Qubits and invariant theory

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Collaborative work with Peter Jarvis and Trevor Welsh

Invariants of a 2-qubit density matrix

Players in the game:

- Richard Davis, Hobart, Tasmania;
- Bob Delbourgo, Hobart, Tasmania;
- Peter Jarvis, Hobart, Tasmania;
- Brian Wybourne, Torun, Poland;
- Ron King, Southampton, UK;
- Trevor Welsh, Southampton, UK;
- Chris Cummins, Montreal, Canada.

Key publications

- M Grassl, M Rotteler and T Beth, *Computing local invariants of qubit quantum systems* Phys Rev A 58 1833-1839 (1998)
- R I A Davis, R Delbourgo and P D Jarvis *Covariance, correlation and entanglement* J Phys A33 1895-1914 (2000)
- R I A Davis, R Delbourgo and P D Jarvis *Integrity bases for local invariants of composite quantum systems* J Phys A33 3723-3725 (2000)
- Y Makhlin, *Nonlocal properties of two-qubit gates and mixed states and optimization of quantum computations* Quantum Info Proc 1 243-252 (2002)

Introduction and motivation

- Entanglement of a two qubit system is a non-local property
- Measures of entanglement should be independent of local transformations
- A mixed two qubit system is described by its density matrix
- Non-local entangling properties must be described by local invariants
- A complete set of local invariants is provided by the polynomial invariants

Hierarchy of problems

- Determine number of invariants of given degree
  - Count number of invariants of specific degrees
  - Calculate the corresponding generating function - that is evaluate Molien series
- Construct invariants explicitly
  - Identify fundamental invariants - known to be finite by Hilbert's Theorem
  - Distinguish between primary and secondary invariants
  - Determine complete structure of ring of invariants: known to be Cohen-Macauley

Statement of problem

- Qubit: a quantum state ψ_i with $i = 0, 1$.
- Two-qubit: $\Psi_{a,i} = \psi_a(1)\psi_i(2)$ with $a, i = 0, 1$.
- Density matrix: $\rho = (\rho_{a,i}^{b,j})$ with $a, i, b, j = 0, 1$.

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- Local transformations of qubits:
  $U : \psi_i \;\to\; \sum_{j=0}^{1} \psi_j \, U_{ji}$ for $U \in SU(2)$.
- Local transformations of the density matrix:
  $U \otimes V : \rho_{a,i}^{b,j} \;\to\; \sum_{c,k,d,l} U_{bd}^{-1} \, V_{jl}^{-1} \; \rho_{c,k}^{d,l} \, U_{ca} \, V_{ki}$
  for $U \otimes V \in SU(2) \times SU(2)$.

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  for $U \otimes V \in SU(2) \times SU(2)$.
- Determine all polynomial functions of the 16 components of ρ that are invariant under the local transformations $U \otimes V$ of $SU(2) \otimes SU(2)$, viewed as a subgroup of $SU(16)$.

Group theoretic approach

- The 16 components, $\rho_{a,i}^{b,j}$, of the mixed 2-qubit density matrix form the basis of the defining irrep V of $SU(16)$.
- The polynomials of degree m form the basis of the mth fold symmetrised power irrep $V^{\{m\}}$ of $SU(16)$.
- In terms of Schur functions $s_\lambda = \{\lambda\}$, the irreps V and $V^{\{m\}}$ have characters $\{1\}$ and $\{m\}$.

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- Let n_m be the number of independent polynomial $(SU(2) \times SU(2))$-invariants of degree m.
- Then n_m is the multiplicity of the identity irrep V_0 in the restriction of $V^{\{m\}}$ from $SU(16)$ to $SU(2) \times SU(2)$.

Group-subgroup chain and branching rules

- Group-subgroup chain: $SU(16) \supset SU(4) \times SU(4) \supset (SU(2) \times SU(2)) \times (SU(2) \times SU(2)) \supset SU(2) \times SU(2)$.

