Exactness

Kirchberg and Wasserman have shown that a discrete group Γ is exact if and only if the reduced $\mathbb{C}^*$-algebra $\mathbb{C}_r^*(\Gamma)$ is exact.

This means that for every exact sequence of C^*-algebras

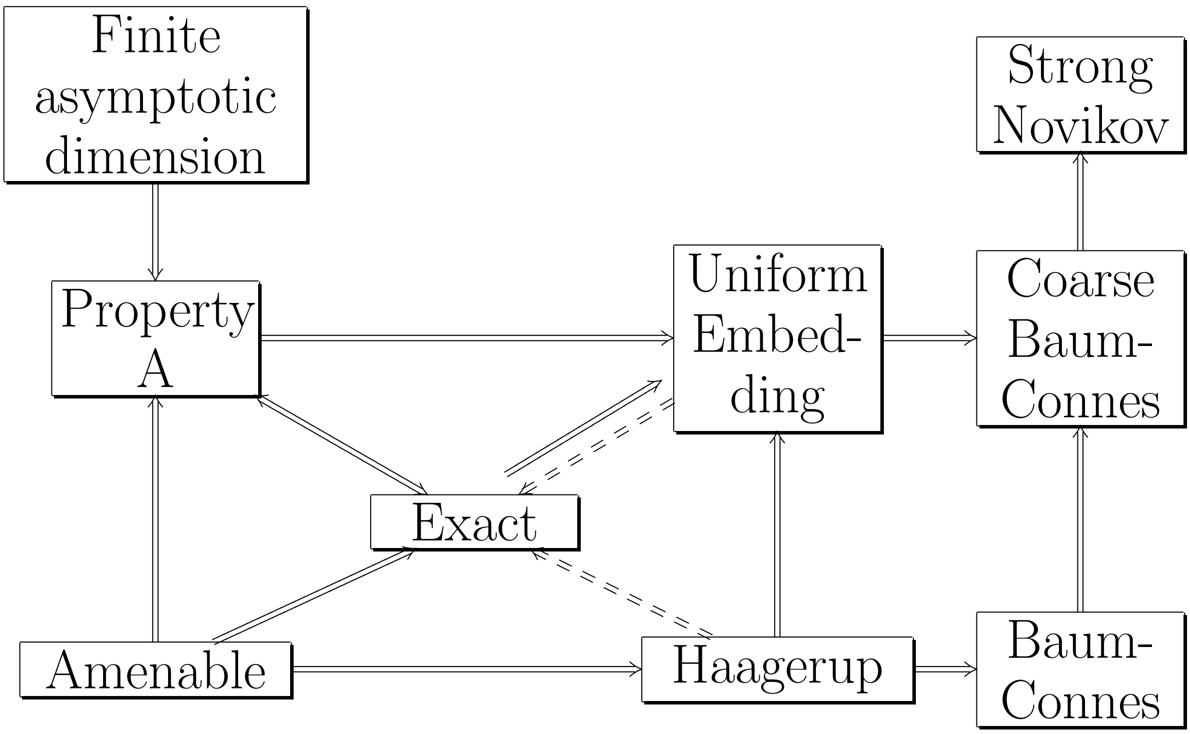
$$0 \to B \to C \to D \to 0$$

the sequence

$$0 \to C_r^\star(\Gamma) \otimes B \to C_r^\star(\Gamma) \otimes C \to C_r^\star(\Gamma) \otimes D \to 0$$

is exact.

Exactness also links many geometric and analytic properties of groups. This is represented by the following diagram:



Amenability

Definition: A locally compact group G is amenable if there is a left invariant mean on G, i.e. a state on $\mathbf{L}^\infty(G)$. (A state is a positive linear functional of unit norm.)

Examples: All finite groups and all Abelian groups are amenable. Compact groups are amenable as the Haar measure is an invariant mean.

The notion of amenability has been extended to many equivalent conditions. The one we are interested in is Folners Condition, although we will use it in a slightly different form:

Theorem: A group G is amenable iff there exists a sequence $\{G_n\}$ of subsets of G such that $\forall g \in G$,

$$\lim_{n\to\infty} \frac{|gG_n \triangle G_n|}{|G_n|} = 0$$

Exactness and Amenability

It is known that amenable groups are exact. Lance has proved that a group G is amenable iff $C^*(G)$ is nuclear, which implies that it is exact.

We will use a very different method which relies on the Folner property of amenable groups.

Exactness and Property (O)

Ozawa (C.R.A.S. 2000) introduced a property which we shall call Property O and showed that for a discrete group G, the following three statements are equivalent:

1. The reduced $\mathbb{C}^*$-algebra $\mathbb{C}^*_r(G)$ is exact.
2. G has Property O.
3. The uniform Roe algebra $UC^*(G)$ is nuclear.

It thus follows from the result of Kirchberg and Wassermann that any discrete group G which has Property O is exact.

