Littlewood-Richardson coefficients and the hive model

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Presented at:

Seoul National University
Seoul, South Korea: August 2008
Key sources

- **Littlewood-Richardson coefficients**
    Oxford: Clarendon Press, 1940

- **The hive model**
Key sources

- Horn inequalities

- Puzzles
Key sources

Polynomial property of stretched LR-coefficients


Overview

Schur functions

- Let $n$ be a fixed positive integer and $x = (x_1, x_2, \ldots, x_n)$ a sequence of indeterminates.

- Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ be a partition of weight $|\lambda|$ and length $\ell(\lambda) \leq n$, so that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0$.

- Then the Schur function $s_\lambda(x)$ is defined by:

$$s_\lambda(x) = \left| \frac{x_i^{n+\lambda_j-j}}{x_i^{n-j}} \right|_{1 \leq i,j \leq n}.$$

- The Schur functions form a $\mathbb{Z}$-basis of $\Lambda_n$, the ring of polynomial symmetric functions of $x_1, \ldots, x_n$.

- Each Schur function $s_\lambda(x)$ may be interpreted as the character $\text{ch } V^\lambda(x)$ of an irrep of $gl(n)$. 

LR-coefficients

Any product of Schur functions can be expressed as a linear sum of Schur functions:

$$s_\lambda(x) \cdot s_\mu(x) = \sum_\nu c^\nu_{\lambda\mu} \cdot s_\nu(x)$$

The coefficients $c^\nu_{\lambda\mu}$ are the multiplicities appearing in the decomposition of the tensor product of irreps of $\mathfrak{gl}(n)$:

$$V^\lambda \otimes V^\mu = \sum_\nu c^\lambda_{\lambda\mu} \cdot V^\nu$$

Each Littlewood-Richardson coefficient $c^\nu_{\lambda\mu}$ is a non-negative integer that may be evaluated by means of the Littlewood-Richardson rule.
Young diagrams

- Each partition $\lambda$ specifies a Young diagram $F^\lambda$ consisting of $|\lambda|$ boxes arranged in $\ell(\lambda)$ left adjusted rows of lengths $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_{\ell(\lambda)} > 0$.

- The partition $\lambda'$ conjugate to $\lambda$ is such that $F^{\lambda'}$ is obtained from $F^\lambda$ by interchanging rows and columns.

- Ex: If $\lambda = (4, 3, 1)$ then $|\lambda| = 8$, $\ell(\lambda) = 3$, $\lambda' = (3, 2, 2, 1)$, with

\[ F^\lambda = \begin{array}{c}
\hline
| & | & | & | \\
| & | & | & \\
| & | & | \\
| & | & \\
\hline
\end{array} \quad F^{\lambda'} = \begin{array}{c}
\hline
| & | & | & | \\
| & | & | \\
| & | \\
\hline
\end{array} \]
Skew Young diagrams

- Given partitions $\lambda$ and $\nu$ such that all boxes of $F^\lambda$ are contained in $F^\nu$ we write $\lambda \subseteq \nu$.
- Removing the boxes of $F^\lambda$ from $F^\nu$ leaves the skew Young diagram $F^{\nu/\lambda}$
- **Ex:** If $\nu = (5, 4, 2)$ and $\lambda = (3, 1)$ then $F^{\nu/\lambda} = \begin{array}{ccc} * & * & * \\ & * & \\ & & \\ \end{array}$
- The corresponding skew Schur function is such that

$$s_{\nu/\lambda}(x) = \sum_{\mu} c_{\lambda\mu}^{\nu} s_\mu(x)$$
Littlewood-Richardson rule

1. Fill the boxes of the Young diagram $F^\lambda$ with 0’s.
2. Then fill the boxes of the skew Young diagram $F^{\nu/\lambda}$ with $\mu_i$ entries $i$ for $i = 1, 2, \ldots, n$.
3. $c_{\lambda \mu}^{\nu}$ is the number of such diagrams with entries
   - weakly increasing across rows from left to right
   - strictly increasing down columns from top to bottom
   - satisfying the lattice permutation rule - i.e. at every stage in the sequence of non-zero entries read from right to left across rows taken in turn from top to bottom $\#1's \geq \#2's \geq \cdots \geq \#n's$
Application of the LR-rule

**Ex:** \( n = 4, \lambda = (4, 2), \mu = (4, 3, 2), \nu = (6, 5, 3, 1) \)

- The only valid LR-diagrams

```
0 0 0 0 1 1
0 0 1 1 2
2 2 3
3
```

```
0 0 0 0 1 1
0 0 1 2 2
1 2 3
3
```

```
0 0 0 0 1 1
0 0 1 2 2
1 1 3
3
```

- Some invalid diagrams

```
0 0 0 0 1 1
0 0 1 2 2
1 3 2
3
```

```
0 0 0 0 1 1
0 0 1 1 2
2 3 3
2
```

```
0 0 0 0 1 1
0 0 1 1 3
2 2 2
3
```

- Hence \( c_\lambda^\nu \mu = 3 \).
Integer hives

- Knutson & Tao [99], as described by Buch [00]
- An integer $n$-hive is a triangular graph with vertex labels $a_{ij} \in \mathbb{Z}$ for $0 \leq i, j, i + j \leq n$.

**Ex:** $n = 4$

- Vertex labels increase along each edge from left to right
Relation between vertex and edge labels

- Edge labels are the non-negative differences between neighbouring vertex labels

\[ \alpha = a_{i,j+1} - a_{ij}, \quad \beta = a_{i+1,j-1} - a_{ij}, \quad \gamma = a_{i+1,j} - a_{ij} \]

- Vertex and edge labels for the two types of elementary triangle

\[ \sigma = q - p \geq 0, \quad \tau = r - q \geq 0, \quad \rho = r - p \geq 0 \]

so that automatically we have \( \sigma + \tau = \rho \) in any such triangle
Hive conditions

- Three distinct types of rhombi with vertex labels:

  ![Rhombus 1](image1)
  ![Rhombus 2](image2)
  ![Rhombus 3](image3)

  The **hive condition** for each rhombus: \(b + c \geq a + d\)

- Three distinct types of rhombi with edge labels:

  ![Rhombus 4](image4)
  ![Rhombus 5](image5)
  ![Rhombus 6](image6)

  The **hive condition** for each rhombus: \(\alpha \geq \gamma \) and \(\beta \geq \delta\)

- **Note:** The triangle edge condition implies \(\alpha + \delta = \beta + \gamma\).
**LR-hives vertex labels**

**Definition** An LR-hive is an integer $n$-hive for which

- all rhombi satisfy the hive conditions;
- boundaries determined by partitions $\lambda, \mu, \nu$ with $\ell(\lambda), \ell(\mu), \ell(\nu) \leq n$ and $|\lambda| + |\mu| = |\nu|$;
- boundary vertex labels as shown:
**LR-hives edge labels**

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- boundary edge labels as shown:
Bijection between LR-diagrams and LR-hives

Example: \( n = 3, \lambda = (320), \mu = (210) \) and \( \nu = (431) \).

- \( D \) = Littlewood-Richardson diagram;
- \( G \) = Generalised Gelfand-Zetlin pattern;
- \( Z \) = Zeros and cumulative row sums of \( G \);
- \( H \) = LR-hive = reorientation of lower triangular part of \( Z \).

\[
D = \begin{bmatrix}
0 & 0 & 0 & 1 \\
0 & 0 & 2 \\
1
\end{bmatrix} \iff G = \begin{bmatrix}
4 & 3 & 1 \\
4 & 3 & 1 \\
4 & 2 & 1 \\
3 & 2 & 0
\end{bmatrix}
\]

\[
Z = \begin{bmatrix}
0 & 4 & 7 & 8 \\
0 & 4 & 6 & 7 \\
0 & 3 & 5 & 5 \\
0 & 3 & 5 & 5
\end{bmatrix} \iff H = \begin{bmatrix}
5 \\
3 & 6 & 8 \\
0 & 4 & 7 & 8
\end{bmatrix}
\]
**Theorem**

- **Lemma**  A bijection between LR integer $n$-hives, $H$, and LR-diagrams, $D$, is provided by the formula:

\[ a_{ij} = \# \text{ of entries } \leq i \text{ in the first } i + j \text{ rows of } D \]

for the $(i, j)$th vertex label in $H$, for all $i, j$ such that $0 \leq i, j, i+j \leq n$.