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- Branching rules:

$$SU(16) \supset SU(4) \times SU(4) : \ \{m\} \ \to \sum_{\lambda \vdash m; \ell(\lambda) \leq 4} \{\lambda\}, \{\lambda\};$$

$$SU(4) \supset SU(2) \times SU(2) : \ \{\lambda\} \ \to \sum_{\mu,\nu \vdash m; \ell(\mu), \ell(\nu) \leq 2} k^{\lambda}_{\mu\nu} \{\mu\}, \{\overline{\nu}\};$$

$$SU(4) \supset SU(2) \times SU(2) : \ \{\lambda\} \ \to \sum_{\sigma,\tau \vdash m; \ell(\sigma), \ell(\tau) \leq 2} k^{\lambda}_{\sigma\tau} \{\sigma\}, \{\overline{\tau}\};$$

$$SU(2) \times SU(2) \supset SU(2) : \ \{\mu\}, \{\overline{\nu}\} \ \to \ + \cdots + \delta_{\mu\nu}\{0\};$$

$$SU(2) \times SU(2) \supset SU(2) : \ \{\sigma\}, \{\overline{\tau}\} \ \to \ + \cdots + \delta_{\sigma\tau}\{0\}.$$

Number n_m of invariants of degree m

- Molien series $M(q) = \sum_{m=0}^{\infty} n_m \, q^m$.
- $n_m = \sum_{\lambda \vdash m; \ell(\lambda) \leq 4} \left(\sum_{\mu \vdash m; \ell(\mu) \leq 2} k_{\mu\mu}^{\lambda} \right)^2$ with $\chi_\kappa^\mu \, \chi_\kappa^\nu = \sum_\lambda k_{\mu\nu}^\lambda \, \chi_\kappa^\lambda$.

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- Results obtained using SCHUR (BGW)

$$
\begin{aligned}
M(q) = {} & 1+q+4q^2+6q^3+16q^4+23q^5+52q^6+77q^7\\
& +150q^8+224q^9+396q^{10}+583q^{11}+964q^{12}+1395q^{13}\\
& +2180q^{14}+3100q^{15}+4639q^{16}+6466q^{17}+9344q^{18}\\
& +12785q^{19}+17936q^{20}+24121q^{21}+33008q^{22}+43674q^{23}\\
& +58512q^{24}+76277q^{25}+100312q^{26}+129009q^{27}\\
& +166932q^{28}+212022q^{29}+270448q^{30}+O(q^{31}).
\end{aligned}
$$

Number of invariants in case $m = 4$

$$
\begin{aligned}
SU(16) &\rightarrow SU(4) \times SU(4) \\
\{4\} &\rightarrow \{4\},\{4\} + \{31\},\{31\} + \{22\},\{22\} \\
&\quad +\{211\},\{211\} + \{1111\},\{1111\} \\
SU(4) &\rightarrow SU(2) \times SU(2) \\
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\end{aligned}
$$

For each $\{\lambda\}$ sum coefficients of terms $\{\mu\},\{\overline{\mu}\}$ and then sum squares to give $n_4 = 3^2 + 1^2 + 2^2 + 1^2 + 1^2 = 16$.

Molien's Theorem

- Molien's Theorem Let G be a compact continuous group, with elements g and Haar measure $d\mu(g)$. Then

$$M(q) = \int_{g \in G} \frac{d\mu(g)}{\det(I - q\,g)}\,.$$

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- For $G = SO(3)$ and g realised by 3×3 matrices:
  - Any $g \in G$ can be diagonalised
  - Eigenvalues $\{e^{i\theta}, 1, e^{-i\theta}\}$
  - Measure $d\mu(g) = \sin^2\theta\, d\theta$
  - Setting $z = e^{i\theta}$ integral is around $|z| = 1$

Calculation of Molien series

- In our case $G = SU(2) \times SU(2) \sim SO(3) \times SO(3)$ and g is realised by the tensor product of two 4×4 matrices.
  - Any $g \in G$ can be diagonalised
  - Eigenvalues $\{1, e^{i\theta}, 1, e^{-i\theta}\} \times \{1, e^{i\phi}, 1, e^{-i\phi}\}$
  - Measure $d\mu(g) = \sin^2\theta \, \sin^2\phi \, d\theta \, d\phi$
  - Set $z = e^{i\theta}, w = e^{i\phi}$. Integrate around $|z| = 1, |w| = 1$.

