Yu's Property A

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Slides will be available later on my webpage.

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Amenability

Amenability was first described as a measure theoretic property as follows:

Let Γ be a locally compact group and consider $\mathbf{L}^\infty(\Gamma)$.

This is the Banach space of all essentially bounded functions $f : \Gamma \to \mathbb{R}$ with respect to the Haar measure.

Definition. *Let* $\mathbf{L}_g$ *be the left action of* $g \in \Gamma$ *on* $f \in \mathbf{L}^\infty(\Gamma)$. $(\mathbf{L}_g f)(h) = f(gh)$.

A *mean* is a linear functional on $\mathbf{L}^\infty(\Gamma)$ which maps the constant function $f(g) = 1$ to 1 and non-negative functions to non-negative numbers.

A mean is said to be *left invariant* if $\mu(\mathbf{L}_g f) = \mu(f)$ for all $g \in \Gamma$ and $f \in \mathbf{L}^\infty(\Gamma)$.

Definition. *A locally compact group is amenable if there is a left (or right) invariant mean on* $\mathbf{L}^\infty(\Gamma)$.

Amenability for discrete groups

In some sense amenability is easier to define for discrete groups with no topological structure.

A discrete group G is amenable if there is a measure function which assigns to each subset of G a number from 0 to 1 such that:

- The measure is a probability measure.
- The measure is finitely additive.
- The measure is left invariant: the measure of A= the measure of gA.

This definition is equivalent to the one in terms of $\mathbf{L}^{\infty}(G)$

All finite and abelian groups satisfy this property.

Følner's criterion

Another definition of an amenable group is the existence of Følner sets.

These can be viewed as a sequence of subsets of the group with a particular relationship between the size of the intersection of two sets and the size of the set itself. This is defined rigorously as follows:

Theorem. *(Folner's condition)A group G is amenable if there exists a sequence $\{G_n\}$ of subsets of G such that $\forall g \in G$,*

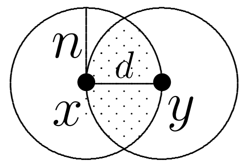
$$\lim_{n\to\infty} \frac{|gG_n \triangle G_n|}{|G_n|} = 0$$

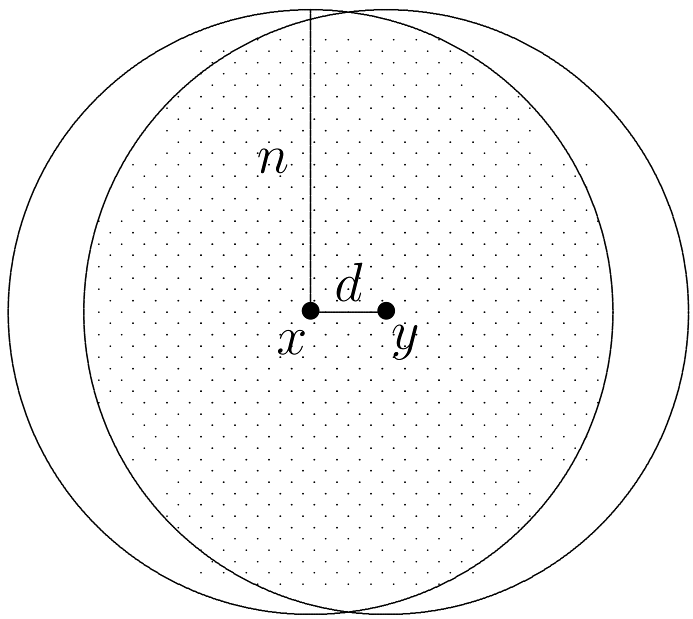
Example: Groups of subexponential growth

Definition. *Let G be a group with generating set A. Let $\beta_A(n)$ be the number of vertices in the closed ball of radius n about 1 in the Cayley graph of the group generated by A. The growth function of G with respect to A is $n \to \beta_A(n)$.*

Definition. *G has subexponential growth if $\beta_A(n) \leq e^{\sqrt{n}}$ for all $n \in \mathbb{N}$.*

Examples of groups of subexponential growth include finite groups, abelian groups and nilpotent groups.





