{\infty} dq_f \, dq_i \, \psi_f^*(q_f) \langle q_f, t_f | q_i, t_i\rangle \psi_i(q_i)$$

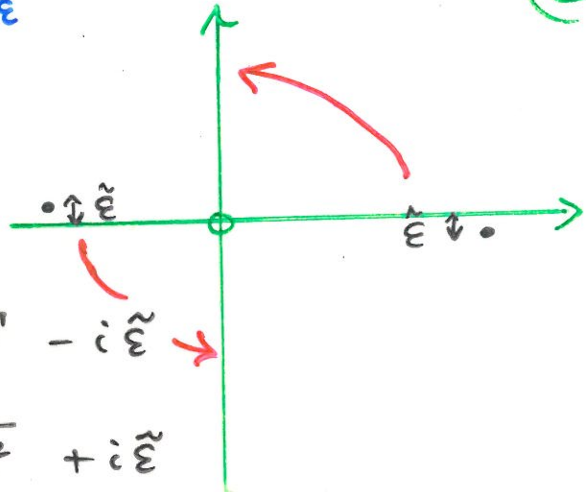
Interlude: Wick Rotation.

Recall scalar propagator

$$i\Delta_F(x) \equiv \int d^4p \, \frac{i}{p^2 - m^2 + i\varepsilon} \, e^{-ip.x} \qquad (32\text{-}1)$$

$(E = p^0)$

Complex energy plane:



Feynman $i\varepsilon$ prescription puts poles at $E = \sqrt{\underline{p}^2 + m^2} - i\tilde{\varepsilon}$

$$= -\sqrt{\underline{p}^2 + m^2} + i\tilde{\varepsilon}$$

Equivalently poles are such that 1st & 3rd quadrants of complex energy plane are singularity free. Therefore can deform integral along energy contour to point up imaginary axis. Then write $p_0 = E = ip_4$

$p_4 \in \mathbb{R}$

$x_0 = -ix_4$

After this you need to change vars in (32-1) as $p_4 \to -p_4$. Result: $\boxed{\text{Ex.}}$

$$\text{Propagator} = \int d^4p \, \frac{1}{p^2 + m^2} \, e^{-ip.x}$$

where.

well behaved – no singul

now

$$p^2 = p_1^2 + p_2^2 + p_3^2 + p_4^2$$

and $p \cdot x = \sum_{i=1}^{4} p_i x_i$

i.e. we have Euclidean 4-space with O(4) symmetry!

Correct singularity structure so enforced by "Wick rotating" back to real axis with the assumption that no singularities are encountered on the way.

Wick rotation of Path Integral

$$S = \int_{t_i}^{t_f} dt \left\{ \frac{1}{2} m \left(\frac{dq}{dt}\right)^2 - V(q) \right\}$$

Write $x_4 = \tau$ where $\tau = it$ then

$$= i \underbrace{\int_{\tau_i}^{\tau_f} d\tau \left\{ \frac{1}{2} m \left(\frac{dq}{d\tau}\right)^2 + V(q) \right\}}_{S_E}$$

$\leftarrow$ well behaved (S_E bounded from below)

$$\langle q_f, \tau_f | q_i, \tau_i \rangle = \frac{1}{Z_0} \int \mathcal{D}q \; e^{-S_E}$$

like a probability density

N. Wiggly paths are damped out because

$e^{-S_E} << 1$

$$\int \mathcal{D}q \; e^{-\int d\tau \frac{1}{2} m \left(\frac{dq}{d\tau}\right)^2}$$

<u>Wiener measure</u>

→ well defined construct.

<u>Typical Paths.</u>

Wiggliness controlled by the above term.

In discretized case

$$e^{-S_E^{\text{kinetic}}} \sim e^{-\sum_\alpha \frac{1}{2} m \frac{(q_{\alpha+1} - q_\alpha)^2}{\delta\tau_\alpha}}$$

Gaussian width $\sim \sqrt{m \delta\tau_\alpha}$

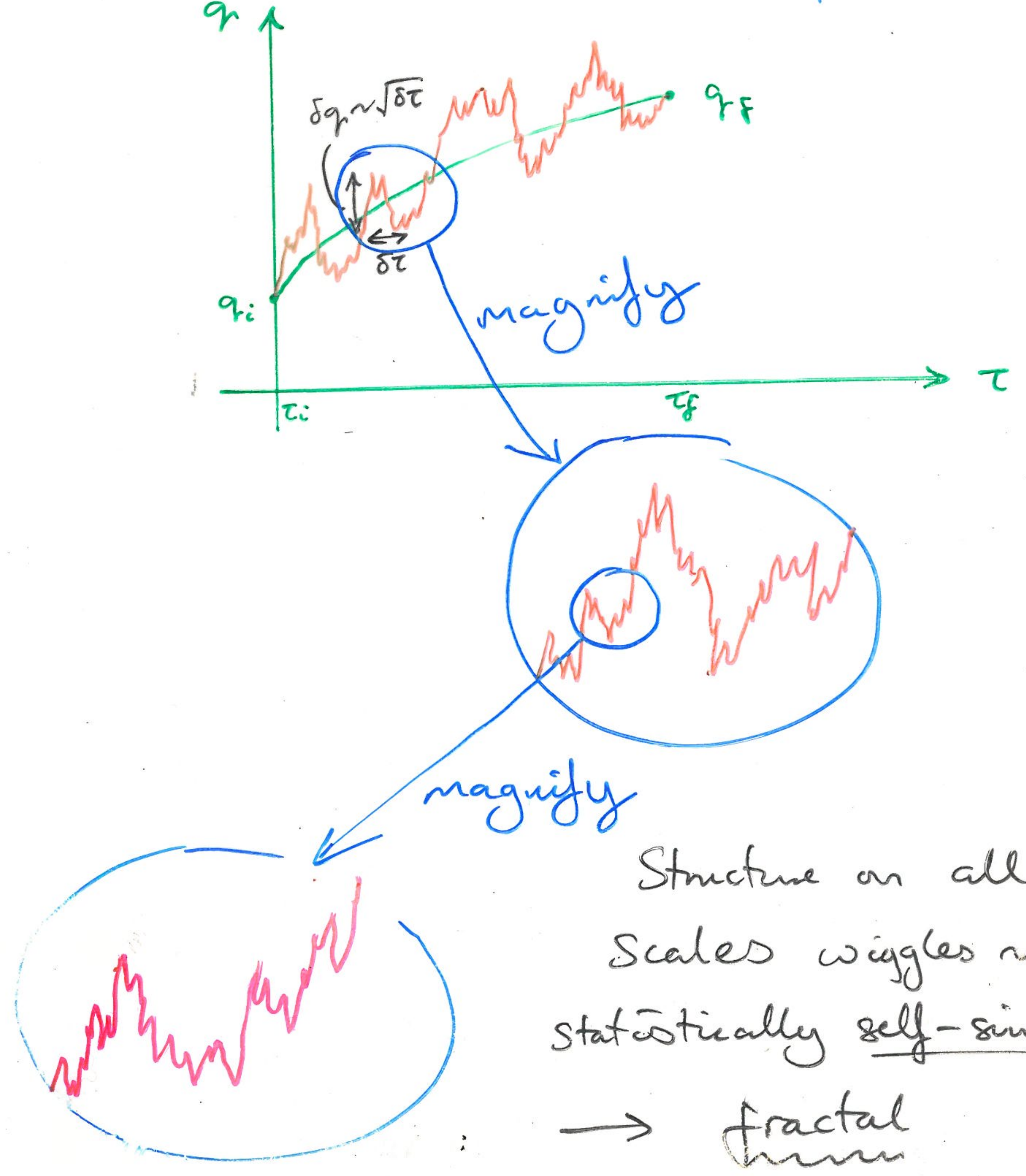
Thus typical ~~set~~ step $\delta q = q_{\alpha+1} - q_\alpha \sim \sqrt{\delta\tau_\alpha}$

As $\delta\tau \to 0$, $\delta q \to 0$

Paths are <u>continuous</u>

But $\dfrac{\delta q}{\delta \tau} \sim \dfrac{1}{\sqrt{\delta\tau}} \longrightarrow \infty$

Paths are <u>nowhere differentiable</u>.