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  - Measure $d\mu(g) = \sin^2\theta \, \sin^2\phi \, d\theta \, d\phi$
  - Set $z = e^{i\theta}, w = e^{i\phi}$. Integrate around $|z| = 1, |w| = 1$.
- Molien's Theorem then gives $M(q) =$

$$\oint_{|z|=1} \oint_{|w|=1} \frac{(1-z)^2(1-w)^2 z^{-2} w^{-2}}{(1-q)^4(1-qz)^2(1-q/z)^2(1-qw)^2(1-q/w)^2} \frac{dw\, dz}{(1-qzw)(1-qz/w)(1-qw/z)(1-q/zw)}.$$

Generating function for the Molien series

- The repeated use of Cauchy's residue theorem yields (Grassl et .al):

$$M(q) = \frac{1 - q^2 - q^3 + 2q^4 + 2q^5 + 2q^6 - q^7 - q^8 + q^{10}}{(1-q)^{10}(1+q)^6(1+q^2)^2(1+q+q^2)^3} \, .$$

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- This can be recast in the form RCK

$$M(q) = \frac{1 + q^4 + q^5 + 3q^6 + 2q^7 + 2q^8 + 3q^9 + q^{10} + q^{11} + q^{15}}{(1-q)(1-q^2)^3(1-q^3)^2(1-q^4)^3(1-q^6)} \, .$$

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- Its expansion as a power series confirms (and may be used to extend) BGW's calculations:

$$M(q) = 1 + q + 4q^2 + 6q^3 + 16q^4 + 23q^5 + 52q^6 + 77q^7 + 150q^8 + 224q^9 + 396q^{10} + 583q^{11} + 964q^{12} + \cdots$$

Properties of the ring $\mathcal{R}$ of invariants

- Theorem (Hilbert): $\mathcal{R}$ is generated by a finite set of fundamental invariants $\{I_1, I_2, \ldots, I_N\}$.

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- An arbitrary invariant: $I = \sum_{k=0}^{r} J_k \, P_k(K_1, K_2, \dots, K_n)$ with P_k polynomial in the K_m.

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- An arbitrary invariant: $I = \sum_{k=0}^{r} J_k \, P_k(K_1, K_2, \ldots, K_n)$ with P_k polynomial in the K_m.
- Primary invariants K_m are algebraically independent.
- Secondary invariants J_k are linearly independent, but are algebraic functions of the K_m.

Properties of the ring $\mathcal{R}$ of invariants

- Molien series for $\mathcal{R}$:

$$M(q) = \frac{\sum_{k=0}^{r} q^{\deg J_k}}{\prod_{m=1}^{n} (1 - q^{\deg K_m})}$$

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- This implies that:
  - there are 10 primary invariants with degrees $1, 2, 2, 2, 3, 3, 4, 4, 4, 6$.
  - and 15 secondary invariants with degrees $4, 5, 6, 6, 6, 7, 7, 8, 8, 9, 9, 9, 10, 11, 15$.

Parametrisation of the density matrix

- Group-subgroup chain
$$SU(16) \supset SU(2) \times SU(2) \sim SO(3) \times SO(3).$$
- Two-qubit density matrix
$\rho = t\,\hat{I} \otimes \hat{I} + s_a\,\hat{\sigma}_a \otimes \hat{I} + p_i\,\hat{I} \otimes \hat{\sigma}_i + b_{ai}\,\hat{\sigma}_a \otimes \hat{\sigma}_i$,
where $\hat{I}$ is the 2×2 unit matrix
and $\hat{\sigma}_k$ for $k = 1, 2, 3$ are the Pauli matrices
and the repeated indices a, i are summed over $1, 2, 3$.
Let $s = (s_1, s_2, s_3)$ and $p = (p_1, p_2, p_3)$
with $b = (b_{ai})_{1 \le a,i \le 3}$ and $b^T = (b_{ia})_{1 \le a,i \le 3}$.