- **Theorem** The LR-coefficient $c_{\lambda\mu}^\nu$ is the number of LR-hives with boundary labels determined by $\lambda$, $\mu$ and $\nu$.

- **Note**: Neither this theorem nor the Littlewood-Richardson rule allows us to see whether or not a given LR-coefficient $c_{\lambda\mu}^\nu$ is non-zero.
Example of bijection

Ex: \( n = 4, \lambda = (753), \mu = (742), \nu = (9964) \)

\[ a_{ij} : \]

\[
\begin{array}{cccc}
15 \\
15 & 22 \\
12 & 21 & 26 \\
7 & 16 & 24 & 28 \\
0 & 9 & 18 & 24 & 28 \\
\end{array}
\]

\[
\begin{array}{cccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 1 & 1 & 2 & 2 \\
0 & 0 & 0 & 1 & 1 & 2 \\
1 & 2 & 3 & 3 \\
\end{array}
\]
Example of bijection

Ex: \( n = 4, \lambda = (753), \mu = (742), \nu = (9964) \)

\[ a_{ij} \):

\[
\begin{array}{ccccccccc}
7 & 7 & 6 & 4 & 9 & 5 & 5 & 2 & 1 \\
& & 3 & & 1 & & 0 & & \\
& & & & & & 0 & & \\
\end{array}
\]

\[
\begin{array}{ccccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 1 & 1 & 2 & 2 \\
0 & 0 & 0 & 1 & 1 & 2 & & \\
1 & 2 & 3 & 3 & & & & & \\
\end{array}
\]

\[ \iff \]
LR-hives showing that \( c_{753,742}^{9964} = 6 \)

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Non-zero conditions

We know that $c_{\lambda\mu}^{\nu}$ is the number of LR-hives with boundary labels determined by $\lambda$, $\mu$ and $\nu$

For given $\lambda$, $\mu$ and $\nu$ we would like some way of determining if $c_{\lambda\mu}^{\nu}$ is non-zero

Horn [62] defined a set of inequalities and conjectured that they gave necessary and sufficient conditions for the solution of a problem later realised to be equivalent to that of LR-coefficients being non-zero

The validity of Horn’s conjecture was proved by the efforts of Klyachko [98], Knutson and Tao [99], Belkale [01], and Knutson, Tao and Woodward [04]

For a comprehensive review see Fulton [00]
Partial sums

- Let \( N = \{1, 2, \ldots, n\} \), then for fixed \( r \), with \( 1 \leq r \leq n \), let \( I = \{i_1, i_2, \ldots, i_r\} \subseteq N \) and \( \bar{I} = N \setminus I \).

- For any partition \( \lambda \) let:
  \[
  ps(\lambda)_I = \lambda_{i_1} + \lambda_{i_2} + \cdots + \lambda_{i_r}.
  \]

- If \( i_1 < i_2 < \cdots < i_r \) then let
  \[
  part(I) = (i_r - r, \ldots, i_2 - 2, i_1 - 1).
  \]

- Let \( T^n_r \) be the set of triples \((I, J, K)\) with \( I, J, K \subset N \) and \( \#I = \#J = \#K = r \) with \( c_{part(K)}^{part(I)part(J)} > 0 \).

- Let \( R^n_r \) be the set of triples \((I, J, K)\) with \( I, J, K \subset N \) and \( \#I = \#J = \#K = r \) with \( c_{part(K)}^{part(I)part(J)} = 1 \).
Non-zero conditions

**Theorem:** The LR-coefficient $c_{\nu\lambda\mu}$ is non-zero if and only if

$$|\nu| = |\lambda| + |\mu|$$

and Horn’s inequalities,

$$ps(\nu)_K \leq ps(\lambda)_I + ps(\mu)_J,$$

are satisfied for all $r = 1, 2, \ldots, n - 1$ and all $(I, J, K) \in T^n_r$.

**Note:** Not all of Horn’s inequalities are essential. Horn’s essential inequalities are those for which $(I, J, K) \in R^n_r$ - but where do they come from?
Definition A **puzzle** is a diagram on a triangular lattice in which edges are distinguished so that it is composed of copies of the following pieces oriented in any way so as to fit:
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Definition A puzzle is a diagram on a triangular lattice in which edges are distinguished so that it is composed of copies of the following pieces oriented in any way so as to fit:
Hive plan or labyrinth

Definition A hive plan is made up of corridors, dark rooms and light rooms obtained by deleting interior edges of a puzzle:
Hive plan or labyrinth

Definition A hive plan is made up of shaded corridors, dark rooms and light rooms obtained by deleting interior edges of a puzzle:
Hive plan or labyrinth

Definition: A hive plan is made up of corridors, blue rooms and red rooms obtained by deleting interior edges of a puzzle:
Link between puzzles and Horn triples

- $(I, J, K)$ is Horn triple if it specifies the positions of the thick edges on the boundary of any puzzle. It is essential if the puzzle with these boundary thick edges is unique.

- For $I = (1, 2, 4)$, $J = (2, 3, 4)$ and $K = (2, 3, 5)$ we have:
Each Horn triple defines an inequality

$$\nu_2 + \nu_3 + \nu_5 \leq (\nu_2 + \nu_3) + \gamma_4 = (\alpha_1 + \alpha_2 + \beta_1 + \beta_2) + \gamma_4$$

$$\leq \lambda_1 + \alpha_2 + \beta_1 + \beta_2 + \gamma_4 \leq \lambda_1 + \lambda_2 + \beta_1 + \beta_2 + \gamma_4$$

$$\leq \lambda_1 + \lambda_2 + \beta_3 + \beta_2 + \gamma_4 \leq \lambda_1 + \lambda_2 + (\beta_3 + \beta_4 + \gamma_4)$$

$$= \lambda_1 + \lambda_2 + (\alpha_4 + \mu_2 + \mu_3 + \mu_4) \leq \lambda_1 + \lambda_2 + \lambda_4 + \mu_2 + \mu_3 + \mu_4$$

That is  \( ps (\nu)_K \leq ps (\lambda)_I + ps (\mu)_J. \)
The same procedure applied to thin-edge inequalities gives

\[ \nu_1 + \nu_4 \geq \gamma_1 + \nu_4 = \gamma_1 + (\alpha_5 + \beta_5) \]

\[ \geq \gamma_1 + \alpha_3 + \beta_5 \geq (\gamma_1 + \alpha_3) + \mu_5 = (\lambda_3 + \gamma_2) + \mu_5 \]

\[ \geq \lambda_3 + \gamma_3 + \mu_5 = \lambda_3 + \lambda_5 + \mu_1 + \mu_5 \]

That is  \[ ps (\nu)_{\overline{K}} \geq ps (\lambda)_{\overline{T}} + ps (\mu)_{\overline{T}} \]
Significance of inequalities derived from puzzles

- Each inequality $p_s(\nu)_{K} \leq p_s(\lambda)_{I} + p_s(\mu)_{J}$ derived from a puzzle must be satisfied if $c^{\nu}_{\lambda\mu}$ is to be non-zero.

- To show that a puzzle triple $(I, J, K)$ is a Horn triple, and the inequality a Horn inequality, we must make a connection between puzzles with thick boundary edges specified by $(I, J, K)$ and $c_{part}(K)$ and $c_{part}(I), part(J)$.

- First note the connection between (for example) $M = \{1, 3, 6, 8\} \subseteq N = \{1, 2, \ldots, 10\}$ and $F_{part}(M) = F_{431}$.
How may puzzles are there?

**Theorem**  The number of puzzles with the positions of the thick edges on the boundary specified by \((I, J, K)\) is given by

\[
\binom{\text{part}(K)}{\text{part}(I),\text{part}(J)}
\]

**Ex:** \(n = 5, r = 2, I = (1, 2, 4), J = (2, 3, 4), K = (2, 3, 5)\)

- In this case \(\binom{\text{part}(K)}{\text{part}(I),\text{part}(J)} = \binom{311}{1,111} = 1\)
- and there exists just the one puzzle identified earlier
A map from puzzles to hives

Let the thick edges of the puzzle be specified by \((I, J, K)\).