Structure on all scales wiggles $\sim \sqrt{\delta\tau}$, statistically <u>self-similar</u>

$\longrightarrow$ <u>fractal</u>

- N.B. fractal looks like Brownian motion.

No accident: it is Brownian motion.
(ie. Wick rotated)

Schrödinger eq^n in Euclidean time:

$$\frac{1}{2m}\frac{\partial^2 \psi}{\partial q^2}\left(+V\psi\right)=\frac{\partial \psi}{\partial \tau}$$

← Diffusion eq^n!

- QM: Cannot know pos^n (q) and mom^{tm} ($p = m\dot{q}$) simultaneously. But for each path in path integral we know $q(t)$ with certainty ∴ uncertainty in mom^{tm} $\Delta p = \infty$.

∴ Actual value of p with probability $= 1$ ("almost certain") is $p = \pm\infty \Leftrightarrow \underline{\underline{\dot{q} = \pm\infty}}$, as we have already shown!

- N.N.B. See Feynman and Hibbs: if we replace this analysis with the relativistic one we would deduce $\dot{q} = \pm c$. Feynman shows that the path integral then gives sol^n to Dirac eq^n in one dim^n !!!

- <u>Connection to Thermodynamics.</u>

$$\langle q_f, \tau_f \mid q_i, \tau_i \rangle = \frac{1}{Z_0} \int \mathcal{D}q \, e^{-S_E}$$

where $S_E = \int_{\tau_i}^{\tau_f} d\tau \left\{ \underbrace{\frac{1}{2} m \left(\frac{dq}{d\tau}\right)^2 + V(q)}_{\mathcal{L}_E = H \ !} \right\}$

Partition function of thermodynamics:

$$Z = \sum_{\substack{\text{Energy} \\ \text{Eigenstates } i}} e^{-\beta E_i} \qquad \beta = \frac{1}{kT}$$

Boltzmann's const.

$$= \sum_i \langle i | e^{-\beta H} | i \rangle$$

$$= \int_{-\infty}^{\infty} dq' \sum_i \langle i | q' \rangle \langle q' | e^{-\beta H} | i \rangle$$

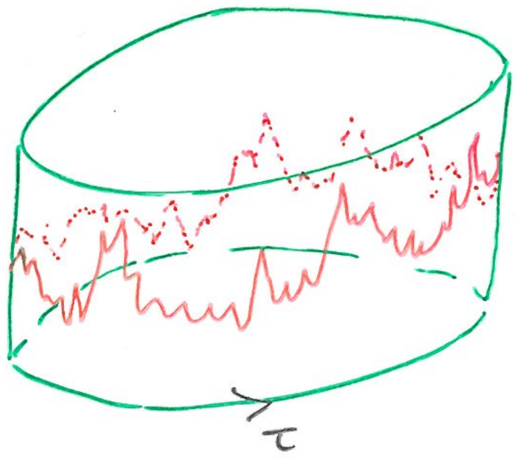
$$= \int_{-\infty}^{\infty} dq' \underbrace{\langle q' | e^{-\beta H} | q' \rangle}_{\langle q', -i\beta \mid q', 0 \rangle}$$

[Use (24.6) & then c.c.]

$$= \int_{-\infty}^{\infty} dq' \frac{1}{Z_0} \int \mathcal{D}q \, e^{-S_E}$$

where 'sum' is over all paths, lasting for Euclidean time β & starting and finishing at the same point.

i.e.



Euclidean time is compactified on a circle.

← circumference $= \beta$

This is

Equilibrium thermodynamics ; real time not a relevant concept. In QFT find a similar picture : real time direction replaced by compactified Euclidean time. For very hot QFT $\beta \to 0$ and dimensional reduction takes place to a Euclidean QFT with one less dim^n. I.e. very hot 4D QFT has properties (correlation f^{ns} etc) which are those of 3D Euclidean QFT!

Remark: Fermions turn out to require antiperiodic boundary cond^{ns} (Möbius strip).

- Factorization property of path integrals.

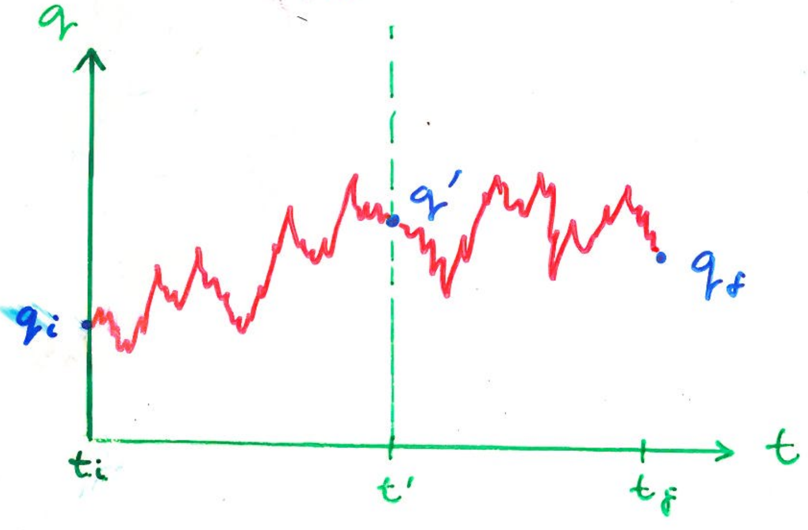
$$\langle q_f, t_f | q_i, t_i \rangle = \frac{1}{Z_0} \int_{q(t_i)=q_i}^{q(t_f)=q_f} \mathcal{D}q \; e^{iS}$$

But, by their construction / or directly we can write:

$$\langle q_f, t_f | q_i, t_i \rangle = \int_{-\infty}^{\infty} dq' \, \langle q_f, t_f | q', t' \rangle \langle q', t' | q_i, t_i \rangle$$

where $t_f > t' > t_i$

$$= \int_{-\infty}^{\infty} dq' \, \frac{1}{Z_0^{(1)} Z_0^{(2)}} \int_{q(t_i)=q_i}^{q(t')=q'} \mathcal{D}q \; e^{iS} \int_{q(t')=q'}^{q(t_f)=q_f} \mathcal{D}q \; e^{iS}$$



Recall (29.7): • A little f^nal analysis..

$$\langle q_f, t_f | q_i, t_i \rangle = \frac{1}{Z_0} \int \mathcal{D}[q(t), p(t)]\, e^{iS[q,p]} \qquad (40.1)$$

where $S[q,p] = \int_{t_i}^{t_f} dt \, \{ \dot{q}(t)\, p(t) - H(q,p) \}$

As an exercise in functional calculus let's show this is equivalent to $\langle q_f, t_f | q_i, t_i \rangle = \frac{1}{Z_0} \int \mathcal{D}q \, e^{iS[q]}$

i.e. (29.1).

$$H = \frac{p^2}{2m} + V(q)$$

so (40.1) is

$$\langle q_f, t_f | q_i, t_i \rangle = \frac{1}{Z_0} \int \mathcal{D}[p,q]\, e^{i\int dt \left\{ -\frac{1}{2m}(p - m\dot{q})^2 + \frac{m}{2}\dot{q}^2 - V(q) \right\}}$$

by completing the square.