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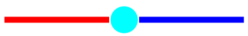
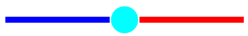
- Aim: to construct $SO(3) \times SO(3)$ scalars as polynomials in the components of t, s, p and b.

Building blocks

- Scalar t 
- Vectors s_a and p_i  
- Matrices b_{ai} and b^T_{ia}  

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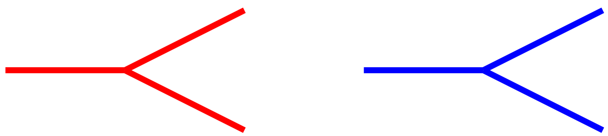
- Symmetric metric tensors δ_{ab} and δ_{ij}
- Antisymmetric tensors ϵ_{abc} and ϵ_{ijk}



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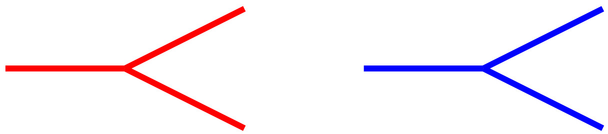
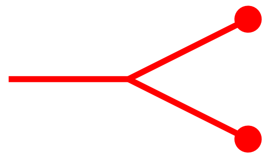


- s-vectors: $s_a, \quad b_{ai}p_i, \quad b_{ai}b^T_{ib}s_b, \; \cdots$

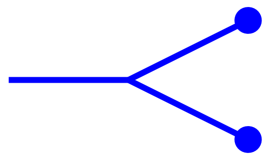
- p-vectors: $p_i, \quad b^T_{ia}s_a, \quad b^T_{ia}b_{aj}p_j, \; \cdots$

Combinations to be avoided

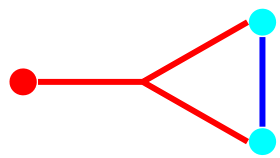
- $\epsilon_{abc}\, s_a\, s_b = 0$

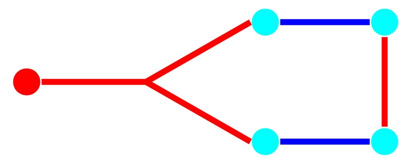


- $\epsilon_{ijk}\, p_i\, p_j = 0$

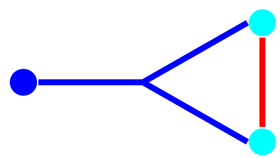


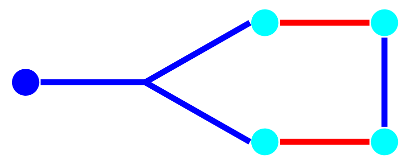
- $\epsilon_{abc}\, b_{ai} b^T_{id} \cdots b_{dj} b^T_{jb} = 0$





- $\epsilon_{ijk}\, b^T_{ia} b_{al} \cdots b^T_{lb} b_{bj} = 0$





- $$\epsilon_{abc}\epsilon_{def} = \delta_{ad}\delta_{be}\delta_{cf} + \delta_{ae}\delta_{bf}\delta_{cd} + \delta_{af}\delta_{bd}\delta_{ce} - \delta_{ad}\delta_{bf}\delta_{ce} - \delta_{ae}\delta_{bd}\delta_{cf} - \delta_{af}\delta_{be}\delta_{cd}.$$

- $$\epsilon_{ijk}\epsilon_{lmn} = \delta_{il}\delta_{jm}\delta_{kn} + \delta_{im}\delta_{jn}\delta_{kl} + \delta_{in}\delta_{jl}\delta_{km} - \delta_{il}\delta_{jn}\delta_{km} - \delta_{im}\delta_{jl}\delta_{kn} - \delta_{in}\delta_{jm}\delta_{kl}.$$

