1) Abelian gauge symmetry e the EM field.

2) <u>S-matrix e LSZ reduction</u>

<u>QED</u>

electric charge of electron

electron Dirac field

$$S_{QED} = \int d^4x \, \mathcal{L}$$

where $\mathcal{L} = -\frac{1}{4} F_{\mu\nu}^2 + \bar{\psi}(i\not{\partial} - e\not{A} - m)\psi$

$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$

$$\left[Z[J,\eta,\bar{\eta}] = \int \mathcal{D}(A_\mu, \psi, \bar{\psi}) \, e^{iS_{QED} + iS_{source}} \right]$$

at top of board

where $S_{source} = \int d^4x \left\{ \bar{\eta}\psi + \bar{\psi}\eta + J^\mu A_\mu \right\}$

every field a source so J^μ of x of course

Apparently straight-forward generalisation of what we had before, so e.g. pass to interaction rep^n by writing: $\mathcal{L} = \mathcal{L}_{free} + \mathcal{L}_{int}$

$\mathcal{L}_{free} = -\frac{1}{4} F_{\mu\nu}^2 + \bar{\psi}(i\not{\partial} - m)\psi$

$\mathcal{L}_{int}(A_\mu, \psi, \bar{\psi}) = -e\bar{\psi}\not{A}\psi$ and then

$$Z = e^{-ie\int \mathcal{L}_{int}\left(\frac{-i\delta}{\delta J^\mu}, \frac{-i\delta}{\delta\bar{\eta}}, \frac{i\delta}{\delta\eta}\right)} Z_{free}[J,\eta,\bar{\eta}]$$

But $\mathcal{L}$ has a gauge invariance (Abelian)

small, & real ; group of phases so $U(1)$ which is Abelian.

$$\delta \psi(x) = -ie\,\Omega(x)\,\psi(x)$$ (local phase change $\psi \xrightarrow{\Omega} e^{-ie\Omega(x)}\psi(x)$)

$$\Rightarrow \delta\bar{\psi} = +ie\,\bar{\psi}\,\Omega$$

$$\delta A_\mu(x) = \partial_\mu \Omega(x)$$ Clearly $\delta\left[\bar{\psi}(i\partial\!\!\!/ - eA\!\!\!/ - m)\psi\right] = 0$ &

Clearly $\delta F_{\mu\nu} = \partial_\mu\partial_\nu\Omega - \partial_\nu\partial_\mu\Omega = 0.$

Rearrange $\int d^4x \left(-\frac{1}{4}F_{\mu\nu}^2\right)$

$$= -\frac{1}{2}F_{\mu\nu}\,\partial^\mu A^\nu = -\frac{1}{2}\partial_\mu A_\nu\,\partial^\mu A^\nu + \frac{1}{2}\partial_\nu A_\mu\,\partial^\mu A^\nu$$

$$\equiv -\frac{1}{2}A_\nu\left(-\Box g^{\nu\lambda} + \partial^\nu\partial^\lambda\right)A_\lambda$$

by parts

Want to use the completing-the-square trick so to call $\Delta^{-1\,\nu\lambda}$

But $\Delta^{-1\,\nu\lambda}\,\partial_\lambda\Omega = -\Box\partial^\nu\Omega + \Box\partial^\nu\Omega = 0$ (of course)

so actually this kernel has a zero eigenvalue and $\therefore$ no inverse!

[In momentum space $\Delta^{-1\,\nu\lambda} \rightarrow p^2 g^{\nu\lambda} - p^\nu p^\lambda$

and $\left(p^2 g^{\nu\lambda} - p^\nu p^\lambda\right)p_\lambda = 0$]

Basically the problem is that

$$\int \mathcal{D} A_\mu \sim \int \mathcal{D} \tilde{A}_\mu \mathcal{D} \Omega$$ ← this integration is redundant.

where $A_\mu = \tilde{A}_\mu + \partial_\mu \Omega$ $\quad (\psi = e^{-ie\Omega} \tilde{\psi})$

Define split by making a choice of gauge eg.

$\partial^\mu \tilde{A}_\mu = 0$ (Lorentz gauge)

then $\partial^\mu A_\mu = \cancel{\partial^\mu \tilde{A}_\mu}^{\,0} + \Box \Omega$

Physical results independent of choice of gauge. (Ward Identities)

so $\Omega = \frac{1}{\Box} \partial^\mu A_\mu$ $\quad \left(\Omega(p) = \frac{i}{p^2} p^\mu A_\mu(p) \right)$

<u>Fadeev-Popov trick:</u> insert $1 = \int \mathcal{D} f \, \delta[\underbrace{\partial_\mu A_\mu(x) - f(x)}_{\text{f}^{\text{nal}}\ \delta f^{\text{n}}}]$

$$\int \mathcal{D} A_\mu = \int \mathcal{D} A_\mu \int \mathcal{D} f \, \delta[\partial_\mu A_\mu - f]$$

$$= \underbrace{\int \mathcal{D} f}_{} \underbrace{\int \mathcal{D} A_\mu \, \delta[\partial_\mu A_\mu - f]}_{= \int \mathcal{D} \tilde{A}_\mu}$$

$f = \Box \Omega$ so $\int \mathcal{D}\Omega \underbrace{\operatorname{Det} \Box}$

Fadeev-Popov determinant. Just a constant in QED but not in QCD.