Change vars $p' = p - m\dot{q} \Rightarrow$

$$= \frac{1}{\tilde{Z}_0} \int \mathcal{D}[p', q]\, e^{i\int dt \left\{ -\frac{1}{2m} p'^2 + L(q, \dot{q}) \right\}}$$

But $\int \mathcal{D}p' \, e^{-i\int dt \frac{p'^2}{2m}} \propto \prod_\alpha \int_{-\infty}^{\infty} dp'_\alpha \, e^{-\frac{i}{2m}\delta t_\alpha (p'_\alpha)^2} = \text{const.}$

$$\therefore \quad = \frac{1}{Z_0} \int \mathcal{D}q \, e^{i\int dt\, L}$$

by redef^n of overall const.

Operator Insertion & Time Ordering

Recall that LSZ reduction allows to relate S matrix elements to correlators $\langle 0 | T \{ \varphi(x_1) \varphi(x_2) \cdots \varphi(x_n) \} | 0 \rangle$

Thermodynamics!

More generally, its useful to compute

$$\langle 0 | T \, \mathcal{O}_1(x_1) \, \mathcal{O}_2(x_2) \cdots \mathcal{O}_n(x_n) | 0 \rangle$$

vacuum in far past

vacuum in far future.

where the $\mathcal{O}_i$ are operators made out of the fields

eg. $T_{\mu\nu}(x)$ stress-energy tensor.

$J_\mu(x) \sim \bar{\psi}(x) \gamma_\mu \psi(x)$ etc.

Quantum mechanical analogue is:

$$\langle 0 | T \, \mathcal{O}_1(t_1) \, \mathcal{O}_2(t_2) \cdots \mathcal{O}_n(t_n) | 0 \rangle$$

where $\mathcal{O}_i(t_i)$ are f^{ns} of q evaluated at t_i (local operators).

$$= \iint_{-\infty}^{\infty} dq_i \, dq_f \; \underbrace{\langle 0 | q_f, \infty \rangle}_{\psi_0^*(q_f)} \underbrace{\langle q_f, \infty | T \, \mathcal{O}_1 \cdots \mathcal{O}_n | q_i, -\infty \rangle}_{} \underbrace{\langle q_i, -\infty | 0 \rangle}_{\psi_0(q_i)} \qquad (41.10)$$

Concentrate on this.

For given times $t_1, \dots, t_n$ we have $t_{\alpha_1} > t_{\alpha_2} > \dots > t_{\alpha_n}$ for some rearrangement $\alpha_1, \dots, \alpha_n$ of the integers $1, \dots, n$. Then

$$\langle q_f, \infty | T\, \mathcal{O}_1(t_1) \cdots \mathcal{O}_n(t_n) | q_i, -\infty \rangle$$

$$= \langle q_f, \infty | \mathcal{O}_{\alpha_1}(t_{\alpha_1}) \cdots \mathcal{O}_{\alpha_n}(t_{\alpha_n}) | q_i, -\infty \rangle$$

Insert complete sets of pos^n states:

$$= \int_{-\infty}^{\infty} dq_1 \cdots dq_n \, dq_1' \cdots dq_n' \; \langle q_f, \infty | q_{\alpha_1}, t_{\alpha_1} \rangle \langle q_{\alpha_1}, t_{\alpha_1} | \mathcal{O}_{\alpha_1}(t_{\alpha_1}) | q'_{\alpha_1}, t_{\alpha_1} \rangle$$

$$\langle q'_{\alpha_1}, t_{\alpha_1} | q_{\alpha_2}, t_{\alpha_2} \rangle \langle q_{\alpha_2}, t_{\alpha_2} | \mathcal{O}_{\alpha_2}(t_{\alpha_2}) | q'_{\alpha_2}, t_{\alpha_2} \rangle \langle q'_{\alpha_2}, t_{\alpha_2} | q_{\alpha_3}, t_{\alpha_3} \rangle$$

$$\cdots \langle q'_{\alpha_{n-1}}, t_{\alpha_{n-1}} | q_{\alpha_n}, t_{\alpha_n} \rangle \langle q_{\alpha_n}, t_{\alpha_n} | \mathcal{O}_{\alpha_n}(t_{\alpha_n}) | q'_{\alpha_n}, t_{\alpha_n} \rangle$$

$$\langle q'_{\alpha_n}, t_{\alpha_n} | q_i, -\infty \rangle$$

But each $\langle q_j, t_j | \mathcal{O}_j(t_j) | q'_j, t_j \rangle$

$$= \mathcal{O}_j(t_j) \; \delta(q_j - q_j')$$

All q_j' integrals may be done & result $q_j' \to q_j$

a f^n of q_j

and each $\langle q'_{\alpha_k}, t_{\alpha_k} | q_{\alpha_{k+1}}, t_{\alpha_{k+1}} \rangle = \frac{1}{Z_0^{(\alpha_k)}} \int_{q(t_{\alpha_{k+1}}) = q_{\alpha_{k+1}}}^{q(t_{\alpha_k}) = q_{\alpha_k}} \mathcal{D}q \; e^{iS}$

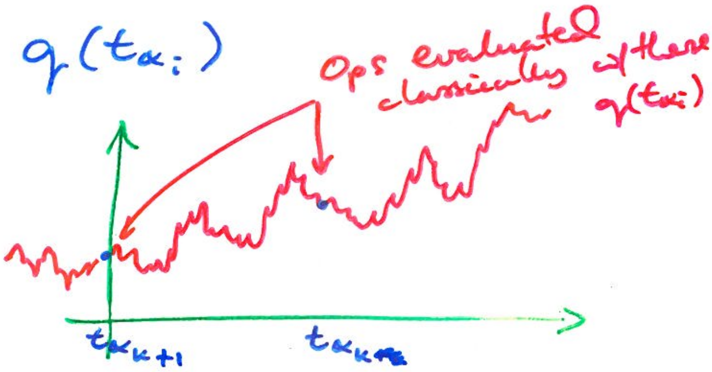
$q'_{\alpha_k} \to q_{\alpha_k}$

Thus 'sewing' back together, using the factorization property, cf p 39:

$$\langle q_f, \infty | T\, \mathcal{O}_1 \cdots \mathcal{O}_n | q_i, -\infty \rangle$$

$$= \frac{1}{Z_0} \int_{q(-\infty)=q_i}^{q(\infty)=q_f} \mathcal{D}q \; \mathcal{O}_{\alpha_1}(t_{\alpha_1}) \cdots \mathcal{O}_{\alpha_n}(t_{\alpha_n})\, e^{iS}$$

where the $\mathcal{O}_{\alpha_i}(t_{\alpha_i})$ are functions (not operators any longer) of $q(t_{\alpha_i})$



But therefore we can write:

$$= \frac{1}{Z_0} \int_{q(-\infty)=q_i}^{q(\infty)=q_f} \mathcal{D}q \; \mathcal{O}_1(t_1) \cdots \mathcal{O}_n(t_n)\, e^{iS}$$

Insertions of operators are automatically time ordered!