For \(M = I, J, K\) in turn, label each thick boundary edge of the puzzle by the corresponding row length of \(\mathcal{F}^\text{part}(M)\).

Scale the length of all thin edges by \(t\) and let \(t \to 0\).
A map from puzzles to hives contd

- In each thick edged room set all parallel edge labels equal using the triangle condition wherever required.
- Reflect the resulting diagram in its vertical axis of symmetry to obtain a hive.

**Lemma** This map provides a bijection between puzzles with thick edges specified by \((I, J, K)\) and LR-hives with boundary specified by \(part(J), part(I), part(K)\).
A map from puzzles to LR-hives

Ex: If \( n = 5, \ r = 2, \ I = (1, 2, 4), \ J = (2, 3, 4), \ K = (2, 3, 5) \)
we have \( \text{part} (I) = (1), \ \text{part} (J) = (1, 1, 1), \ \text{part} (K) = (2, 1, 1) \)
Corollaries

- Each triple \((I, J, K)\) is a Horn triple if and only if it specifies the thick boundary edges of a puzzle.
- The corresponding inequality \(ps(\nu)_K \leq ps(\lambda)_I + ps(\mu)_J\) defined by the puzzle is a Horn inequality.
- The number of puzzles specified by \((I, J, K)\) is equal to \(c_{\text{part}(K)} c_{\text{part}(J),\text{part}(I)} = c_{\text{part}(I),\text{part}(J)}\).
- The puzzle is said to be rigid if it is unique, that is \(c_{\text{part}(K)} c_{\text{part}(J),\text{part}(I)} = 1\), and the corresponding Horn inequality is essential.
- The LR-coefficient \(c_{\lambda\mu}^\nu\) is non-zero if and only if every essential Horn inequality is satisfied.
Consequences of any Horn equality

All sequences of inequalities become equalities.

\[
\nu_2 + \nu_3 + \nu_5 = (\nu_2 + \nu_3) + \gamma_4 = (\alpha_1 + \alpha_2 + \beta_1 + \beta_2) + \gamma_4 \\
= \lambda_1 + \alpha_2 + \beta_1 + \beta_2 + \gamma_4 = \lambda_1 + \lambda_2 + \beta_1 + \beta_2 + \gamma_4 \\
= \lambda_1 + \lambda_2 + \beta_3 + \beta_2 + \gamma_4 = \lambda_1 + \lambda_2 + (\beta_3 + \beta_4 + \gamma_4) \\
= \lambda_1 + \lambda_2 + (\alpha_4 + \mu_2 + \mu_3 + \mu_4) = \lambda_1 + \lambda_2 + \lambda_4 + \mu_2 + \mu_3 + \mu_4.
\]
Edge label equalities

\[ \nu_1 + \nu_4 = \gamma_1 + \nu_4 = \gamma_1 + (\alpha_5 + \beta_5) = \gamma_1 + \alpha_3 + \beta_5 = (\gamma_1 + \alpha_3) + \mu_5 = (\lambda_3 + \gamma_2) + \mu_5 = \lambda_3 + \gamma_3 + \mu_5 = \lambda_3 + \lambda_5 + \mu_1 + \mu_5 \]

These equalities imply:

\[\nu_5 = \gamma_4, \quad \alpha_1 = \lambda_1, \quad \alpha_2 = \lambda_2, \quad \beta_1 = \beta_3, \quad \beta_2 = \beta_4, \quad \alpha_4 = \lambda_4, \quad \beta_5 = \mu_5, \quad \nu_1 = \gamma_1, \quad \gamma_2 = \gamma_3.\]
Factorisation of LR-hives
Illustration of $H_n$ and subhives $H_r, H_{n-r}$
LR-coefficient factorisation

- **Lemma** In the case of any Horn equality and a corresponding puzzle, the deletion of redundant corridors from any LR-hive $H_n$ gives a pair of LR-subhives $H_r$ and $H_{n-r}$.

- **Lemma** In the case of any essential Horn equality, this map from the LR-hives $H_n$ to pairs of LR-hives $H_r$ and $H_{n-r}$ is a bijection.

- **Theorem** If an essential Horn inequality is saturated then $c^{\nu}_{\lambda \mu}$ factorises.

- **Definition** If all essential Horn inequalities are strict $c^{\nu}_{\lambda \mu}$ is said to be primitive.
LR factorisation example

Ex: \( n = 5, \ r = 3, \ n - r = 2: \)

- \( \lambda = (9, 7, 6, 2, 0), \ \mu = (13, 5, 3, 1, 0), \ \nu = (14, 12, 11, 5, 4). \)
- \( I = \{1, 2, 4\}, \ J = \{2, 3, 4\}, \ K = \{2, 3, 5\}. \)
- \( \lambda_I = (9, 7, 2), \ \mu_J = (5, 3, 1), \ \nu_K = (12, 11, 4) \)
- \( \lambda_I = (6, 0), \ \mu_J = (13, 0), \ \nu_K = (14, 5) \)
- \( ps(\nu)_K = 27 = 18 + 9 = ps(\lambda)_I + ps(\mu)_J \)

Hence

\[
\binom{(14,12,11,5,4)}{(9,7,6,2,0),(13,5,3,1,0)} = \binom{(12,11,4)}{(9,7,2),(5,3,1)} \binom{(14,5)}{(6,0),(13,0)} = 2 \cdot 1 = 2
\]

Note: This is an example of the reduction of an LR-coefficient, since

\[
\binom{(14,12,11,5,4)}{(9,7,6,2,0),(13,5,3,1,0)} = \binom{(12,11,4)}{(9,7,2),(5,3,1)} = 2
\]
Proof of factorisation

To be shown:

- the corridors $R_n$ of $H_n$ are redundant;
- the dark rooms constitute an LR-hive $H_r$;
- the light rooms constitute an LR-hive $H_{n-r}$;
- any LR hives $H_r, H_{n-r}$ joined by $R_n$ gives an LR-hive $H_n$.

To be checked that the Horn equality implies:

- all corridor edge labels fixed;
- LR hive conditions for any rhombus split by corridor;
- LR hive conditions across corridor/dark room boundary;
- LR hive conditions across corridor/light room boundary.
Corridor edges fixed

- Hive conditions: \( \gamma_1 \leq \sigma_1 \leq \tau_1 \leq \alpha_1, \quad \gamma_2 \leq \sigma_2 \leq \tau_2 \leq \alpha_2. \)

- Horn inequality: \( \gamma_1 + \gamma_2 \leq \alpha_1 + \alpha_2. \)
Corridor edges fixed

- Hive conditions: $\gamma_1 \leq \sigma_1 \leq \tau_1 \leq \alpha_1$, $\gamma_2 \leq \sigma_2 \leq \tau_2 \leq \alpha_2$.

- Horn equality: $\gamma_1 + \gamma_2 = \alpha_1 + \alpha_2$.

- Implies: $\gamma_1 = \alpha_1$ and $\gamma_2 = \alpha_2$.

- Implies: $\gamma_1 = \sigma_1 = \tau_1 = \alpha_1$ and $\gamma_2 = \sigma_2 = \tau_2 = \alpha_2$. 
Deletion of corridor

- Initial hive conditions: \( \gamma \leq \sigma, \sigma \leq \tau, \tau \leq \alpha. \)

- Implies final hive condition: \( \gamma \leq \alpha. \)

- Horn equality \( \alpha - \beta = \rho = \gamma - \delta \) implies \( \alpha + \delta = \beta + \gamma. \)
Paths - gentle and good

- **Path**: a continuous sequence of connected corridor walls with dark rooms, thick-edged 0-regions, on the right and light rooms, thin-edged 1-regions, on the left.

- **Gentle path**: at each vertex the deviation is 0 or $\pm \pi/3$.

- **Gentle loop**: a gentle path that forms a closed interior loop.

- An edge is **good** if it forms the short diagonal of a rhombus satisfying the hive condition, otherwise it is **bad**.

- **Good path**: gentle path along which all the edges are **good**, ie with the hive condition satisfied across each edge.
Good paths and factorisation

Observations

- When subdividing two LR subhives, $H_r$ and $H_{n-r}$, and inserting corridors to create a hive $H_n$, this hive will be an LR hive if and only if each edge of every internal corridor wall of the corresponding hive plan is good.