So throw away $\int \mathcal{D} f$ and use $\int \mathcal{D} \tilde{A}_\mu$ instead of $\int \mathcal{D} A_\mu$ in Z.

$Z \mapsto \tilde{Z}[J, \eta, \bar{\eta}, f]$ now depends on f (!)

but physical results are independent of f.

Awkward to deal with – so another trick:

<u>'t Hooft averaging</u>

Define instead
$$Z = \int \mathcal{D}f \; e^{-\frac{i}{2\xi}\int d^4x\, f^2(x)} \; \tilde{Z}[J, \eta, \bar{\eta}, f]$$

gauge fixing parameter

physical results are independent of ξ.

$$= \int \mathcal{D}(A_\mu, \psi, \bar{\psi}) \; e^{i(S_{QED} + S_{gf} + S_{source})}$$

where $S_{gf} = -\frac{1}{2\xi}\int d^4x\, (\partial_\mu A_\mu)^2$

$$\int d^4x \, -\tfrac{1}{4} F_{\mu\nu}^2 + S_{gf} = -\tfrac{1}{2}\int d^4x\, A_\mu \underbrace{\{-\Box g^{\mu\nu} + (1 - \tfrac{1}{\xi})\partial^\mu \partial^\nu\}}_{\Delta^{-1\,\mu\nu}} A_\nu$$

$\Delta^{-1\,\mu\nu}$ can now be inverted.

$$\Delta^{-1\,\mu\nu}(p)\, p_\nu = \left(p^2 g^{\mu\nu} + (\tfrac{1}{\xi} - 1)\, p^\mu p^\nu\right) p_\nu = \tfrac{1}{\xi}\, p^2 p^\mu$$

Simplest in <u>Feynman gauge</u> $\xi = 1$

Feynman propagator $i\Delta^{\mu\nu} = \dfrac{-ig^{\mu\nu}}{-\Box} = \int d^4p \; e^{-ip\cdot(x-y)} \left(\dfrac{-ig^{\mu\nu}}{p^2}\right)$

N.B. Wick rotate into Euclidean space

$$g^{\mu\nu} = \begin{pmatrix} +1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix} \longmapsto -\delta^{\mu\nu}$$

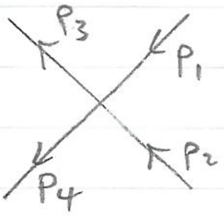
then photon propagator is $i\delta^{\mu\nu}/p^2$

(LSZ)

Lehmann - Symanzik - Zimmermann Reduction

Consider just the example of $2 \to 2$ scattering we had earlier and the connected graph

$$\langle 0 | T\, \varphi(p_1)\, \varphi(p_2)\, \varphi(-p_3)\, \varphi(-p_4) | 0 \rangle$$

$=$  $= -i\lambda \dfrac{i}{p_1^2 - m^2} \dfrac{i}{p_2^2 - m^2} \dfrac{i}{p_3^2 - m^2} \dfrac{i}{p_4^2 - m^2}$

$$\times\, \delta(p_1 + p_2 - p_3 - p_4)$$

We want to compute the amplitude e.g.

$$\mathcal{A} = \langle p_3, p_4 |_{t=\infty}\, T e^{-i\int_{-\infty}^{\infty} dt\, H_{int}}\, | p_1, p_2 \rangle_{t=-\infty} \qquad (\text{interaction rep}^{n})$$

$$H_{int} = \int d^3x\, \frac{\lambda}{4!} \varphi^4(x)$$

$$\varphi(x) = \int \frac{d^3k}{2k^0} \left\{ a(\underline{k})\, e^{ik\cdot x} + a^\dagger(\underline{k})\, e^{-ik\cdot x} \right\}$$

$$k^0 = \sqrt{\underline{k}^2 + m^2}$$

$$| p_1, p_2 \rangle = a^\dagger(p_1)\, a^\dagger(p_2) | 0 \rangle$$

$$\langle p_3, p_4 | = \langle 0 | a(p_3)\, a(p_4)$$

$$[a(\underline{k}), a^\dagger(\underline{p})] = 2p^0\, \delta(\underline{k} - \underline{p})$$

(Ex!) Net result for this contribution:

$$\mathcal{A} = -i\lambda\, \delta(p_1 + p_2 - p_3 - p_4)$$

so $$\mathcal{A} = \underbrace{\lim_{p_i^2 \to m^2}}_{\substack{\text{go} \\ \text{"on shell"}}} \underbrace{\prod_{i=1}^{4} \left(\frac{p_i^2 - m^2}{i}\right)}_{} \langle 0 | T \varphi(p_1)\varphi(p_2)\varphi(p_3)\varphi(p_4) | 0 \rangle$$

inverse external propagators to strip off external propagators

<u>then</u>

N.B. Can't go on shell for $\langle 0 | T \varphi(p_1) \cdots \varphi(p_4) | 0 \rangle$ without this since result would be ∞ (ill-defined).

This is the general result: to go from correlator to amplitude, strip external propagators and then go on shell.