Littlewood-Richardson coefficients and the hive model

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Presented at:

Seoul National University

Seoul, South Korea: August 2008

Key sources

- Littlewood-Richardson coefficients
  - D.E. Littlewood and A. Richardson, *Group characters and algebra*, Phil. Trans. Roy. Soc. London A **233** (1934) 99–141
  - D.E. Littlewood, *Theory of Group Characters*, 2nd Ed. Oxford: Clarendon Press, 1940
- The hive model
  - A. Knutson and T. Tao, *The honeycomb model of $GL_n(\mathbb{C})$ tensor products. I: Proof of the saturation conjecture*, J. Amer. Math. Soc. **12** (1999) 1055–1090
  - A.S. Buch, *The saturation conjecture (after A. Knutson and T. Tao). With an appendix by W. Fulton*, Enseign. Math. **46** (2000) 43–60

Key sources

- Horn inequalities
  - A. Horn, *Eigenvalues of sums of Hermitian matrices*, Pacific J. Math. **12** (1962) 225–241
- Puzzles
  - A. Knutson, T. Tao and C. Woodward, *The honeycomb model of $GL_n(\mathbb{C})$ tensor products. II: Puzzles determine facets of the Littlewood-Richardson cone*, Amer. Math. Soc. **17** (2004) 19–48
  - V.I. Danilov and G.A. Koshevoy, *Diskretnaya vognutost' i ermitovy matritsy*, Trudy Math. Inst. Steklova **241** (2003) 68–90, in Russian. Title translation: *Discrete convexity and Hermitian matrices*

Key sources

- Polynomial property of stretched LR-coefficients
  - E. Ehrhart, *Sur les polyedres rationnels homothetiques a n'dimensions*, C. R. Acad. Sci. Paris Ser A **254** (1962) 616-618
  - H. Derksen and J. Weyman, *On the Littlewood-Richardson polynomials*, J. Algebra. **255** (2002) 247–257
  - E. Rassart, *A polynomial property of Littlewood-Richardson coefficients*, J Comb. Theo. Ser. A **107** (2004) 161–179
- Overview
  - W. Fulton, *Eigenvalues, invariant factors, highest weights, and Schubert calculus*, Bull. Amer. Math. Soc. **37** (2000) 209–249

Schur functions

- Let n be a fixed positive integer and $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ a sequence of indeterminates.
- Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ be a partition of weight $|\lambda|$ and length $\ell(\lambda) \leq n$, so that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0$.
- Then the Schur function $s_\lambda(\mathbf{x})$ is defined by:

$$s_\lambda(\mathbf{x}) = \frac{\left| x_i^{n+\lambda_j-j} \right|_{1 \leq i,j \leq n}}{\left| x_i^{n-j} \right|_{1 \leq i,j \leq n}}.$$

- The Schur functions form a $\mathbb{Z}$-basis of Λ_n, the ring of polynomial symmetric functions of $x_1, \ldots, x_n$.
- Each Schur function $s_\lambda(\mathbf{x})$ may be interpreted as the character $\mathrm{ch}\, V^\lambda(\mathbf{x})$ of an irrep of $gl(n)$.

LR-coefficients

- Any product of Schur functions can be expressed as a linear sum of Schur functions:

$$s_\lambda(\mathbf{x})\; s_\mu(\mathbf{x}) = \sum_\nu \; c^\nu_{\lambda\mu}\; s_\nu(\mathbf{x})$$

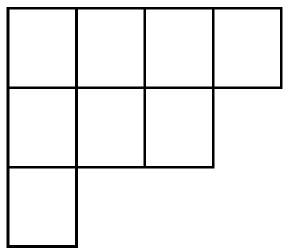
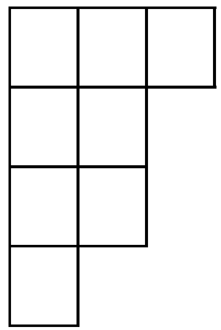
- The coefficients $c^\nu_{\lambda\mu}$ are the multiplicities appearing in the decomposition of the tensor product of irreps of $gl(n)$:

$$V^\lambda \otimes V^\mu = \sum_\nu \; c^\lambda_{\lambda\mu}\; V^\nu$$

- Each Littlewood-Richardson coefficient $c^\nu_{\lambda\mu}$ is a non-negative integer that may be evaluated by means of the Littlewood-Richardson rule

Young diagrams

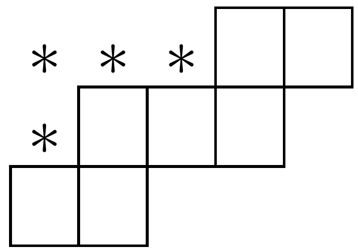
- Each partition λ specifies a Young diagram F^λ consisting of $|\lambda|$ boxes arranged in $\ell(\lambda)$ left adjusted rows of lengths $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_{\ell(\lambda)} > 0$.
- The partition λ' conjugate to λ is such that $F^{\lambda'}$ is obtained from F^λ by interchanging rows and columns.
- Ex: If $\lambda = (4,3,1)$ then $|\lambda| = 8$, $\ell(\lambda) = 3$, $\lambda' = (3,2,2,1)$, with

$F^\lambda =$  $F^{\lambda'} =$ 

Skew Young diagrams

- Given partitions λ and ν such that all boxes of F^λ are contained in F^ν we write $\lambda \subseteq \nu$.
- Removing the boxes of F^λ from F^ν leaves the skew Young diagram $F^{\nu/\lambda}$
- Ex: If $\nu = (5,4,2)$ and $\lambda = (3,1)$ then

$$F^{\nu/\lambda} =$$



- The corresponding skew Schur function is such that

$$s_{\nu/\lambda}(\mathbf{x}) = \sum_{\mu} c^{\nu}_{\lambda\mu}\, s_\mu(\mathbf{x})$$

Littlewood-Richardson rule

- Fill the boxes of the Young diagram F^λ with 0's
- Then fill the boxes of the skew Young diagram $F^{\nu/\lambda}$ with μ_i entries i for $i = 1, 2, \ldots, n$.
- $c^\nu_{\lambda\mu}$ is the number of such diagrams with entries
  - weakly increasing across rows from left to right
  - strictly increasing down columns from top to bottom
  - satisfying the lattice permutation rule -
    i.e. at every stage in the sequence of non-zero entries read from right to left across rows taken in turn from top to bottom $\#1's \geq \#2's \geq \cdots \geq \#n's$

Application of the LR-rule

Ex: $n = 4$, $\lambda = (4, 2)$, $\mu = (4, 3, 2)$, $\nu = (6, 5, 3, 1)$

- The only valid LR-diagrams

0	0	0	0	1	1
0	0	1	1	2	
2	2	3			
3					

0	0	0	0	1	1
0	0	1	2	2	
1	2	3			
3					

0	0	0	0	1	1
0	0	1	2	2	
1	1	3			
3					

- Some invalid diagrams

0	0	0	0	1	1
0	0	1	2	2	
1	3	2			
3					

0	0	0	0	1	1
0	0	1	1	2	
2	3	3			
2					

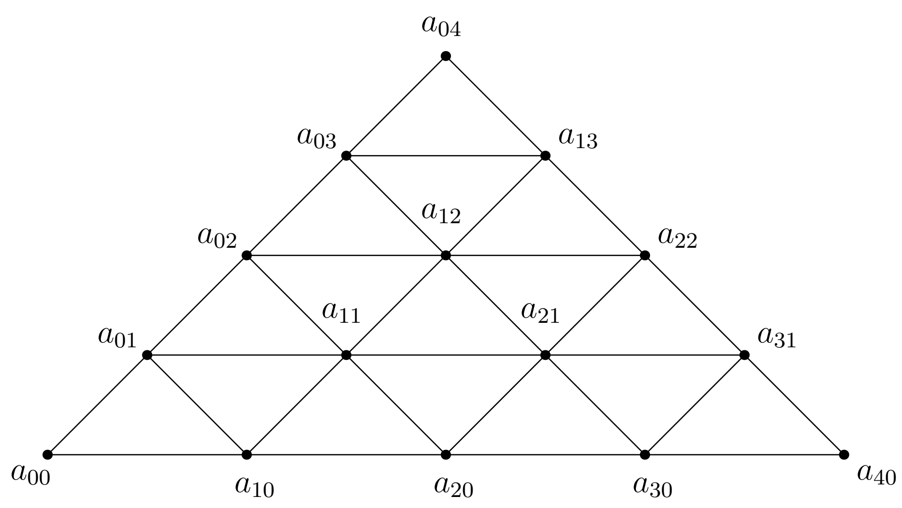
0	0	0	0	1	1
0	0	1	1	3	
2	2	2			
3					

- Hence $c^{\nu}_{\lambda\mu} = 3$.

Integer hives

- Knutson & Tao [99], as described by Buch [00]
- An integer n-hive is a triangular graph with vertex labels $a_{ij} \in \mathbb{Z}$ for $0 \leq i, j, i+j \leq n$.

Ex: $n = 4$



- Vertex labels increase along each edge from left to right

Relation between vertex and edge labels

- Edge labels are the non-negative differences between neighbouring vertex labels

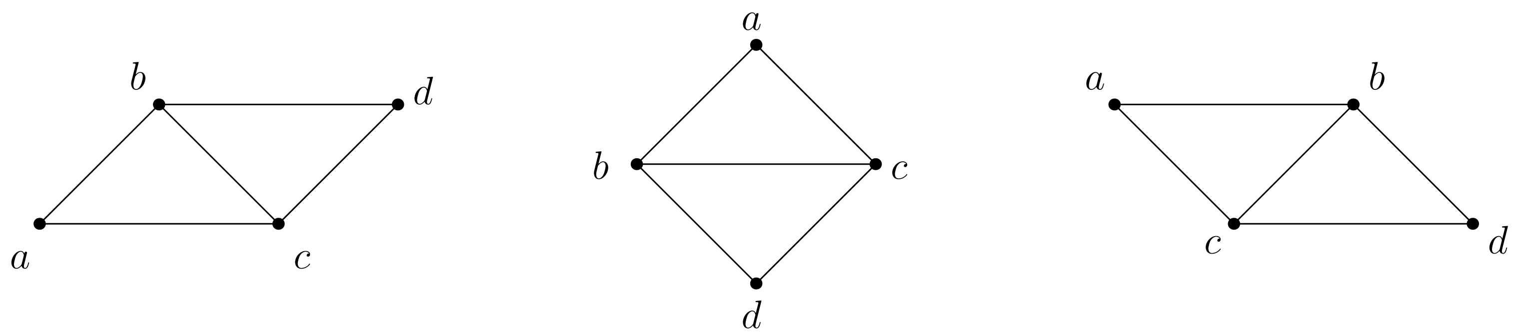
$$\alpha = a_{i,j+1} - a_{ij}, \quad \beta = a_{i+1,j-1} - a_{ij}, \quad \gamma = a_{i+1,j} - a_{ij}$$

- Vertex and edge labels for the two types of elementary triangle

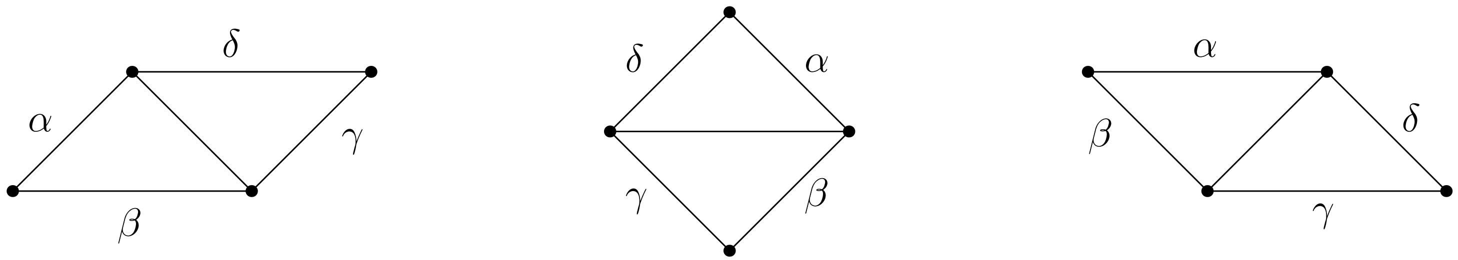
- In each case $\sigma = q - p \geq 0$, $\tau = r - q \geq 0$, $\rho = r - p \geq 0$ so that automatically we have $\sigma + \tau = \rho$ in any such triangle

Hive conditions

- Three distinct types of rhombi with vertex labels:



- The hive condition for each rhombus: $b + c \geq a + d$
- Three distinct types of rhombi with edge labels:

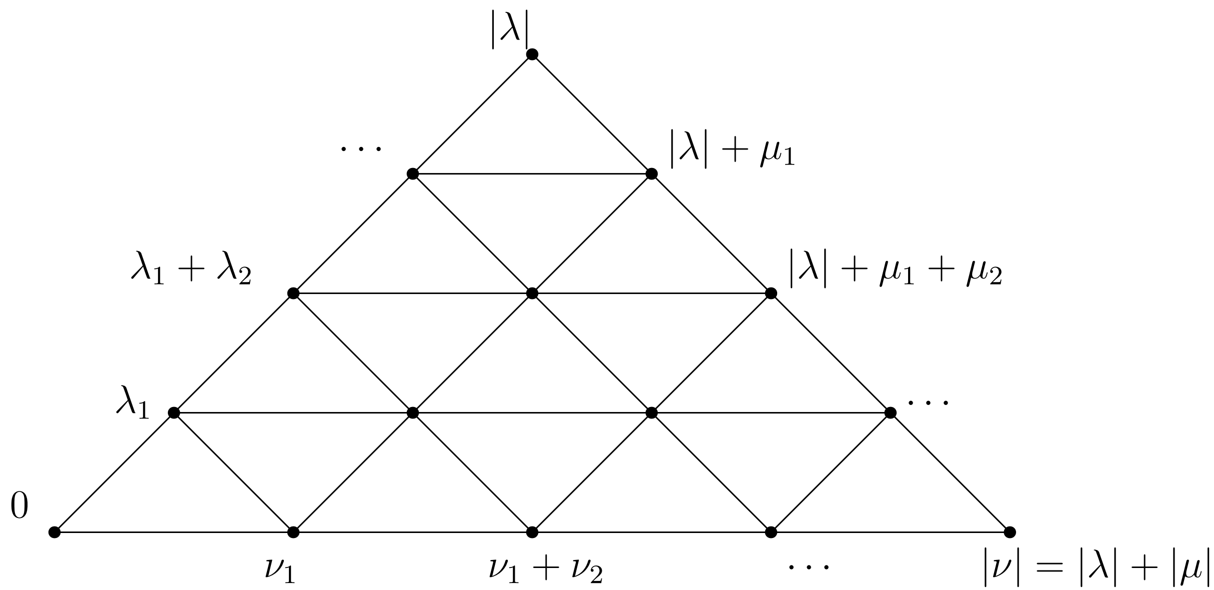


- The hive condition for each rhombus: $\alpha \geq \gamma$ and $\beta \geq \delta$
- Note: The triangle edge condition implies $\alpha + \delta = \beta + \gamma$.

LR-hives vertex labels

Definition An LR-hive is an integer n-hive for which

- all rhombi satisfy the hive conditions;
- boundaries determined by partitions λ, μ, ν with $\ell(\lambda), \ell(\mu), \ell(\nu) \leq n$ and $|\lambda| + |\mu| = |\nu|$;
- boundary vertex labels as shown:



LR-hives edge labels

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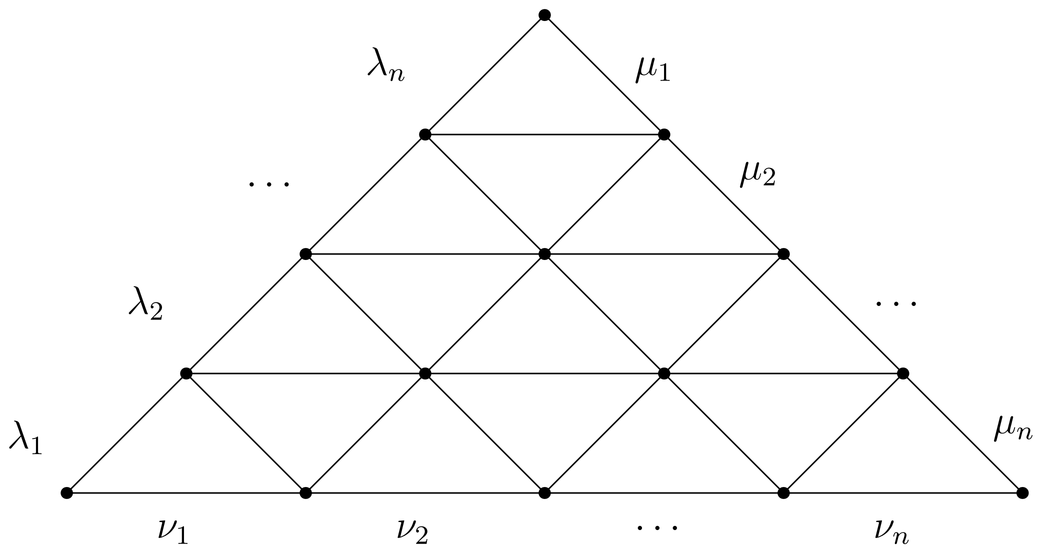

Bijection between LR-diagrams and LR-hives

Example: $n = 3$, $\lambda = (320)$, $\mu = (210)$ and $\nu = (431)$.

- $D =$ Littlewood-Richardson diagram;
- $G =$ Generalised Gelfand-Zetlin pattern;
- $Z =$ Zeros and cumulative row sums of G;
- $H =$ LR-hive $=$ reorientation of lower triangular part of Z.

$$D = \begin{array}{|c|c|c|c|}\hline 0 & 0 & 0 & 1 \\ \hline 0 & 0 & 2 & \multicolumn{1}{c}{} \\ \cline{1-3} 1 & \multicolumn{3}{c}{} \\ \cline{1-1}\end{array} \quad \Longleftrightarrow \quad G = \begin{array}{ccccccc} 4 & & 3 & & 1 & & \\ & 4 & & 3 & & 1 & \\ & & 4 & & 2 & & 1 \\ & & & 3 & & 2 & & 0 \end{array}$$

$$\Longleftrightarrow \quad Z = \begin{array}{ccccccccccc} 0 & & 4 & & 7 & & 8 & & & & \\ & 0 & & 4 & & 7 & & 8 & & & \\ & & 0 & & 4 & & 6 & & 7 & & \\ & & & 0 & & 3 & & 5 & & 5 & \end{array} \quad \Longleftrightarrow \quad H = \begin{array}{ccccccc} & & & 5 & & & \\ & & 5 & & 7 & & \\ & 3 & & 6 & & 8 & \\ 0 & & 4 & & 7 & & 8 \end{array}$$

Theorem

- Lemma A bijection between LR integer n-hives, H, and LR-diagrams, D, is provided by the formula:

$$a_{ij} = \#\text{ of entries} \leq i \text{ in the first } i+j \text{ rows of } D$$

for the (i,j)th vertex label in H, for all i,j such that $0 \leq i, j, i+j \leq n$.

- Theorem The LR-coefficient $c^{\nu}_{\lambda\mu}$ is the number of LR-hives with boundary labels determined by λ, μ and ν.

- Note: Neither this theorem nor the Littlewood-Richardson rule allows us to see whether or not a given LR-coefficient $c^{\nu}_{\lambda\mu}$ is non-zero.

Example of bijection

Ex: $n = 4,\ \lambda = (753),\ \mu = (742),\ \nu = (9964)$

a_{ij}:

$$\begin{array}{ccccccccc} & & & & 15 & & & & \\ & & & 15 & & 22 & & & \\ & & 12 & & 21 & & 26 & & \\ & 7 & & 16 & & 24 & & 28 & \\ 0 & & 9 & & 18 & & 24 & & 28 \end{array} \quad \Longleftrightarrow \quad \begin{array}{|c|c|c|c|c|c|c|c|c|} \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ \hline 0 & 0 & 0 & 0 & 0 & 1 & 1 & 2 & 2 \\ \hline 0 & 0 & 0 & 1 & 1 & 2 \\ \cline{1-6} 1 & 2 & 3 & 3 \\ \cline{1-4} \end{array}$$

Example of bijection

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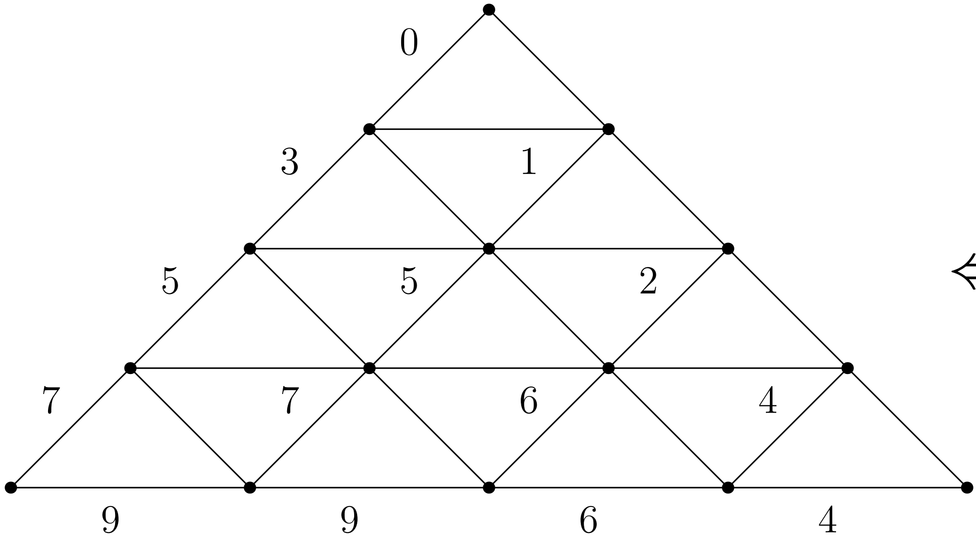
a_{ij}:



$\Longleftrightarrow$

0	0	0	0	0	0	0	1	1
0	0	0	0	0	1	1	2	2
0	0	0	1	1	2			
1	2	3	3					

LR-hives showing that $c^{9964}_{753,742} = 6$

$$\begin{array}{ccccccccc} & & & & 15 & & & & \\ & & & 15 & & 22 & & & \\ & & 12 & & 21 & & 26 & & \\ & 7 & & 16 & & 24 & & 28 & \\ 0 & & 9 & & 18 & & 24 & & 28 \end{array}$$

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Non-zero conditions

- We know that $c^{\nu}_{\lambda\mu}$ is the number of LR-hives with boundary labels determined by λ, μ and ν
- For given λ, μ and ν we would like some way of determining if $c^{\nu}_{\lambda\mu}$ is non-zero
- Horn [62] defined a set of inequalities and conjectured that they gave necessary and sufficient conditions for the solution of a problem later realised to be equivalent to that of LR-coefficients being non-zero
- The validity of Horn's conjecture was proved by the efforts of Klyachko [98], Knutson and Tao [99], Belkale [01], and Knutson, Tao and Woodward [04]
- For a comprehensive review see Fulton [00]

Partial sums

- Let $N = \{1, 2, \ldots, n\}$, then for fixed r, with $1 \leq r \leq n$, let $I = \{i_1, i_2, \ldots, i_r\} \subseteq N$ and $\overline{I} = N \backslash I$.
- For any partition λ let:

$$ps\,(\lambda)_I = \lambda_{i_1} + \lambda_{i_2} + \cdots + \lambda_{i_r}.$$

- If $i_1 < i_2 < \cdots < i_r$ then let

$$part\,(I) = (i_r - r, \ldots, i_2 - 2, i_1 - 1).$$

- Let T_r^n be the set of triples (I, J, K) with $I, J, K \subset N$ and $\#I = \#J = \#K = r$ with $c_{part\,(I) part\,(J)}^{part\,(K)} > 0$.
- Let R_r^n be the set of triples (I, J, K) with $I, J, K \subset N$ and $\#I = \#J = \#K = r$ with $c_{part\,(I) part\,(J)}^{part\,(K)} = 1$.

Non-zero conditions

Theorem: The LR-coefficient $c^{\nu}_{\lambda\mu}$ is non-zero if and only if

$$|\nu| = |\lambda| + |\mu|$$

and Horn's inequalities,

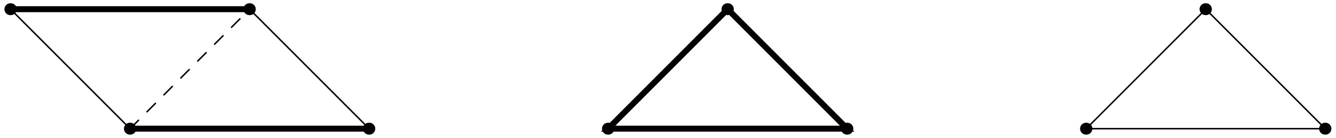
$$ps\,(\nu)_K \leq ps\,(\lambda)_I + ps\,(\mu)_J\,,$$

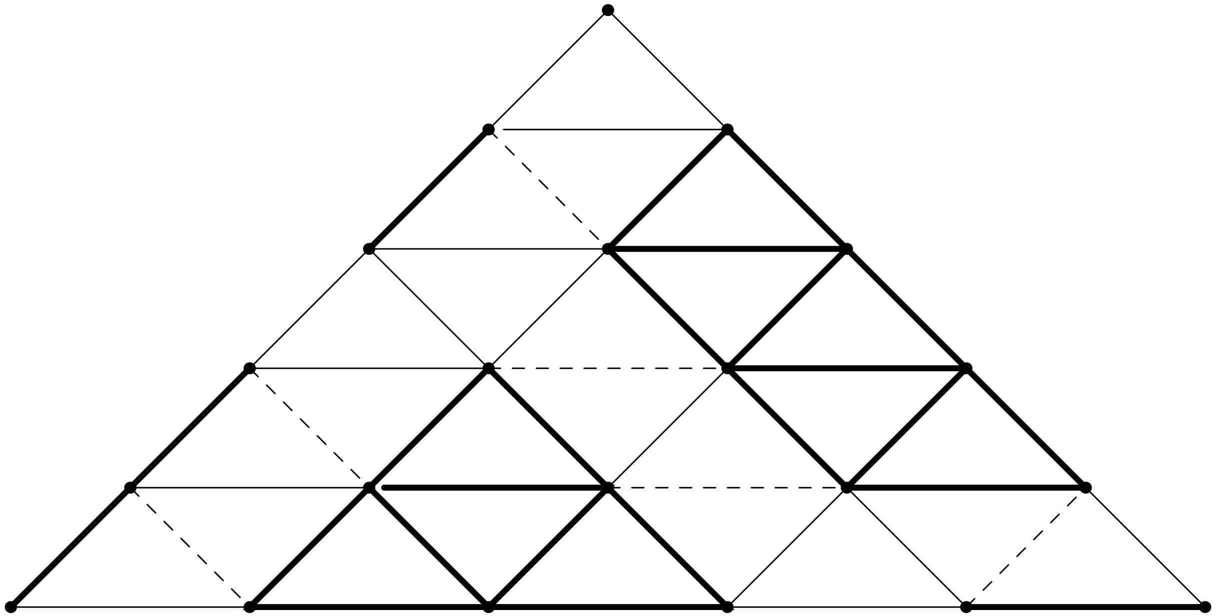
are satisfied for all $r = 1, 2, \ldots, n-1$ and all $(I, J, K) \in T^n_r$

Note: Not all of Horn's inequalities are essential. Horn's essential inequalities are those for which $(I, J, K) \in R^n_r$ - but where do they come from?

