Exactness

We say that a group Γ is exact if the operation of taking the reduced crossed product with Γ preserves exactness of short exact sequences of Γ-C^*-algebras.

Exact groups satisfy the coarse Baum Connes conjecture and the property was first made prominent by the work of Kirchberg and Wassermann and studied by several authors. Examples of exact groups include amenable groups, and groups of finite asymptotic dimension.

Establishing exactness of a group has traditionally been done by analytic methods. However, recent work by Guentner, Kaminker and Ozawa enable us to use a more geometric approach.

Hilbert Space Compression

Guentner and Kaminker have recently introduced the concept of Hilbert Space Compression.

We shall first recall a couple of definitions:

Let X and Y be metric spaces.

Definition: A function $f : X \to Y$ is large-scale Lipschitz if there exist $C > 0$ and $D \geq 0$ such that

$$d_Y(f(x), f(y)) \leq C d_X(x, y) + D$$

Definition: Following Gromov, the compression $\rho(f)$ of $f \in Lip^{ls}(X, Y)$ is

$$\rho_f(r) = \inf_{d_X(x,y) \geq r} d_Y(f(x), f(y))$$

Hilbert Space Compression

Definition: Let X be a metric space with an unbounded metric.

1. The asymptotic compression R_f of a large scale Lipschitz map $f \in Lip^{ls}(X,Y)$ is
$$R_f = \liminf_{r \to \infty} \frac{\log \rho_f^*(r)}{\log r}$$
where $\rho_f^*(r) = \max\{\rho_f(r), 1\}$.
2. The compression of X in Y is
$$R(X,Y) = \sup\{R_f : f \in Lip^{ls}(X,Y)\}.$$
3. If Y is a Hilbert space, then the Hilbert space compression of X is
$$R(X) = R(X,Y).$$

Link to Exactness

This is linked to exactness by the following theorem:

Theorem: Let Γ be a finitely generated discrete group regarded as a metric space via the word metric. If the Hilbert space compression of Γ is greater than $1/2$ then Γ is exact.

This means that if we can find a Large Scale Lipschitz function for which the asymptotic compression is greater than $1/2$ then Γ is exact.

At first glance, this approach may seem very analytic. But in fact, it can be approached in a geometric way. We need to produce a function from a metric space to a Hilbert space which satisfies certain conditions. But the properties of the function will depend directly on the geometry of the space X.

Example: Trees

Guentner and Kaminker produced such a function from a the Cayley graph of a free group to a Hilbert Space, thus showing that free groups are exact.

Their method was to produce a family of functions dependent on some $0 < \epsilon \leq 1/2$ whose asymptotic compression tended to 1.

Let $X = (V, E)$ be the Cayley graph of a finite dimensional free group. This is a tree. Fix some vertex v. For every vertex $s \in X$ look at the unique path from s to v, denote the edges $e_1, e_2, \ldots, e_k$ and define f_ϵ as follows:

$$f_\epsilon(s) = 1^\epsilon \delta_{e_1(s)} + 2^\epsilon \delta_{e_2(s)} + \ldots + k^\epsilon \delta_{e_k(s)}$$

where δ_e is the Dirac function of the edge e.

Large Scale Lipschitz

We need to show that f_ϵ is large scale Lipschitz. To do this it is sufficient to show that there exists $C > 0$ such that

$$d(s,t) = 1 \Rrightarrow \|f_\epsilon(s) - f_\epsilon(t)\|^2 \leq C \;\; \forall s,t \in X$$

And so we have that

$$\|f_\epsilon(s) - f_\epsilon(t)\|^2 = 1^\epsilon + (2^\epsilon - 1^\epsilon)^2 + \ldots + [(k+1)^\epsilon - k^\epsilon]^2$$

And since $\sum_{i=2}^{\infty}[i^\epsilon - (i-1)^\epsilon]^2$ is finite, the above sum is bounded as required.

Asymptotic compression $R_{f_\epsilon} > 1/2$

Recall that the compression $\rho(f)$ of $f \in Lip^{ls}(X,Y)$ is

$$\rho_f(r) = \inf_{d_X(x,y)\geq r} d_Y(f(x), f(y))$$

It is sufficient to show that $\forall s,t \in X$ with $d(s,t) \geq r$

$$\|f_\epsilon(s) - f_\epsilon(t)\|^2 \geq C_\epsilon r^{1+2\epsilon}$$

Let $s,t \in X$ be such that $d(s,t) \geq r$. Denote by $\sharp(r)$ the smallest integer greater than $r/2$.

Thus, the edges $e_1(t), e_2(t), \ldots, e_{\sharp(r)}$ apppear in the expression for $f_\epsilon(t)$ but not in $f_\epsilon(s)$.

And so we get the following inequality:

$$\|f_\epsilon(s) - f_\epsilon(t)\|^2 \geq 1^{2\epsilon} + 2^{2\epsilon} + \ldots + \sharp r^{2\epsilon}$$

but

$$1^{2\epsilon}+2^{2\epsilon}+\ldots+\sharp r^{2\epsilon} \geq \int_0^{r/2} x^\epsilon dx = \frac{r^{1+2\epsilon}}{(2^{1+2\epsilon})(2\epsilon+1)}$$

Hence f_ϵ is large scale lipschitz and has asymptotic compression greater than 1/2. It satisfies all our requirements and free groups are exact.

Extension to Groups acting properly and cocompactly on $CAT(0)$ cube complexes

The previous method relied heavily on two properties of the metric space X. The first was the existence of unique paths from one point to another. The second was the median property.

At first glance $CAT(0)$ cube complexes do not satisfy these properties as for example, edge paths between points are not unique. But it is possible to find a way round this. Instead of using a function based on the edges, we must use a function based on the hyperplanes separating one point from another. These can be partially ordered via normal cube paths.

Work by Niblo, Roller and Reeves show that the hyperplanes on these cube paths satisfy conditions which mean that this ordering in conjunction with the median property is enough to construct a new function which satisfies all the properties we need.

Thus

Theorem: Groups which act properly and cocompactly on a $CAT(0)$ cube complex are exact.