Invariants involving no ϵ s

$$K_1 := t$$

$$K_2 := b_{ai} b^T_{ia}$$

$$K_3 := s_a s_a$$

$$K_4 := p_i p_i$$

$$K_6 := s_a b_{ai} p_i$$

$$K_7 := b_{ai} b^T_{ib} b_{bj} b^T_{ja}$$

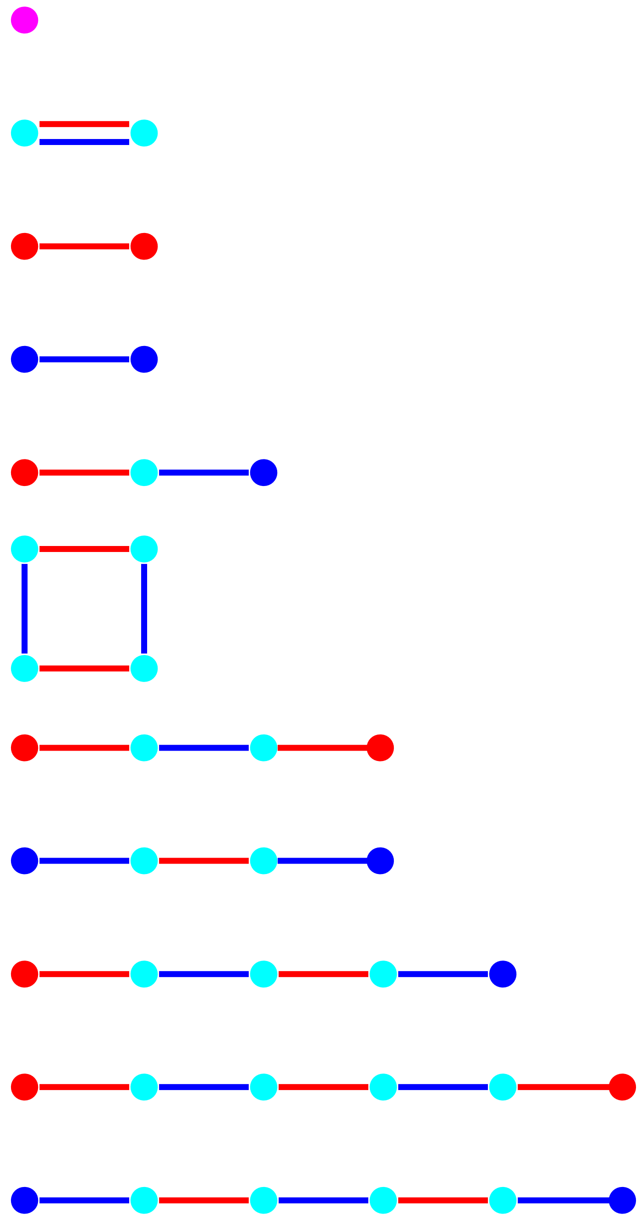
$$K_8 := s_a b_{ai} b^T_{ib} s_b$$

$$K_9 := p_i b^T_{ia} b_{aj} p_j$$

$$U_2 := s_a b_{ai} b^T_{ib} b_{bj} p_j$$

$$X_1 := s_a b_{ai} b^T_{ib} b_{bj} b^T_{jc} s_c$$

$$X_2 := p_i b^T_{ia} b_{aj} b^T_{jb} b_{bk} p_k$$



Invariants involving one ϵ

$$W_1 := \epsilon_{abc}\ s_a\ b_{bi} p_i\ b_{cj} b^T_{jd} s_d$$

$$V_1 := \epsilon_{ijk}\ p_i\ b^T_{ja} s_a\ b^T_{kb} b_{bl} p_l$$