(Folner's condition)A group G is amenable if there exists a sequence $\{G_n\}$ of subsets of G such that $\forall g \in G$,

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Yu's Property A

Yu introduced Property A as a generalisation of amenability which holds for hyperbolic groups.

It is similar enough to amenability to be of considerable use.

For example if X has property A then X satisfies the Novikov conjecture.

The original definition is as follows:

Definition. *A discrete metric space X has* Property *A if for all $R, \epsilon > 0$ there exists $S > 0$ and a family of finite non-empty subsets A_x of $X \times \mathbb{N}$, indexed by x in X, such that*

- *for all x, x' with $d(x, x') < R$ we have*

$$\frac{|A_x \Delta A_{x'}|}{|A_x|} < \epsilon$$

- *for all (x', n) in A_x we have $d(x, x') \leq S$.*

Reduced C^*-algebras

Let Γ be a discrete group.

Definition. $\ell^2(\Gamma)$ *is the space of square summable functions on* Γ*.*

$$\ell^2(\Gamma) = \{f : \Gamma \to \mathbb{C} \mid \sum_{g \in \Gamma} \mid f(g) \mid^2 < \infty\}$$

By left regular representation, each element f of the group ring is assigned a bounded operator $\lambda(f)$ which acts on any element $\xi \in \ell^2(\Gamma)$ by convolution.

$$\lambda(f)(\xi) = f * \xi$$

$\mathbb{B}(l^2(\Gamma))$ is the algebra of bounded operators on the space $l^2(\Gamma)$. The image of the group ring under the above representation is a *-subalgebra of it.

Definition. *The closure of this representation in the* C^* *norm of* $\mathbb{B}(l^2(\Gamma))$ *is the reduced* C^**-algebra of* Γ *and is denoted* $C^*_r(\Gamma)$*.*

Theorem. *(Kirchberg, Wassermann) A group* Γ *is exact if and only if its reduced* C^**-algebra is exact.*

Yu's Property A and reduced C^*-algebras

Ozawa (C.R.A.S. 2000) introduced a property which we shall call Property O:

A discrete group G is said to have Ozawa's Property O if for any finite subset $E \subset G$ and any $\epsilon > 0$, there are a finite subset $F \subset G$ and u: $G \times G \to \mathbb{R}$ such that

- u is a positive kernel
- $u(x, y) \neq 0$ only if $x^{-1}y \in F$
- $|1 - u(x, y)| < \epsilon$ if $x^{-1}y \in E$

and showed that for a discrete group G, the following three statements are equivalent:

1. The reduced $\mathbb{C}^*$-algebra $\mathbb{C}^*_r(G)$ is exact.
2. G has Property O.
3. The Uniform Roe algebra $UC^*(G)$ is nuclear.

Nuclearity of the Uniform Roe Algebra implies exactness

Finite width operators are the set of $A : \Gamma \times \Gamma \to \mathbb{C}$ satisfying:

1. $\exists M > 0$ such that $\mid A(s,t) \mid \leq M \; \forall s,t \in \Gamma$
2. $\exists R > 0$ such that $A(s,t) = 0$ if $d(s,t) > R$

Definition. *The Uniform Roe Algebra of* Γ, $UC^*(\Gamma)$ *is the closure of the* $*$*-algebra of finite width operators. It is a* C^**-algebra.*

Definition. *A* C^**-algebra* A *is nuclear iff for any* C^**-algebra* B, $\| \quad \|_{max} = \| \quad \|_{min}$ *on the algebraic tensor product* $A \odot B$.

Theorem. *Any nuclear algebra is also exact.*

The reduced C^*-algebra of the group is a closed subalgebra of the Roe algebra (Guentner,Kaminker, 2000) and so if the Uniform Roe algebra is exact, then the reduced C^*-algebra must also be exact.