- Let the boundary labels $\lambda, \mu, \nu$ of the hive $H_n$ be such that for a given puzzle specified by $(I, J, K)$ the corresponding Horn inequality is saturated, i.e. $ps(\nu)_K = ps(\lambda)_I + ps(\mu)_J$. If in the corresponding hive plan each edge of every internal corridor wall lies on a good path, then the LR-coefficient $c_{\lambda\mu}^\nu$ factorises.
Lemma  The first edge of any path starting from any boundary is **good**

- Each boundary has edges specified by a partition: \( \alpha \geq \gamma \).
- Horn equality applied to corridors: \( \beta = \alpha \) or \( \beta = \gamma \).
- Hence \( \beta = \alpha \geq \gamma \) or \( \beta = \gamma \leq \alpha \) so that in both cases \( OP \) is **good**.
Path along an interior corridor wall

- Initial hive conditions: $\alpha \leq \beta$
- Horn equality applied to corridors: $\delta = \gamma$.
- $PO \text{ good } \Rightarrow \gamma \leq \alpha \Rightarrow \delta = \gamma \leq \alpha \leq \beta \Rightarrow OR \text{ good}$
- $OR \text{ bad } \Rightarrow \delta > \beta \Rightarrow \gamma = \delta > \beta \geq \alpha \Rightarrow PO \text{ bad}$
Path reaching a vertex

- Initial hive condition: \( \alpha \leq \beta \); Horn equality \( \delta = \beta \)
- \( PO \) good \( \Rightarrow \gamma \leq \alpha \Rightarrow \gamma \leq \alpha \leq \beta = \delta \Rightarrow OR \) good
- \( OR \) bad \( \Rightarrow \gamma > \delta \Rightarrow \gamma > \delta = \beta \geq \alpha \Rightarrow PO \) bad
Intersecting paths

- Assume $PO$ good, so that $\beta \geq \alpha$.
- Horn equality $\gamma = \delta$
- $QO$ good $\Rightarrow \gamma \geq \beta \Rightarrow \delta = \gamma \geq \beta \geq \alpha \Rightarrow OS$ good
- $OS$ bad $\Rightarrow \delta < \alpha \Rightarrow \gamma = \gamma = \delta < \alpha \leq \beta \Rightarrow QO$ bad
Good paths cover all interior corridor walls

All LR hive conditions satisfied

Hence we have factorisation
Bad edges and gentle loops

- Good paths may not cover all interior corridor edges
- If there exists a bad edge then its predecessor on some gentle path must also be bad
- A reverse gentle path of bad edges cannot reach the boundary, since all gentle paths start from the boundary with a good edge
- A reverse gentle path of bad edges must therefore continue indefinitely
- Since there are only a finite number of edges, it follows that there may only be bad edges if there exists a gentle loop
Example exhibiting a gentle loop

- $n = 10, \ r = 5$.
- $I = (1, 2, 4, 6, 8), \ J = (1, 3, 4, 7, 9), \ K = (2, 4, 6, 8, 10)$. 
Example of a gentle loop

A gentle loop is any closed interior gentle path.
Good paths do not include all interior corridor walls.
Obstruction to good paths

Good paths do not include all interior corridor walls.
Rigid puzzles and gentle loops

Theorem [KTW 04] The hive plan of a puzzle has no gentle loops if only if the puzzle is rigid

- A puzzle is rigid if and only if the corresponding Horn inequality is essential
- All gentle paths in the hive plan of a rigid puzzle are good, i.e., have no bad edges
- This implies that in the case of any essential Horn equality, the map from the LR-hives $H_n$ to pairs of LR-hives $H_r$ and $H_{n-r}$ is a bijection
- Hence we have proved:

Theorem If an essential Horn inequality is saturated then $c^\nu_{\lambda\mu}$ factorises.
Example exhibiting a gentle loop

Ex: \( n = 6, \ r = 3, \ I = J = (1, 3, 5) \) and \( K = (2, 4, 6) \).

There exist two puzzles, each exhibiting a gentle loop.

\[ P1: \]

\[ P2: \]

- The corresponding inessential saturated Horn inequality
  \[ \nu_2 + \nu_4 + \nu_6 = \lambda_1 + \lambda_3 + \lambda_5 + \mu_1 + \mu_3 + \mu_5 \]
  is satisfied by \( \lambda = \mu = (221100) \) and \( \nu = (332211) \).

- There can be no corresponding factorisation since
  \[ c_{\lambda\mu}^\nu = c_{221100,221100}^{332211} = 3 \neq 2 \cdot 2 = c_{210,210}^{321} c_{210,210}^{321} = c_{\lambda_1\mu,J}^{\nu_K} c_{\lambda_1\mu,J}^{\nu_K} \]
Maps from 6-hives to pairs of 3-hives

The puzzles $P_1$ and $P_2$ provide the same maps in the following two cases:

$P_1, P_2$
Maps from 6-hives to pairs of 3-hives

The puzzles $P_1$ and $P_2$ provide two different maps in the following case:

Thus both $P_1$ and $P_2$ map all three possible 6-hives to three pairs of 3-hives

However there exists four pairs of 3-hives
Maps from a pair of 3-hives to 6-hives

The puzzles $P_1$ and $P_2$ provide two different maps:

The resulting 6 hive is not an LR-hive as can be seen from the edges labelled 2 3 1 2 from left to right across the centre of the hive.
A 6-hive exhibiting a bad gentle loop

Our resulting 6-hive takes the form

The hive conditions are violated in the case of 12 rhombi

All edges are bad along the hexagonal gentle loop
Stretched LR coefficients

- Littlewood-Richardson coefficient $c_{\lambda\mu}^{\nu}$
- Partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ stretching parameter $t \in \mathbb{N}$
- Stretched partition $t\lambda = (t\lambda_1, t\lambda_2, \ldots, t\lambda_n)$
- Stretched Littlewood-Richardson coefficient $c_{t\lambda,t\mu}^{t\nu}$

**Ex:** $n = 3$, $\lambda = (2, 1, 0)$, $\mu = (3, 2, 0)$, $\nu = (4, 3, 1)$

- $t = 1$: $c_{21,32}^{431} = 2$
- $t = 2$: $c_{42,64}^{862} = 3$
- $t = 3$: $c_{63,94}^{1293} = 4$
- ...

suggests $c_{t\lambda,t\mu}^{t\nu} = t + 1$. 
LR coefficients and polynomials

Ex: Let $c_{421,532}^\nu = c$ and $c_{t(421),t(532)}^{t\nu} = P(t)$.

- $c = 1, \nu = (953)$ \quad $P(t) = 1$
- $c = 2, \nu = (9431)$ \quad $P(t) = (t + 1)$
- $c = 3, \nu = (8441)$ \quad $P(t) = (t + 1)(t + 2)/2$
- $c = 4, \nu = (8531)$ \quad $P(t) = (t + 1)(t + 2)(t + 3)/6$
- $c = 4, \nu = (7442)$ \quad $P(t) = (t + 1)^2$
- $c = 5, \nu = (7541)$ \quad $P(t) = (t + 1)(t + 2)(2t + 3)/6$
- $c = 6, \nu = (7532)$ \quad $P(t) = (t + 1)^2(t + 2)/2$
- $c = 7, \nu = (74321)$ \quad $P(t) = (t + 1)(t + 2)(t^2 + 3t + 6)/6$
Generating function for LR-polynomials

Ex: Let \( F(z) = \frac{G(z)}{(1 - z)^{d+1}} = \sum_{t=0}^{\infty} P(t) z^t \).

\[
\begin{align*}
  c &= 1 \quad \nu = (953) \quad d = 1 \quad G(z) = 1 \\
  c &= 2 \quad \nu = (9431) \quad d = 2 \quad G(z) = 1 \\
  c &= 3 \quad \nu = (8441) \quad d = 3 \quad G(z) = 1 \\
  c &= 4 \quad \nu = (8531) \quad d = 4 \quad G(z) = 1 \\
  c &= 4 \quad \nu = (7442) \quad d = 3 \quad G(z) = 1 + z \\
  c &= 5 \quad \nu = (7541) \quad d = 4 \quad G(z) = 1 + z \\
  c &= 6 \quad \nu = (7532) \quad d = 4 \quad G(z) = 1 + 2z \\
  c &= 7 \quad \nu = (74321) \quad d = 5 \quad G(z) = 1 + 2z + z^2
\end{align*}
\]
**Further example**