$$W_2 := \epsilon_{abc}\ s_a\ b_{bj} p_j\ b_{ck} b^T_{kd} b_{dl} p_l$$

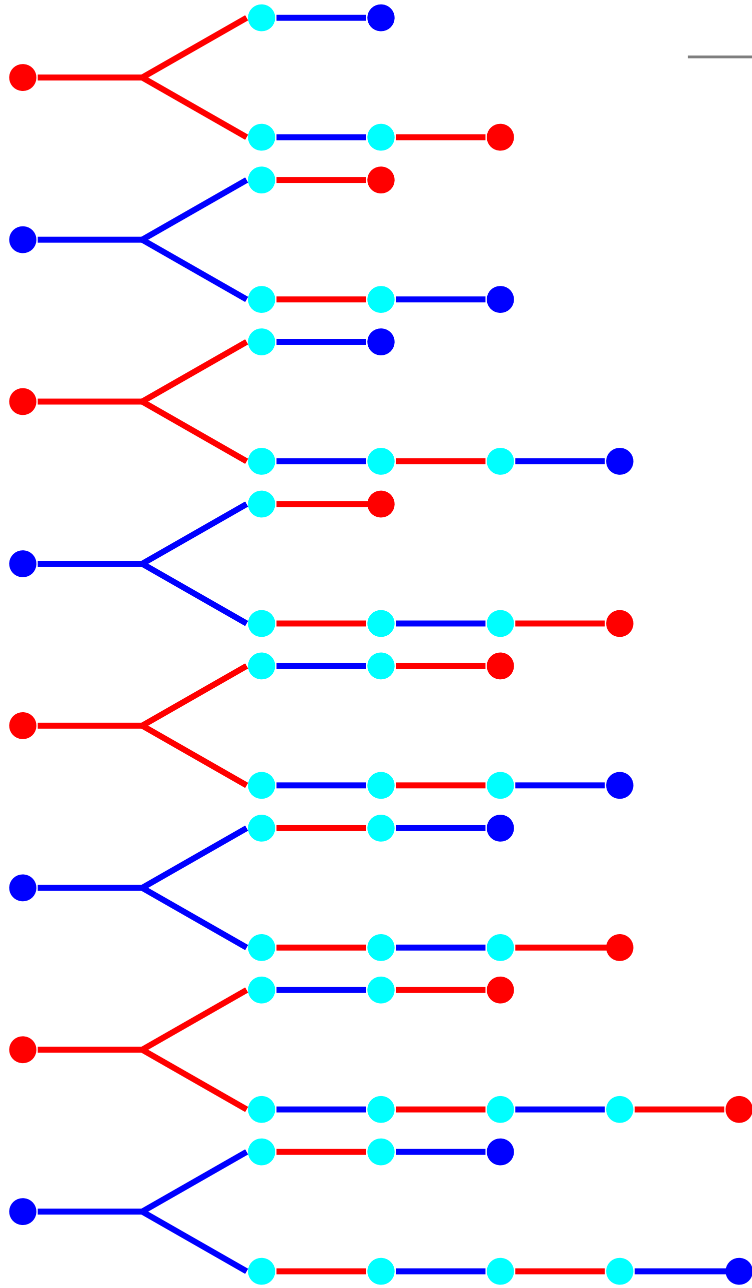
$$V_2 := \epsilon_{ijk}\ p_i\ b^T_{ja} s_a\ b^T_{kb} b_{bl} b^T_{lc} s_c$$

$$V_3 := \epsilon_{abc}\ s_a\ b_{bi} b^T_{id} s_d\ b_{cj} b^T_{je} b_{ek} p_k$$

$$W_3 := \epsilon_{ijk}\ p_i\ b^T_{ja} b_{al} p_l\ b^T_{kb} b_{bm} b^T_{mc} s_c$$

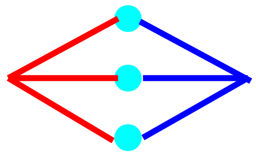
$$V_4 := \epsilon_{abc}\ s_a\ b_{bi} b^T_{id} s_d\ b_{cj} b^T_{je} b_{ek} b^T_{kf} s_f$$

$$W_4 := \epsilon_{ijk}\ p_i\ b^T_{ja} b_{al} p_l\ b^T_{kb} b_{bm} b^T_{mc} b_{cn} p_n$$