Comments

1) More generally could have let $\mathcal{O}_i$ be f^n of $q(t_i)$ and $p(t_i)$ operators. In this case we need to insert appropriately

$$1 = \int_{-\infty}^{\infty} dp_1 \; |p_1, t_1\rangle\langle p_1, t_1|$$

& use the $\mathcal{D}(p,q)$ version (29.7). End result after integrating over p's is that $\mathcal{O}_i$ turns into an operator which is a f^n of $q(t_i)$ & $\dot{q}(t_i)$.

2) Intuitively, the weighting for the end pts of paths at q_i & q_f, cf. (41.10), does not matter much (being at $t = \pm\infty$). This is often correct especially if we can work directly in Euclidean space since

$$\langle q_f, \infty | q_i, -\infty\rangle = \lim_{\tau\to\infty} \langle q_f | e^{-\tau H} | q_i\rangle$$

$$= \lim_{\tau\to\infty} \sum_n \langle q_f | n\rangle e^{-\tau E_n} \langle n | q_i\rangle$$

↖ energy eigenstates

$$\sim \langle q_f | 0\rangle\langle 0 | q_i\rangle e^{-\tau E_0} = \langle q_f|0\rangle\langle 0|q_i\rangle\langle 0| e^{-\tau H} |0\rangle$$

i.e. for fixed q_i & q_f result is $\propto$ vacuum correlator. True also for correlator w/ operator insertion.

3) Helpful to introduce sources $J(t)$ to generate all time ordered correlators, e.g. for correlators in $q(t)$. Write

$$Z[J] = \frac{1}{Z_0}\int \mathcal{D}q \; e^{iS[q] + \underbrace{i\int_{-\infty}^{\infty} dt\, J(t) q(t)}_{\text{effectively extra piece in action: external field } J.}} \tag{45.5}$$

$$\langle 0 | T q(t_1) \cdots q(t_n) | 0 \rangle = \left(-i\frac{\delta}{\delta J(t_1)}\right)\left(-i\frac{\delta}{\delta J(t_2)}\right)\cdots\left(\frac{-i\delta}{\delta J(t_n)}\right) Z[J]\Big|_{J=0} \tag{45.6}$$

(4) Note $Z[0] = \langle 0|0\rangle = 1$.

Thus $Z_0 = \int \mathcal{D}q \; e^{iS[q]}$ (45.8)

Conventionally define partition f^n unnormalised as $Z[J] = \int \mathcal{D}q \; e^{iS[q] + i\int dt\, J(t) q(t)}$

and write (45.6) then as

$$\langle 0 | T q(t_1) \cdots q(t_n) | 0 \rangle = \frac{1}{Z_0}\left(-i\frac{\delta}{\delta J(t_1)}\right)\cdots\left(\frac{-i\delta}{\delta J(t_n)}\right) Z[J]\Big|_{J=0} \tag{45.10}$$

Z_0 defined as in (45.8) & now $= Z[0]$

Generalisation to multidimensional QM:

$$\langle 0 | T\, q_{i_1}(t_1) \cdots q_{i_n}(t_n) | 0 \rangle$$

$$= \frac{1}{Z_0} \left(-i \frac{\delta}{\delta J_{i_1}(t_1)} \right) \cdots \left(-i \frac{\delta}{\delta J_{i_n}(t_n)} \right) Z[\underline{J}] \Big|_{\underline{J} = \underline{0}}$$

$$Z[\underline{J}] = \int \mathcal{D}\underline{q}\; e^{\, i S[\underline{q}] + i \sum_i \int dt\, J_i(t) q_i(t)}$$

$$\sum_i \frac{m}{2} \dot{q}_i^2 - V(\underline{q})$$

$$S[\underline{q}] = \int dt\; L(q_i, \dot{q}_i)$$

$\boxed{\text{Ex: If in doubt check!}}$

Generalisation to QFT:

QFT is QM with an ∞ # degrees of freedom:

$$q_i(t) \longrightarrow \varphi_{\underline{x}}(t) \equiv \varphi(\underline{x}, t)$$

sums $\longrightarrow$ integrals

thus eg. $\sum_i J_i(t) q_i(t) \rightarrow \int d^3x\, \varphi_{\underline{x}}(t) J_{\underline{x}}(t)$

$$= \int d^3x\, \varphi(\underline{x}, t) J(\underline{x}, t)$$

Explicit construction goes as follows..

$$\prod_{\underline{x}} |\varphi_j(\underline{x}, t_j)\rangle$$

$$\varphi(\underline{y}, t_j) | \varphi_j(*, t_j) \rangle = \varphi_j(\underline{y}, t_j) | \varphi_j(*, t_j) \rangle$$

Commuting set of 'posⁿ' ops $[\varphi(\underline{y}, t), \varphi(\underline{x}, t)] = 0$

$$\int \mathcal{D}\varphi_i(\underline{x}, t_i)\; |\varphi_i(*, t_i)\rangle \langle \varphi_i(*, t_i)| = \mathbb{1}$$

Thus,

$x_1^\mu = (\underline{x}_1, t_1)$

$$\langle 0 | T\varphi(x_1)\cdots\varphi(x_n)|0\rangle$$

$$= \frac{1}{\mathcal{Z}_0}\left(-i\frac{\delta}{\delta J(x_1)}\right)\cdots\left(-i\frac{\delta}{\delta J(x_n)}\right)\mathcal{Z}[J]\Big|_{J=0} \qquad (47.1)$$

$$\mathcal{Z}[J] = \int \mathcal{D}\varphi \; e^{iS[\varphi] + i\int d^4x\, J(x)\varphi(x)}$$

Functional Integral

Integral over all 4 dim^nl field configs.

$$\mathcal{Z}_0 = \mathcal{Z}[0] \quad (\text{still}) \qquad (47.3)$$

$$S[\varphi] = \int dt\, L[\varphi] = \int d^4x\, \mathcal{L}(\varphi, \partial_\mu\varphi)$$

and eg. $\mathcal{L} = \frac{1}{2}(\partial_\mu\varphi)^2 - \frac{1}{2}m^2\varphi^2 - V(\varphi)$

Propagator & Wicks Theorem via f^nl integrals.

Propagator.