Exactness implies property O

Given a Hilbert space H and any exact C^*-algebra A of $\mathbb{B}(H)$ Ozawa shows:

There exists a unital, positive, finite rank operator $\theta : A \longrightarrow \mathbb{B}(H)$ such that:

1. $\|\theta(x) - x\| \leq \epsilon$ for all $x \in E$.
2. There exist $f_k \in \mathcal{V}(H_0)$ and operators y_k in $\mathbb{B}(H)$ such that $\theta(x) = \sum_{k=1}^{d} f_k(x) y_k$, $x \in E$

where E is finite and $\subseteq A$, H_0 is a total subset of H and $\mathcal{V}(H_0)$ it's linear span.

Given a group G, take H to be $\mathbb{B}((l)^2(G))$ and assume θ exists.

Define $u(s,t) = \langle \delta_s, \theta(\delta_{st^{-1}})\delta_t \rangle$.

This satisfies the conditions of property O and such a kernel exists for any choice or E or ϵ. Thus G has Property O.

Property O implies nuclearity of the Uniform Roe Algebra:

To show the final step Ozawa assumes there exists a net of functions u_i and sets E_i which form a family of Ozawa kernels.

He constructs a set of Schur multipliers θ_i associated to each u_i.

He uses the fact that they are positive contractions (Paulsen, 1986) to prove that given a unital C^*-algebra B,

$$UC^*(G) \otimes_{min} B = UC^*(G) \otimes_{max} B$$

The Roe algebra $UC^*(G)$ is nuclear.

Examples of groups with property A (and thus exact):

Free groups, amenable groups, word hyperbolic groups, discrete subgroups of connected Lie groups, and groups acting properly on CAT(0) cube complexes of finite dimension.

The Haagerup Property

Definition. *A locally compact group has the Haagerup property if and only if it has a proper isometric action on a Hilbert space.*

Remark: All amenable groups are Haagerup.

The relationship between property A and the Haagerup property is not yet understood. However there are no known Haagerup groups without property A.

There is one famous test group which is known to be Haagerup but for which the question of property A remains open: Thompson's group F.

Thompson's Group F

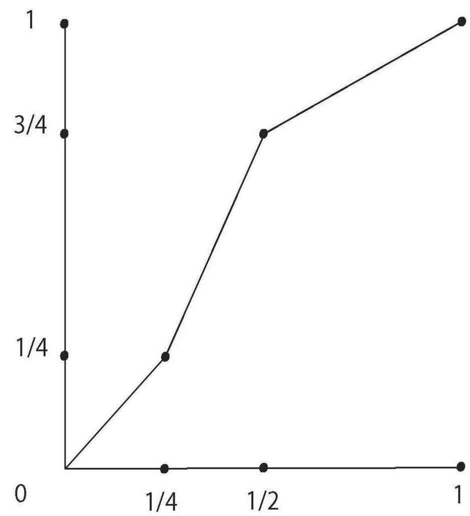
First introduced in 1965, Thompson's group F is generally known as the group with presentation

$$\langle x_0, x_1, x_2, ... | x_n x_k = x_k x_{n+1} \text{ for } n > k \rangle$$

It is also better known as follows:

Definition. *Thompson's group F is the set of orientation preserving piecewise linear homeomorphisms from the closed unit interval* $[0, 1]$ *to itself that are differentiable except at finitely many dyadic rational numbers and such that on intervals of differentiability the derivatives are powers of 2.*