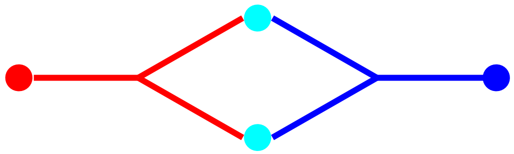


Invariants involving both ϵ s

$$K_5 := \quad \epsilon_{abc}\, \epsilon_{ijk}\, b_{ai}\, b_{bj}\, b_{ck}$$



$$U_3 := \quad \epsilon_{abc}\, \epsilon_{ijk}\, s_a\, b_{bj}\, b_{ck}\, p_i$$



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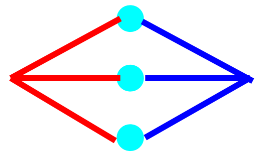

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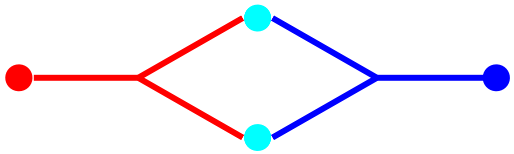
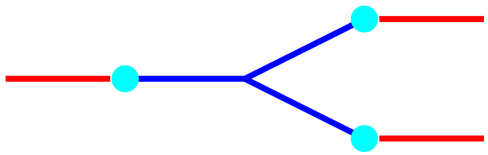


- Fundamental invariants and their degrees

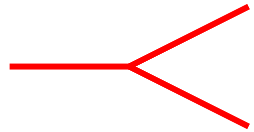
Deg		Deg	
1	K_1	6	X_1, X_2, V_1, W_1
2	$K_2, K_3, K_4,$	7	V_2, W_2
3	K_5, K_6	8	V_3, W_3
4	K_7, K_8, K_9, U_3	9	V_4, W_4
5	U_2		

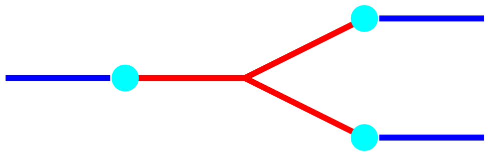
Elimination of other invariants

- Note: $K_5 = 6\ \det(b)$ and



$= \det(b) \times$





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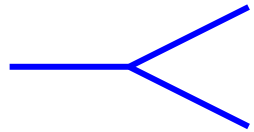