Puzzles - Knutson, Tao & Woodward [04]

Definition A puzzle is a diagram on a triangular lattice in which edges are distinguished so that it is composed of copies of the following pieces oriented in any way so as to fit:





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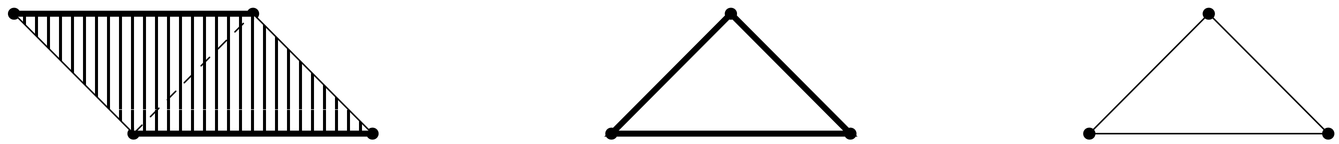

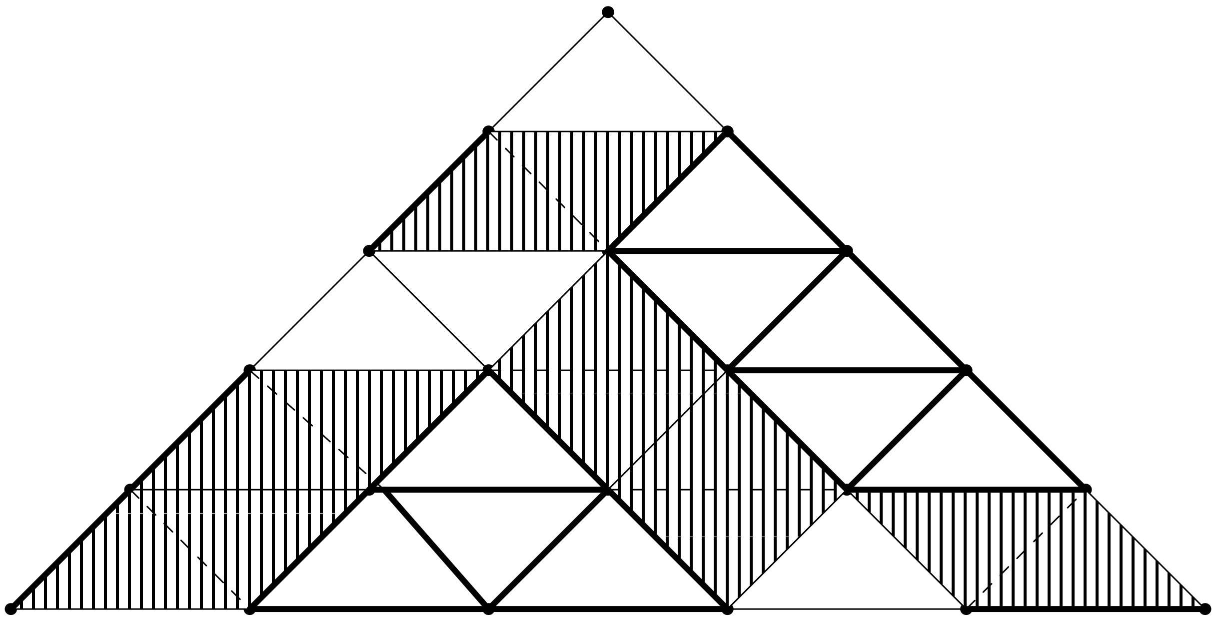

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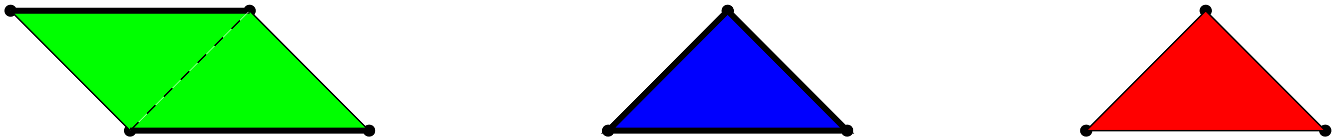
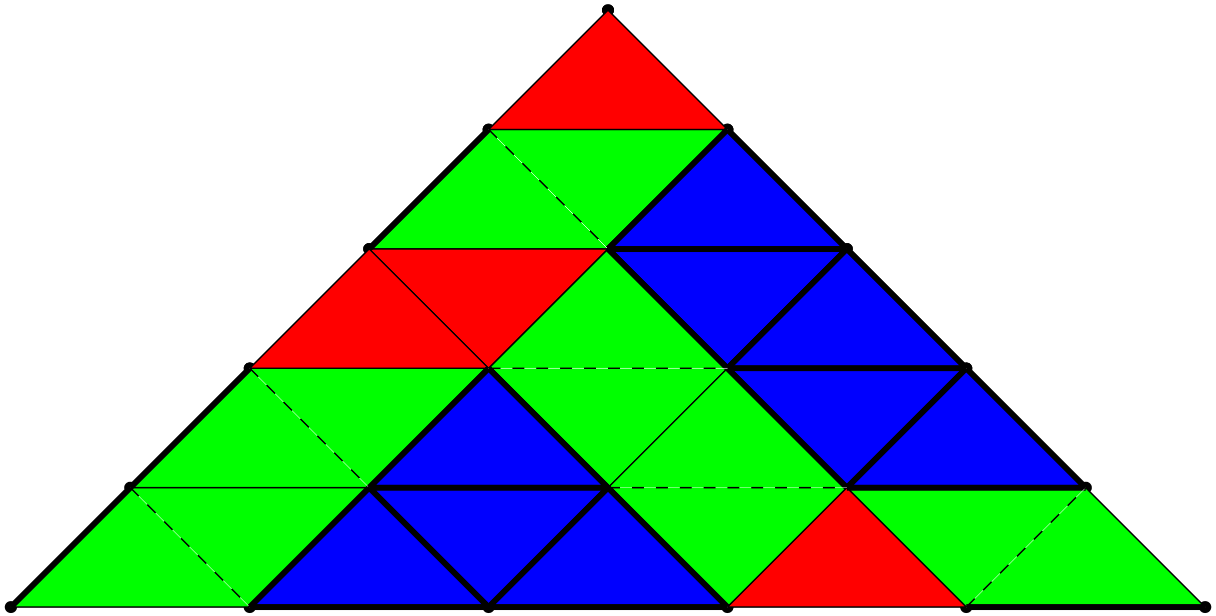
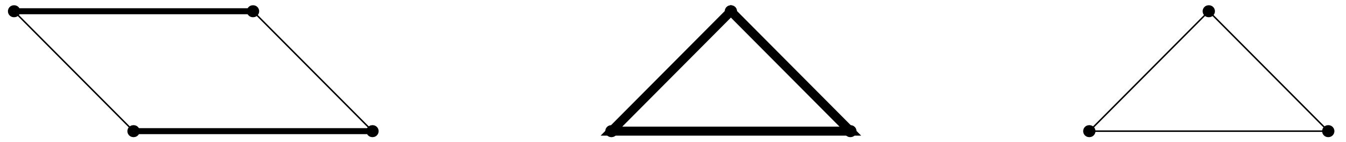
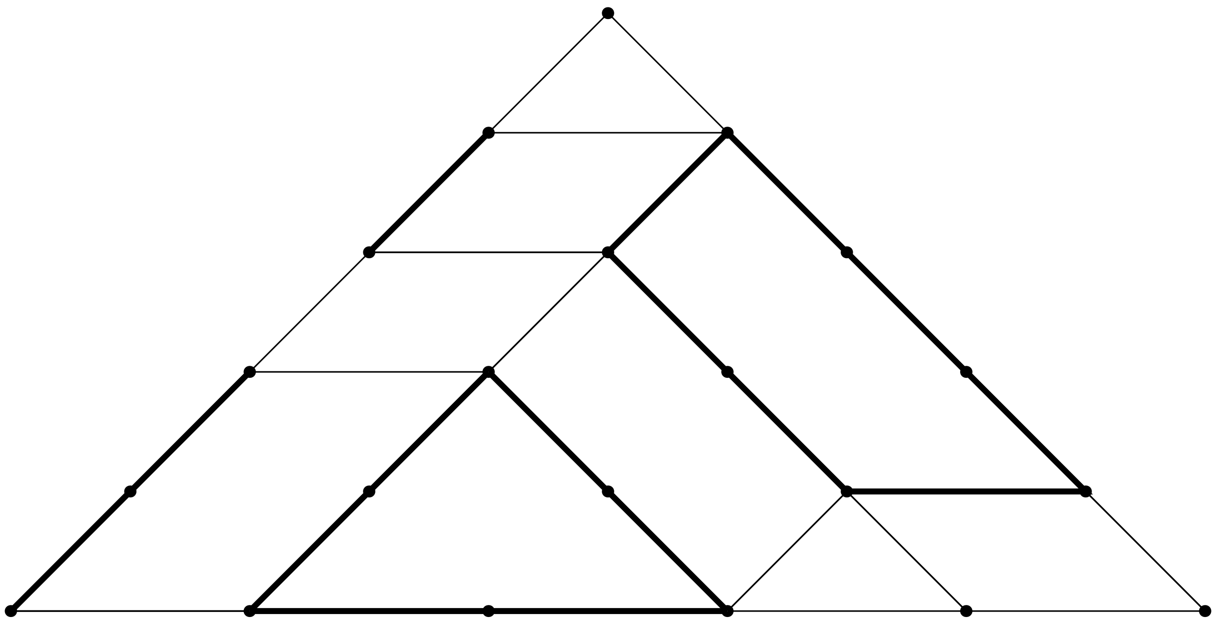




Hive plan or labyrinth

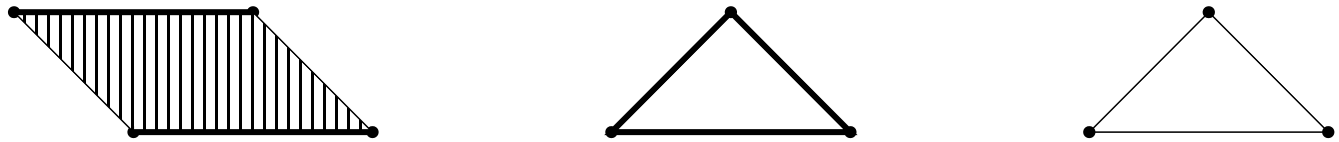
Definition A hive plan is made up of corridors, dark rooms and light rooms obtained by deleting interior edges of a puzzle:

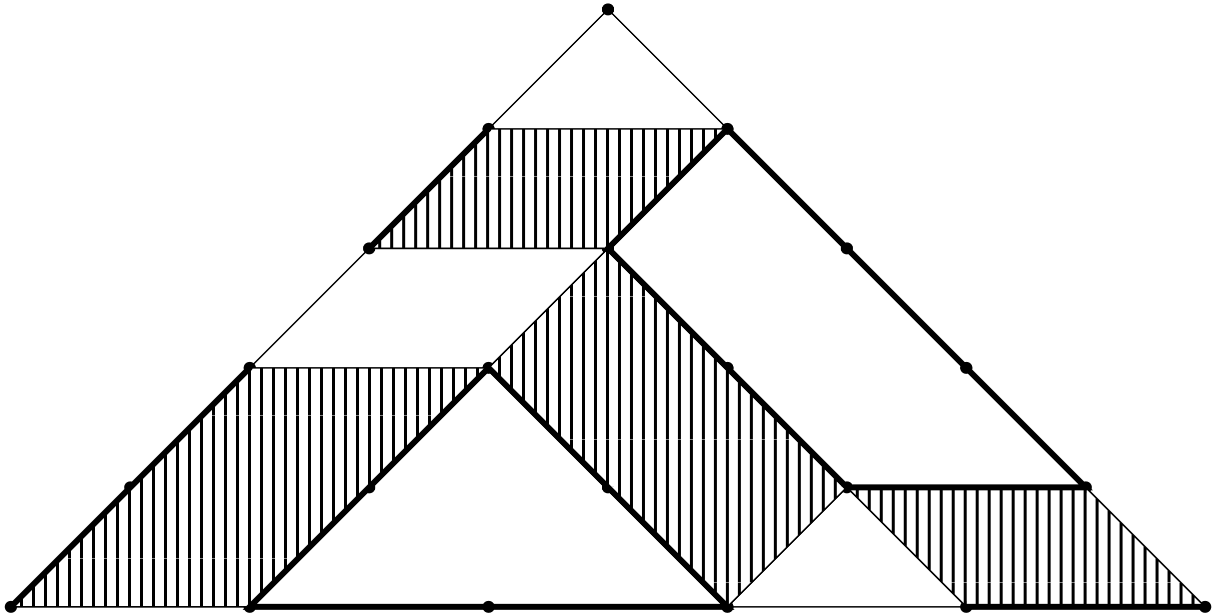




Hive plan or labyrinth

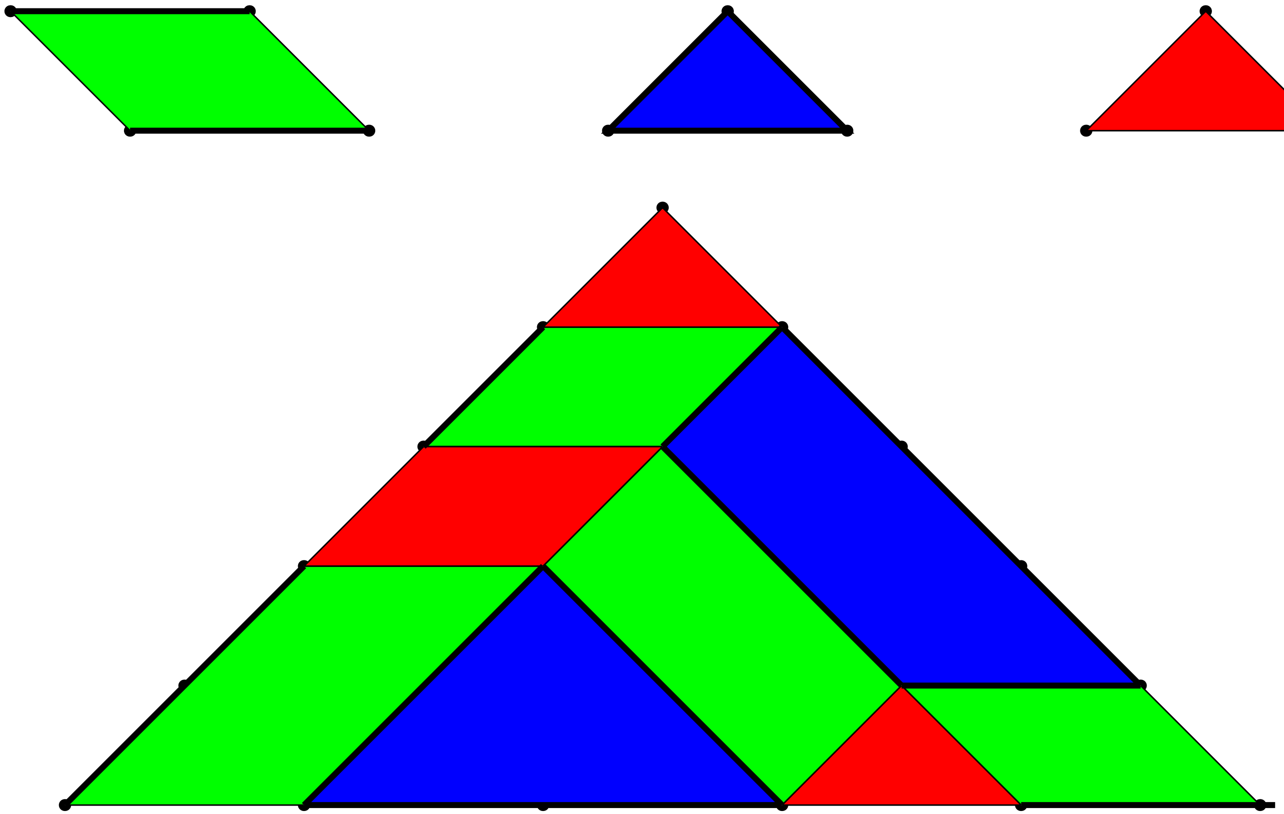
Definition A hive plan is made up of shaded corridors, dark rooms and light rooms obtained by deleting interior edges of a puzzle:





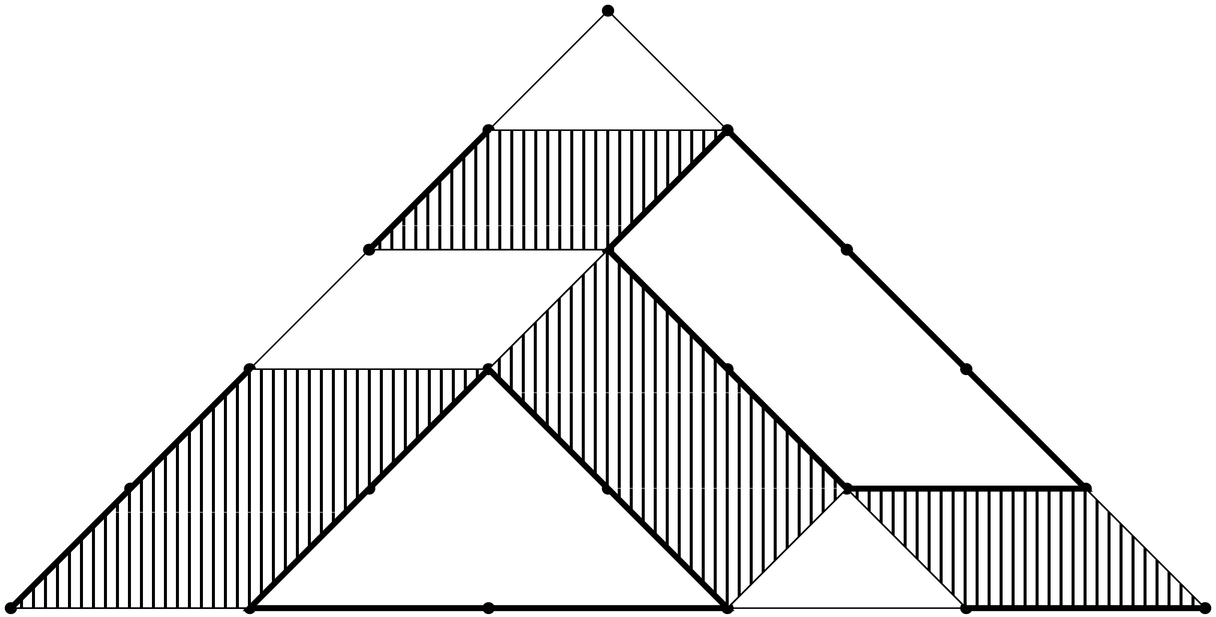
Hive plan or labyrinth

Definition A hive plan is made up of corridors, blue rooms and red rooms obtained by deleting interior edges of a puzzle:

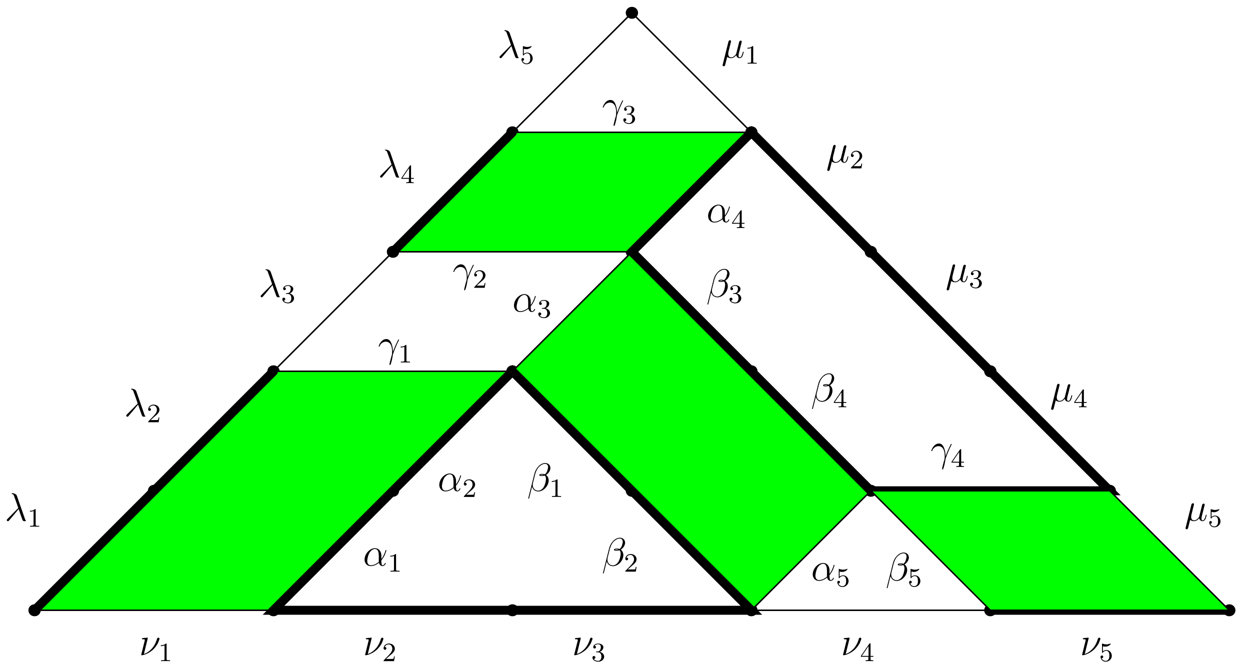


Link between puzzles and Horn triples

- (I, J, K) is Horn triple if it specifies the positions of the thick edges on the boundary of any puzzle. It is essential if the puzzle with these boundary thick edges is unique.
- For $I = (1, 2, 4)$, $J = (2, 3, 4)$ and $K = (2, 3, 5)$ we have:



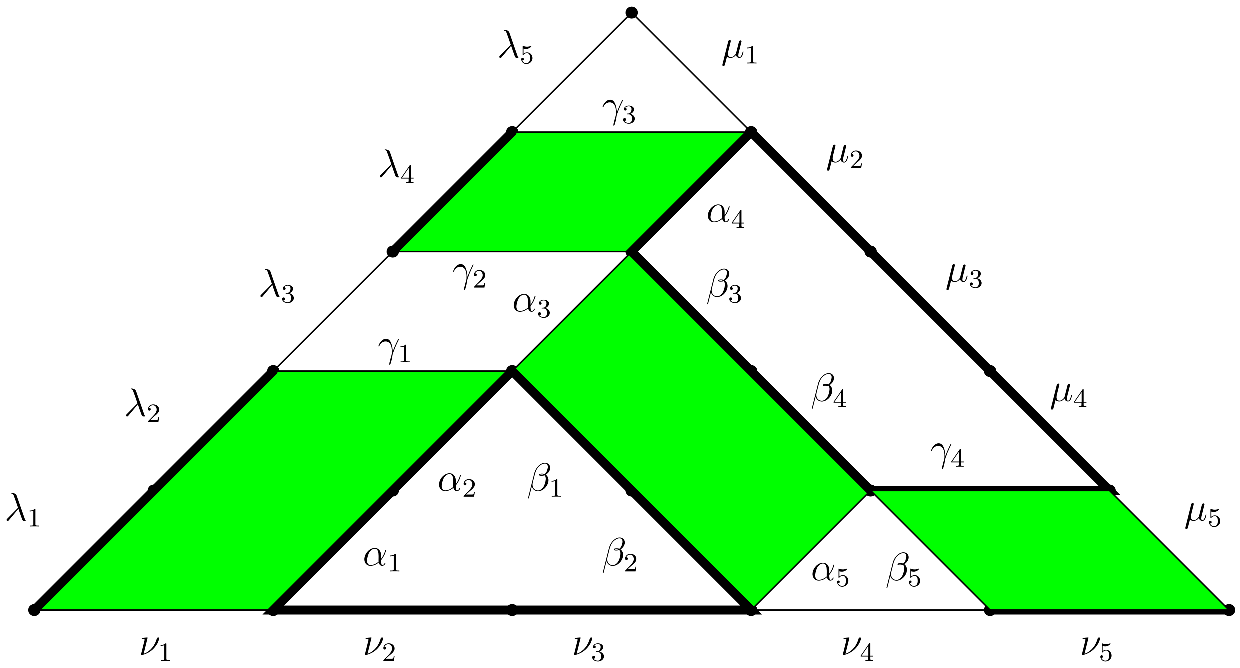
Each Horn triple defines an inequality



$$\nu_2 + \nu_3 + \nu_5 \leq (\nu_2 + \nu_3) + \gamma_4 = (\alpha_1 + \alpha_2 + \beta_1 + \beta_2) + \gamma_4$$
$$\leq \lambda_1 + \alpha_2 + \beta_1 + \beta_2 + \gamma_4 \leq \lambda_1 + \lambda_2 + \beta_1 + \beta_2 + \gamma_4$$
$$\leq \lambda_1 + \lambda_2 + \beta_3 + \beta_2 + \gamma_4 \leq \lambda_1 + \lambda_2 + (\beta_3 + \beta_4 + \gamma_4)$$
$$= \lambda_1 + \lambda_2 + (\alpha_4 + \mu_2 + \mu_3 + \mu_4) \leq \lambda_1 + \lambda_2 + \lambda_4 + \mu_2 + \mu_3 + \mu_4$$

That is $ps\,(\nu)_K \leq ps\,(\lambda)_I + ps\,(\mu)_J$.