Ex: \( n = 7, \; \lambda = (433210), \; \mu = (432210), \; \nu = (7444321). \)

- LR coefficient \( c_{\lambda \mu}^{\nu} = 13 \)
- LR polynomial

\[
  c_{t\lambda.t\mu}^{t\nu} = \frac{1}{10080} \\
  \times (t + 1)(t + 2)(t + 3)(t + 4)(t + 5) \\
  \times (5t + 21)(t^2 + 2t + 4)
\]

- where \( 10080 = 5! \times 84 \)
- \( d = 8 \) and \( G(z) = 1 + 4z + 12z^2 + 3z^3 \)
**Polynomial behaviour**

**Theorem** For all $\lambda, \mu, \nu$ such that $c_{\lambda \mu}^\nu > 0$ there exists

- a polynomial $P_{\lambda \mu}^\nu(t)$ in $t$ with $P_{\lambda \mu}^\nu(0) = 1$
- such that $P_{\lambda \mu}^\nu(t) = c_{t\lambda, t\mu}^{t\nu}$ for all positive integers $t$.

**Conjectures**

- coefficients in $P_{\lambda \mu}^\nu(t)$ are all rational and non-negative.
- coefficients in $G(z)$ are all positive integers.

**Problems**

- predict degree of polynomial
- explain origin of factors of form $(t + 1)(t + 2) \cdots (t + m)$
- prove (if true) and account for positivity of coefficients
The hive model and convex polytopes

- An LR $n$-hive with fixed boundary labels specified by $\lambda, \mu, \nu$ involves $m = \frac{(n - 1)(n - 2)}{2}$ interior vertex labels $a_{ij}$.
- The hive conditions for any rhombus take the form $a + d \leq b + c$.
- These form a set of linear constraints with integer coefficients on the $m$ interior vertex labels.
- They define a rational convex polytope $\mathcal{P}$ in $\mathbb{R}^m$ known as the hive polytope.
- The LR-coefficient is given by $c_{\lambda\mu}^{\nu} = \#\{\mathcal{P} \cap \mathbb{Z}^m\}$, the number of points of intersection of the hive polytope with the integer lattice.
The hive model and convex polytopes

- Let $\mathcal{P} \in \mathbb{R}^m$ be a convex rational polytope of dimension $d \leq m$.

- Such a convex rational polytope $\mathcal{P} \in \mathbb{R}^m$ is an integer polytope if all of its vertices lie in $\mathbb{Z}^m$.

- Let $t\mathcal{P} = \{tv \mid v \in \mathcal{P}\}$ and $i(\mathcal{P}, t) = \#\{t\mathcal{P} \cap \mathbb{Z}^m\}$, the number of points of intersection of the stretched polytope $t\mathcal{P}$ with the integer lattice.

- The stretched LR-coefficient, which may or may not be polynomial, is given by $c_{t\lambda, t\mu}^{tv} = i(\mathcal{P}, t)$. 
Theorem [Ehrhart 77]
Let \( \mathcal{P} \in \mathbb{R}^m \) be a rational polytope of dimension \( d \leq m \).
Then \( i(\mathcal{P}, t) \) is a quasi-polynomial of degree \( d \) in \( t \),
\[ i(\mathcal{P}, t) = P_s(t) \text{ for } t \equiv s \pmod{r} \]
if \( \mathcal{P} \in \mathbb{R}^m \) is an integer polytope, then \( i(\mathcal{P}, t) \) is a polynomial in \( t \) of degree \( d \).

Corollary
If a hive polytope is an integer polytope, then the corresponding stretched Littlewood-Richardson coefficients \( c_{t\lambda, t\mu}^{t\nu} \) are polynomial in \( t \).
Construction of hive polytopes

Let \( m = (n - 2)(n - 1)/2 \) = \# interior points of an \( n \)-hive

Let \( v = (a_{11}, a_{12}, \ldots) \in \mathbb{R}^m \) be vector of interior labels

Ex: \( \lambda = (753), \mu = (742), \nu = (9964), n = 4, m = 3, \)

\[
\begin{array}{ccc}
15 \\
15 & 22 \\
12 & b & 26 \\
7 & a & c & 28 \\
0 & 9 & 18 & 24 & 28
\end{array}
\]

\( v = (a_{11}, a_{12}, a_{21}) = (a, b, c) \)
Construction of hive polytopes

There are 6 LR-hives, with \( a = 16 \) in all cases and 
\((b, c) = (19, 23), (20, 22), (20, 23), (20, 24), (21, 23), (21, 24)\)

The hive polytope \( \mathcal{P} \) is \( d = 2 \)-dimensional and takes the form

\[ \begin{array}{c}
\text{with integer points} \\
\bullet \quad \bullet \quad \bullet \\
\bullet \\
\end{array} \]

\( \mathcal{P} \) has 5 integer vertices and just one interior integer point
Scaling convex polytope

- Expand $\mathcal{P}$ by scaling with $t$
- Identify and count all integer points to give $\mathcal{P}(t)$

$P(1) = 6, P(2) = 16, P(3) = 31, \ldots, P(t) = \frac{1}{2}(5t^2 + 5t + 2)$. 
Not all hive polytopes are integer polytopes

Ex: For $n = 5$, $\lambda = (32000)$, $\mu = (43210)$ and $\nu = (54321)$
all hives take the form

$$
\begin{array}{cccc}
5 & & & \\
5 & 9 & & \\
5 & 9 & 12 & \\
5 & 9 & 12 & 14 \\
3 & a & b & c & 15 \\
0 & 5 & 9 & 12 & 14 & 15 \\
\end{array}
$$

- The hive conditions fix three interior vertex labels
- The hive polytope $P$ is of dimension 3
- There are 5 distinct LR-hives of this type, so that $c_{\lambda \mu}^\nu = 5$
Not all hive polytopes are integer polytopes

The vertices \( v = (a, b, c) \) of \( \mathcal{P} \) are given by

\[
\begin{align*}
v_1 & = (7, 11, 13) \\
v_2 & = (7, 11, 14) \\
v_3 & = (8, 11, 13) \\
v_4 & = (8, 11, 14) \\
v_5 & = (8, 12, 14) \\
v_6 & = \left(\frac{15}{2}, \frac{21}{2}, \frac{27}{2}\right) = (7.5, 10.5, 13.5)
\end{align*}
\]

The first 5 vertices are integral and specify the 5 LR-hives.

The sixth vertex \( v_6 \) is not integral. The hive polytope is rational but not integer.
Not all hive polytopes are integer polytopes

**Ex:** The hive corresponding to the polytope vertex $v_6$ takes the form

```
   5
  5  9
 5  9  12
 5  9  12  14
 3  15/2  21/2  27/2 15
0  5  9  12  14 15
```

- This hive not an integer hive, let alone an LR-hive
- In this case Ehrhart’s Theorem only asserts that $c_{t\lambda, t\mu}$ is quasi-polynomial in $t$, not polynomial.
**Ex:** For $n = 5$, $\lambda = (32000)$, $\mu = (43210)$ and $\nu = (54321)$ we find the following data on $P(t) = c_{t\lambda, t\mu}^{t\nu}$.

- $t = 1$: $P(1) = 5$
- $t = 2$: $P(2) = 15$
- $t = 3$: $P(3) = 34$
- $t = 4$: $P(3) = 65$
- ...

suggests $P(t) = c_{t\lambda, t\mu}^{t\nu} = (t + 1)(t^2 + 2t + 2)/2$

**Theorem** [Rassart ??]

All stretched LR-coefficients $c_{t\lambda, t\mu}^{t\nu}$ are polynomial in $t$. 
Summary

- **Theorem**
  The LR-coefficient $c^\nu_{\lambda\mu}$ is the number of LR-hives with boundary labels determined by $\lambda$, $\mu$ and $\nu$.