Property (O)

Definition:

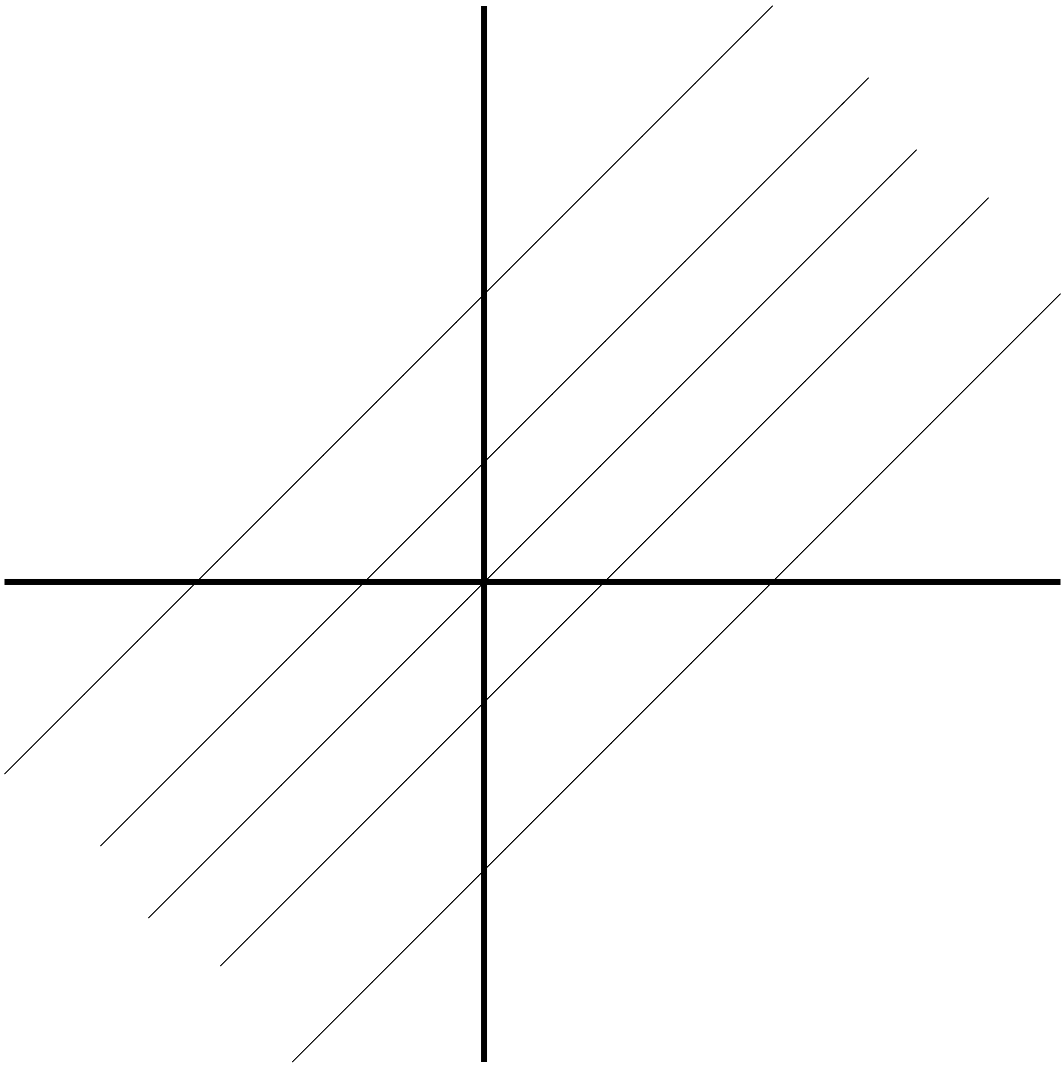
A real positive definite kernel is a function u: $G \times G \to \mathbb{R}$ such that $u(g_i, g_j) = u(g_j, g_i)$ and for any set of $\lambda_i, \lambda_j \in \mathbb{R}$,

$$\sum_{i,j}^{n} \lambda_i \lambda_j u(g_i, g_j) \geq 0$$

Property (O):

A discrete group G is said to have Ozawa's Property (O) if for any finite subset $E \subset G$ and any $\epsilon > 0$, there are a finite subset $F \subset G$ and u: $G \times G \to \mathbb{R}$ such that

- u is a positive definite kernel
- $u(s, t) \neq 0$ only if $s^{-1}t \in F$
- $|1 - u(s, t)| < \epsilon$ if $s^{-1}t \in E$



Folner's Condition

Definition: Any amenable group G satisfies Folner's condition:

For any finite subset E of G and every $\epsilon > 0$, there is another finite subset W of G such that for any $g \in E$, $\frac{|gW \triangle W|}{|W|} \leq \epsilon$

This can be rewritten as follows:

$$\frac{|gW \triangle W|}{|W|} = \frac{|gW \cup W|}{|W|} - \frac{|gW \cap W|}{|W|}$$

$|gW \cup W|$ lies between $|W|$ and $2|W|$, while $|gW \cap W|$ lies between 0 and $|W|$.