Complementary Horn inequalities



The same procedure applied to thin-edge inequalities gives

$$\nu_1 + \nu_4 \geq \gamma_1 + \nu_4 = \gamma_1 + (\alpha_5 + \beta_5)$$

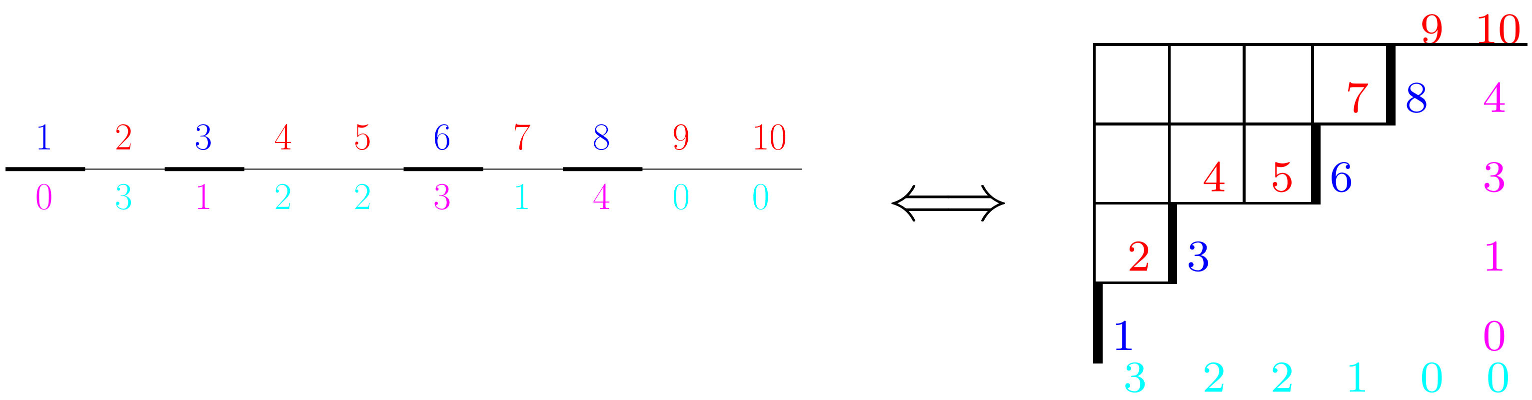
$$\geq \gamma_1 + \alpha_3 + \beta_5 \geq (\gamma_1 + \alpha_3) + \mu_5 = (\lambda_3 + \gamma_2) + \mu_5$$

$$\geq \lambda_3 + \gamma_3 + \mu_5 = \lambda_3 + \lambda_5 + \mu_1 + \mu_5$$

That is $\quad ps\,(\nu)_{\overline{K}} \geq ps\,(\lambda)_{\overline{I}} + ps\,(\mu)_{\overline{J}}$

Significance of inequalities derived from puzzles

- Each inequality $ps\,(\nu)_K \leq ps\,(\lambda)_I + ps\,(\mu)_J$ derived from a puzzle must be satisfied if $c^{\nu}_{\lambda\mu}$ is to be non-zero
- To show that a puzzle triple (I, J, K) is a Horn triple, and the inequality a Horn inequality, we must make a connection between puzzles with thick boundary edges specified by (I, J, K) and $c^{part\,(K)}_{part\,(I),part\,(J)}$
- First note the connection between (for example) $M = \{1, 3, 6, 8\} \subseteq N = \{1, 2, \ldots, 10\}$ and $F^{part\,(M)} = F^{431}$

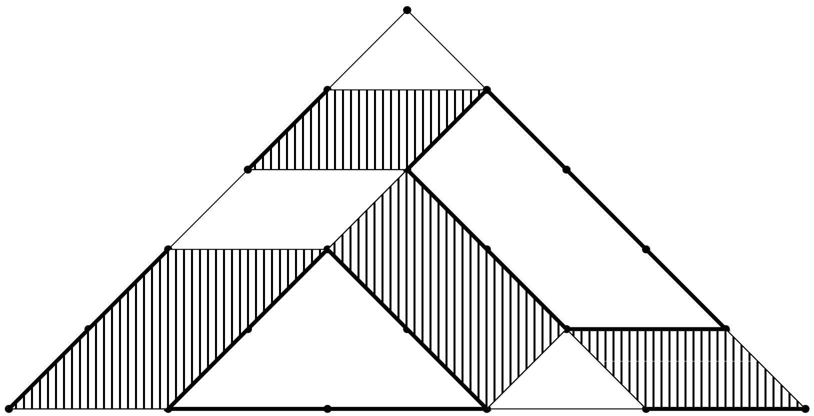


How may puzzles are there?

Theorem The number of puzzles with the positions of the thick edges on the boundary specified by (I, J, K) is given by $c^{part\,(K)}_{part\,(I),part\,(J)}$

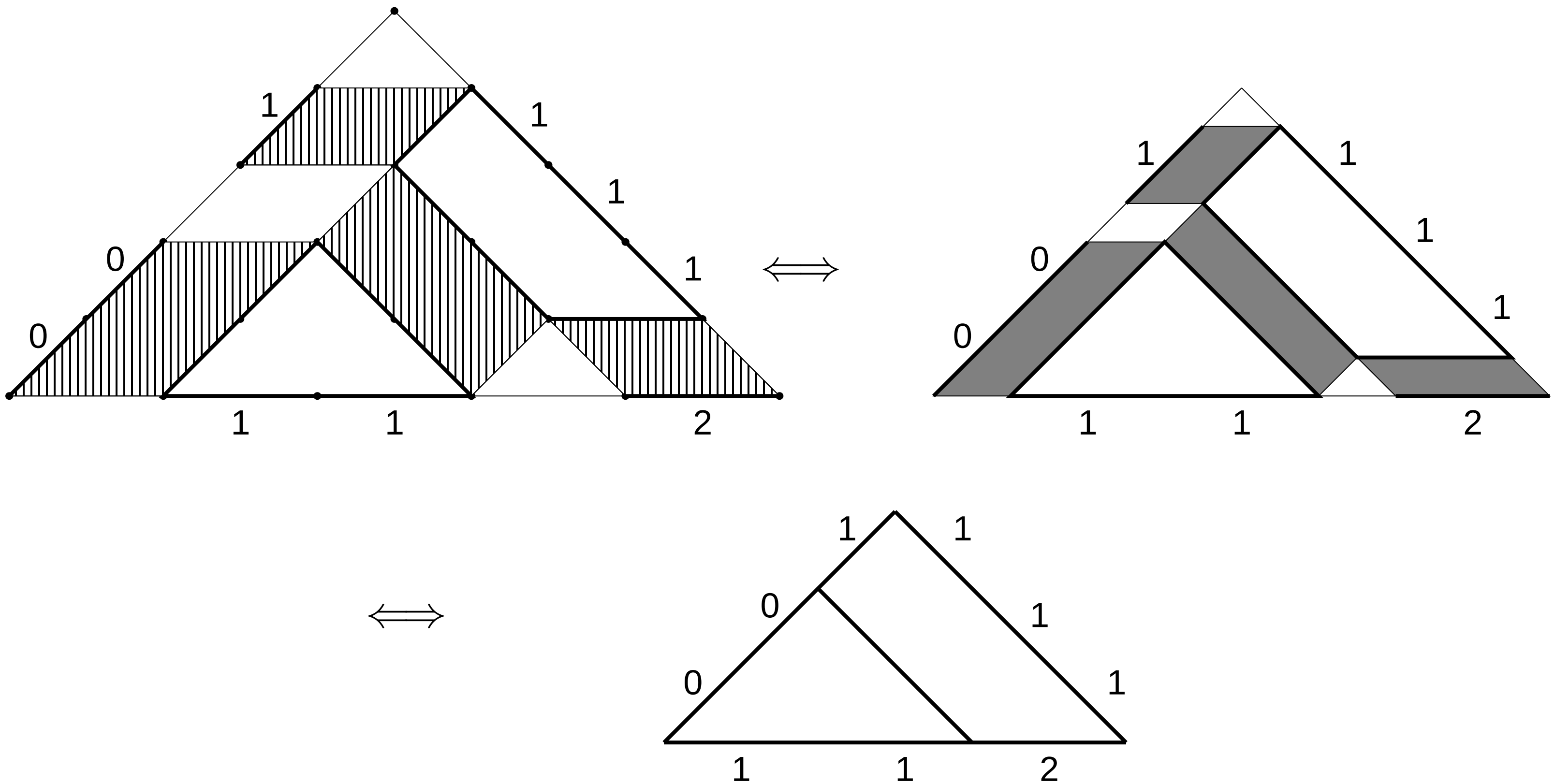
Ex: $n = 5$, $r = 2$, $I = (1, 2, 4)$, $J = (2, 3, 4)$, $K = (2, 3, 5)$

- In this case $c^{part\,(K)}_{part\,(I),part\,(J)} = c^{311}_{1,111} = 1$
- and there exists just the one puzzle identified earlier



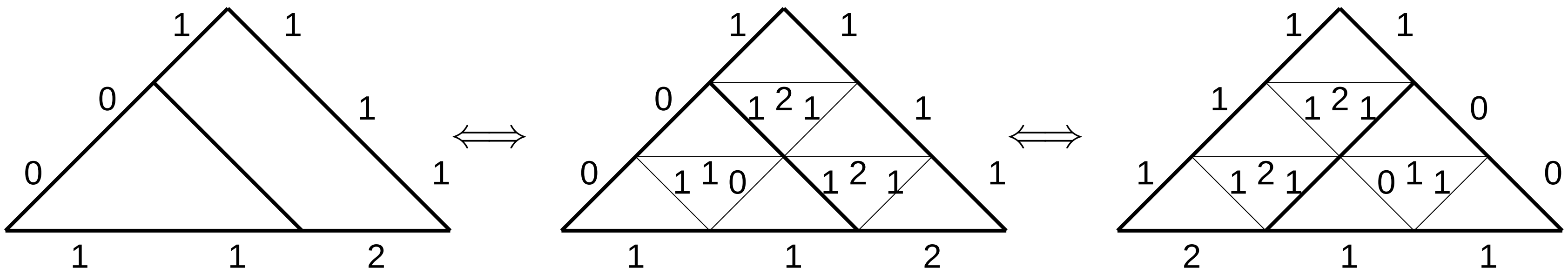
A map from puzzles to hives

- Let the thick edges of the puzzle be specified by (I,J,K)
- For $M=I,J,K$ in turn, label each thick boundary edge of the puzzle by the corresponding row length of $F^{part\,(M)}$
- Scale the length of all thin edges by t and let $t \to 0$



A map from puzzles to hives contd

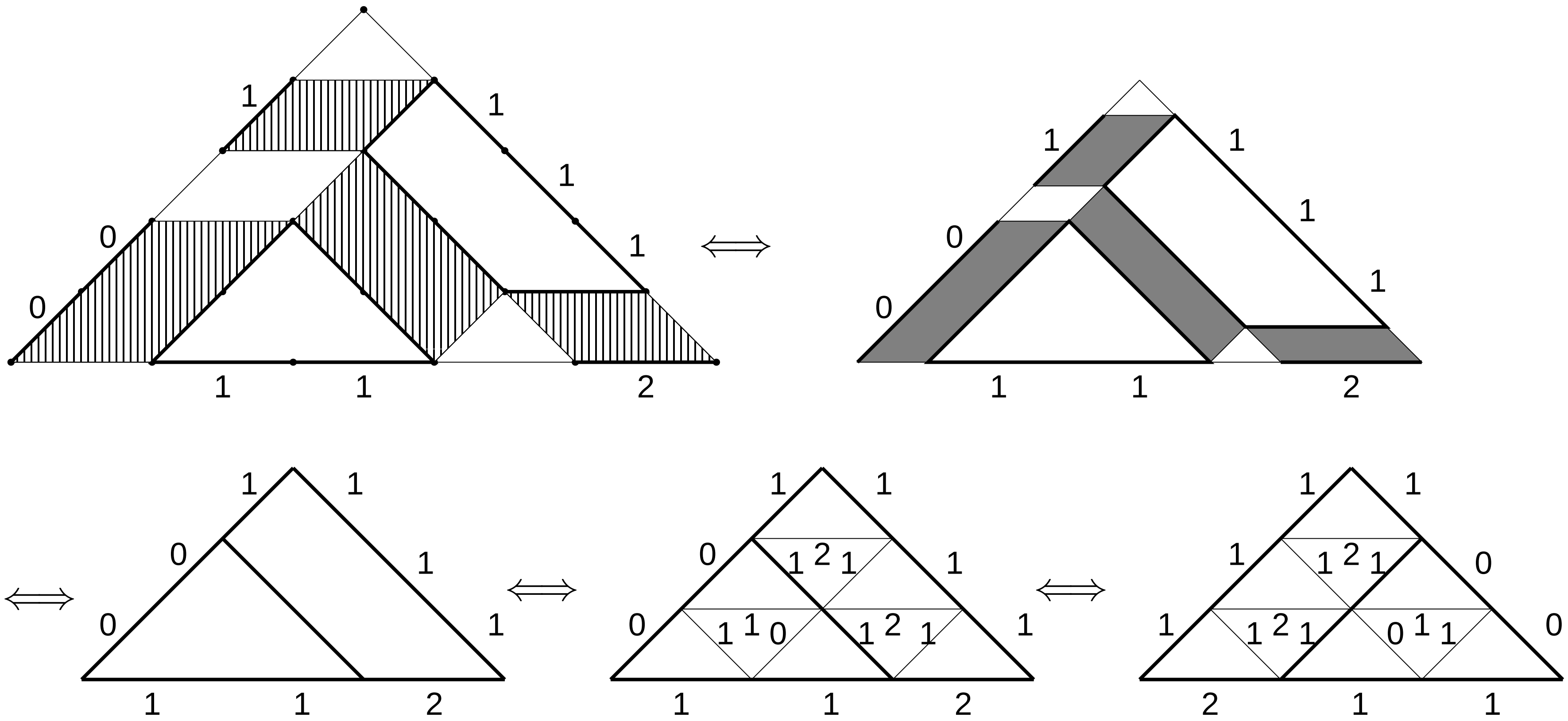
- In each thick edged room set all parallel edge labels equal using the triangle condition wherever required
- Reflect the resulting diagram in its vertical axis of symmetry to obtain a hive



Lemma This map provides a bijection between puzzles with thick edges specified by (I, J, K) and LR-hives with boundary specified by $part\,(J), part\,(I), part\,(K)$

A map from puzzles to LR-hives

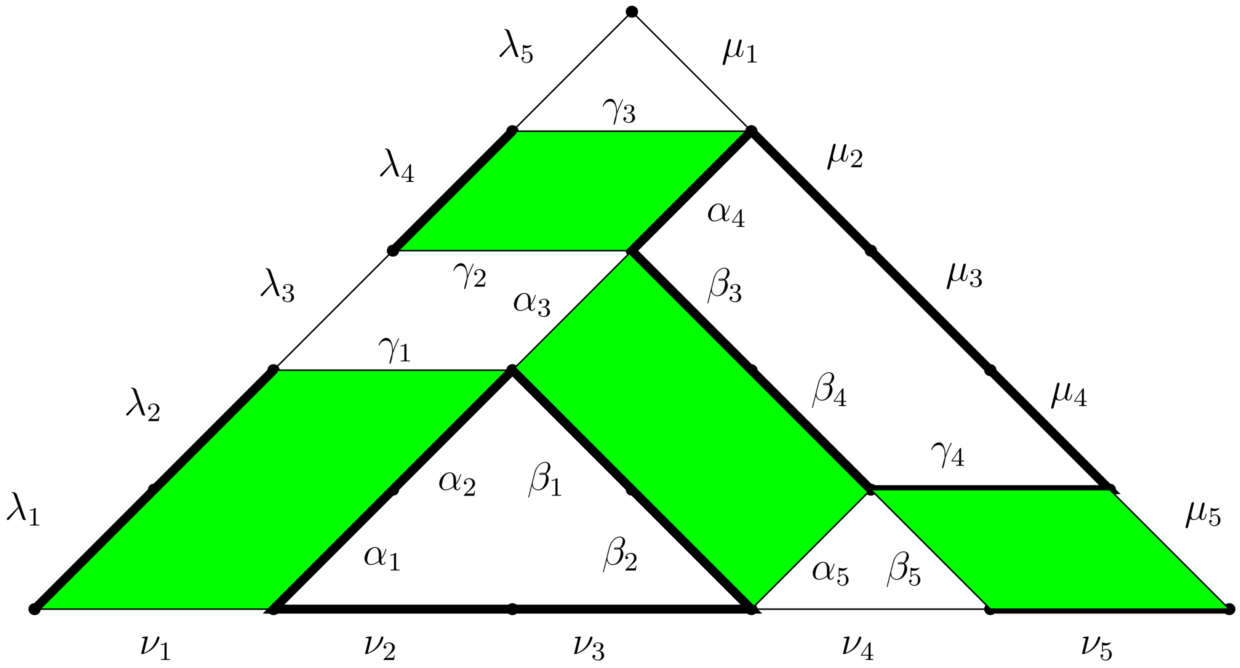
Ex: If $n = 5$, $r = 2$, $I = (1, 2, 4)$, $J = (2, 3, 4)$, $K = (2, 3, 5)$
we have $part\,(I) = (1)$, $part\,(J) = (1, 1, 1)$, $part\,(K) = (2, 1, 1)$



Corollaries

- Each triple (I, J, K) is a Horn triple if and only if it specifies the thick boundary edges of a puzzle
- The corresponding inequality $ps\,(\nu)_K \leq ps\,(\lambda)_I + ps\,(\mu)_J$ defined by the puzzle is a Horn inequality
- The number of puzzles specified by (I, J, K) is equal to $c^{part\,(K)}_{part\,(J),part\,(I)} = c^{part\,(K)}_{part\,(I),part\,(J)}$
- The puzzle is said to be rigid if it is unique, that is $c^{part\,(K)}_{part\,(J),part\,(I)} = 1$, and the corresponding Horn inequality is essential
- The LR-coefficient $c^{\nu}_{\lambda\mu}$ is non-zero if and only if every essential Horn inequality is satisfied

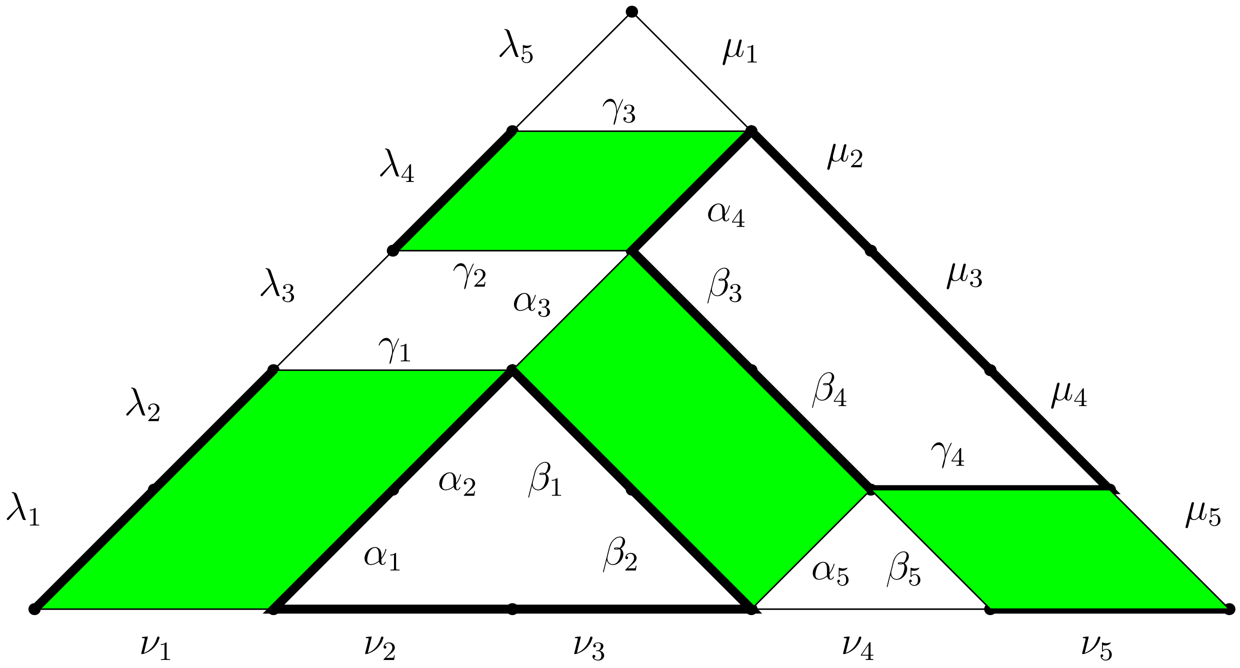
Consequences of any Horn equality



All sequences of inequalities become equalities.

- $$\nu_2+\nu_3+\nu_5 = (\nu_2+\nu_3)+\gamma_4 = (\alpha_1+\alpha_2+\beta_1+\beta_2)+\gamma_4$$
  $$= \lambda_1+\alpha_2+\beta_1+\beta_2+\gamma_4 = \lambda_1+\lambda_2+\beta_1+\beta_2+\gamma_4$$
  $$= \lambda_1+\lambda_2+\beta_3+\beta_2+\gamma_4 = \lambda_1+\lambda_2+(\beta_3+\beta_4+\gamma_4)$$
  $$= \lambda_1+\lambda_2+(\alpha_4+\mu_2+\mu_3+\mu_4) = \lambda_1+\lambda_2+\lambda_4+\mu_2+\mu_3+\mu_4.$$

Edge label equalities



- $\nu_1 + \nu_4 = \gamma_1 + \nu_4 = \gamma_1 + (\alpha_5 + \beta_5)$
  $= \gamma_1 + \alpha_3 + \beta_5 = (\gamma_1 + \alpha_3) + \mu_5 = (\lambda_3 + \gamma_2) + \mu_5$
  $= \lambda_3 + \gamma_3 + \mu_5 = \lambda_3 + \lambda_5 + \mu_1 + \mu_5$
- These equalities imply: $\nu_5 = \gamma_4$, $\alpha_1 = \lambda_1$, $\alpha_2 = \lambda_2$,
  $\beta_1 = \beta_3$, $\beta_2 = \beta_4$, $\alpha_4 = \lambda_4$, $\beta_5 = \mu_5$, $\nu_1 = \gamma_1$, $\gamma_2 = \gamma_3$.