- **Corollary**
  The LR-polynomial $P^\nu_{\lambda\mu}(t)$ can be identified as the Ehrhart quasi-polynomial $i(\mathcal{P}, t) = \#\{t\mathcal{P} \cap \mathbb{Z}^m\}$, of a rational convex polytope $\mathcal{P}$ defined by the LR-hive boundary conditions and the set of LR-hive inequalities: $a + d \leq b + c$ for each rhombus.

- **Note**: Even though $\mathcal{P}$ may be rational but not integer the Ehrhart quasi-polynomial $i(\mathcal{P}, t)$ is polynomial.
Linear factors

Origin of some linear factors in LR-polynomials.

- Let $\mathcal{P}$ be an LR hive polytope, and $\overline{\mathcal{P}}$ its interior.
- For $t \in \mathbb{N}$: $P_{\lambda \mu}^{\nu}(t) = i(\mathcal{P}, t) = \# \{ t\mathcal{P} \cap \mathbb{Z}^d \}$.
- Ehrhart reciprocity: $i(\mathcal{P}, -t) = (-1)^d \# \{ t\overline{\mathcal{P}} \cap \mathbb{Z}^d \}$.
- For $m \in \mathbb{N}$: $P_{\lambda \mu}^{\nu}(-m) = i(\mathcal{P}, -m) = (-1)^d \# \{ m\overline{\mathcal{P}} \cap \mathbb{Z}^d \}$.
- Hence $P_{\lambda \mu}^{\nu}(-m) = 0$ and $P_{\lambda \mu}^{\nu}(t)$ contains a factor $(t + m)$ if and only if $m\mathcal{P}$ contains no interior integer points.

Corollary $P_{\lambda \mu}^{\nu}(t)$ contains $(t + 1)(t + 2) \cdots (t + m)$ as a factor if $m\mathcal{P}$ contains no interior integer points.

Problem: predict maximum value of $m$. 
Construction of convex polytopes

Ex: $\lambda = (210), \mu = (320), \nu = (431), n = 3, d = 1$

\[
\begin{array}{cccc}
5 \\
5 & 7 & \text{with } a = 6, 7 \\
3 & a & 8 \\
0 & 4 & 7 & 8
\end{array}
\]

- $\mathcal{P} \cap \mathbb{Z} =$ \bullet \bullet \quad \text{no interior points}
- $2\mathcal{P} \cap \mathbb{Z} =$ \bullet \bullet \bullet \quad \text{one interior point}
- implies $P(t)$ contains a factor $(t + 1)$ but no factor $(t + 2)$. In fact $P(t) = (t + 1)$. 
Construction of convex polytopes

Ex: $\lambda = (753), \mu = (742), \nu = (9964), n = 4, d = 2$

$15$
$15 \ 22$
$12 \ b \ 26$
$7 \ 16 \ c \ 28$
$0 \ 9 \ 18 \ 24 \ 28$

with $(b, c) = \begin{cases} (21, 24) & (21, 23) \\ (20, 24) & (20, 23) \\ (20, 22) & (19, 23) \end{cases}$

$P \cap \mathbb{Z}^2$:  

\[ P \cap \mathbb{Z}^2: \bullet \bullet \bullet \quad \text{one interior point} \]

\[ \bullet \]

\[ \bullet \text{ implies no factor } (t + m). \text{ In fact } P(t) = \frac{1}{2}(5t^2 + 5t + 2). \]
Degrees of LR-polynomials

For $c_{\lambda\mu}^\nu > 0$ the LR-rule implies $\ell(\lambda), \ell(\mu) \leq \ell(\nu)$.

$c_{\lambda\mu}^\nu$ is the number of LR $n$-hives with $n = \ell(\nu)$, boundary labels linear in the parts of $\lambda, \mu, \nu$, interior vertex labels subject to linear inequalities (HCs).

For $t \in \mathbb{N}$, $P_{\lambda\mu}^\nu(t)$ is the number of scaled LR $n$-hives with boundary labels scaled by $t$ and interior vertex labels subject to the same scaled linear inequalities.

The range of each vertex label is at most linear in $t$.

An $n$-hive has $(n - 1)(n - 2)/2$ interior vertices.

Degree bound $\deg P_{\lambda\mu}^\nu(t) \leq (n - 1)(n - 2)/2$ with $n = \ell(\nu)$. 
**First example**

**Ex:** $n = 5$, degree bound $(n - 1)(n - 2)/2 = 6$.

- $\lambda = (9, 7, 6, 2, 0)$, $\mu = (13, 5, 3, 1, 0)$, $\nu = (14, 12, 11, 5, 4)$.

- $P^\nu_{\lambda\mu}(t) = (t + 1)$ so that $\deg P^\nu_{\lambda\mu}(t) = 1$.

**Origin of mismatch - factorisation**

- $P^\nu_{\lambda\mu}(t) = P^{\nu K}_{\lambda I, \mu J}(t) P^{\nu \overline{K}}_{\lambda \overline{I}, \mu \overline{J}}(t)$.

- LR-hives for $n = 5$ are fixed by two smaller subhives of sizes $r = 3$ and $n - r = 2$. 

![Diagram](image-url)
LR factorisation example

Ex: \( n = 5, \ r = 3, \ n - r = 2 \):

- \( \lambda = (9, 7, 6, 2, 0), \ \mu = (13, 5, 3, 1, 0), \ \nu = (14, 12, 11, 5, 4) \).
- \( I = \{1, 2, 4\}, \ J = \{2, 3, 4\}, \ K = \{2, 3, 5\} \).
- \( \lambda_I = (9, 7, 2), \ \mu_J = (5, 3, 1), \ \nu_K = (12, 11, 4) \)
- \( \lambda_I = (6, 0), \ \mu_J = (13, 0), \ \nu_K = (14, 5) \)

LR-coefficient:

\[
c^{(14,12,11,5,4)}_{(9,7,6,2,0),(13,5,3,1,0)} = c^{(12,11,4)}_{(9,7,2),(5,3,1)} c^{(14,5)}_{(6,0),(13,0)} = 2 \cdot 1 = 2.
\]

LR-polynomial:

\[
P^{(14,12,11,5,4)}_{(9,7,6,2,0),(13,5,3,1,0)}(t) = P^{(12,11,4)}_{(9,7,2),(5,3,1)}(t) P^{(14,5)}_{(6,0),(13,0)}(t) = (t + 1) \cdot 1 = (t + 1).
\]
Degree bound for a primitive example

Ex: $n = 6$, degree bound $(n - 1)(n - 2)/2 = 10$.

- $\lambda = (4, 3, 3, 1, 0, 0)$, $\mu = (4, 2, 1, 1, 1, 0)$, $\nu = (6, 5, 4, 2, 2, 1)$.
- $P^{\nu}_{\lambda \mu}(t) = (t + 1)(t + 2)(t + 3)(t + 4)/24$.
- deg $P^{\nu}_{\lambda \mu}(t) = 4$.

- All essential Horn inequalities are strict.
- The LR-polynomial does not factorise.
- The factorised degree bound is not saturated.

Origin of mismatch - partitions have equal parts.
Five-vertex equal edge constraints

- Equal edge constraints on 5-vertex subdiagrams

- In each case $\alpha \geq \beta \geq \alpha$ so that $\beta = \alpha$.

- Consecutive equal edges force neighbouring equal edge.

- Retain skeleton consisting of only equal edges.
Skeleton of an LR-hive and degree bounds

- Apply 5-vertex equal edge procedure to LR \( n \)-hive.
- Work inwards from boundaries specified by \( \lambda, \mu, \nu \).
- Invoke triangular hive condition \( \alpha + \beta = \gamma \):

\[
\begin{array}{c}
\alpha \\
\beta \\
\gamma
\end{array}
\]

- Result is skeletal graph \( G_{n;\lambda\mu\nu} \) of hive.
- Let \( d(G_{n;\lambda\mu\nu}) \) be number of components of \( G_{n;\lambda\mu\nu} \) not connected to the boundary.