To get this we need only free field theory:

$$S_{free}[\varphi] = \int d^4x \left\{ \frac{1}{2}(\partial_\mu\varphi)^2 - \frac{1}{2}m^2\varphi^2 \right\}$$

This is of the form

$$S_{free}[\varphi] = \frac{1}{2}\varphi_x \Delta^{-1}_{xy}\varphi_y \qquad (47.4)$$

'Summation' convention $\left(\varphi_x J_x \equiv \int d^4x\, \varphi(x)J(x)\right)$

for a 'matrix' (continuous $\infty \times$ continuous ∞):

$$\Delta^{-1}_{xy} = (-\Box_x - m^2)\,\delta^{(4)}(x-y) \qquad (48.1)$$

We need to do the integral:

$$Z_{free}[J] = \int \mathcal{D}\varphi \; e^{i\left\{\frac{1}{2}\varphi_x \Delta^{-1}_{xy}\varphi_y + J_x\varphi_x\right\}}$$

Change basis to diagonalize matrix:

$$\varphi(x) = \int đ^4p \; \underset{\uparrow}{e^{-ip.x}} \; \underset{\uparrow}{\tilde{\varphi}(p)}$$

basis vector — components in this basis.

$$\mathcal{D}\varphi = \mathcal{D}\tilde{\varphi}\left|\frac{\delta\varphi}{\delta\tilde{\varphi}}\right|$$

$$\text{Jacobian} = \left|\operatorname{Det}\left(\frac{\delta\varphi(x)}{\delta\tilde{\varphi}(p)}\right)\right| = 1$$

> Proof: Show transf^n $\tilde{\varphi}_p \to \varphi_x$ is unitary 'matrix'. And, show that the modulus of determinant of any unitary matrix is unity.

Now $S_{free}[\varphi] = S_{free}[\tilde{\varphi}] = \frac{1}{2}\int đ^4p\; \tilde{\varphi}(p)\,\tilde{\varphi}^*(p)\,(p^2-m^2)$

By $\varphi^*(x) = \varphi(x)$: $\tilde{\varphi}(-p) = \tilde{\varphi}^*(p)$ Ex.

Now it is easy to do the integrals because everything factorizes:

$$\int \mathcal{D}\tilde{\varphi} \underset{\uparrow}{\sim} \prod_p \int d^2\varphi(p)$$

$$e^{iS_{free}[\tilde{\varphi}] + i\int d^4p\, J(-p)\varphi(p)}$$

$$\sim \prod_p e^{\frac{i}{2}(p^2-m^2)|\tilde{\varphi}(p)|^2 + iJ(-p)\varphi(p)}$$

(discretize – there are overall const factors $\to \tilde{Z}_0$).

$\boxed{\text{Ex. Follow through this derivation to arrive at (50.2)}}$

Alternative & slicker derivation:

write (47.10) etc. fully in matrix notⁿ:

$$Z_{free}[J] = \int \mathcal{D}\varphi\, e^{i\{\frac{1}{2}\varphi^T\Delta^{-1}\varphi + J^T\varphi\}}$$

Complete square for matrix / bilinear product:

N.B. $\Delta^T = \Delta$
• Prove it from [48.1]
• Show it must be true w.l.o.g.

$$\frac{1}{2}\varphi^T\Delta^{-1}\varphi + J^T\varphi$$

$$= \frac{1}{2}\underbrace{(\varphi^T + J^T\Delta)}_{(\varphi+\Delta J)^T}\Delta^{-1}(\varphi + \Delta J) - \frac{1}{2}J^T\Delta J$$

Change variables to $\varphi' = \varphi + \Delta J$

Jacobian $\left|\frac{\delta \varphi}{\delta \varphi'}\right| = \left|\mathrm{Det}\, \mathbb{1}\right| = 1$

Thus,

$$\underset{free}{Z}[J] = \underbrace{\int \mathcal{D}\varphi' \, e^{\frac{i}{2}\varphi'^T \Delta^{-1} \varphi'}} \, e^{-\frac{i}{2} J^T \Delta J} \qquad (50.2)$$

Just a number
– by definition $= Z_0^{free}$

Propagator by def^n is $\langle 0 | T \varphi(x) \varphi(y) | 0 \rangle$.

Recall [47.1]

$$\langle 0 | T \varphi(x) \varphi(y) | 0 \rangle = \frac{1}{Z_0} \left(-i \frac{\delta}{\delta J(x)}\right)\left(-i \frac{\delta}{\delta J(y)}\right) Z[J] \Big|_{J=0}$$

$$= i \Delta_{xy} = i \Delta_{yx} \qquad [50.2]$$

What is Δ_{xy}?

By def^n $\Delta^{-1}_{xy} \Delta_{yz} = \delta_{xz}$

i.e. $\int d^4y \, \Delta^{-1}_{xy} \Delta_{yz} = \delta^{(4)}(x - z)$

Substitute [48.1] then:

$$\underline{(\Box_x + m^2) \Delta_{xz} = -\delta^{(4)}(x - z)}$$

But this is just the def^n of the usual propagator

Recall, solving by FT :

$$i\Delta_{xy} = \int d^4p \; e^{-ip.(x-y)} \frac{i}{p^2-m^2}$$

Functional Determinants

[50.2] shows that $Z_0^{free} = \int \mathcal{D}\varphi \; e^{\frac{i}{2}\varphi^T \Delta^{-1}\varphi}$.

Sometimes it is useful to know how to evaluate this (eg. Δ^{-1} might itself be dependent on some other field - so yield not just a const.).

More generally $\underline{\underline{\int \mathcal{D}\varphi \; e^{-\frac{1}{2}\varphi^T M \varphi}}} \propto \underline{\underline{\mathcal{D}et^{-\frac{1}{2}} M}}$

const.

symmetric

[so here $Z_0 \propto \mathcal{D}et^{-\frac{1}{2}} i\Delta^{-1} = \mathcal{D}et^{\frac{1}{2}}(-i\Delta)$.]

Functional Determinant.

Proof:

Recall p48: Rotate to orthonormal eigenbasis v_α :

$$\varphi(x) = \text{"}\sum_\alpha\text{"} \; a_\alpha v_\alpha(x)$$

$$\int d^4y \; M(x,y) \, v_\alpha(y) = \lambda_\alpha v_\alpha(x)$$

and $\quad v_\alpha^T v_\beta = \text{"}\delta_{\alpha\beta}\text{"}$

Jacobian = 1 (cf. p48) so -

$$\int \mathcal{D}\varphi \, e^{-\frac{1}{2}\varphi^T M \varphi} = \int \mathcal{D}a \, e^{-\frac{1}{2}\sum_\alpha a_\alpha^2 \lambda_\alpha}$$

$$\sim \prod_\alpha \Big(\underbrace{\int_{-\infty}^{\infty} da_\alpha \, e^{-\frac{1}{2}\lambda_\alpha a_\alpha^2}}_{\sqrt{2\pi/\lambda_\alpha}} \Big)$$

$$\propto \Big(\underbrace{\prod_\alpha \lambda_\alpha}_{} \Big)^{-\frac{1}{2}}$$

By def^n $\mathrm{Det}\, M$. $\square$.