Factorisation of LR-hives

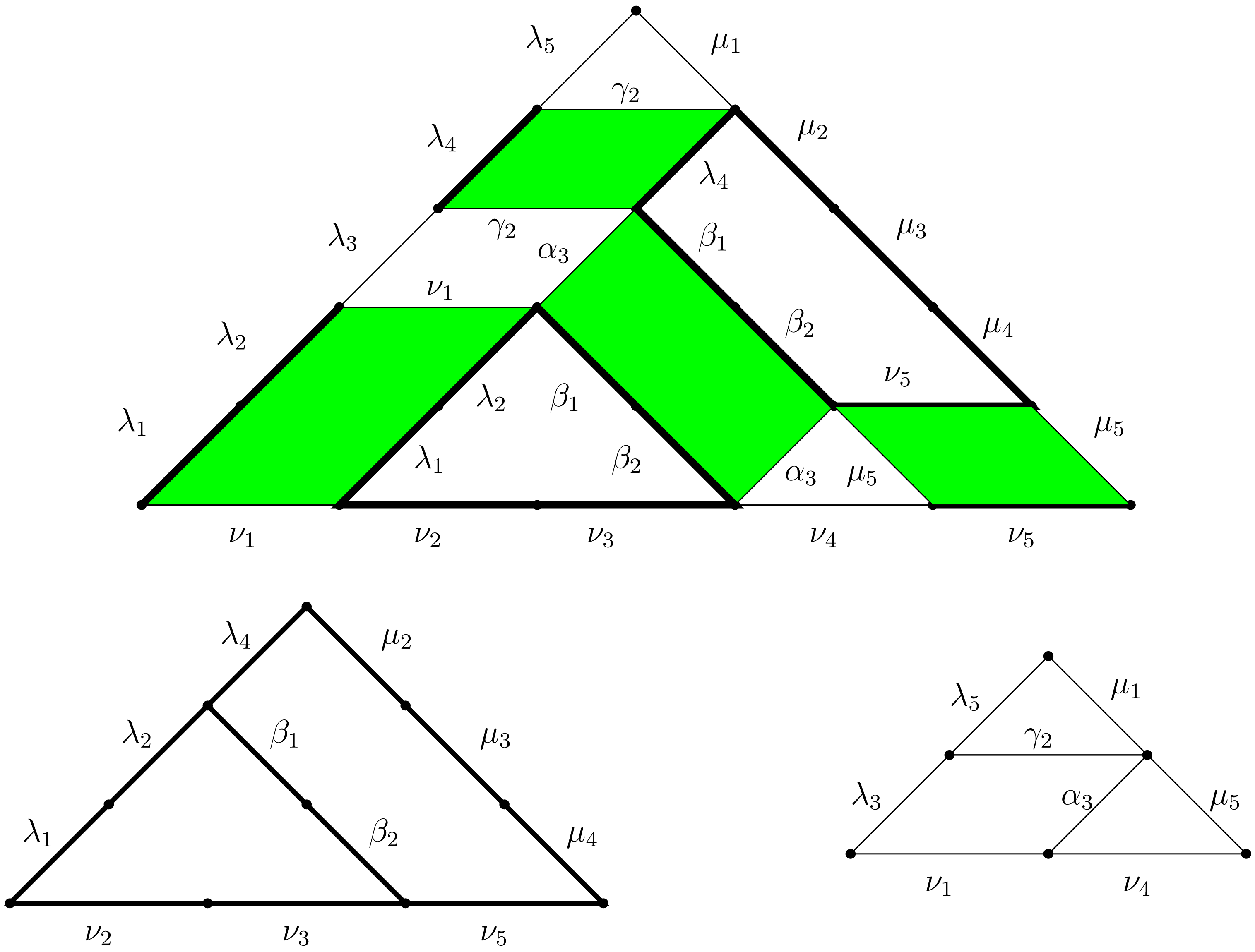
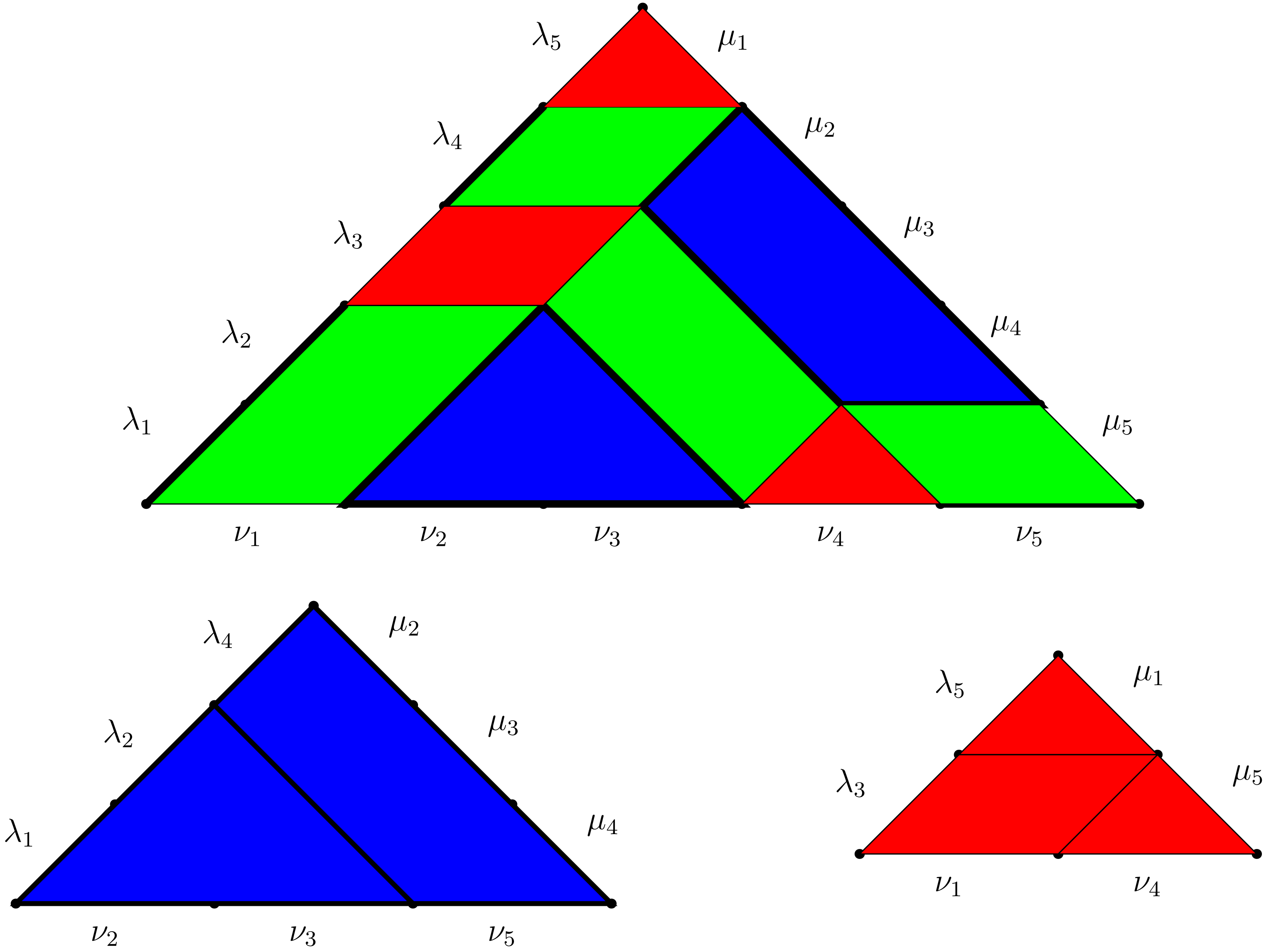


Illustration of H_n and subhives H_r,H_{n-r}



LR-coefficient factorisation

- Lemma In the case of any Horn equality and a corresponding puzzle, the deletion of redundant corridors from any LR-hive H_n gives a pair of LR-subhives H_r and H_{n-r}.
- Lemma In the case of any essential Horn equality, this map from the LR-hives H_n to pairs of LR-hives H_r and H_{n-r} is a bijection.
- Theorem If an essential Horn inequality is saturated then $c_{\lambda\mu}^{\nu}$ factorises.
- Definition If all essential Horn inequalities are strict $c_{\lambda\mu}^{\nu}$ is said to be primitive.

LR factorisation example

Ex: $n=5$, $r=3$, $n-r=2$:

- $\lambda=(9,7,6,2,0)$, $\mu=(13,5,3,1,0)$, $\nu=(14,12,11,5,4)$.
- $I=\{1,2,4\}$, $J=\{2,3,4\}$, $K=\{2,3,5\}$.
- $\lambda_I=(9,7,2)$, $\mu_J=(5,3,1)$, $\nu_K=(12,11,4)$
- $\lambda_{\overline{I}}=(6,0)$, $\mu_{\overline{J}}=(13,0)$, $\nu_{\overline{K}}=(14,5)$
- $ps\,(\nu)_K=27=18+9=ps\,(\lambda)_I+ps\,(\mu)_J$

Hence

$$c^{(14,12,11,5,4)}_{(9,7,6,2,0),(13,5,3,1,0)}=c^{(12,11,4)}_{(9,7,2),(5,3,1)}\;c^{(14,5)}_{(6,0),(13,0)}=2\cdot 1=2$$

Note: This is an example of the reduction of an LR-coefficient, since $c^{(14,12,11,5,4)}_{(9,7,6,2,0),(13,5,3,1,0)}=c^{(12,11,4)}_{(9,7,2),(5,3,1)}=2$

Proof of factorisation

To be shown:

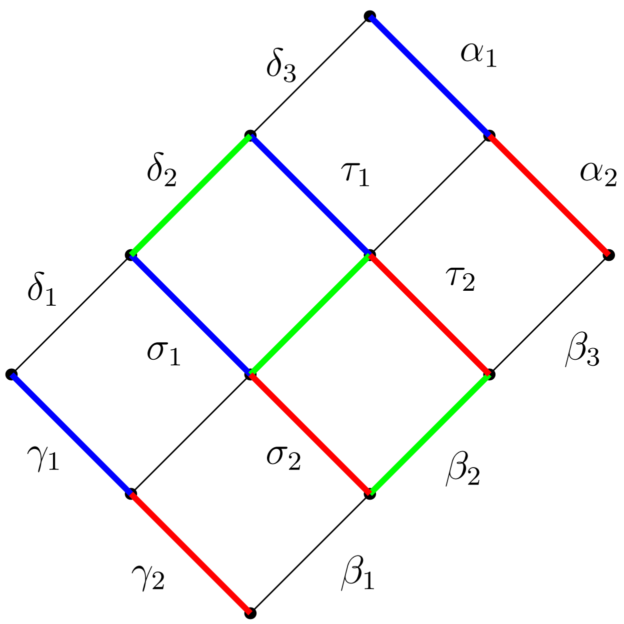
- the corridors R_n of H_n are redundant;
- the dark rooms constitute an LR-hive H_r;
- the light rooms constitute an LR-hive H_{n-r};
- any LR hives H_r,H_{n-r} joined by R_n gives an LR-hive H_n.

To be checked that the Horn equality implies:

- all corridor edge labels fixed;
- LR hive conditions for any rhombus split by corridor;
- LR hive conditions across corridor/dark room boundary;
- LR hive conditions across corridor/light room boundary.

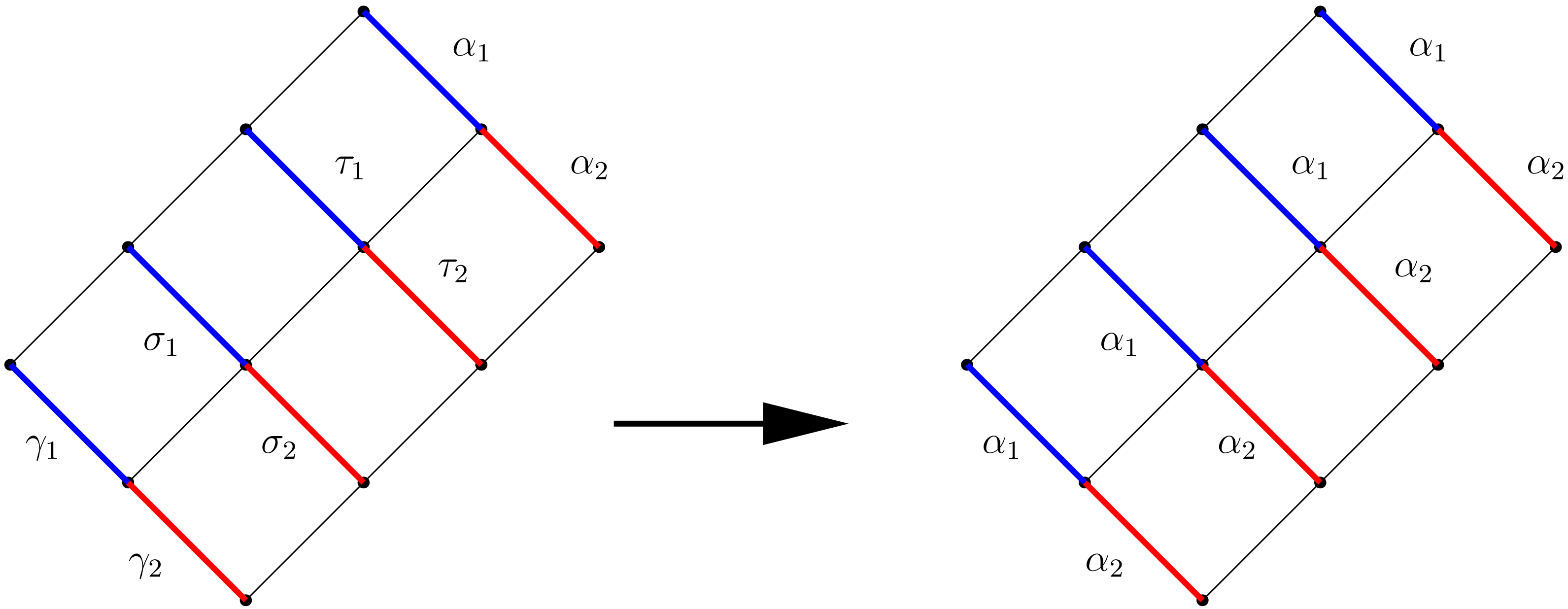
Corridor edges fixed

- Hive conditions: $\gamma_1 \leq \sigma_1 \leq \tau_1 \leq \alpha_1, \quad \gamma_2 \leq \sigma_2 \leq \tau_2 \leq \alpha_2$.
- Horn inequality: $\gamma_1 + \gamma_2 \leq \alpha_1 + \alpha_2$.



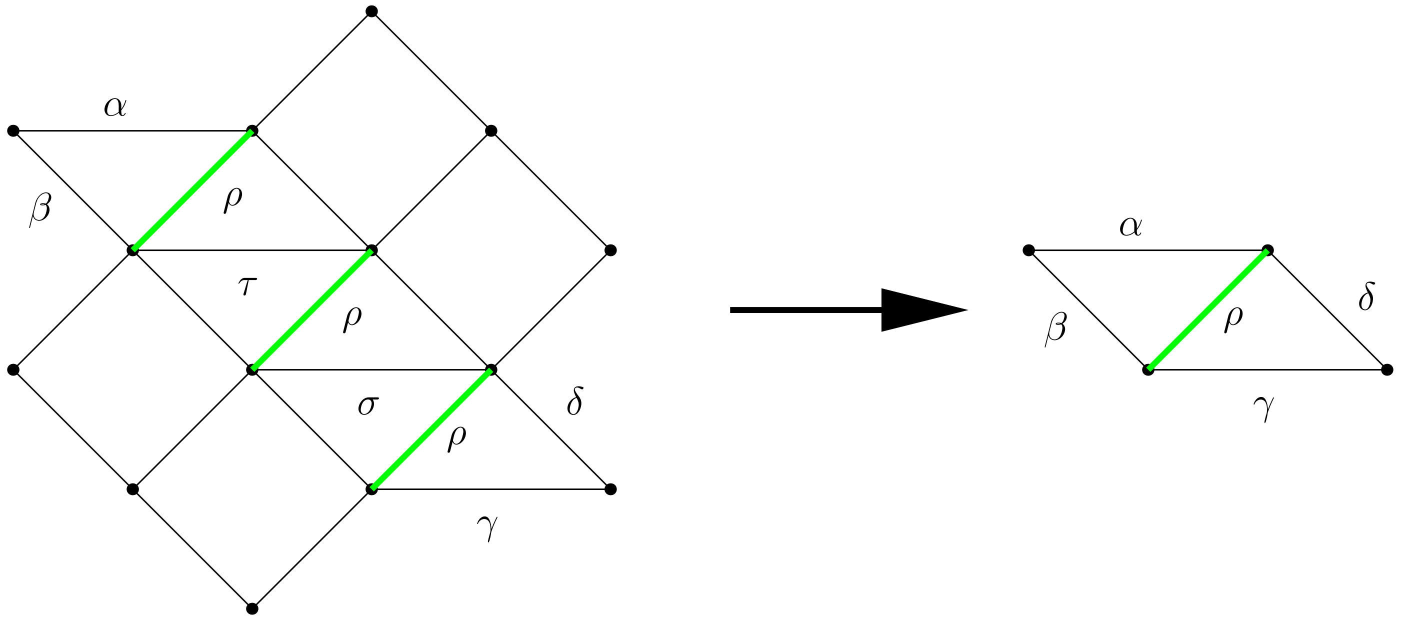
Corridor edges fixed

- Hive conditions: $\gamma_1 \le \sigma_1 \le \tau_1 \le \alpha_1, \quad \gamma_2 \le \sigma_2 \le \tau_2 \le \alpha_2$.
- Horn equality: $\gamma_1 + \gamma_2 = \alpha_1 + \alpha_2$.
- Implies: $\gamma_1 = \alpha_1$ and $\gamma_2 = \alpha_2$.
- Implies: $\gamma_1 = \sigma_1 = \tau_1 = \alpha_1$ and $\gamma_2 = \sigma_2 = \tau_2 = \alpha_2$.



Deletion of corridor

- Initial hive conditions: $\gamma \leq \sigma, \ \sigma \leq \tau, \ \tau \leq \alpha$.
- Implies final hive condition: $\gamma \leq \alpha$.



- Horn equality $\alpha - \beta = \rho = \gamma - \delta$ implies $\alpha + \delta = \beta + \gamma$.

Paths - gentle and good

- Path: a continuous sequence of connected corridor walls with dark rooms, thick-edged 0-regions, on the right and light rooms, thin-edged 1-regions, on the left.
- Gentle path: at each vertex the deviation is 0 or $\pm\pi/3$.
- Gentle loop: a gentle path that forms a closed interior loop.
- An edge is good if it forms the short diagonal of a rhombus satisfying the hive condition, otherwise it is bad.
- Good path: gentle path along which all the edges are good, ie with the hive condition satisfied across each edge.

Good paths and factorisation

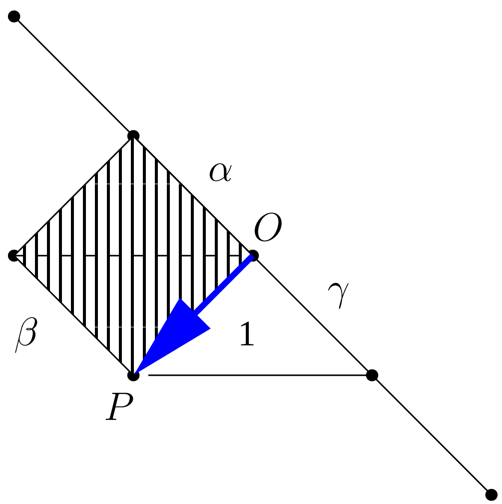
Observations

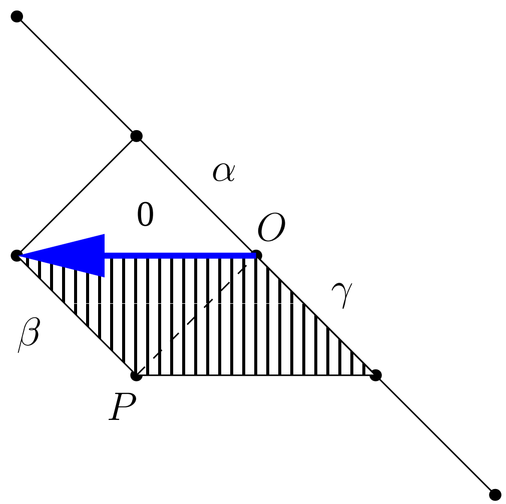
- When subdividing two LR subhives, H_r and H_{n-r}, and inserting corridors to create a hive H_n, this hive will be an LR hive if and only if each edge of every internal corridor wall of the corresponding hive plan is good.
- Let the boundary labels λ, μ, ν of the hive H_n be such that for a given puzzle specified by (I, J, K) the corresponding Horn inequality is saturated, ie. $ps\,(\nu)_K = ps\,(\lambda)_I + ps\,(\mu)_J$. If in the corresponding hive plan each edge of every internal corridor wall lies on a good path, then the LR-coefficient $c_{\lambda\mu}^{\nu}$ factorises.

Start of good paths

Lemma The first edge of any path starting from any boundary is good

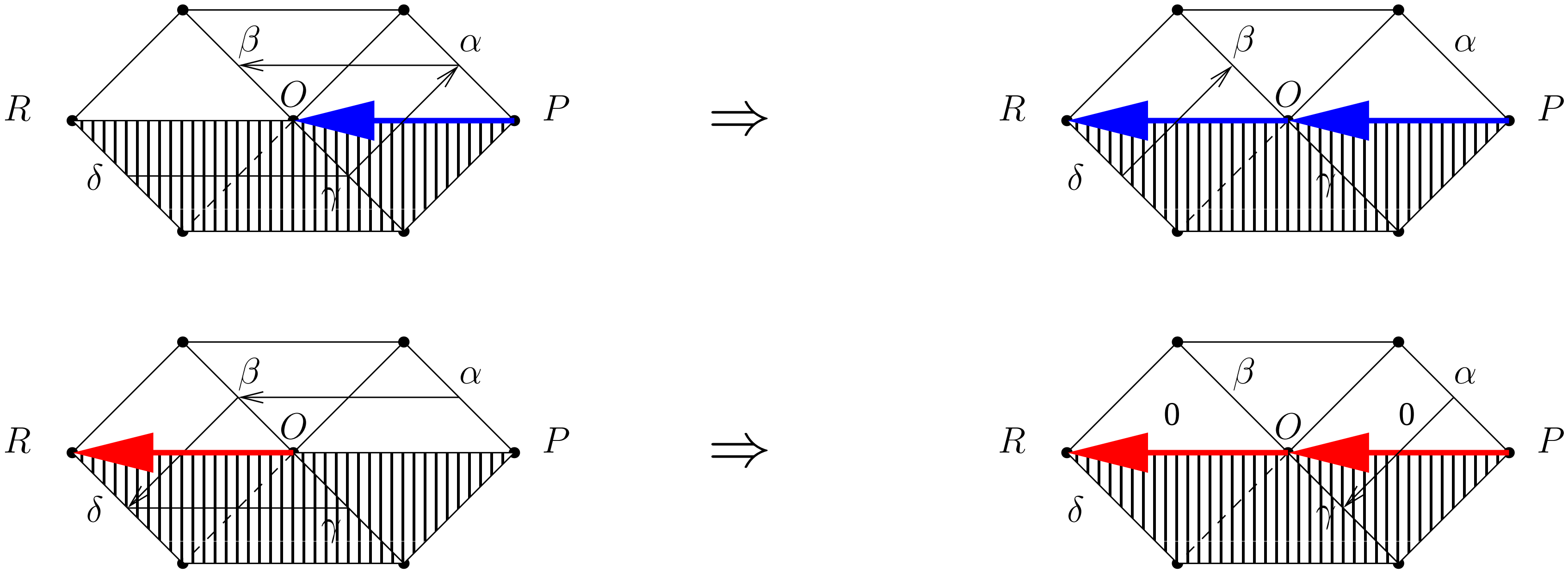
- Each boundary has edges specified by a partition: $\alpha \geq \gamma$.
- Horn equality applied to corridors: $\beta = \alpha$ or $\beta = \gamma$.
- Hence $\beta = \alpha \geq \gamma$ or $\beta = \gamma \leq \alpha$ so that in both cases OP is good.





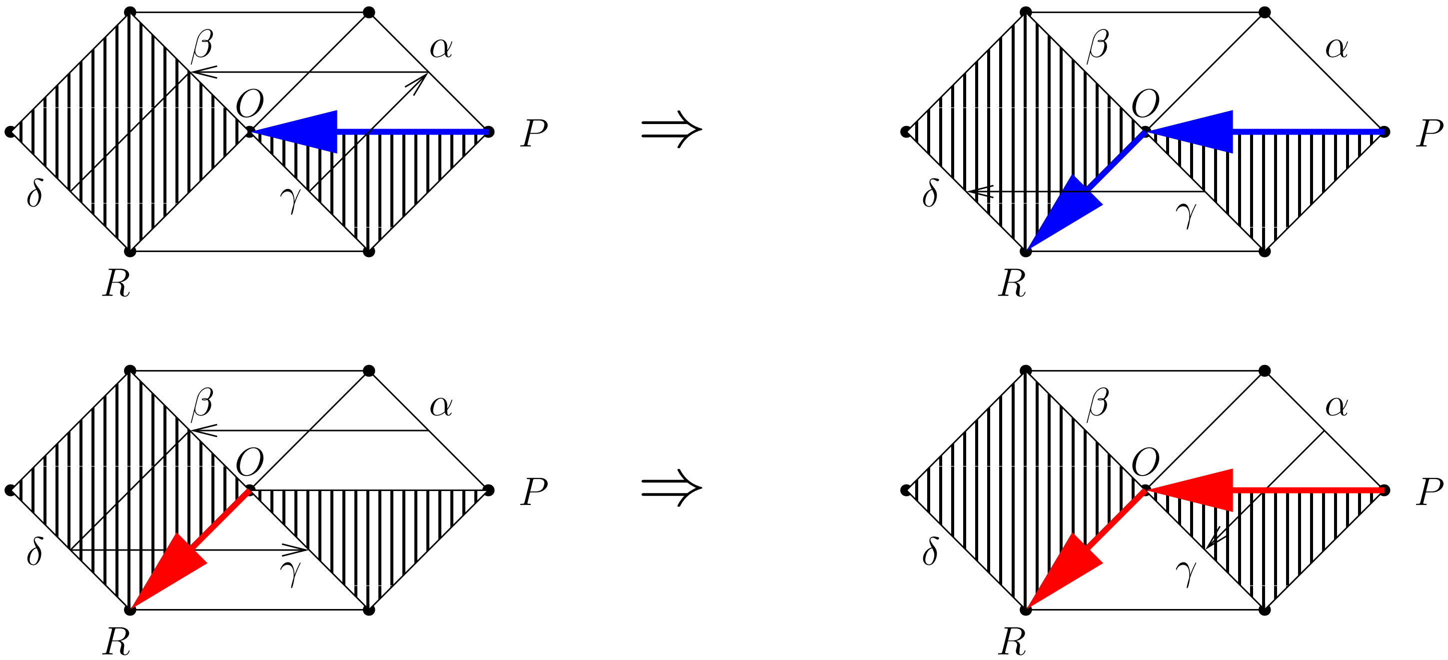
Path along an interior corridor wall

- Initial hive conditions: $\alpha \leq \beta$
- Horn equality applied to corridors: $\delta = \gamma$.
- PO good $\Rightarrow \gamma \leq \alpha \Rightarrow \delta = \gamma \leq \alpha \leq \beta \Rightarrow OR$ good
- OR bad $\Rightarrow \delta > \beta \Rightarrow \gamma = \delta > \beta \geq \alpha \Rightarrow PO$ bad