So $1 \leq \frac{|gW \cup W|}{|W|} \leq 2$ and $0 \leq \frac{|gW \cap W|}{|W|} \leq 1$.

Since the difference between them is less than ϵ, we have $\left|1 - \frac{|gW \cap W|}{|W|}\right| < \epsilon$.

The function $u(x, y)$

Given an amenable group G, a finite subset E, some $\epsilon > 0$ and the associated finite subset W, consider the function

$$u(x, y) = \frac{|xW \cap yW|}{|W|}$$

We will show that $u(x, y)$ has the following properties:

- $u(x, y)$ is a positive kernel
- If $x^{-1}y \in E$, then $|u(x, y) - 1| < \epsilon$
- There exists a finite subset F such that $u(x, y) \neq 0$ only if $x^{-1}y \in F$

$u(x, y)$ is a positive definite kernel

An element $g \in G$ belongs to the intersection $xW \cap yW$ only if $g \in xW$ and $g \in yW$. This is equivalent to $x \in gW^{-1}$ and $y \in gW^{-1}$.

So we can rewrite $u(x, y)$ as a sum over all elements $g \in G$

$$u(x, y) = \frac{1}{|W|} \sum_{g \in G} \chi_{gW^{-1}}(x) \chi_{gW^{-1}}(y)$$

Thus:

$$\begin{aligned}
\sum_{i,j}^{n} \lambda_i \lambda_j u(x_i, x_j) &= \sum_{i,j}^{n} \lambda_i \lambda_j \frac{1}{|W|} \sum_{g \in G} \chi_{gW^{-1}}(x_i) \chi_{gW^{-1}}(x_j) \\
&= \frac{1}{|W|} \sum_{g \in G} \left(\sum_{i}^{n} \lambda_i \chi_{gW^{-1}}(x_i) \sum_{j}^{n} \lambda_j \chi_{gW^{-1}}(x_j) \right) \\
&= \frac{1}{|W|} \sum_{g \in G} \left(\sum_{i}^{n} \lambda_i \chi_{gW^{-1}}(x_i) \right)^2 \\
&\geq 0
\end{aligned}$$

If $x^{-1}y \in E$, then $|1 - u(x, y)| < \epsilon$

From Folners Condition we have that for any $g \in E$ there exists W such that

$$\left|1 - \frac{|gW \cap W|}{|W|}\right| < \epsilon$$

So now let g be equal to $x^{-1}y$ and the condition becomes

$$\left|1 - \frac{|x^{-1}yW \cap W|}{|W|}\right| = \left|1 - \frac{|xW \cap yW|}{|W|}\right| = |1-u(x, y)| < \epsilon$$

And so we have as required that if $x^{-1}y \in E$, then $|1 - u(x, y)| \leq \epsilon$

There exists a finite set F such that $u(x, y) \neq 0$ only if $x^{-1}y \in F$

Since W is finite, it is contained within a ball of diameter r.

Let F be the ball of radius r around the origin.

If $d(x, y) > 2r$, ie $x^{-1}y \notin F$ there is no intersection between xW and yW and so $u(x, y) = 0$ as required.

Amenable groups are exact

We have shown that $u(x, y)$ has the following properties:

- $u(x, y)$ is a positive kernel
- If $x^{-1}y \in E$, then $|u(x, y) - 1| < \epsilon$
- There exists a finite subset F such that $u(x, y) \neq 0$ only if $x^{-1}y \in F$

In other words, $u(x, y)$ is an Ozawa kernel and so any amenable group G is an exact group.

Example: Groups of subexponential growth

Definition: Let G be a group with generating set A. Let $\beta_A(n)$ be the number of vertices in the closed ball of radius n about 1 in the Cayley graph of the group generated by A. The growth function of G with respect to A is $n \to \beta_A(n)$.

Definition: G has subexponential growth if $\beta_A(n) \leq e^{\sqrt{n}}$ for all $n \in \mathbb{N}$.

Examples of groups of subexponential growth include finite groups, abelian groups and nilpotent groups.

Groups of subexponential growth are amenable and so satisfy Folners condition.

In fact, it can be shown that balls of radius n in the Cayley graph of G are Folner sets, where the radius n will depend on the chosen ϵ and finite set E.

In this case, the Ozawa kernel $u(x, y)$ is simply the size of the intersection of the balls of radius n centred at x and y scaled by the size of a ball of radius n.

$$u(x, y) = \frac{|xB_n \cap yB_n|}{|B_n|}$$

Example: The integers

The integers form an amenable group and so satisfy Folners condition.

In fact, it can be shown that balls of radius n in the Cayley graph of G are Folner sets, where the radius n will depend on the chosen ϵ and finite set E.

In this case, the Ozawa kernel $u(x, y)$ is simply the size of the intersection of the balls of radius n centred at x and y scaled by the size of a ball of radius n.

$$u(x, y) = \frac{|xB_n \cap yB_n|}{|B_n|}$$