Wick's Thm:

Recall that for free field theory,

$$\langle 0 | T \varphi(x_1)\varphi(x_2) \cdots \varphi(x_{2n}) | 0 \rangle$$

$$= \underbrace{\langle 0 | T \varphi(x_1)\varphi(x_2) | 0 \rangle}_{i\Delta_{x_1,x_2}\ [50.2]} \langle 0 | T \varphi(x_3)\varphi(x_4) | 0 \rangle \cdots$$

$$\cdots \langle 0 | T \varphi(x_{2n-1})\varphi(x_{2n}) | 0 \rangle$$

$$+ \langle 0 | T \varphi(x_1)\varphi(x_3) | 0 \rangle \langle 0 | T \varphi(x_2)\varphi(x_4) | 0 \rangle \cdots \text{(rest same)}$$

$+ \cdots$ all other possible contractions $\cdots$

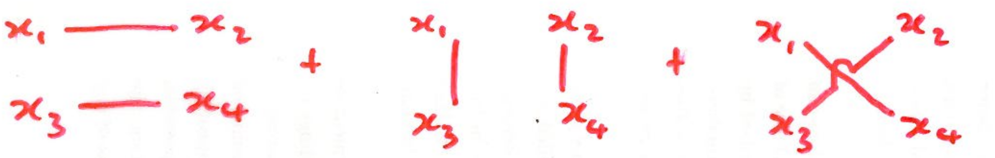
To see that Wicks theorem falls out of the present formalism, (very easily!) compute an example:

$$\langle 0 | T\, \varphi(x_1)\, \varphi(x_2)\, \varphi(x_3)\, \varphi(x_4) | 0 \rangle$$ [47-1]

$$= \left(-i\frac{\delta}{\delta J(x_1)}\right)\left(-i\frac{\delta}{\delta J(x_2)}\right)\left(-i\frac{\delta}{\delta J(x_3)}\right)\left(-i\frac{\delta}{\delta J(x_4)}\right) \underbrace{Z_{free}[J] \big/ Z_0^{free}}_{e^{-\frac{i}{2}\iint d^4x\, d^4y\, J(x)\, \Delta_{xy}\, J(y)}} \Big|_{J=0}$$

[50.2]

$$= i\Delta_{x_1 x_2}\, i\Delta_{x_3 x_4} + i\Delta_{x_1 x_3}\, i\Delta_{x_2 x_4} + i\Delta_{x_1 x_4}\, i\Delta_{x_2 x_3}$$



$\boxed{\text{Ex!}}$

Interactions & Perturbation Theory.

Now $S[\varphi] = S_{free}[\varphi] - \int d^4x\, V(\varphi)$ (for example)

Using J we can reduce the f^{nl} integral to to free case & thus compute it!:

Thus

$$Z[J] = \int \mathcal{D}\varphi \; e^{iS_{free} - i\int V(\varphi) + i\int J\varphi}$$

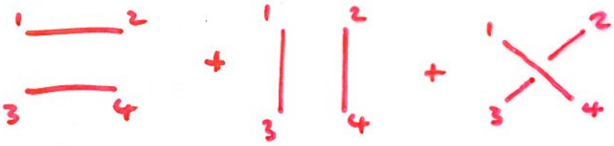
$$= e^{-i\int V(-i\frac{\delta}{\delta J})} \int \mathcal{D}\varphi \; e^{iS_{free} + i\int J\varphi}$$

$$= e^{-i\int V(-i\frac{\delta}{\delta J})} \; Z_{free}[J] \qquad (54.1)$$

Can now expand this exponential perturbatively in V (for example). Computing derivatives just amounts to performing Wicks theorem on the $V(\varphi \mapsto -i\frac{\delta}{\delta J})$. Thus [54.1] is equivalent to $\langle 0 | T \, e^{-i\int V(\varphi)} | 0 \rangle$ in interaction representation – the standard starting point for pert$^{\text{bn}}$ theory in the operator formalism. (We should have insertions of operators here too – eg. $\varphi(x_1) \cdots \varphi(x_n)$, otherwise we are just computing the "vacuum to vacuum amplitude").

Example.

$\langle 0|T\,\varphi(x_1)\varphi(x_2)\varphi(x_3)\varphi(x_4)|0\rangle$ in $V=\frac{\lambda}{(4!)}\varphi^4$ theory.

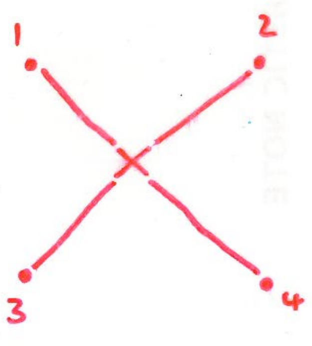
Lowest order $= O(\lambda^0)$ i.e. free $=$ 

as we've already seen. Next order:

$$\langle 0|T\varphi(x_1)\varphi(x_2)\varphi(x_3)\varphi(x_4)|0\rangle\Big|_{O(\lambda)} = \prod_{i=1}^{4}\left(\frac{-i\delta}{\delta J(x_i)}\right)\frac{-i\lambda}{4!}\int d^4x\left(\frac{i\delta}{\delta J(x)}\right)^4 \frac{Z_{free}[J]}{Z_0}\Bigg|_{J=0} \qquad (55\cdot3)$$

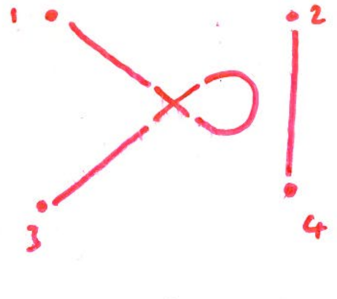
N.B. NOT Z_0^{free}!

But from here we just Wick contract, with each line representing a propagator $i\Delta$:

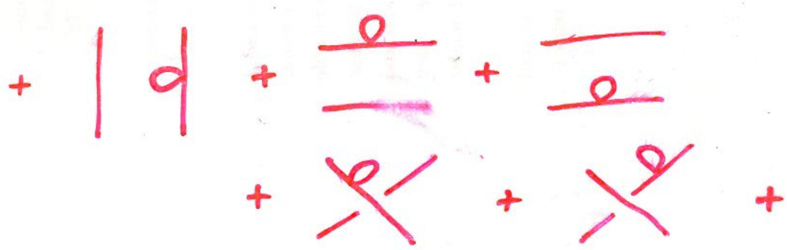
4.3.2.1 

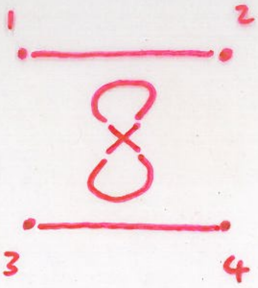
$$= (-i\lambda)\int d^4x\,(i\Delta_{x_1x})(i\Delta_{x_2x})(i\Delta_{x_3x})(i\Delta_{x_4x}) \qquad (55\cdot5)$$

$+$ 

4.3 

$$= i\Delta_{24}\left(\frac{-i\lambda}{2}\right)\int d^4x\,(i\Delta_{1x})(i\Delta_{3x})(i\Delta_{xx}) \qquad (55\cdot6)$$



$+3.2$  $= \overbrace{i\Delta_{12}\, i\Delta_{34} \left(\frac{-i\lambda}{8}\right) \int d^4x\, i\Delta_{xx}\, i\Delta_{xx}}^{\text{8}}$

$+$  (56-1)

But we have still to evaluate

$$\mathcal{Z}_0 = \mathcal{Z}_0^{free} - \frac{i\lambda}{4!}\int d^4x \left(-i\frac{\delta}{\delta J(x)}\right)^4 \mathcal{Z}_{free}[J]\Big|_{J=0}$$