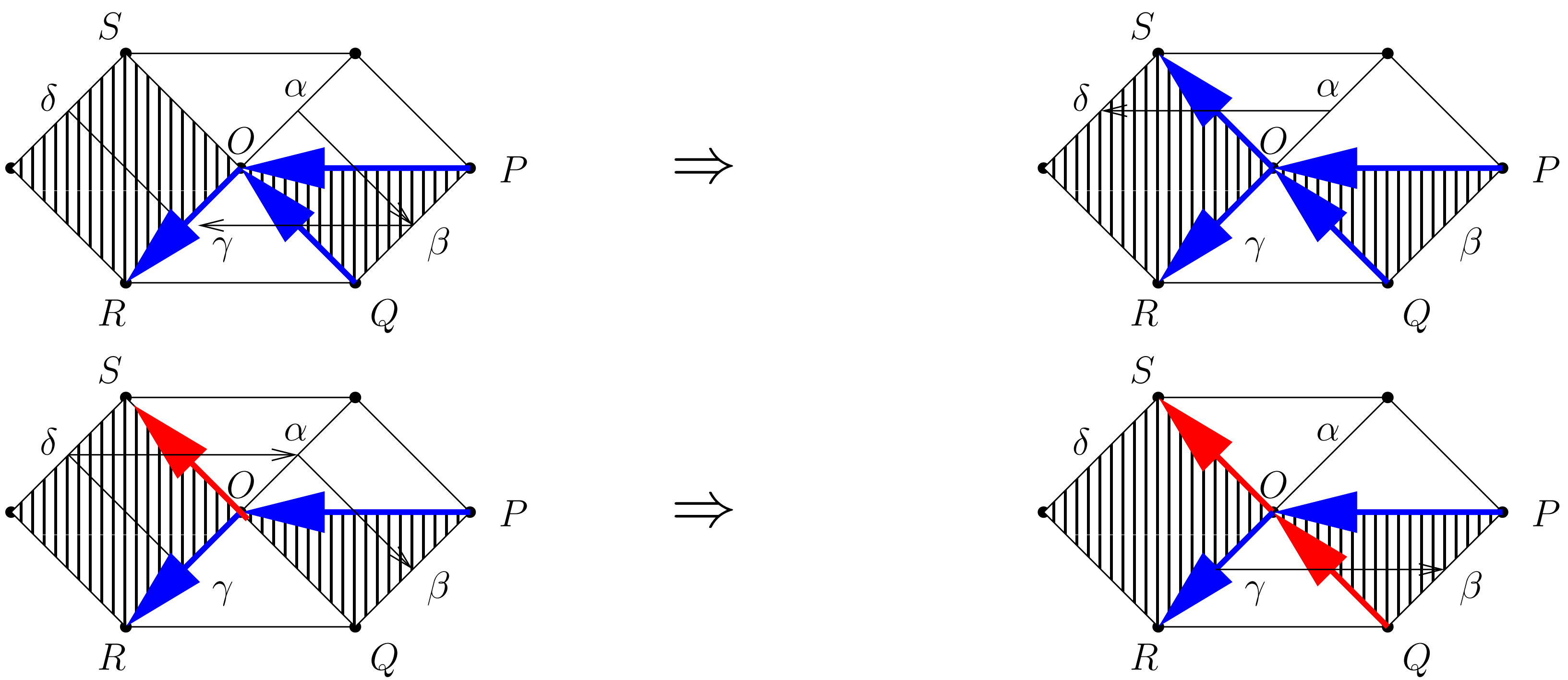
Path reaching a vertex

- Initial hive condition: $\alpha \leq \beta$; Horn equality $\delta = \beta$
- PO good $\Rightarrow \gamma \leq \alpha \Rightarrow \gamma \leq \alpha \leq \beta = \delta \Rightarrow OR$ good
- OR bad $\Rightarrow \gamma > \delta \Rightarrow \gamma > \delta = \beta \geq \alpha \Rightarrow PO$ bad

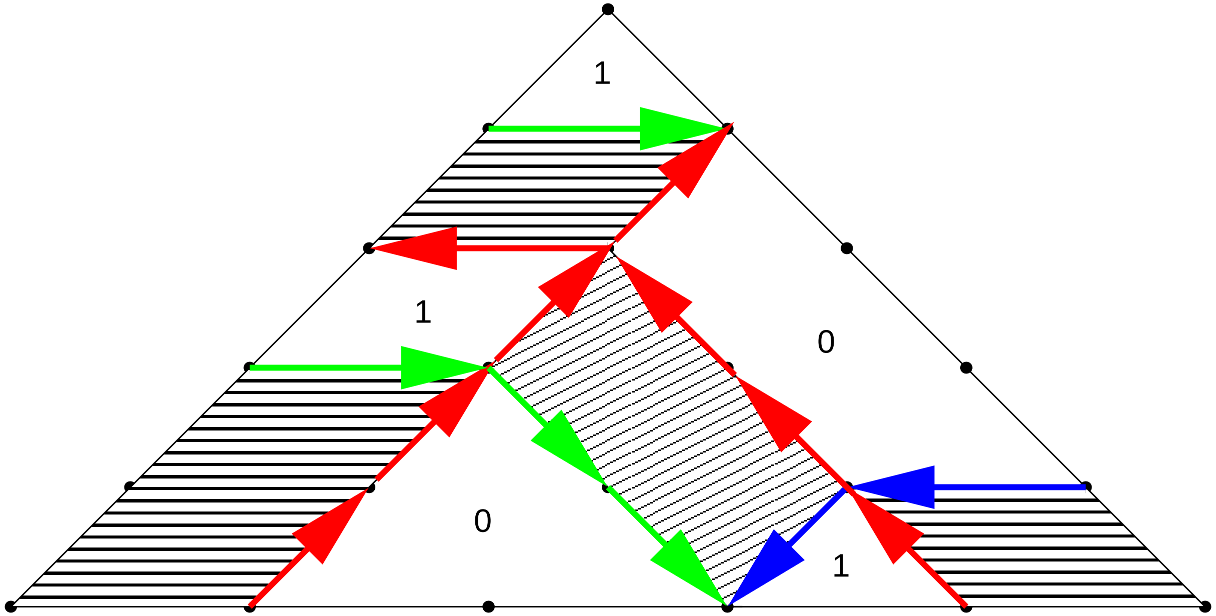


Intersecting paths

- Assume PO good , so that $\beta \geq \alpha$.
- Horn equality $\gamma = \delta$
- QO good $\Rightarrow \gamma \geq \beta \Rightarrow \delta = \gamma \geq \beta \geq \alpha \Rightarrow OS$ good
- OS bad $\Rightarrow \delta < \alpha \Rightarrow \gamma = \gamma = \delta < \alpha \leq \beta \Rightarrow QO$ bad



Union of good paths



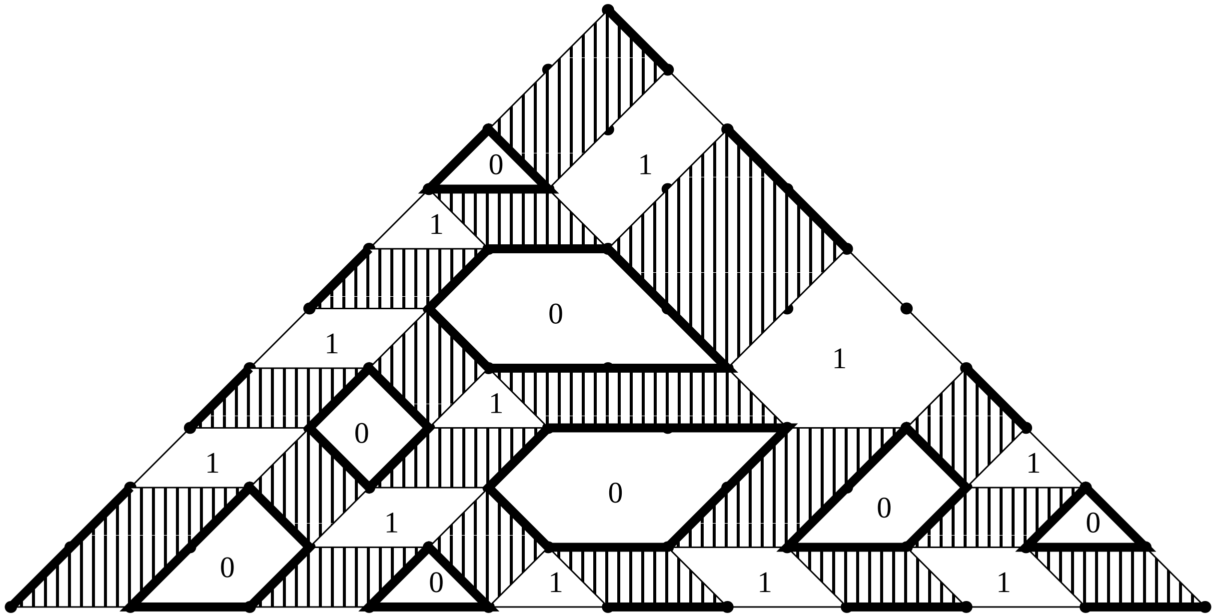
- Good paths cover all interior corridor walls
- All LR hive conditions satisfied
- Hence we have factorisation

Bad edges and gentle loops

- Good paths may not cover all interior corridor edges
- If there exists a bad edge then its predecessor on some gentle path must also be bad
- A reverse gentle path of bad edges cannot reach the boundary, since all gentle paths start from the boundary with a good edge
- A reverse gentle path of bad edges must therefore continue indefinitely
- Since there are only a finite number of edges, it follows that there may only be bad edges if there exists a gentle loop

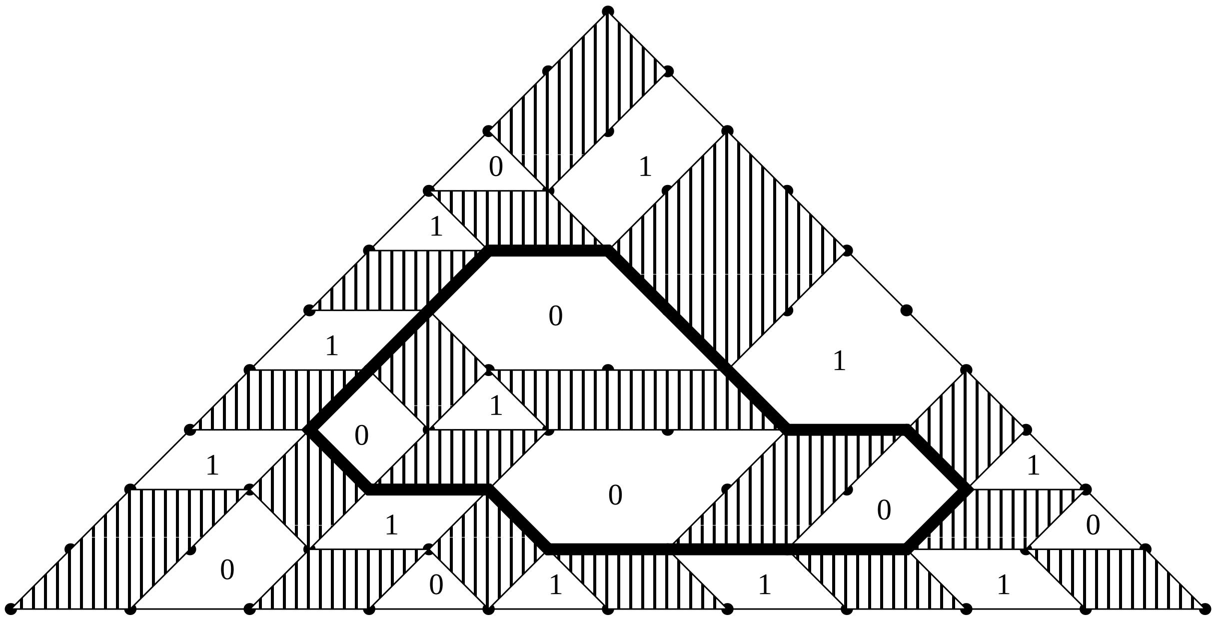
Example exhibiting a gentle loop

- $n = 10$, $r = 5$.
- $I = (1, 2, 4, 6, 8), J = (1, 3, 4, 7, 9), K = (2, 4, 6, 8, 10)$.



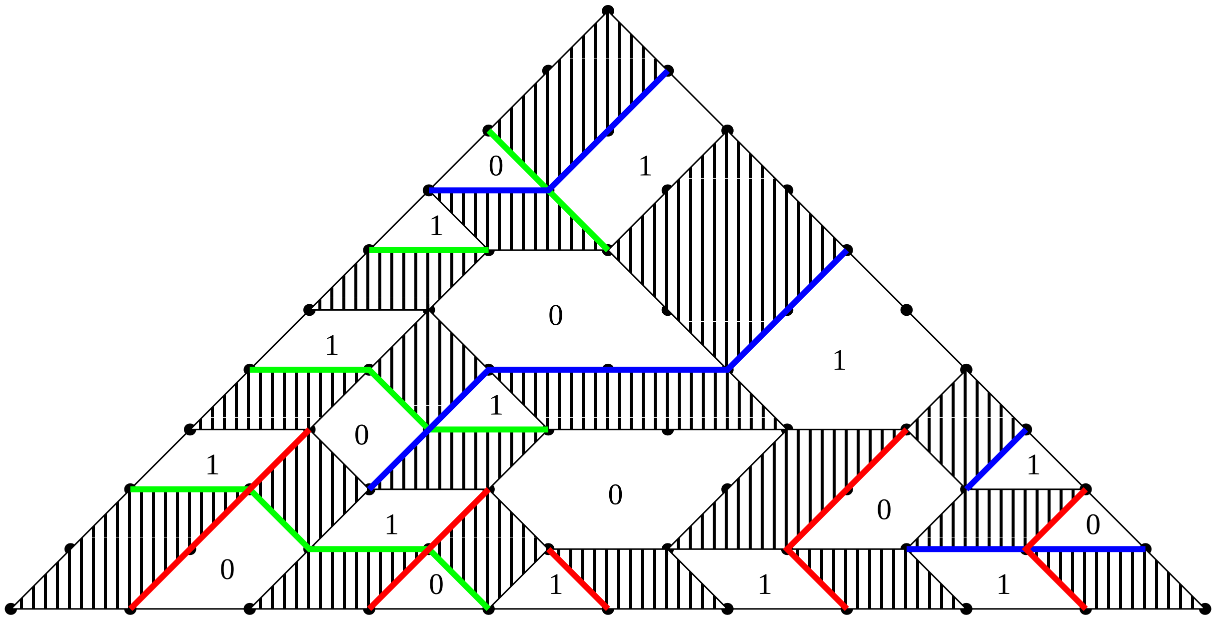
Example of a gentle loop

- Gentle loop is any closed interior gentle path.



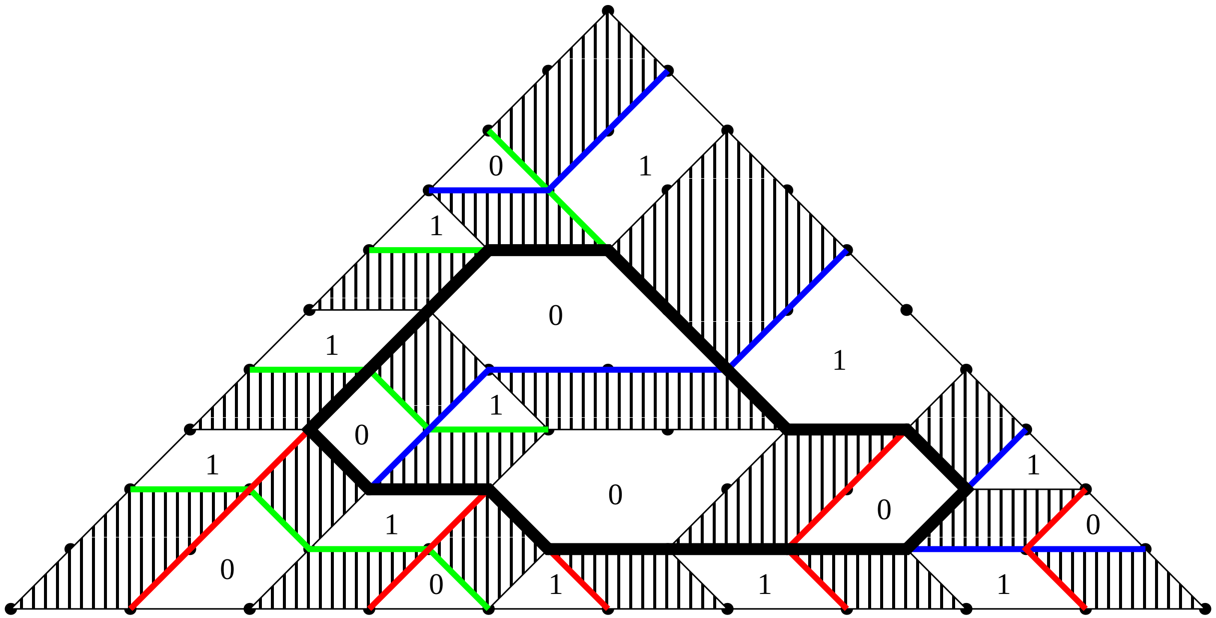
Good paths

- Good paths do not include all interior corridor walls.



Obstruction to good paths

- Good paths do not include all interior corridor walls.



Rigid puzzles and gentle loops

Theorem [KTW 04] The hive plan of a puzzle has no gentle loops if only if the puzzle is rigid

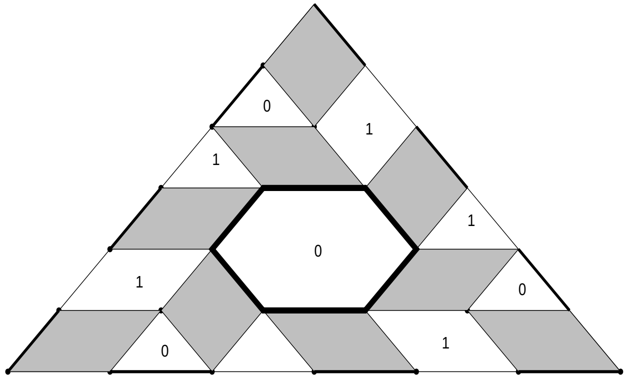
- A puzzle is rigid if and only if the corresponding Horn inequality is essential
- All gentle paths in the hive plan of a rigid puzzle are good, ie have no bad edges
- This implies that in the case of any essential Horn equality, the map from the LR-hives H_n to pairs of LR-hives H_r and H_{n-r} is a bijection
- Hence we have proved:
  Theorem If an essential Horn inequality is saturated then $c_{\lambda\mu}^{\nu}$ factorises.

Example exhibiting a gentle loop

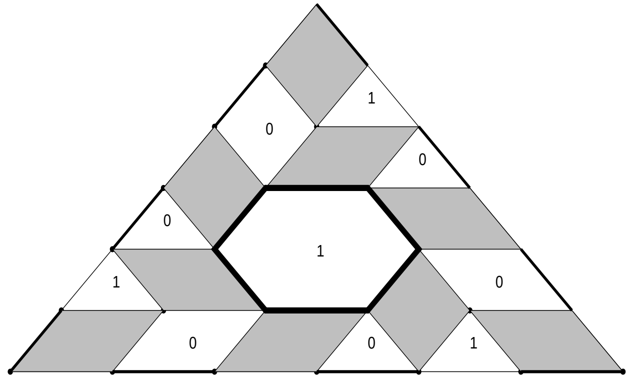
Ex: $n = 6$, $r = 3$, $I = J = (1, 3, 5)$ and $K = (2, 4, 6)$.

There exist two puzzles, each exhibiting a gentle loop

$P1$:



$P2$:



- The corresponding inessential saturated Horn inequality

$$\nu_2 + \nu_4 + \nu_6 = \lambda_1 + \lambda_3 + \lambda_5 + \mu_1 + \mu_3 + \mu_5$$

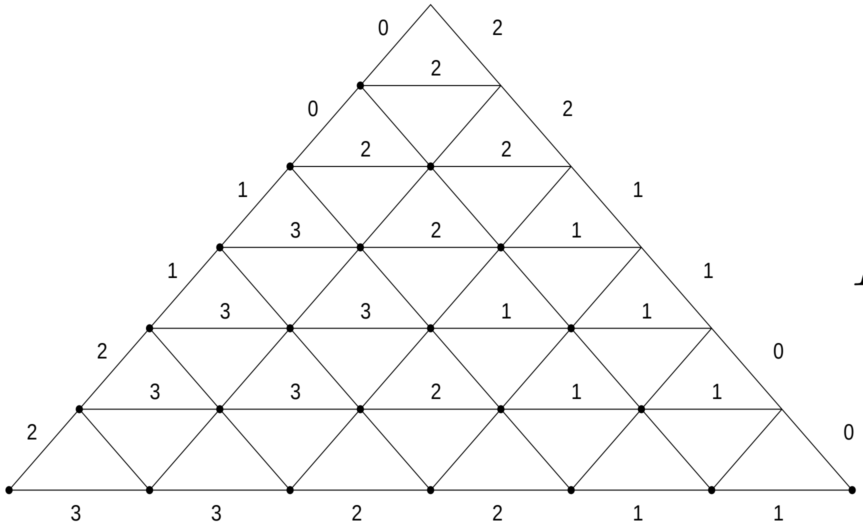
is satisfied by $\lambda = \mu = (221100)$ and $\nu = (332211)$

- There can be no corresponding factorisation since

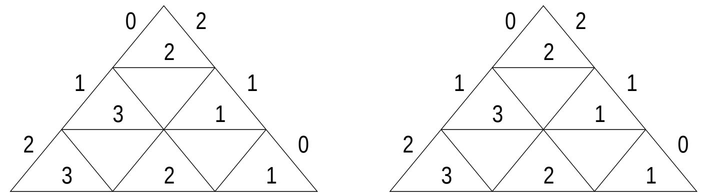
$$c_{\lambda\mu}^{\nu} = c_{221100,221100}^{332211} = 3 \neq 2 \cdot 2 = c_{210,210}^{321}\, c_{210,210}^{321} = c_{\lambda_I \mu_J}^{\nu_K}\, c_{\lambda_{\overline{I}} \mu_{\overline{J}}}^{\nu_{\overline{K}}}$$

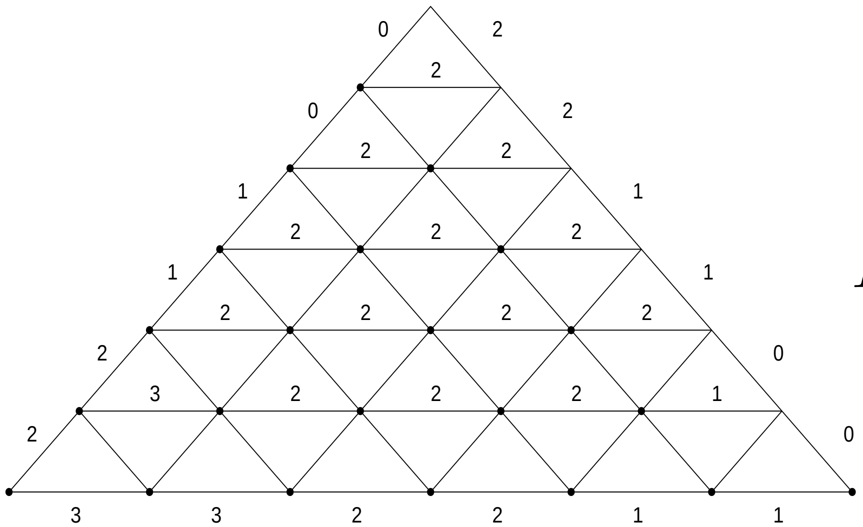
Maps from 6-hives to pairs of 3-hives

The puzzles P_1 and P_2 provide the same maps in the following two cases:

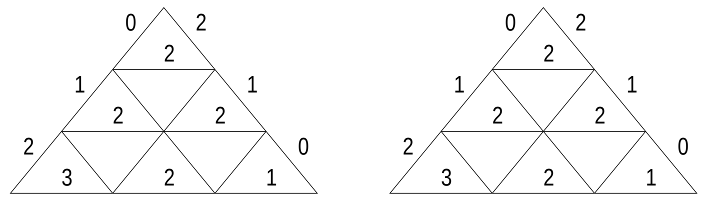


$P1, P2$



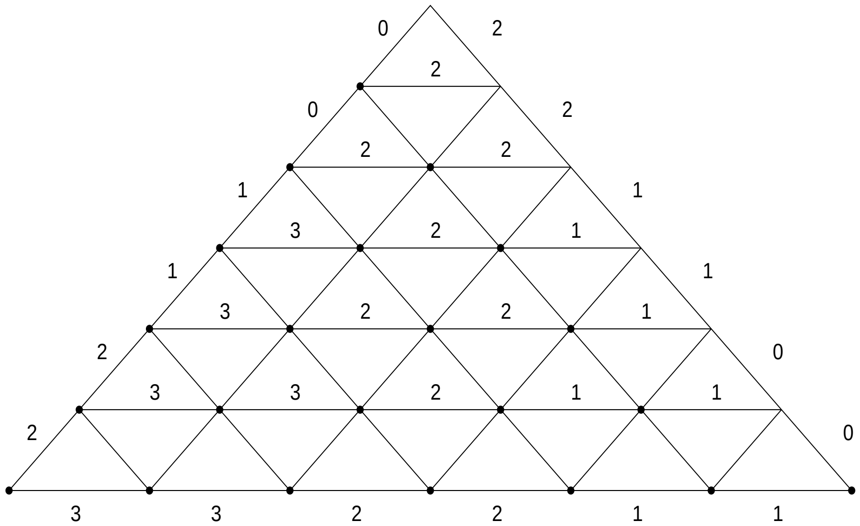


$P1, P2$

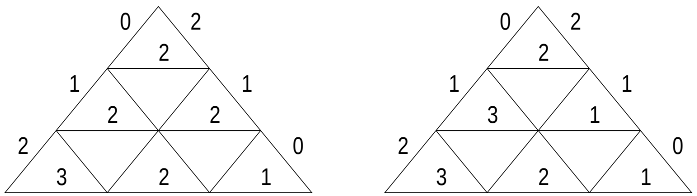


Maps from 6-hives to pairs of 3-hives

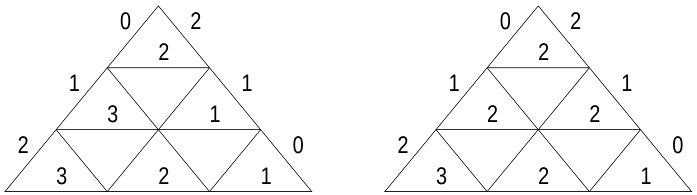
The puzzles P_1 and P_2 provide two different maps in the following case:



$P1$



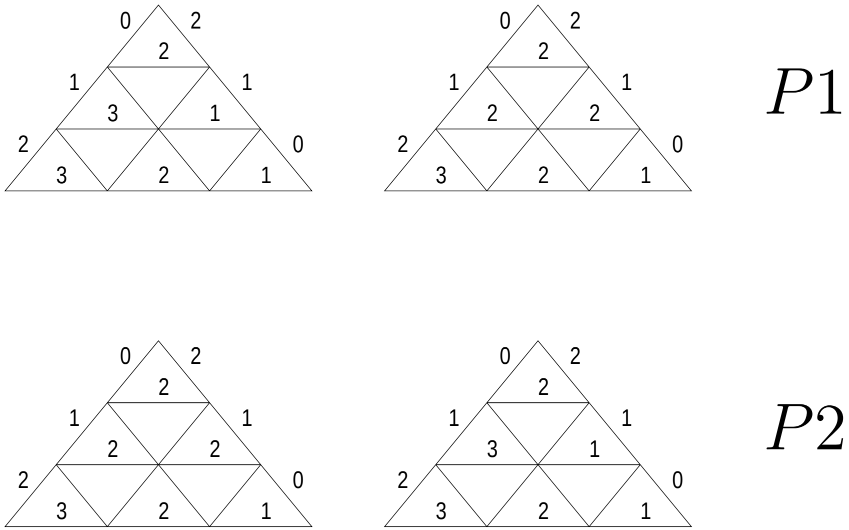
$P2$

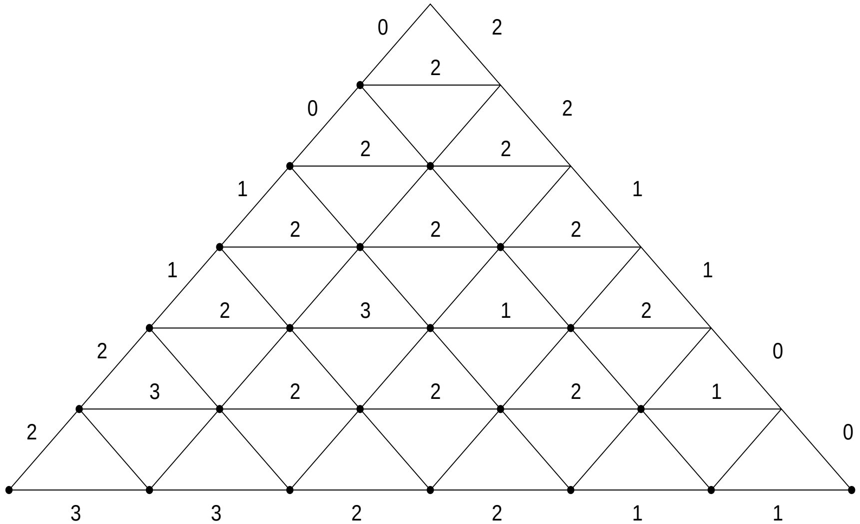


- Thus both P_1 and P_2 map all three possible 6-hives to three pairs of 3-hives
- However there exists four pairs of 3-hives

Maps from a pair of 3-hives to 6-hives

The puzzles P_1 and P_2 provide two different maps:

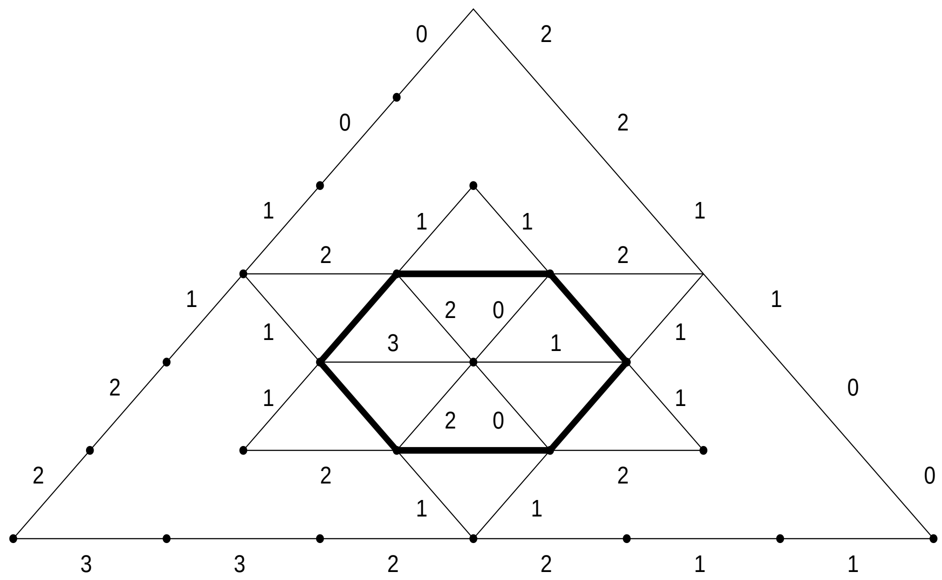




- The resulting 6 hive is not an LR-hive as can be seen from the edges labelled $2\ 3\ 1\ 2$ from left to right across the centre of the hive

A 6-hive exhibiting a bad gentle loop

- Our resulting 6-hive takes the form



- The hive conditions are violated in the case of 12 rhombi
- All edges are bad along the hexagonal gentle loop

Stretched LR coefficients

- Littlewood-Richardson coefficient $c_{\lambda\mu}^{\nu}$
- Partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ stretching parameter $t \in \mathbb{N}$
- Stretched partition $t\lambda = (t\lambda_1, t\lambda_2, \ldots, t\lambda_n)$
- Stretched Littlewood-Richardson coefficient $c_{t\lambda,t\mu}^{t\nu}$

- Ex: $n = 3$, $\lambda = (2, 1, 0)$, $\mu = (3, 2, 0)$, $\nu = (4, 3, 1)$
  - $t = 1$: $c_{21,32}^{431} = 2$
  - $t = 2$: $c_{42,64}^{862} = 3$
  - $t = 3$: $c_{63,94}^{12\,93} = 4$
  - $\cdots$
  - suggests $c_{t\lambda,t\mu}^{t\nu} = t + 1$.

LR coefficients and polynomials

Ex: Let $c^{\nu}_{421,532} = c$ and $c^{t\nu}_{t(421),t(532)} = P(t)$.

$$c = 1 \quad \nu = (953) \quad P(t) = 1$$

$$c = 2 \quad \nu = (9431) \quad P(t) = (t+1)$$

$$c = 3 \quad \nu = (8441) \quad P(t) = (t+1)(t+2)/2$$

$$c = 4 \quad \nu = (8531) \quad P(t) = (t+1)(t+2)(t+3)/6$$

$$c = 4 \quad \nu = (7442) \quad P(t) = (t+1)^2$$

$$c = 5 \quad \nu = (7541) \quad P(t) = (t+1)(t+2)(2t+3)/6$$

$$c = 6 \quad \nu = (7532) \quad P(t) = (t+1)^2(t+2)/2$$

$$c = 7 \quad \nu = (74321) \quad P(t) = (t+1)(t+2)(t^2+3t+6)/6$$

Generating function for LR-polynomials

Ex: Let $F(z) = G(z)/(1-z)^{d+1} = \sum_{t=0}^{\infty} P(t)\, z^t$.

$$c=1 \quad \nu=(953) \quad d=1 \quad G(z)=1$$
$$c=2 \quad \nu=(9431) \quad d=2 \quad G(z)=1$$
$$c=3 \quad \nu=(8441) \quad d=3 \quad G(z)=1$$
$$c=4 \quad \nu=(8531) \quad d=4 \quad G(z)=1$$
$$c=4 \quad \nu=(7442) \quad d=3 \quad G(z)=1+z$$
$$c=5 \quad \nu=(7541) \quad d=4 \quad G(z)=1+z$$
$$c=6 \quad \nu=(7532) \quad d=4 \quad G(z)=1+2z$$
$$c=7 \quad \nu=(74321) \quad d=5 \quad G(z)=1+2z+z^2$$

Further example

Ex: $n = 7,\ \lambda = (433210),\ \mu = (432210),\ \nu = (7444321)$.

- LR coefficient $c^{\nu}_{\lambda\mu} = 13$
- LR polynomial

$$
\begin{aligned}
c^{t\nu}_{t\lambda.t\mu} &= 1/10080 \\
&\times (t+1)(t+2)(t+3)(t+4)(t+5) \\
&\times (5t+21)(t^2+2t+4)
\end{aligned}
$$

- where $10080 = 5!\ 84$
- $d = 8$ and $G(z) = 1 + 4z + 12z^2 + 3z^3$

Polynomial behaviour

Theorem For all λ, μ, ν such that $c^{\nu}_{\lambda\mu} > 0$ there exists

- a polynomial $P^{\nu}_{\lambda\mu}(t)$ in t with $P^{\nu}_{\lambda\mu}(0) = 1$
- such that $P^{\nu}_{\lambda\mu}(t) = c^{t\nu}_{t\lambda,t\mu}$ for all positive integers t.

Conjectures

- coefficients in $P^{\nu}_{\lambda\mu}(t)$ are all rational and non-negative.
- coefficients in $G(z)$ are all positive integers.

Problems

- predict degree of polynomial
- explain origin of factors of form $(t+1)(t+2)\cdots(t+m)$
- prove (if true) and account for positivity of coefficients

The hive model and convex polytopes

- An LR n-hive with fixed boundary labels specified by λ, μ, ν involves $m = (n-1)(n-2)/2$ interior vertex labels a_{ij}
- The hive conditions for any rhombus take the form $a + d \leq b + c$
- These form a set of linear constraints with integer coefficients on the m interior vertex labels
- They define a rational convex polytope $\mathcal{P}$ in $\mathbb{R}^m$ known as the hive polytope
- The LR-coefficient is given by $c^{\nu}_{\lambda\mu} = \#\{\mathcal{P} \cap \mathbb{Z}^m\}$, the number of points of intersection of the hive polytope with the integer lattice

The hive model and convex polytopes

- Let $\mathcal{P} \in \mathbb{R}^m$ be a convex rational polytope of dimension $d \leq m$
- Such a convex rational polytope $\mathcal{P} \in \mathbb{R}^m$ is an integer polytope if all of its vertices lie in $\mathbb{Z}^m$
- Let $t\mathcal{P} = \{tv \,|\, v \in \mathcal{P}\}$ and $i(\mathcal{P},t) = \#\{t\,\mathcal{P} \cap \mathbb{Z}^m\}$, the number of points of intersection of the stretched polytope $t\,\mathcal{P}$ with the integer lattice
- The stretched LR-coefficient, which may or may not be polynomial, is given by $c_{t\lambda,\,t\mu}^{t\nu} = i(\mathcal{P},t)$

The hive model and convex polytopes

Theorem [Ehrhart 77]

Let $\mathcal{P} \in \mathbb{R}^m$ be a rational polytope of dimension $d \leq m$.
Then $i(\mathcal{P}, t)$ is a quasi-polynomial of degree d in t,
ie. there exists $r \geq 1$ and polynomials $P_s(t)$ of
degree d in t such that $i(\mathcal{P}, t) = P_s(t)$ for $t \equiv s (\operatorname{mod} r)$.

If $\mathcal{P} \in \mathbb{R}^m$ is an integer polytope, then $i(\mathcal{P}, t)$ is a
polynomial in t of degree d.

Corollary

If a hive polytope is an integer polytope, then the
corresponding stretched Littlewood-Richardson
coefficients $c_{t\lambda, t\mu}^{t\nu}$ are polynomial in t.

Construction of hive polytopes

- Let $m = (n-2)(n-1)/2 \; = \; \#$ interior points of an n-hive
- Let $v = (a_{11}, a_{12}, ...) \in \mathbb{R}^m$ be vector of interior labels

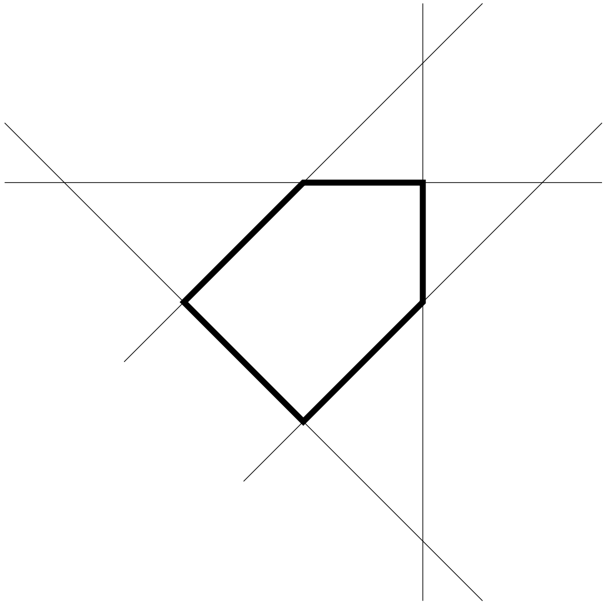
Ex: $\lambda = (753)$, $\mu = (742)$, $\nu = (9964)$, $n = 4$, $m = 3$,

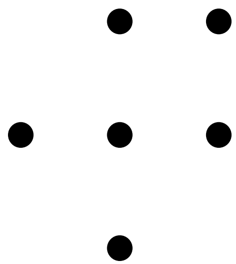
$$
\begin{array}{ccccccccc}
 & & & & 15 & & & & \\
 & & & 15 & & 22 & & & \\
 & & 12 & & b & & 26 & & \\
 & 7 & & a & & c & & 28 & \\
0 & & 9 & & 18 & & 24 & & 28
\end{array}
$$

- $v = (a_{11}, a_{12}, a_{21}) = (a, b, c)$

Construction of hive polytopes

- There are 6 LR-hives, with $a = 16$ in all cases and $(b, c) = (19, 23), (20, 22), (20, 23), (20, 24), (21, 23), (21, 24)$
- The hive polytope $\mathcal{P}$ is $d = 2$-dimensional and takes the form

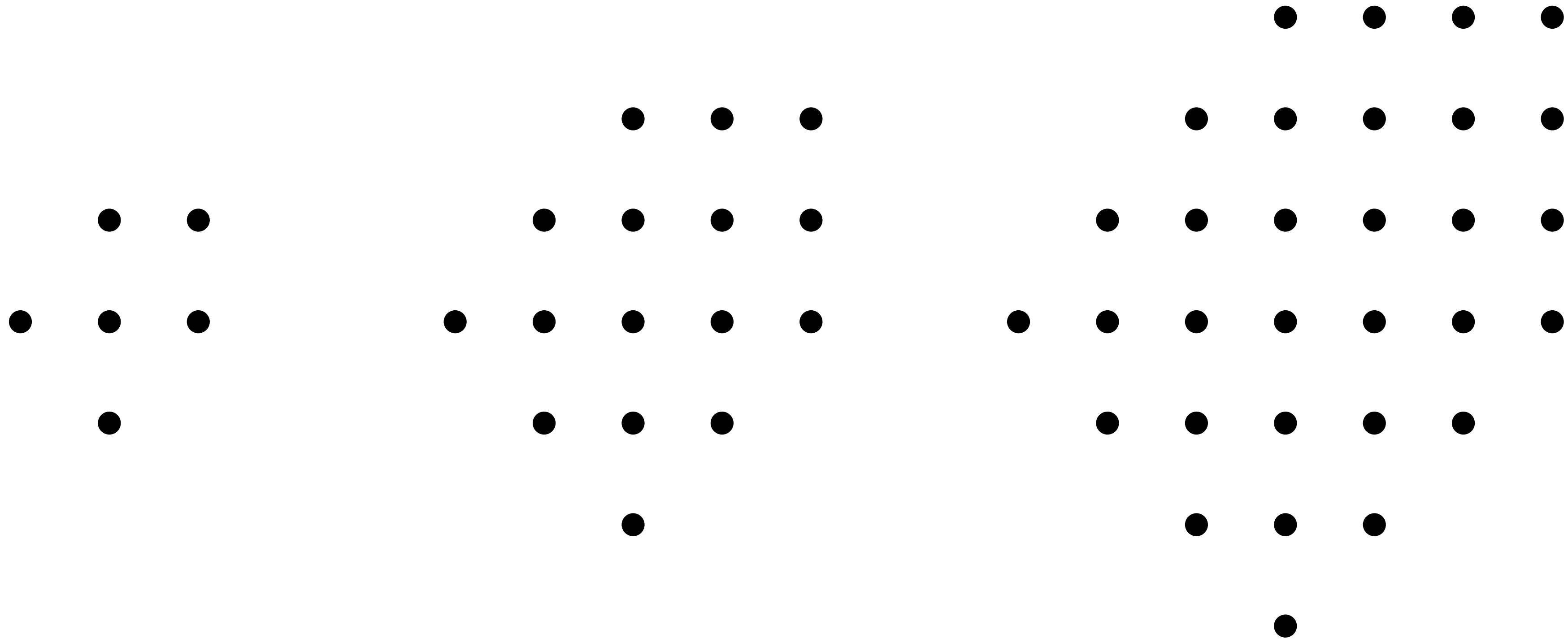


with integer points 

- $\mathcal{P}$ has 5 integer vertices and just one interior integer point

Scaling convex polytope

- Expand $\mathcal{P}$ by scaling with t
- Identify and count all integer points to give $\mathcal{P}(t)$



- $P(1) = 6$, $P(2) = 16$, $P(3) = 31$, $\cdots$, $P(t) = \frac{1}{2}(5t^2 + 5t + 2)$.

Not all hive polytopes are integer polytopes

Ex: For $n = 5$, $\lambda = (32000)$, $\mu = (43210)$ and $\nu = (54321)$ all hives take the form

$$
\begin{array}{ccccccccccc}
 & & & & & 5 & & & & & \\
 & & & & 5 & & 9 & & & & \\
 & & & 5 & & 9 & & 12 & & & \\
 & & 5 & & 9 & & 12 & & 14 & & \\
 & 3 & & a & & b & & c & & 15 & \\
0 & & 5 & & 9 & & 12 & & 14 & & 15
\end{array}
$$

- The hive conditions fix three interior vertex labels
- The hive polytope $\mathcal{P}$ is of dimension 3
- There are 5 distinct LR-hives of this type, so that $c^{\nu}_{\lambda\mu} = 5$

Not all hive polytopes are integer polytopes

- The vertices $v = (a, b, c)$ of $\mathcal{P}$ are given by

$$
\begin{aligned}
v_1 &= (7, 11, 13) \\
v_2 &= (7, 11, 14) \\
v_3 &= (8, 11, 13) \\
v_4 &= (8, 11, 14) \\
v_5 &= (8, 12, 14) \\
v_6 &= (\tfrac{15}{2}, \tfrac{21}{2}, \tfrac{27}{2}) = (7.5, 10.5, 13.5)
\end{aligned}
$$

- The first 5 vertices are integral and specify the 5 LR-hives
- The sixth vertex v_6 is not integral. The hive polytope is rational but not integer.

Not all hive polytopes are integer polytopes

Ex: The hive corresponding to the polytope vertex v_6 takes the form

$$\begin{array}{ccccccccccc}
 & & & & & 5 & & & & & \\
 & & & & 5 & & 9 & & & & \\
 & & & 5 & & 9 & & 12 & & & \\
 & & 5 & & 9 & & 12 & & 14 & & \\
 & 3 & & \frac{15}{2} & & \frac{21}{2} & & \frac{27}{2} & & 15 & \\
0 & & 5 & & 9 & & 12 & & 14 & & 15
\end{array}$$

- This hive not an integer hive, let alone an LR-hive
- In this case Ehrhart's Theorem only asserts that $c_{t\lambda,\,t\mu}^{t\nu}$ is quasi-polynomial in t, not polynomial.

Polynomial behaviour

Ex: For $n = 5$, $\lambda = (32000)$, $\mu = (43210)$ and $\nu = (54321)$ we find the following data on $P(t) = c^{t\nu}_{t\lambda,\,t\mu}$.

- $t = 1$: $P(1) = 5$
- $t = 2$: $P(2) = 15$
- $t = 3$: $P(3) = 34$
- $t = 4$: $P(3) = 65$
- $\cdots$
- suggests $P(t) = c^{t\nu}_{t\lambda,\,t\mu} = (t+1)(t^2+2t+2)/2$

Theorem [Rassart ??]
All stretched LR-coefficients $c^{t\nu}_{t\lambda,\,t\mu}$ are polynomial in t.

Summary

- Theorem

  The LR-coefficient $c^{\nu}_{\lambda\mu}$ is the number of LR-hives with boundary labels determined by λ, μ and ν.

- Corollary

  The LR-polynomial $P^{\nu}_{\lambda\mu}(t)$ can be identified as the Ehrhart quasi-polynomial $i(\mathcal{P},t) = \#\{t\mathcal{P} \cap \mathbb{Z}^m\}$, of a rational convex polytope $\mathcal{P}$ defined by the LR-hive boundary conditions and the set of LR-hive inequalities: $a + d \leq b + c$ for each rhombus.

- Note: Even though $\mathcal{P}$ may be rational but not integer the Ehrhart quasi-polynomial $i(\mathcal{P},t)$ is polynomial.

Linear factors

Origin of some linear factors in LR-polynomials.

- Let $\mathcal{P}$ be an LR hive polytope, and $\overline{\mathcal{P}}$ its interior.
- For $t \in \mathbb{N}$: $P^{\nu}_{\lambda\mu}(t) = i(\mathcal{P}, t) = \#\{t\mathcal{P} \cap \mathbb{Z}^d\}$.
- Ehrhart reciprocity: $i(\mathcal{P}, -t) = (-1)^d \#\{t\overline{\mathcal{P}} \cap \mathbb{Z}^d\}$.
- For $m \in \mathbb{N}$: $P^{\nu}_{\lambda\mu}(-m) = i(\mathcal{P}, -m) = (-1)^d \#\{m\overline{\mathcal{P}} \cap \mathbb{Z}^d\}$.
- Hence $P^{\nu}_{\lambda\mu}(-m) = 0$ and $P^{\nu}_{\lambda\mu}(t)$ contains a factor $(t+m)$ if and only if $m\mathcal{P}$ contains no interior integer points.

Corollary $P^{\nu}_{\lambda\mu}(t)$ contains $(t+1)(t+2)\cdots(t+m)$ as a factor if $m\mathcal{P}$ contains no interior integer points.

Problem: predict maximum value of m.

Construction of convex polytopes

Ex: $\lambda = (210)$, $\mu = (320)$, $\nu = (431)$, $n = 3$, $d = 1$

$$\begin{array}{ccccccc} & & & 5 & & & \\ & & 5 & & 7 & & \\ & 3 & & a & & 8 & \\ 0 & & 4 & & 7 & & 8 \end{array}$$

with $a = 6, 7$

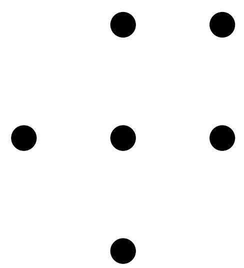
- $\mathcal{P} \cap \mathbb{Z} =$ $\bullet \quad \bullet$ no interior points
- $2\mathcal{P} \cap \mathbb{Z} =$ $\bullet \quad \bullet \quad \bullet$ one interior point
- implies $P(t)$ contains a factor $(t+1)$ but no factor $(t+2)$. In fact $P(t) = (t+1)$.

Construction of convex polytopes

Ex: $\lambda = (753)$, $\mu = (742)$, $\nu = (9964)$, $n = 4$, $d = 2$

$$\begin{array}{ccccccccc} & & & & 15 & & & & \\ & & & 15 & & 22 & & & \\ & & 12 & & b & & 26 & & \\ & 7 & & 16 & & c & & 28 & \\ 0 & & 9 & & 18 & & 24 & & 28 \end{array}$$

with $(b,c) = \begin{cases} (21,24) & (21,23) \\ (20,24) & (20,23) \\ (20,22) & (19,23) \end{cases}$

- $\mathcal{P} \cap \mathbb{Z}^2$:  one interior point
- implies no factor $(t+m)$. In fact $P(t) = \frac{1}{2}(5t^2 + 5t + 2)$.

Degrees of LR-polynomials

- For $c^{\nu}_{\lambda\mu} > 0$ the LR-rule implies $\ell(\lambda), \ell(\mu) \leq \ell(\nu)$.
- $c^{\nu}_{\lambda\mu}$ is the number of LR n-hives with $n = \ell(\nu)$, boundary labels linear in the parts of λ, μ, ν, interior vertex labels subject to linear inequalities (HCs).
- For $t \in \mathbb{N}$, $P^{\nu}_{\lambda\mu}(t)$ is the number of scaled LR n-hives with boundary labels scaled by t and interior vertex labels subject to the same scaled linear inequalities.
- The range of each vertex label is at most linear in t.
- An n-hive has $(n-1)(n-2)/2$ interior vertices.

Degree bound $\deg P^{\nu}_{\lambda\mu}(t) \leq (n-1)(n-2)/2$ with $n = \ell(\nu)$.

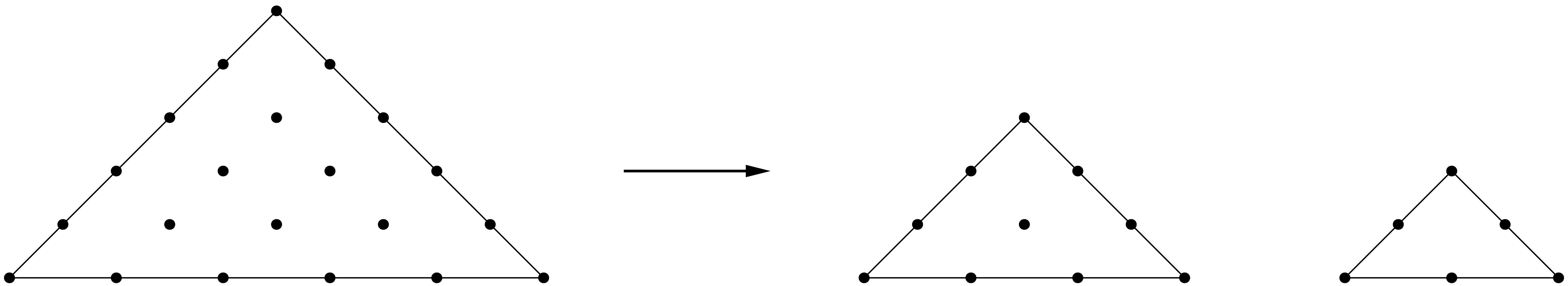
First example

Ex: $n = 5$, degree bound $(n-1)(n-2)/2 = 6$.