**Theorem** \( \deg P_{\lambda\mu}^\nu(t) \leq d(G_{n;\lambda\mu\nu}) \).

**Conjecture** If \( P_{\lambda\mu}^\nu(t) \) is primitive then \( \deg P_{\lambda\mu}^\nu(t) = d(G_{n;\lambda\mu\nu}) \).
Theorem \( \deg P_{\lambda\mu}^\nu(t) \leq d(G_{n;\lambda\mu\nu}). \)

Ex: \( n = 6, \lambda = (433100), \mu = (421110), \nu = (654221). \)

\[
P_{\lambda\mu}^\nu(t) = \frac{(t + 1)(t + 2)(t + 3)(t + 4)}{24}.
\]

\( \deg P_{\lambda\mu}^\nu(t) = 4 = d(G_{n;\lambda\mu\nu}). \)

Skeleton graph degree bound is saturated.
Degree of LR-polynomial

Ex: \( n = 7, \lambda = (4332100), \mu = (4322100), \nu = (7444321) \).

\[ P_{\lambda\mu}^{\nu}(t) = (t + 1)(t + 2)(t + 3)(t + 4)(t + 5) \times (5t + 21)(t^2 + 2t + 4)/10080. \]

\[ \deg P_{\lambda\mu}^{\nu}(t) = 8 = d(G_{n;\lambda\mu\nu}). \]
Degree of primitive LR-polynomial

Ex: $n = 6$, $\lambda = (221100)$, $\mu = (221100)$, $\nu = (332211)$.

\[ P^\nu_{\lambda\mu}(t) = \frac{1}{2} (t + 1)(t + 2). \]

Now construct skeleton - multi-stage process

\[ \deg P^\nu_{\lambda\mu}(t) = 2 = d(G_n;\lambda\mu) \].
Counterexample to skeleton degree bound

Ex: $n = 8$, $\lambda = (76531000)$, $\mu = (65553000)$, $\nu = (88886422)$.

- $P_{\lambda \mu}^\nu(2) = (t + 1)(t + 2)(t + 3)(t + 4)/24$
- Now construct skeleton

\[ \text{deg } P_{\lambda \mu}^\nu(t) = 4 < 5 = d(G_n; \lambda \mu \nu). \]
Linear factors

Origin of linear factors in LR-polynomials.

- Let $\mathcal{P}$ be an LR hive polytope, and $\overline{\mathcal{P}}$ its interior.
- For $t \in \mathbb{N}$: $P_{\chi \mu}^\nu(t) = i(\mathcal{P}, t) = \#\{t\mathcal{P} \cap \mathbb{Z}^d\}$.
- Ehrhart reciprocity: $i(\mathcal{P}, -t) = (-1)^d \#\{t\overline{\mathcal{P}} \cap \mathbb{Z}^d\}$.
- For $m \in \mathbb{N}$: $P_{\chi \mu}^\nu(-m) = i(\mathcal{P}, -m) = (-1)^d \#\{m\overline{\mathcal{P}} \cap \mathbb{Z}^d\}$.
- Hence $P_{\chi \mu}^\nu(-m) = 0$ and $P_{\chi \mu}^\nu(t)$ must contain a linear factor $(t + m)$ if $m\mathcal{P}$ contains no interior integer points.

Anticipate: $P_{\chi \mu}^\nu(t)$ may contain $(t + 1)(t + 2) \cdots (t + M)$.

Problem: Determine $M$. 
Possible continuation in $t$

- For $x = (x_1, x_2, \ldots, x_n)$ let $\overline{x} = (\overline{x}_1, \overline{x}_2, \ldots, \overline{x}_n)$ with $\overline{x}_i = x_i^{-1}$ for $i = 1, 2, \ldots, n$.

- For $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ let $\tilde{\lambda} = (\lambda_n, \ldots, \lambda_2, \lambda_1)$.

- $s_{t\lambda}(x) = \begin{vmatrix} t\lambda_j + n - j \\ \overline{x}_i^{n-j} \end{vmatrix} \quad \Longrightarrow \quad s_{-m\lambda}(x) = \begin{vmatrix} -m\lambda_j + n - j \\ \overline{x}_i^{n-j} \end{vmatrix}$.

- This gives $s_{-m\lambda}(x) = \begin{vmatrix} m\lambda_{n-k+1} + n - k \\ \overline{x}_i^{n-k} \end{vmatrix} = s_{m\tilde{\lambda}}(\overline{x})$.

Definition For $c_{\lambda\mu}^\nu > 0$ and any positive integer $m$, let

$$c_{-m\lambda, -m\mu}^{m\nu} = c_{m\tilde{\lambda}, m\tilde{\mu}}^{m\nu}.$$
LR polynomials for negative $t$

**Conjecture:** Let $c_{\lambda \mu}^\nu > 0$ be simple, all Horn inequalities strict, then

$$P_{\lambda \mu}^\nu (-m) = c_{m\tilde{\lambda}, m\tilde{\mu}}^{m\tilde{\nu}},$$

where

$$s_{m\tilde{\lambda}}(x) s_{m\tilde{\mu}}(x) = \sum_{\nu} c_{m\tilde{\lambda}, m\tilde{\mu}}^{m\tilde{\nu}} s_{m\tilde{\nu}}(x).$$

**Standardization:**

- $s_{m\tilde{\lambda}}(x) = 0$ or $\pm s_\rho(x)$ for some partition $\rho$.
- $s_{m\tilde{\mu}}(x) = 0$ or $\pm s_\sigma(x)$ for some partition $\sigma$.
- $s_{m\tilde{\nu}}(x) = 0$ or $\pm s_\tau(x)$ for some partition $\tau$.

**Two types of zero:**

- $s_{m\tilde{\lambda}}(\bar{x}) = 0$, $s_{m\tilde{\mu}}(\bar{x}) = 0$, $s_{m\tilde{\nu}}(\bar{x}) = 0$.
- $c_\rho^\tau = 0$. 
Simple example

Ex: \( n = 7, \lambda = (433210), \mu = (432210), \nu = (7444321) \).

\[
\begin{align*}
P^{\nu}_{\lambda\mu}(t) &= (t + 1)(t + 2)(t + 3)(t + 4)(t + 5) \\
&\quad \cdot (5t + 21)(t^2 + 2t + 4)/10080.
\end{align*}
\]

Type one zeros for \( m = 1, 2, 3 \) since:

- \( s_{m\lambda}(\overline{x}) = s_{m\mu}(\overline{x}) = 0 \) for \( m = 1, 2 \).
- \( s_{m\nu}(\overline{x}) = 0 \) for \( m = 1, 2, 3 \).

Type two zeros for \( m = 4, 5 \) since:

- \( c^\tau_{\rho\sigma} = 0 \) for \( m = 4, 5 \).

No more zeros for \( m > 5 \) since for \( m = 6 \): \( c^\tau_{\rho\sigma} = 3 \).
Simple and non-simple examples

**Simple:** \( n = 7, \lambda = (433210), \mu = (432210), \nu = (7444321). \)

- \( P_{\lambda\mu}^\nu(t) = (t + 1)(t + 2)(t + 3)(t + 4)(t + 5) \nonumber \)
  \( \cdot (5t + 21)(t^2 + 2t + 4)/10080. \)
- \( c_{m\lambda,m\mu}^{m\nu} = 0, 0, 0, 0, 3, 39, 247 \) for \( m = 1, 2, 3, 4, 5, 6, 7, 8. \)
- \( P_{\lambda\mu}^\nu(-m) = 0, 0, 0, 0, 0, 3, 39, 247 \) for \( m = 1, 2, 3, 4, 5, 6, 7, 8. \)

**Non-simple:** \( n = 6, \lambda = (221100), \mu = (221100), \nu = (332211). \)

- \( P_{\lambda\mu}^\nu(t) = (t + 1)(t + 2)/2. \)
- \( c_{m\lambda,m\mu}^{m\nu} = 0, 0, 0, 3, 6, \) for \( m = 1, 2, 3, 4, 5. \)
- \( P_{\lambda\mu}^\nu(-m) = 0, 0, 3, 6, 10 \) for \( m = 1, 2, 3, 4, 5. \)
Non-primitive example

Non-primitive: $n = 5$, $\lambda = (9, 7, 6, 2, 0)$, $\mu = (13, 5, 3, 1, 0)$, $\nu = (14, 12, 11, 5, 4)$, $P_{\lambda \mu}^\nu(t) = (t + 1)$.

- $c_{m \lambda, m \mu}^{m \nu} = 0, 0, 0, \ldots$ for $m = 1, 2, 3, \ldots$.
- $P_{\lambda \mu}^\nu(-m) = 0, 1, 2, \ldots$ for $m = 1, 2, 3, \ldots$.