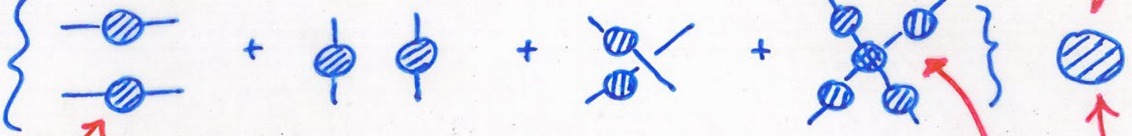
[47-3]

$1 \quad + \quad \infty \quad + \dots$

$$= \mathcal{Z}_0^{free}\left(1 - \frac{i\lambda}{8}\int d^4x\, i\Delta_{xx}\, i\Delta_{xx} + O(\lambda^2)\right) \qquad (56\text{-}3)$$

But to the order in which we are working,

$1 + 8 + \dots$

$(56.1) =$  (56-4)

"Full" propagator = $+ \dots$

full vacuum to vacuum amplitude "<unit operator>"

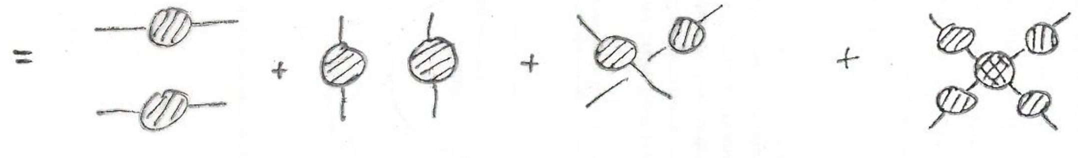
a similarly corrected vertex

( $+ \dots$)

(→ of course this statement is actually valid to all orders)

Thus substituting (56·4) & (56·3) into (55·3) we see that Z_0 just cancels out the multiplicative vacuum-to-vac corrections. The general result is then:

$$\langle 0 | T\, \varphi(x_1)\, \varphi(x_2)\, \varphi(x_3)\, \varphi(x_4) | 0 \rangle$$

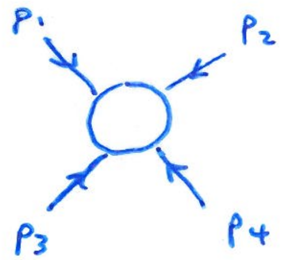


[These statements will be developed to all orders in QFT II ...]

Momentum Rep^n

$\boxed{\text{EX!}}$

$$\langle 0 | T\, \varphi(p_1)\, \varphi(p_2)\, \varphi(p_3)\, \varphi(p_4) | 0 \rangle =$$

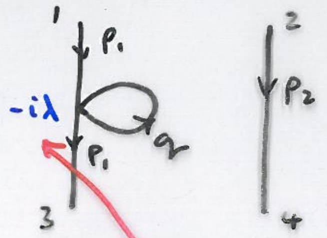


where $\varphi(p_i) = \int d^4x_i \, e^{i p_i \cdot x_i} \varphi(x_i)$

- Show that in this way propagators $i\Delta_{x_i x}$ become $\frac{i}{p_i^2 - m^2}$
- Show that the overall $\int d^4x$ integral in  of [55·5] just enforces overall mom^{tm} conservation.
- Transform [55·6] similarly & see momentum conservation now shows $p_1 = -p_3$ and $p_2 = -p_4$ are required as expected for this process.
- Show that $i\Delta_{xx}$ results in a free mom^{tm} integral $\int d^4q \frac{i}{q^2 - m^2}$

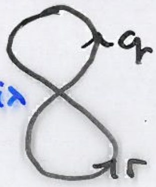
Thus [55.6] can be represented as:

$\frac{4.3}{4!}$ → Combinatorics



Feynman rule.

$\boxed{\text{q integral is a one-loop integral} \to \text{QFT II}}$

- Finally justify  $\equiv \frac{3\cdot 4}{4!}\,(-i\lambda)$ 

$\boxed{\text{2-loop integral} \to \text{QFT II}}$

[N.B. note the existence of an overall "momentum conserving δf^n" $\delta(0) = \int d^4x$.
In actual fact these free pos^n integrals appear so that
$$Z_0 = Z_0^{free} \exp -i \int d^4x\, \mathcal{E}$$

$\mathcal{E} =$  $+$  $+$  $+ \cdots$

where overall $\delta(0) = \int d^4x$ is excluded and $\mathcal{E}$ is the (interaction part of the) vacuum energy.]

Thus:

- Tie up propagators in all possible ways
- Conserve mom^{tm} at each vertex - including free mom^{tm} (loop) integrals as appropriate
- Include a factor of $\underline{iSv}$ where S is permutation symmetry factor ($= 4!$ for φ^4) & v the coupling ($= -\lambda/4!$ for $-\lambda/4!\,\varphi^4$), for each vertex

• Finally compute combinatorics by (e.g.)

$$\Pi \frac{1}{n!} \, \Pi \frac{1}{S} \times (\#\text{ ways joining up vertices \& external legs})$$

from $e^{-i\int V}$ a factor of $\frac{1}{n!}$ if n vertices of same type used.

symmetry factors of each vertex

N.B. Scalars with extra indices and vectors i.e. all bosons – obvious generalisation, thus

$$V = \int d^4x \, \frac{1}{3!} \, d^{abc} \varphi_a \varphi_b \varphi_c$$

totally symmetric w.l.o.g.

→ Feynman rule $\quad -i\, d^{abc}$

→ Propagator $\quad i\, \Delta^{ab}_{x_1 x_2} \quad \left(= i\, \delta^{ab} \Delta_{x_1 x_2} \right)$ [typically]

Fermions.

When we try to set up the path integral formalism for fermions, we encounter a problem. We want to insert a complete set of states specifying the field at every time slice ie, following p46, insert:

$$\int \mathcal{D}\psi'(*,t') \, |\psi'(*,t')\rangle\langle\psi'(*,t')| = \mathbb{1}$$

where $\psi(y,t')|\psi'(*,t')\rangle = \psi'(y,t')\,|\psi'(*,t')\rangle$ (60.2)

and $|\psi'(*,t')\rangle = \prod_{\underline{x},\alpha} |\psi'_\alpha(\underline{x},t')\rangle$

But $\{\psi(y,t) \;\forall y\}$ do not form a commuting set of observables! Rather:

$$\{\psi(\underline{x},t),\psi(y,t)\} = 0$$

The way out is to allow the eigenvalues to be some other sort of numbers:

$0 = \{\psi(\underline{x},t),\psi(y,t)\}\,|\psi'(*,t)\rangle$

$= \psi(\underline{x},t)\,|\psi'(*,t)\rangle\,\psi'(y,t) + \psi(y,t)\,|\psi'(*,t)\rangle\,\psi'(\underline{x},t)$

$= \{\psi'(\underline{x},t)\,\psi'(y,t) + \psi'(y,t)\,\psi'(\underline{x},t)\}\,|\psi'(*,t)\rangle$

[using 60.2]

Recall that:

a (complete) set of simultaneous eigenstates	$\Longleftrightarrow$	Complete set of commuting observables.

We see that this generalises to anticommuting observables only if we introduce 'numbers' that <u>anticommute</u> with each other!

<u>Grassmann Numbers.</u>

<u>Grassmann Algebra</u>

Let a, b, c, d be Grassmann numbers, then:

$ab = -ba$ for any such pair.