- $\lambda = (9,7,6,2,0)$, $\mu = (13,5,3,1,0)$, $\nu = (14,12,11,5,4)$.
- $P^{\nu}_{\lambda\mu}(t) = (t+1)$ so that $\deg P^{\nu}_{\lambda\mu}(t) = 1$.

Origin of mismatch - factorisation

- $P^{\nu}_{\lambda\mu}(t) = P^{\nu_K}_{\lambda_I\,\mu_J}(t)\; P^{\nu_{\overline{K}}}_{\lambda_{\overline{I}}\,\mu_{\overline{J}}}(t)$.
- LR-hives for $n = 5$ are fixed by two smaller subhives of sizes $r = 3$ and $n - r = 2$.



LR factorisation example

Ex: $n = 5$, $r = 3$, $n - r = 2$:

- $\lambda = (9, 7, 6, 2, 0)$, $\mu = (13, 5, 3, 1, 0)$, $\nu = (14, 12, 11, 5, 4)$.
- $I = \{1, 2, 4\}$, $J = \{2, 3, 4\}$, $K = \{2, 3, 5\}$.
- $\lambda_I = (9, 7, 2)$, $\mu_J = (5, 3, 1)$, $\nu_K = (12, 11, 4)$
- $\lambda_{\overline{I}} = (6, 0)$, $\mu_{\overline{J}} = (13, 0)$, $\nu_{\overline{K}} = (14, 5)$

LR-coefficient:

$$c^{(14,12,11,5,4)}_{(9,7,6,2,0),(13,5,3,1,0)} = c^{(12,11,4)}_{(9,7,2),(5,3,1)} \; c^{(14,5)}_{(6,0),(13,0)} = 2 \cdot 1 = 2.$$

LR-polynomial:

$$P^{(14,12,11,5,4)}_{(9,7,6,2,0),(13,5,3,1,0)}(t) = P^{(12,11,4)}_{(9,7,2),(5,3,1)}(t) \; P^{(14,5)}_{(6,0),(13,0)}(t)$$

$$= (t+1) \cdot 1 = (t+1).$$

Degree bound for a primitive example

Ex: $n = 6$, degree bound $(n-1)(n-2)/2 = 10$.

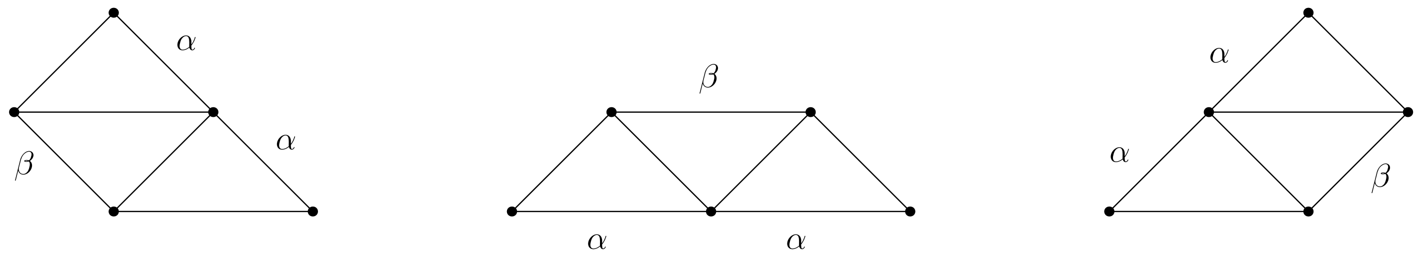
- $\lambda = (4,3,3,1,0,0)$, $\mu = (4,2,1,1,1,0)$, $\nu = (6,5,4,2,2,1)$.
- $P^{\nu}_{\lambda\mu}(t) = (t+1)(t+2)(t+3)(t+4)/24$.
- $\deg P^{\nu}_{\lambda\mu}(t) = 4$.

- All essential Horn inequalities are strict.
- The LR-polynomial does not factorise.
- The factorised degree bound is not saturated.

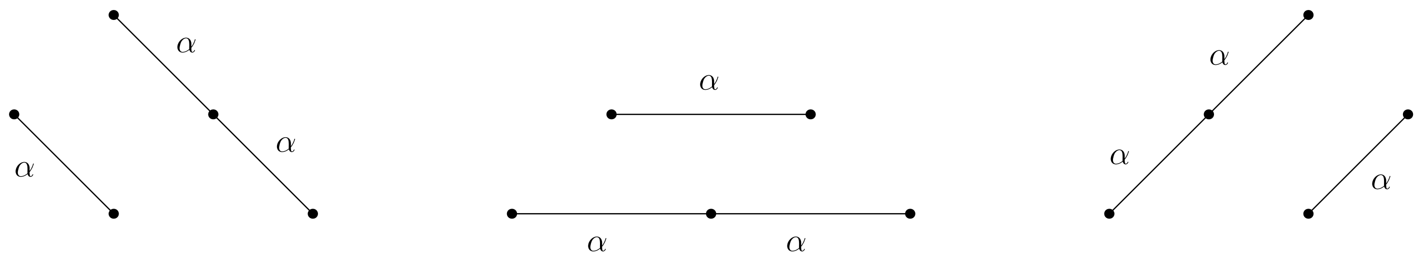
Origin of mismatch - partitions have equal parts.

Five-vertex equal edge constraints

- Equal edge constraints on 5-vertex subdiagrams
- In each case $\alpha \geq \beta \geq \alpha$ so that $\beta = \alpha$.

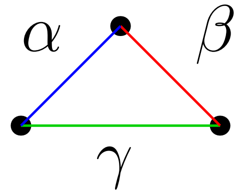


- Consecutive equal edges force neighbouring equal edge.
- Retain skeleton consisting of only equal edges.



Skeleton of an LR-hive and degree bounds

- Apply 5-vertex equal edge procedure to LR n-hive.
- Work inwards from boundaries specified by λ, μ, ν.
- Invoke triangular hive condition $\alpha + \beta = \gamma$:



- Result is skeletal graph $G_{n;\lambda\mu\nu}$ of hive.
- Let $d(G_{n;\lambda\mu\nu})$ be number of components of $G_{n;\lambda\mu\nu}$ not connected to the boundary.

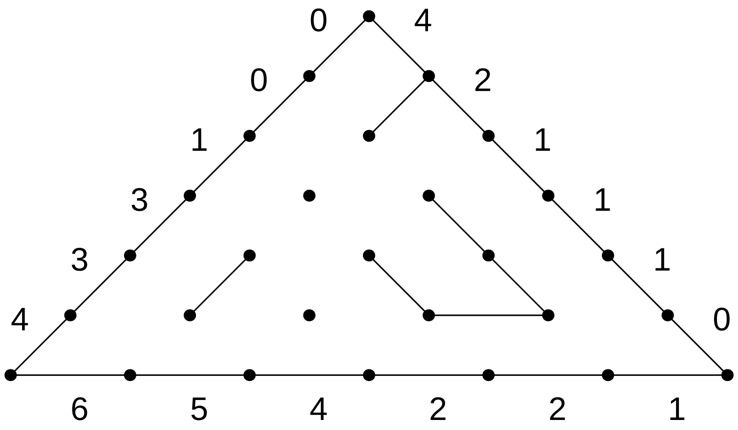
Theorem $\deg P^{\nu}_{\lambda\mu}(t) \leq d(G_{n;\lambda\mu\nu})$.

Conjecture If $P^{\nu}_{\lambda\mu}(t)$ is primitive then $\deg P^{\nu}_{\lambda\mu}(t) = d(G_{n;\lambda\mu\nu})$.

Skeleton graph degree bound

Theorem $\deg P^{\nu}_{\lambda\mu}(t) \leq d(G_{n;\lambda\mu\nu})$.

Ex: $n = 6$, $\lambda = (433100)$, $\mu = (421110)$, $\nu = (654221)$.



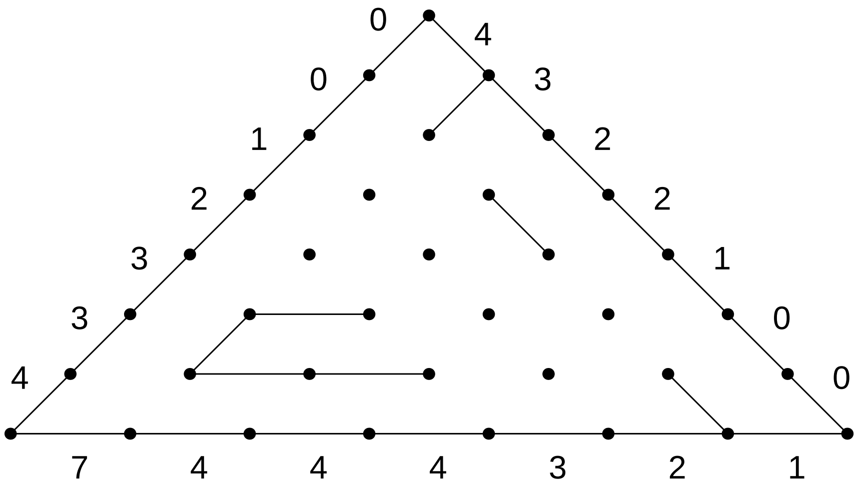
- $P^{\nu}_{\lambda\mu}(t) = (t+1)(t+2)(t+3)(t+4)/24$.
- $\deg P^{\nu}_{\lambda\mu}(t) = 4 = d(G_{n;\lambda\mu\nu})$.

Skeleton graph degree bound is saturated.

Degree of LR-polynomial

Ex: $n = 7$, $\lambda = (4332100)$, $\mu = (4322100)$, $\nu = (7444321)$.

- $P^{\nu}_{\lambda\mu}(t) = (t+1)(t+2)(t+3)(t+4)(t+5)$
  $\times (5t+21)(t^2+2t+4)/10080.$



- $\deg P^{\nu}_{\lambda\mu}(t) = 8 = d(G_{n;\lambda\mu\nu})$.

Degree of primitive LR-polynomial

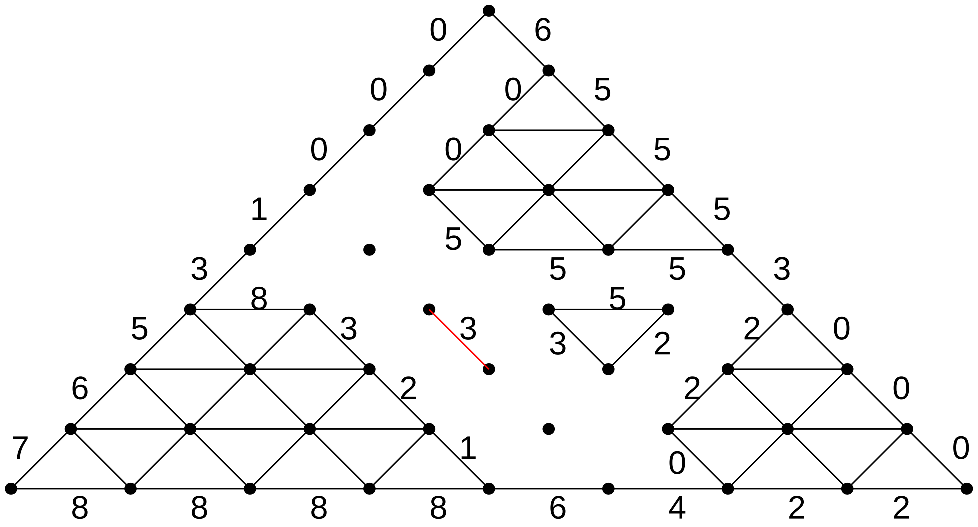
Ex: $n = 6$, $\lambda = (221100)$, $\mu = (221100)$, $\nu = (332211)$.

- $P^{\nu}_{\lambda\mu}(t) = \frac{1}{2}(t+1)(t+2)$.
- Now construct skeleton - multi-stage process
- $\deg P^{\nu}_{\lambda\mu}(t) = 2 = d(G_{n;\lambda\mu\nu})$.

Counterexample to skeleton degree bound

Ex: $n = 8$, $\lambda = (76531000)$, $\mu = (65553000)$, $\nu = (88886422)$.

- $P_{\lambda\mu}^{\nu}(2) = (t+1)(t+2)(t+3)(t+4)/24$
- Now construct skeleton



- $\deg P_{\lambda\mu}^{\nu}(t) = 4 < 5 = d(G_{n;\lambda\mu\nu})$.

Linear factors

Origin of linear factors in LR-polynomials.

- Let $\mathcal{P}$ be an LR hive polytope, and $\overline{\mathcal{P}}$ its interior.
- For $t \in \mathbb{N}$: $P^{\nu}_{\lambda\mu}(t) = i(\mathcal{P}, t) = \#\{t\mathcal{P} \cap \mathbb{Z}^d\}$.
- Ehrhart reciprocity: $i(\mathcal{P}, -t) = (-1)^d \#\{t\overline{\mathcal{P}} \cap \mathbb{Z}^d\}$.
- For $m \in \mathbb{N}$: $P^{\nu}_{\lambda\mu}(-m) = i(\mathcal{P}, -m) = (-1)^d \#\{m\overline{\mathcal{P}} \cap \mathbb{Z}^d\}$.
- Hence $P^{\nu}_{\lambda\mu}(-m) = 0$ and $P^{\nu}_{\lambda\mu}(t)$ must contain a linear factor $(t + m)$ if $m\mathcal{P}$ contains no interior integer points.

Anticipate: $P^{\nu}_{\lambda\mu}(t)$ may contain $(t + 1)(t + 2) \cdots (t + M)$.

Problem: Determine M.

Possible continuation in t

- For $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ let $\overline{\mathbf{x}} = (\overline{x}_1, \overline{x}_2, \ldots, \overline{x}_n)$ with $\overline{x}_i = x_i^{-1}$ for $i = 1, 2, \ldots, n$.
- For $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ let $\tilde{\lambda} = (\lambda_n, \ldots, \lambda_2, \lambda_1)$.
- $s_{t\lambda}(\mathbf{x}) = \dfrac{\left| x_i^{t\lambda_j+n-j} \right|}{\left| x_i^{n-j} \right|} \implies s_{-m\lambda}(\mathbf{x}) = \dfrac{\left| x_i^{-m\lambda_j+n-j} \right|}{\left| x_i^{n-j} \right|}$.
- This gives $s_{-m\lambda}(\mathbf{x}) = \dfrac{\left| \overline{x}_i^{m\lambda_{n-k+1}+n-k} \right|}{\left| \overline{x}_i^{n-k} \right|} = s_{m\tilde{\lambda}}(\overline{\mathbf{x}})$.

Definition For $c_{\lambda\mu}^{\nu} > 0$ and any positive integer m, let

$$c_{-m\lambda,-m\mu}^{-m\nu} = c_{m\tilde{\lambda},m\tilde{\mu}}^{m\tilde{\nu}}.$$

LR polynomials for negative t

Conjecture: Let $c^{\nu}_{\lambda\mu} > 0$ be simple, all Horn inequalities strict,

then $$P^{\nu}_{\lambda\mu}(-m) = c^{m\tilde{\nu}}_{m\tilde{\lambda},m\tilde{\mu}},$$

where $s_{m\tilde{\lambda}}(x)\, s_{m\tilde{\mu}}(x) = \sum_{\nu} c^{m\tilde{\nu}}_{m\tilde{\lambda},m\tilde{\mu}}\, s_{m\tilde{\nu}}(x)$.

Standardization:

- $s_{m\tilde{\lambda}}(\mathbf{x}) = 0$ or $\pm s_{\rho}(\mathbf{x})$ for some partition ρ.
- $s_{m\tilde{\mu}}(\mathbf{x}) = 0$ or $\pm s_{\sigma}(\mathbf{x})$ for some partition σ.
- $s_{m\tilde{\nu}}(\mathbf{x}) = 0$ or $\pm s_{\tau}(\mathbf{x})$ for some partition τ.

Two types of zero:

- $s_{m\tilde{\lambda}}(\overline{x}) = 0$, $s_{m\tilde{\mu}}(\overline{x}) = 0$, $s_{m\tilde{\nu}}(\overline{x}) = 0$.
- $c^{\tau}_{\rho\sigma} = 0$.

Simple example

Ex: $n = 7$, $\lambda = (433210)$, $\mu = (432210)$, $\nu = (7444321)$.

- $P^{\nu}_{\lambda\mu}(t) = (t+1)(t+2)(t+3)(t+4)(t+5)$
$$\cdot (5t+21)(t^2+2t+4)/10080.$$

Type one zeros for $m = 1, 2, 3$ since:

- $s_{m\tilde{\lambda}}(\overline{x}) = s_{m\tilde{\mu}}(\overline{x}) = 0$ for $m = 1, 2$.
- $s_{m\tilde{\nu}}(\overline{x}) = 0$ for $m = 1, 2, 3$.

Type two zeros for $m = 4, 5$ since:

- $c^{\tau}_{\rho\sigma} = 0$ for $m = 4, 5$.

No more zeros for $m > 5$ since for $m = 6$: $c^{\tau}_{\rho\sigma} = 3$.

Simple and non-simple examples

Simple: $n = 7$, $\lambda = (433210)$, $\mu = (432210)$, $\nu = (7444321)$.

- $P^{\nu}_{\lambda\mu}(t) = (t+1)(t+2)(t+3)(t+4)(t+5)$
  $\cdot (5t+21)(t^2+2t+4)/10080$.
- $c^{m\tilde{\nu}}_{m\tilde{\lambda},m\tilde{\mu}} = 0, 0, 0, 0, 0, 3, 39, 247$ for $m = 1, 2, 3, 4, 5, 6, 7, 8$.
- $P^{\nu}_{\lambda\mu}(-m) = 0, 0, 0, 0, 0, 3, 39, 247$ for $m = 1, 2, 3, 4, 5, 6, 7, 8$.

Non-simple: $n = 6$, $\lambda = (221100)$, $\mu = (221100)$, $\nu = (332211)$.

- $P^{\nu}_{\lambda\mu}(t) = (t+1)(t+2)/2$.
- $c^{m\tilde{\nu}}_{m\tilde{\lambda},m\tilde{\mu}} = 0, 0, 0, 3, 6,$ for $m = 1, 2, 3, 4, 5$.
- $P^{\nu}_{\lambda\mu}(-m) = 0, 0, 3, 6, 10$ for $m = 1, 2, 3, 4, 5$.

Non-primitive example

Non-primitive: $n = 5$, $\lambda = (9,7,6,2,0)$, $\mu = (13,5,3,1,0)$, $\nu = (14,12,11,5,4)$, $P^{\nu}_{\lambda\mu}(t) = (t+1)$.

- $c^{m\tilde{\nu}}_{m\tilde{\lambda},m\tilde{\mu}} = 0,0,0,\ldots$ for $m = 1,2,3,\ldots$.
- $P^{\nu}_{\lambda\mu}(-m) = 0,1,2,\ldots$ for $m = 1,2,3,\ldots$.

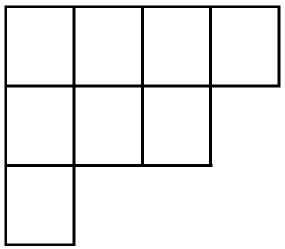
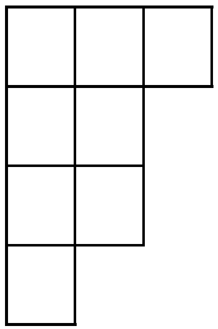
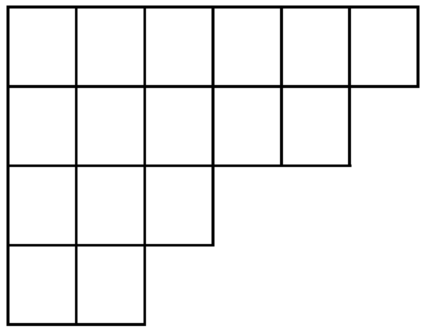
Primitive factors

- $n = 3$, $\lambda_I = (9,7,2)$, $\mu_J = (5,3,1)$, $\nu_K = (12,11,4)$.
- $n = 2$, $\lambda_{\overline{I}} = (6,0)$, $\mu_{\overline{J}} = (13,0)$, $\nu_{\overline{K}} = (14,5)$.

Conjecture: $c^{\nu}_{\lambda\mu} > 0$ is primitive, all essential Horn inequalities strict, if and only if $c^{m\tilde{\nu}}_{m\tilde{\lambda},m\tilde{\mu}} \neq 0$ for some positive integer m.

Some symmetries of LR-coefficients

- Given a partition $\lambda \subseteq m^n$, so that $\ell(\lambda) \leq n$ and $\lambda_1 \leq m$
- Let λ' denote its conjugate
- Let $\tilde{\lambda}$ denote its m^n-complement
  where $\tilde{\lambda} = (m - \lambda_n, \ldots, m - \lambda_2, m - \lambda_1)$
- Ex: If $m = 6$, $n = 4$ and $\lambda = (4, 3, 1, 0)$
  then $\lambda' = (3, 2, 2, 1)$ and $\tilde{\lambda} = (6, 5, 3, 2)$

$F^{\lambda} =$  $F^{\lambda'} =$  $F^{\tilde{\lambda}} =$ 

Some relations between LR-coefficients

Symmetries

- $c^{\nu}_{\mu\,\lambda} = c^{\nu}_{\lambda\,\mu}$
- $c^{\nu'}_{\lambda'\,\mu'} = c^{\nu}_{\lambda\,\mu}$
- $c^{\tilde{\mu}}_{\tilde{\nu}\,\tilde{\lambda}} = c^{\nu}_{\lambda\,\mu}$

Inequalities

- $c^{\nu+(1^c)}_{\lambda+(1^a),\,\mu+(1^b)} \geq c^{\nu}_{\lambda\,\mu}$
- $c^{\nu\cup(c)}_{\lambda\cup(a),\,\mu\cup(b)} \geq c^{\nu}_{\lambda\,\mu}$

  - for all non-negative integers a, b, c, with $c = a + b$
  - where $\lambda + (1^a) = (\lambda_1 + 1, \ldots, \lambda_a + 1, \lambda_{a+1}, \ldots, \lambda_n)$
  - and $\lambda \cup (a) = (\lambda_1, \ldots, \lambda_k, a, \lambda_{k+1}, \ldots, \lambda_n)$
    with $\lambda_k \geq a > \lambda_{k+1}$

Hive based proofs of the symmetry relations

Commutativity $c^{\nu}_{\mu\,\lambda} = c^{\nu}_{\lambda\,\mu}$

Proof: It remains an open problem to find a hive-based proof.