Primitive factors

- $n = 3$, $\lambda_I = (9, 7, 2)$, $\mu_J = (5, 3, 1)$, $\nu_K = (12, 11, 4)$.
- $n = 2$, $\lambda_I = (6, 0)$, $\mu_J = (13, 0)$, $\nu_K = (14, 5)$.

Conjecture: $c_{\lambda \mu}^\nu > 0$ is primitive, all essential Horn inequalities strict, if and only if $c_{m \lambda, m \mu}^{m \nu} \neq 0$ for some positive integer $m$. 
Some symmetries of LR-coefficients

- Given a partition $\lambda \subseteq m^n$, so that $\ell(\lambda) \leq n$ and $\lambda_1 \leq m$
- Let $\lambda'$ denote its conjugate
- Let $\tilde{\lambda}$ denote its $m^n$-complement where $\tilde{\lambda} = (m - \lambda_n, \ldots, m - \lambda_2, m - \lambda_1)$

**Ex:** If $m = 6$, $n = 4$ and $\lambda = (4, 3, 1, 0)$ then $\lambda' = (3, 2, 2, 1)$ and $\tilde{\lambda} = (6, 5, 3, 2)$

- $F^\lambda$
- $F^{\lambda'}$
- $F^{\tilde{\lambda}}$
Some relations between LR-coefficients

Symmetries

\[ c_{\mu \lambda}^\nu = c_{\lambda \mu}^\nu \]
\[ c_{\lambda' \mu'}^{\nu'} = c_{\lambda \mu}^\nu \]
\[ c_{\tilde{\nu} \tilde{\lambda}}^\mu = c_{\lambda \mu}^\nu \]

Inequalities

\[ c_{\lambda+(1^a), \mu+(1^b)}^{\nu+(1^c)} \geq c_{\lambda \mu}^\nu \]
\[ c_{\lambda \cup (a), \mu \cup (b)}^{\nu \cup (c)} \geq c_{\lambda \mu}^\nu \]

for all non-negative integers \( a, b, c \), with \( c = a + b \)

where \( \lambda + (1^a) = (\lambda_1 + 1, \ldots, \lambda_a + 1, \lambda_{a+1}, \ldots, \lambda_n) \)

and \( \lambda \cup (a) = (\lambda_1, \ldots, \lambda_k, a, \lambda_{k+1}, \ldots, \lambda_n) \)

with \( \lambda_k \geq a > \lambda_{k+1} \)
Hive based proofs of the symmetry relations

Commutativity \[ c^\nu_{\mu \lambda} = c^\nu_{\lambda \mu} \]

Proof: It remains an open problem to find a hive-based proof.

We would like a bijection between the LR-hives for \( c^\nu_{\mu \lambda} \) and those for \( c^\nu_{\lambda \mu} \).

The result is by no means obvious from the Littlewood-Richardson rule.

Considerable effort has gone into devising combinatorial proofs.

See for example the work of Benkart, Sotille & Stroomer [96] and of Azenhas [99]
Hive based proofs of the symmetry relations

Conjugacy \( c'_{\lambda' \mu'} = c_{\lambda \mu} \)

Proof

- Let \( N = \{1, 2, \ldots, n\} = M \cup \overline{M} \) with \( M \cap \overline{M} = \emptyset \)

- Recall that if \( \zeta = \text{part}(M) \) then \( \zeta' = \text{part}(\overline{M}) \)

Ex: If \( n = 10, \ M = \{1, 3, 6, 8\}, \ \overline{M} = \{2, 4, 7, 9, 10\} \) we have \( \text{part}(M) = (4, 3, 1) \) and \( \text{part}(\overline{M}) = (3, 2, 2, 1) \)

as illustrated by

\[
\begin{array}{ccccccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\
0 & 3 & 1 & 2 & 2 & 3 & 1 & 4 & 0 & 0 \\
\end{array}
\]

\[
\begin{array}{ccccccc}
\end{array}
\]
Proof of conjugacy contd.

- Recall that the number of $n$-puzzles with thick edges specified by $I, J, K$ is
  $$c_{\text{part}(K), \text{part}(I)} = c_{\text{part}(I), \text{part}(J)}$$

- This was proved by scaling all thick edges by $t$ and allowing $t \to 0$
Proof of conjugacy contd.

- It remains to show that the number of $n$-puzzles with thin edges specified by $\bar{I}, \bar{J}, \bar{K}$ is $\mathcal{C}_{\text{part}(\bar{K})}^{\text{part}(\bar{I}),\text{part}(\bar{J})}$.

- This is proved by scaling all thin edges by $t$ and allowing $t \to 0$.
Hive based proofs of the symmetry relations

Complementarity \[ \tilde{C}_{\nu}^{\mu} \tilde{\lambda} = C^{\nu}_{\lambda \mu} \]

Proof  Consider any LR-hive for \( C^{\nu}_{\lambda \mu} \). Then

- For all edges parallel to the NE, SE, WE boundaries replace the edge labels \( \alpha, \beta, \gamma \) by \( \alpha, m - \beta, m - \gamma \)
- This breaks the triangle and the rhombus hive conditions and the result is not an LR-hive
Complementarity contd.

- Rotate clockwise through $\pi/3$
- The result is an LR-hive for $C_{\tilde{\nu} \tilde{\lambda}}^\mu$

**Lemma** The above sequence of two maps provides a bijection between the LR-hives of $C_{\lambda \mu}^\nu$ and those of $C_{\tilde{\nu} \tilde{\lambda}}^\mu$
Column insertion

Inequality

\[ c^{\nu+(1^c)}_{\lambda+(1^a),\mu+(1^b)} \geq c^\nu_{\lambda,\mu} \]

for all \( a, b, c \), with \( c = a + b \)

Proof

Since \( c_{1^a, 1^b}^{1^c} = 1 \) if \( c = a + b \), there exist a unique LR-hive:

Here each red, magenta and blue edge is labelled 1 and all other edges are labelled 0
Column insertion

The elementary rhombi take one or other of the following forms

However, we can now interpret our multi-coloured LR-hive as being an LR-hive for $c_{\lambda \mu}^{\nu}$ whose red, magenta and blue edge labels have all been increased by 1, with no change to the uncoloured edge labels.
Column insertion

This yields an LR-five for \( c^{\nu+(1^c)}_{\lambda+(1^a), \mu+(1^b)} \) as can be seen by checking the hive conditions in each of the elementary rhombi.

Hence the number of \( c^{\nu+(1^c)}_{\lambda+(1^a), \mu+(1^b)} \) LR-hives is at least as many as the number of \( c^{\nu}_{\lambda, \mu} \) LR-hives.
Row insertion

Inequality \[ \frac{\nu \cup (c)}{\lambda \cup (a), \mu \cup (b)} \geq \frac{\nu}{\lambda} \frac{\mu}{\mu} \quad \text{with} \quad c = a + b \]

- Each LR-hive for \( \frac{\nu}{\lambda} \frac{\mu}{\mu} \) may be divided into regions as shown on the left below.
- Redundant corridors are then inserted as shown on the right to give an LR-hive for \( \frac{\nu \cup (c)}{\lambda \cup (a), \mu \cup (b)} \)
The red line divides the hive into those regions for which NW edge labels are either \( \leq a \) or \( > a \)

The blue line divides the hive into those regions for which SW edge labels are either \( \leq b \) or \( > b \)

The cyan line divides the hive into those regions for which WE edge labels are either \( \leq c \) or \( > c \)
Row insertion

- These lines always intersect at a single point $Z$
- Edges labelled $a, b, c$ are inserted on the boundary
- Each edge along the red, blue and cyan lines is replaced by an appropriate redundant rhombus
- The point $Z$ is replaced by a triangle with edge labels $a, b, c$, and all triangle and rhombus hive conditions are satisfied
Row insertion

- These lines always intersect at a single point $Z$
- Edges labelled $a, b, c$ are inserted on the boundary
- Each edge along the red, blue and cyan lines is replaced by an appropriate redundant rhombus
- The point $Z$ is replaced by a triangle with edge labels $a, b, c$, and all triangle and rhombus hive conditions are satisfied