It is the focus of much research, since it has produced either examples or counter examples to some famous conjectures and remains a possible test case for others.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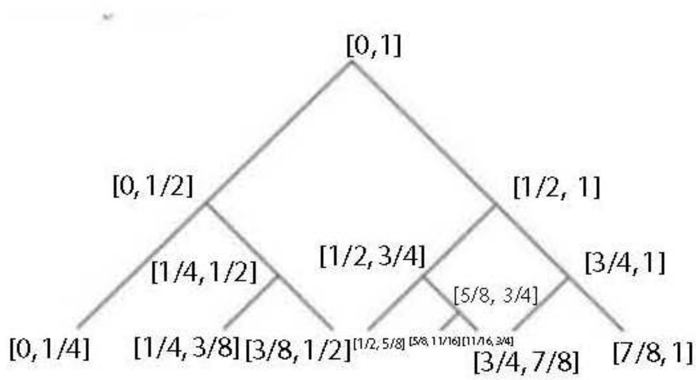
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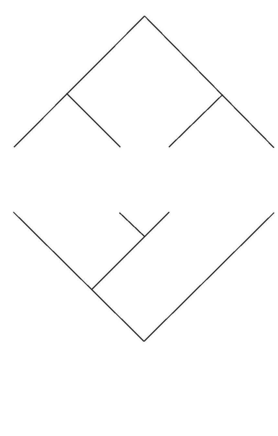

Thompson's Group F and amenability

The most famous question about Thompson's group F is whether or not it is amenable and has remained unsolved since the sixties. It is a fascinating question since either answer would provide an unusual example of a group with certain properties.

- If it is amenable, then it is an example of a finitely presented amenable group which is not elementary amenable. Grigorchuk was the first to construct such a group.
- If on the other hand it is not amenable, then it is an example of a finitely presented group which is not amenable and has no free subgroups of rank > 1. This provides a counter example to the Von Neumann's conjecture that a discrete group is not amenable if and only if it contains a subgroup which is free of rank 2.

Remark: Such a group has recently been constructed by Ol'shanskii and Sapir.

CAT(0) Cube Complexes

Definition. *A cube complex is a polyhedral complex of cells isometric to a Euclidean cell. The gluing of these cells is by isometries. The dimension of a cube complex is the highest dimension of one of its cells.*

Definition. *A* $CAT(0)$ *cube complex is a cube complex which is non positively curved.*

Example 1. *Any tree is a 1-dimensional* $CAT(0)$ *cube complex.*

Example 2. *Euclidean space with vertices at the integer lattice points has the structure of a* $CAT(0)$ *cube complex.*

Some Theorems and Questions

Theorem 1. *(Farley, 2003) Thompson's group F acts properly on a CAT(0) cube complex and hence acts properly on a Hilbert space.*

Theorem 2. *(C, Niblo, 2005) Groups which act properly and cocompactly on a finite dimensional CAT(0) cube complex have property A.*

Unfortunately

- Thompson's group only acts properly on infinite dimensional cube complexes.
- Infinite dimensional cube complexes do not have property A (Nowak, 2007).

Question: What role does the dimension play?

The authors construct functions on the CAT(0) complex. Following Guentner and Kaminker, the properties of these functions imply the existence of Ozawa type kernels.

Their proof relied on approximating functions by elements of a uniform Roe algebra. A key point was controlling the convergence condition and this was done by using the spherical growth function of the group.

Theorem 3. *(Brodzki,C,Guentner,Niblo,Wright, 2008) Let X be a finite dimensional CAT(0) cube complex. Equipped with the geodesic metric, X has property A.*

Corollary. *Groups acting metrically properly on a finite dimensional CAT(0) cube complex have property A.*

Remark: this time there is no assumption that the cube complex is locally finite.

A totally different approach was used here. The authors directly constructed explicit functions of the type given in Yu's original definition on the CAT(0) cube complex, thus giving a much stronger result.

The same argument can also be used for other results.

For example a similar construction shows that stabilisers at infinity are amenable. This theorem is an analog of the better known theorem for buildings for which the stabilizer of a point in the boundary at infinity is amenable. In the case for CAT(0) cube complexes, the boundary at infinity refers to the combinatorial boundary. More formally:

Theorem 4. *(Brodzki, C, Guentner, Niblo, Wright, 2008) Let G be a countable discrete group acting properly on a finite dimensional CAT(0) cube complex X and let z be a vertex at infinity of X. The stabilizer of z in G is amenable.*

Variations on the technique may also be used for other metric spaces with appropriate geometric properties.

Theorem 5. *(C,2008)Affine Buildings have property A.*