- We would like a bijection between the LR-hives for $c^{\nu}_{\mu\,\lambda}$ and those for $c^{\nu}_{\lambda\,\mu}$
- The result is by no means obvious from the Littlewood-Richardson rule.
- Considerable effort has gone into devising combinatorial proofs
- See for example the work of Benkart, Sotille & Stroomer [96] and of Azenhas [99]

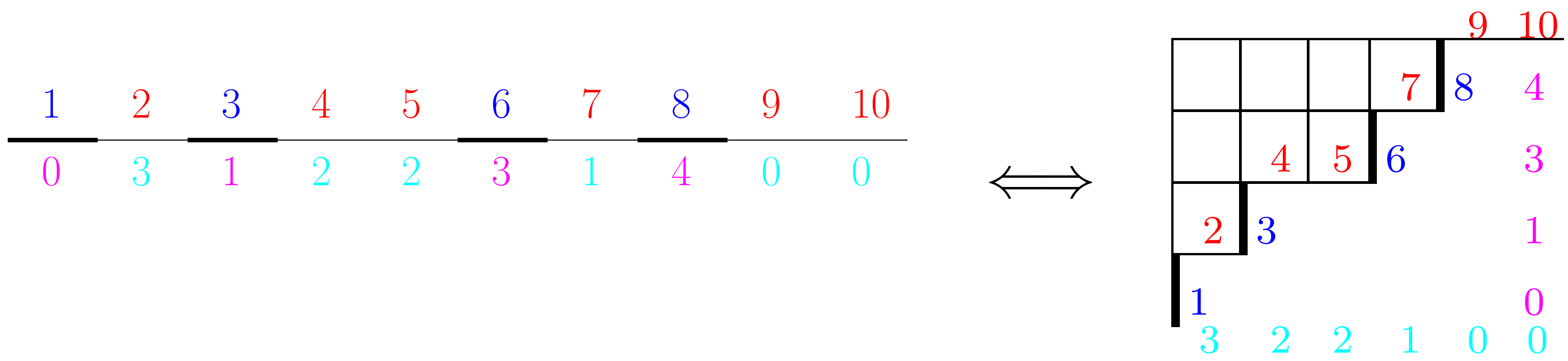
Hive based proofs of the symmetry relations

Conjugacy $c^{\nu'}_{\lambda'\,\mu'} = c^{\nu}_{\lambda\,\mu}$

Proof

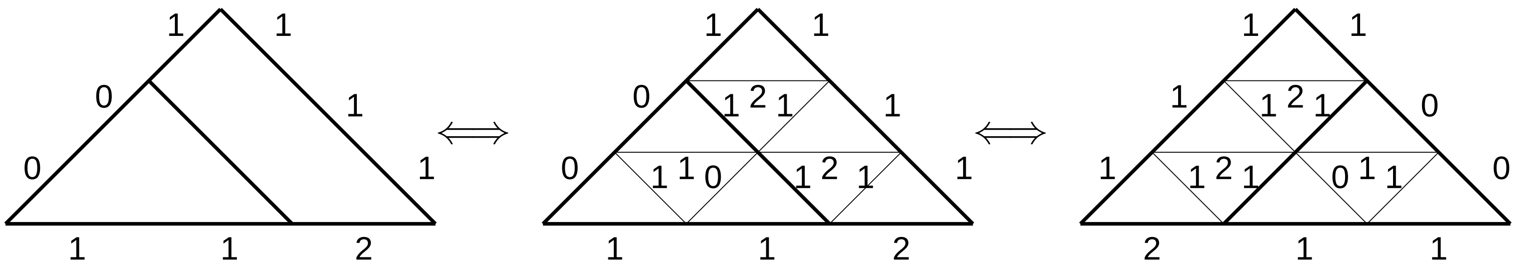
- Let $N = \{1, 2, \ldots, n\} = M \cup \overline{M}$ with $M \cap \overline{M} = \emptyset$
- Recall that if $\zeta = part\,(M)$ then $\zeta' = part\,(\overline{M})$

Ex: If $n = 10$, $M = \{1, 3, 6, 8\}$, $\overline{M} = \{2, 4, 7, 9, 10\}$ we have $part\,(M) = (4, 3, 1)$ and $part\,(\overline{M}) = (3, 2, 2, 1)$ as illustrated by



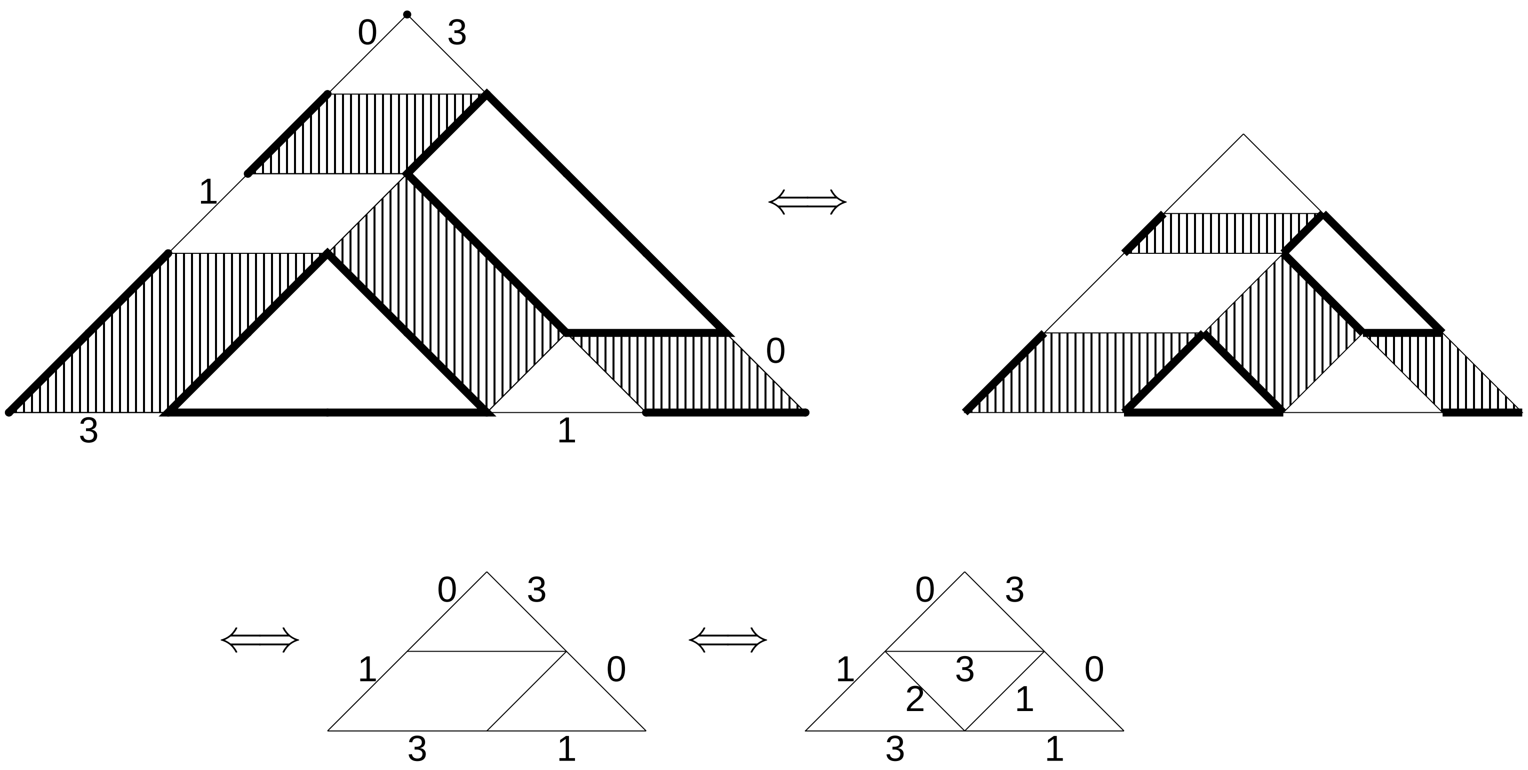
Proof of conjugacy contd.

- Recall that the number of n-puzzles with thick edges specified by I, J, K is $c^{part\,(K)}_{part\,(J),part\,(I)} = c^{part\,(K)}_{part\,(I),part\,(J)}$
- This was proved by scaling all thick edges by t and allowing $t \to 0$



Proof of conjugacy contd.

- It remains to show that the number of n-puzzles with thin edges specified by $\overline{I}, \overline{J}, \overline{K}$ is $c^{part\,(\overline{K})}_{part\,(\overline{I}),part\,(\overline{J})}$
- This is proved by scaling all thin edges by t and allowing $t \longrightarrow 0$

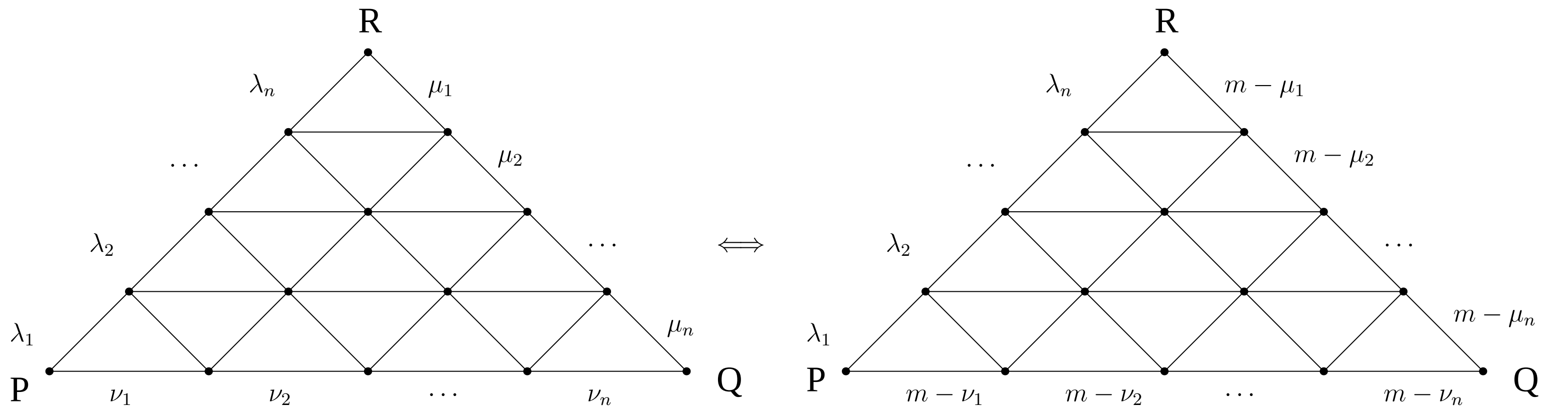


Hive based proofs of the symmetry relations

Complementarity $c^{\tilde{\mu}}_{\tilde{\nu}\,\tilde{\lambda}} = c^{\nu}_{\lambda\,\mu}$

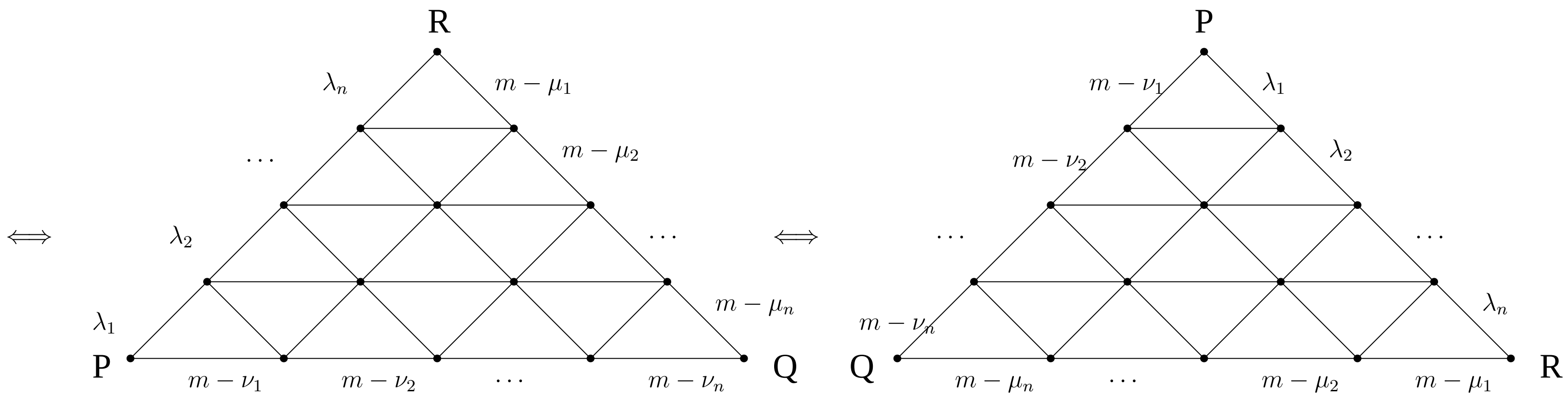
Proof Consider any LR-hive for $c^{\nu}_{\lambda\,\mu}$. Then

- For all edges parallel to the NE,SE,WE boundaries replace the edge labels α,β,γ by α,$m-\beta$,$m-\gamma$
- This breaks the triangle and the rhombus hive conditions and the result is not an LR-hive



Complementarity contd.

- Rotate clockwise through $\pi/3$
- The result is an LR-hive for $c^{\tilde{\mu}}_{\tilde{\nu}\,\tilde{\lambda}}$

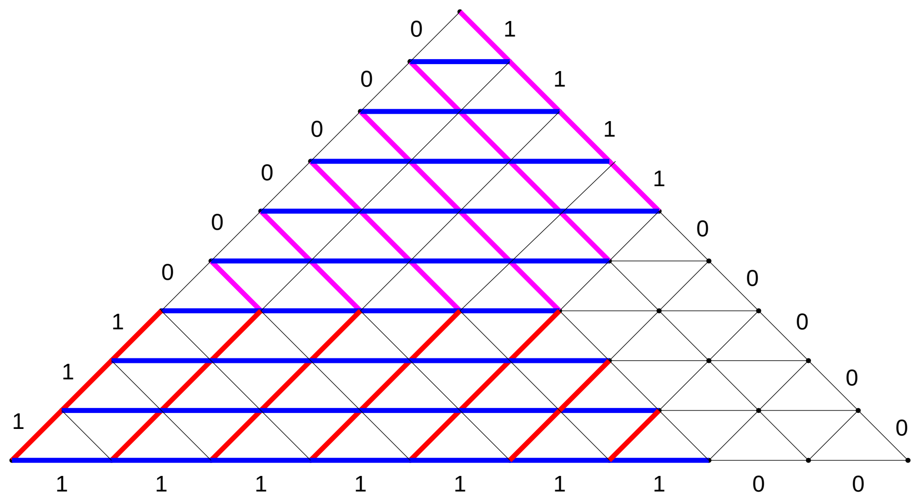


Lemma The above sequence of two maps provides a bijection between the LR-hives of $c^{\nu}_{\lambda\,\mu}$ and those of $c^{\tilde{\mu}}_{\tilde{\nu}\,\tilde{\lambda}}$

Column insertion

Inequality $c^{\nu+(1^c)}_{\lambda+(1^a),\,\mu+(1^b)} \geq c^{\nu}_{\lambda\,\mu}$ for all a, b, c, with $c = a + b$

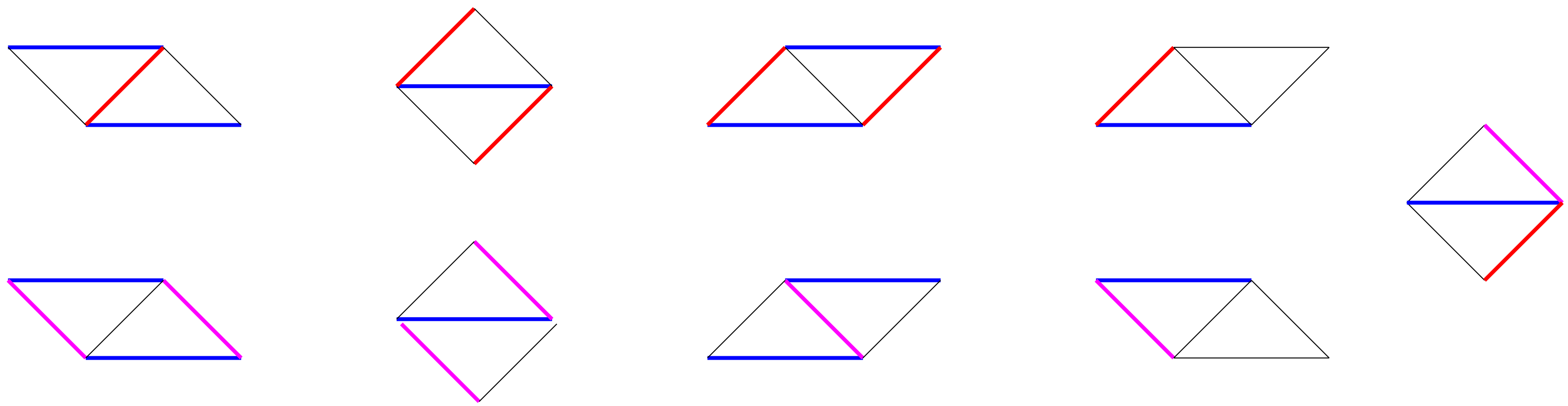
Proof Since $c^{1^c}_{1^a,\,1^b} = 1$ if $c = a + b$, there exist a unique LR-hive:



- Here each red, magenta and blue edge is labelled 1 and all other edges are labelled 0

Column insertion

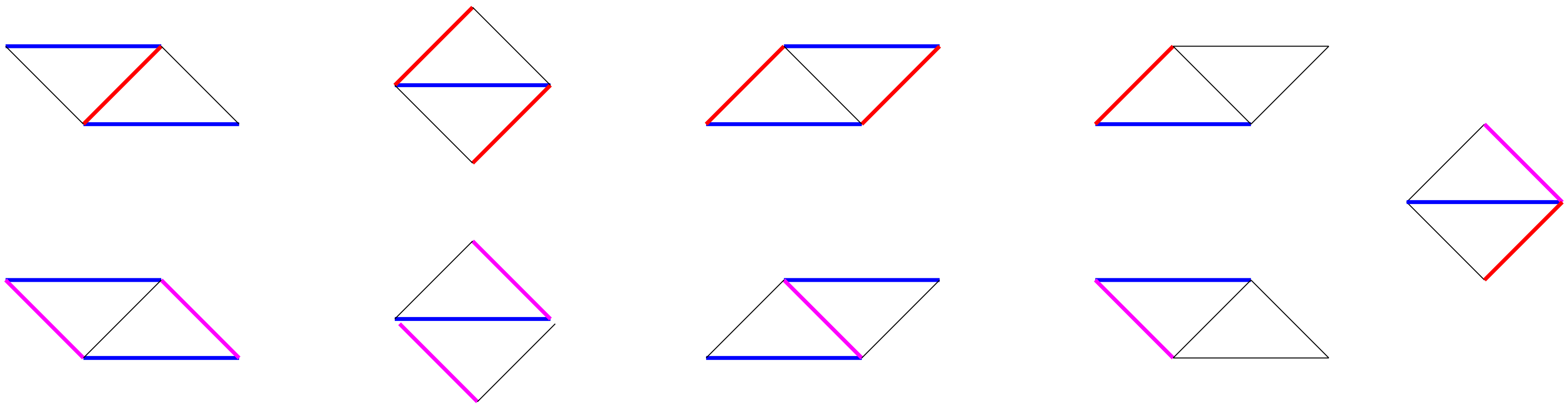
- The elementary rhombi take one or other of the following forms



- However, we can now interpret our multi-coloured LR-hive as being an LR-hive for $c^{\nu}_{\lambda\,\mu}$ whose red, magenta and blue edge labels have all been increased by 1, with no change to the uncoloured edge labels

Column insertion

- This yields an LR-five for $c^{\nu+(1^c)}_{\lambda+(1^a),\,\mu+(1^b)}$ as can be seen by checking the hive conditions in each of the elementary rhombi

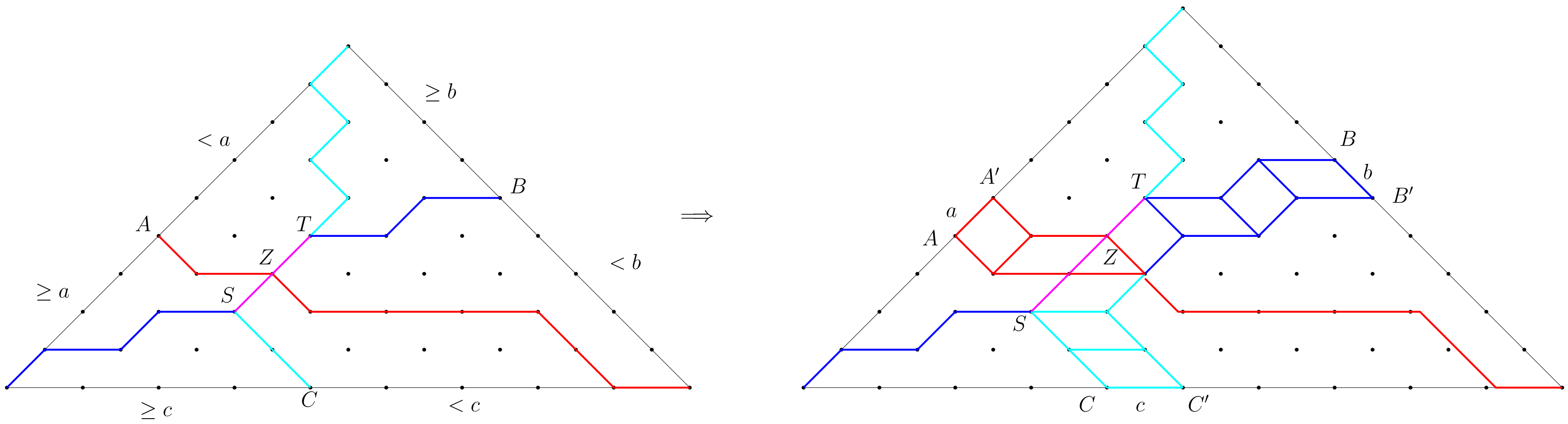


- Hence the number of $c^{\nu+(1^c)}_{\lambda+(1^a),\,\mu+(1^b)}$ LR-hives is at least as many as the number of $c^{\nu}_{\lambda\,\mu}$ LR-hives

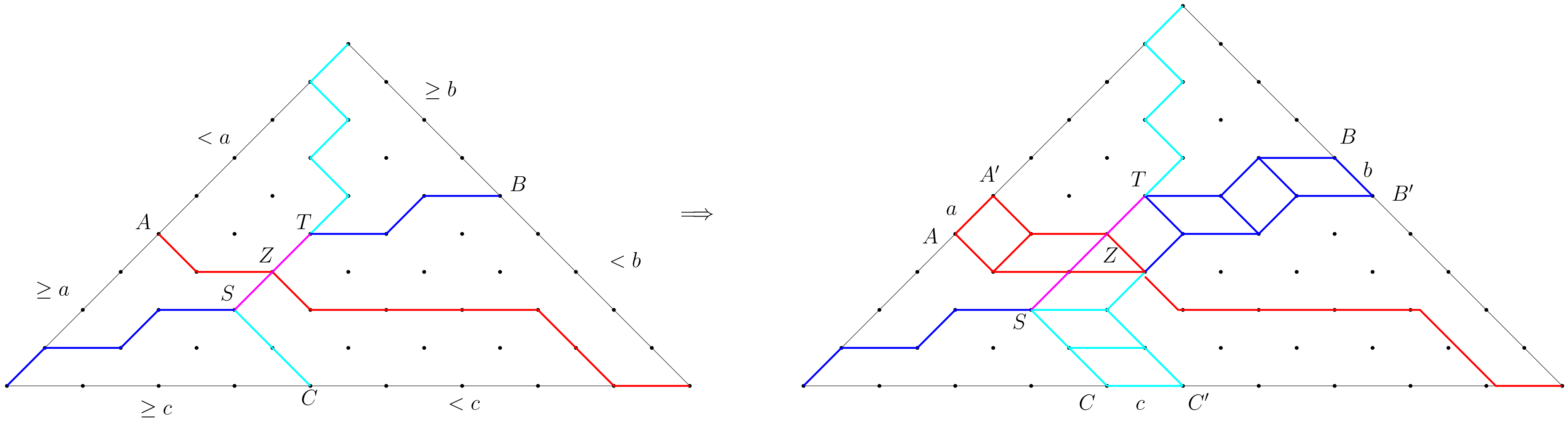
Row insertion

Inequality $\quad c^{\nu\cup(c)}_{\lambda\cup(a),\,\mu\cup(b)} \geq c^{\nu}_{\lambda\,\mu}$ with $c = a + b$

- Each LR-hive for $c^{\nu}_{\lambda\,\mu}$ may be divided into regions as shown on the left below
- Redundant corridors are then inserted as shown on the right to give an LR-hive for $c^{\nu\cup(c)}_{\lambda\cup(a),\,\mu\cup(b)}$

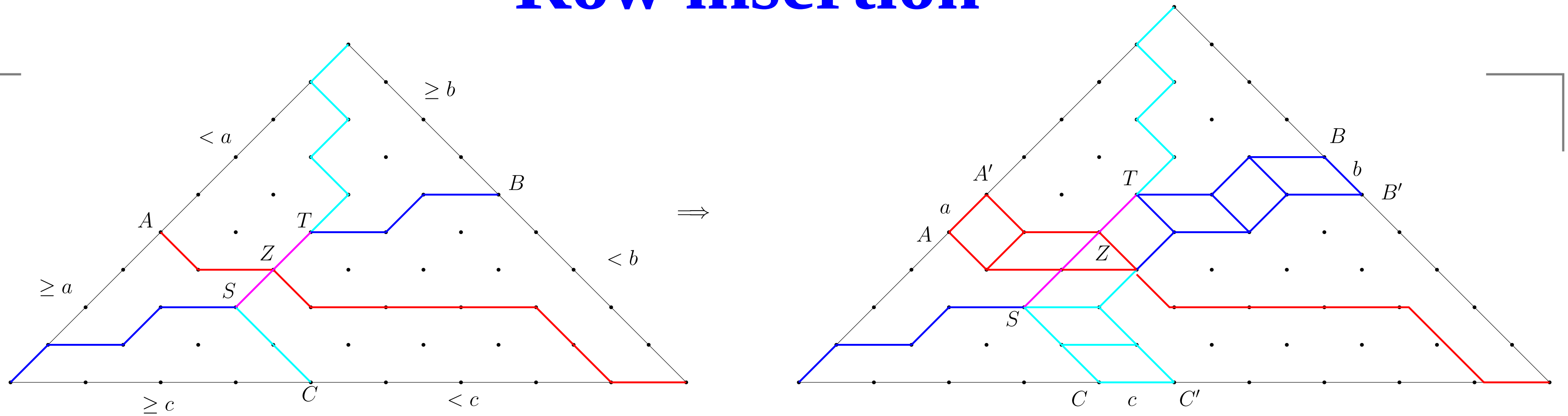


Row insertion



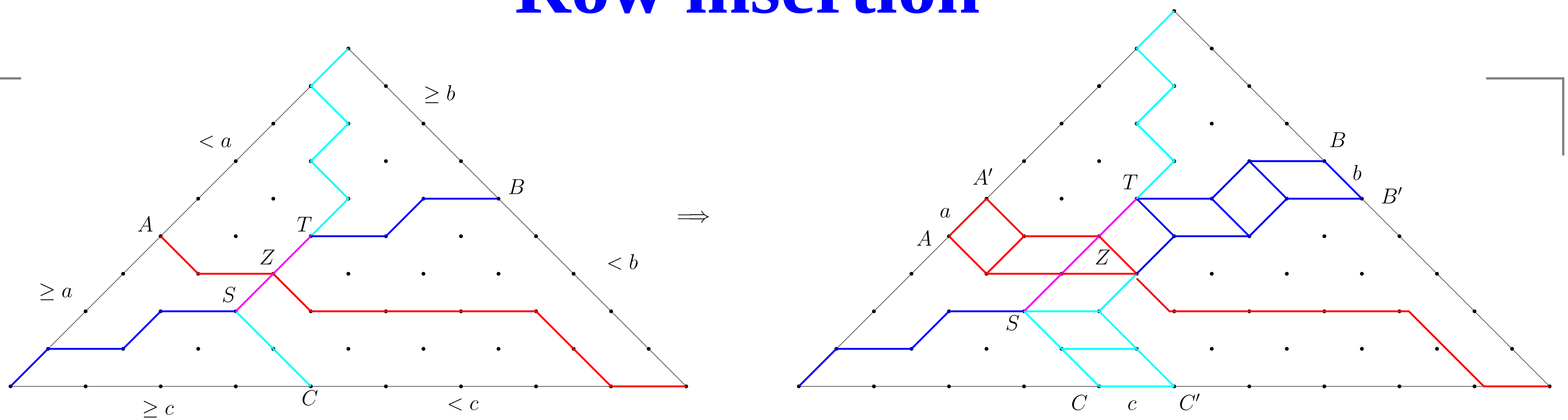
- The red line divides the hive into those regions for which NW edge labels are either $\leq a$ or $> a$
- The blue line divides the hive into those regions for which SW edge labels are either $\leq b$ or $> b$
- The cyan line divides the hive into those regions for which WE edge labels are either $\leq c$ or $> c$

Row insertion



- These lines always intersect at a single point Z
- Edges labelled a, b, c are inserted on the boundary
- Each edge along the red, blue and cyan lines is replaced by an appropriate redundant rhombus
- The point Z is replaced by a triangle with edge labels a, b, c, and all triangle and rhombus hive conditions are satisfied

Row insertion



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