In particular $a^2 = -a^2$ ie. $a^2 = 0$.

This does not mean $a = 0$ because we cannot divide by a ($\nexists\, a^{-1}$).

Let λ and μ be complex #s. We can form new Grassmann #s by linear combination:

eg. $c = \lambda a + \mu b$ complex #'s of course always commute w/ everything

The Grassmann algebra is <u>graded</u>: elements are either <u>even</u> (commute with everything) or <u>odd</u> (anticommute with other odd elements)

Thus a, b, c, d are odd.

An even # multiplied together are even
odd # " " odd.

eg. $d(ab) = -adb = (ab)d$

but $d(abc) = -adbc = abdc = -(abc)d$

Since ab is commuting can it be identified w/ a complex number? NO:

Generically $ab \neq 0$ but

$(ab)^2 = abab = -a^2b^2 = 0.$ (nilpotent)

Grassmann Calculus

To use in the f^{nl} integral we need to be able to integrate and differentiate w/ such #s.

In particular we need to be able to Taylor expand:

$$f(a) = f(0) + a\underline{f'(0)} + \cancel{\frac{a^2}{2} f''(0) + \cdots} \qquad (62.7)$$

All zero because $a^n = 0$ for $n \geq 2$!

Require

$f'(0) = \frac{\partial}{\partial a} f(a) \Big|$

From (62.7) this $\Rightarrow$ $\underline{\underline{\frac{\partial}{\partial a} a = 1}}$ and $\underline{\underline{\frac{\partial}{\partial a} 1 = 0}}$

Note $\frac{\partial}{\partial a}$ must be odd:

even would give "+" here.

$$= \frac{\partial}{\partial a} 0 = \frac{\partial}{\partial a} a^2 = \frac{\partial a}{\partial a} a - a \frac{\partial a}{\partial a}$$

$$= a - a \quad ✓$$

but even gives "+" ✗

Ex: Show $\frac{\partial}{\partial a}(ba) = -b$

$\frac{\partial}{\partial a}\frac{\partial}{\partial b}(ab) = -1$

$\frac{\partial}{\partial a} e^a = 1$

$e^a e^b = e^{a+b+\frac{1}{2}[a,b]}$ (as expected by BCH)

[N.B. e^a is defined by its Taylor expansion]

• We need to do integrals over all Grassmann numbers (eg. $\int \mathcal{D}\bar{\psi}\psi$), but

$$\int da\, f(a) = \int da \{ f(0) + a f'(0) \}$$
$$= \left(\int da\, 1\right) f(0) + \left(\int da\, a\right) f'(0)$$

So we only need to understand how to do these two.

We would like to able shift variables in the integral eg. $a = a' + b$. (Recall derivation of free f^nal integral Z_{free}.) (with unit Jacobian!)

But then $\int da\, a = \int da' \,(a' + b)$

$$= \left(\int da'\, a'\right) + \left(\int da'\, 1\right) b$$

Therefore we must have $\underline{\underline{\int da\, 1 = 0}}$

We also need the result of Grassmann integration to yield just complex numbers or else we will not recover bona fide numerical answers to computations of correlators, scattering matrix elements etc! $\therefore$ we must set $\int da\, a$ to a complex number. Its value is arbitrary (just determines defn of 'measure') & conventionally:

$$\underline{\underline{\int da\, a = 1}}$$

N.B. Consistency requires $\int da$ is <u>odd</u>.

Thus $\int da\, (ab) = b$

$$= -\int da\, (ba) = + b \int da\, a$$

this would "−" if $\int da$ was even

Summary of Grassmann Calculus.

$$\int da \; a = 1 \qquad \frac{\partial}{\partial a} a = 1$$

$$\int da \; 1 = 0 \qquad \frac{\partial}{\partial a} 1 = 0.$$

$$\Rightarrow \quad \int da = \frac{\partial}{\partial a} \quad !$$

F^{nl} Integrals.

These follow through as before. Just be careful about signs! Need Grassmann sources:

$$Z[\eta,\bar\eta] = \int \mathcal{D}(\psi,\bar\psi)\, e^{iS[\psi,\bar\psi] + i\int d^4x\, \bar\eta(x)\psi(x) + i\int d^4x\, \bar\psi(x)\eta(x)}$$

$$S[\psi,\bar\psi] = S_{free}[\psi,\bar\psi] \leftarrow \int d^4x\, \bar\psi(i\not\partial - m)\psi$$

$$+ S^{int}[\psi,\bar\psi] \leftarrow \text{w/ } A_\mu \text{ (e.g.)}$$

or e.g. $g\int d^4x\, (\bar\psi\psi)^2$ Gross-Neveu, Nambu-Jona-Lasinio

$$Z_{free}[\eta,\bar\eta] = \int \mathcal{D}(\psi,\bar\psi) \exp i\{\bar\psi\Delta^{-1}\psi + \bar\eta\psi + \bar\psi\eta\}$$

(matrix notⁿ).

complete square: $(\bar\psi + \bar\eta\Delta)\Delta^{-1}(\psi + \Delta\eta) - \bar\eta\Delta\eta$

$$= Z_0^{free}\, e^{-i\bar\eta\Delta\eta}$$

$$\langle 0| T\{ \psi_{\alpha_1}(x_1) \cdots \psi_{\alpha_n}(x_n) \bar\psi^{\beta_1}(y_1) \cdots \bar\psi^{\beta_n}(y_n)\} |0\rangle$$

$$= \frac{1}{Z[0]}\left(-i\frac{\delta}{\delta\bar\eta^{\alpha_1}(x_1)}\right)\cdots\left(-i\frac{\delta}{\delta\bar\eta^{\alpha_n}(x_n)}\right)\left(+i\frac{\delta}{\delta\eta_{\beta_1}(y_1)}\right)\cdots\left(+i\frac{\delta}{\delta\eta_{\beta_n}(x_n)}\right) Z[\eta,\bar\eta]\Big|_{\eta=\bar\eta=0}$$

Note sign due to Grassmann commutation needed!

→ Find that every closed loop of fermions generates a minus sign.

Extra Feynman rule.

Finally $\int \mathcal{D}(\psi,\bar\psi)\, e^{-\bar\psi M \psi} \propto \operatorname{Det} M$

Proof: Diagonalize as before by unitary change of basis and use $\int d\bar a_\alpha\, da_\alpha\, e^{-\underset{\substack{\uparrow\\ \text{real eigenvalue.}}}{\lambda_\alpha} \bar a_\alpha a_\alpha} = \lambda_\alpha$

$\boxed{\text{Ex.!}}$

Or quicker – discretize and note that

$\boxed{\text{Ex.!}}$

$$\int d\psi_N\, d\psi_{N-1} \cdots d\psi_1\; \psi_{\alpha_1}\psi_{\alpha_2}\cdots\psi_{\alpha_n} = 0 \quad \text{if } n\neq N$$
$$= \varepsilon_{\alpha_1\alpha_2\cdots\alpha_N} \quad n=N$$

and that $\varepsilon_{\alpha_1\cdots\alpha_N} M_{\alpha_1\beta_1} M_{\alpha_2\beta_2}\cdots M_{\alpha_N\beta_N} = \varepsilon_{\beta_1\cdots\beta_N} \operatorname{Det} M$