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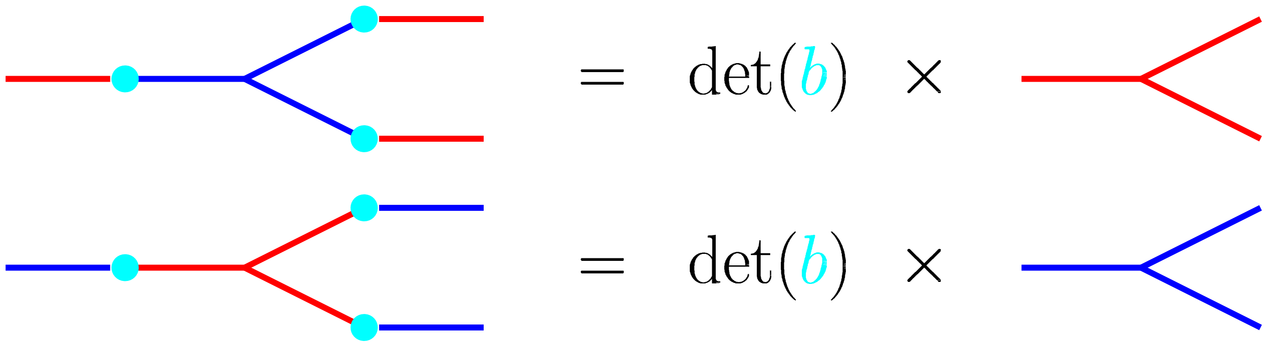
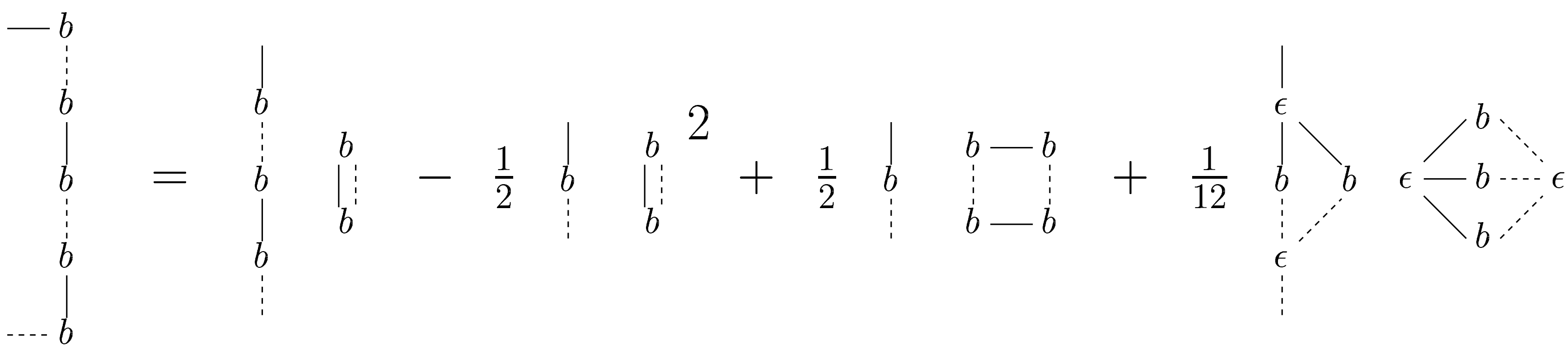
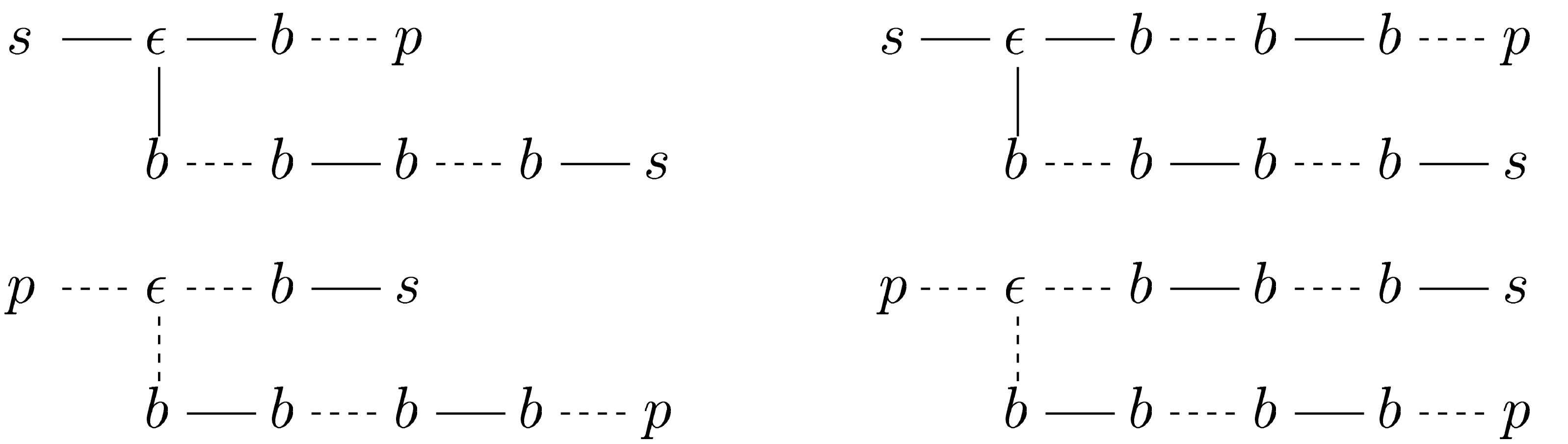


- Any product of 5 or more b s may be eliminated:



Diagrams eliminated by explicit calculation

- One ϵ connected to a pair of particular b-chains.



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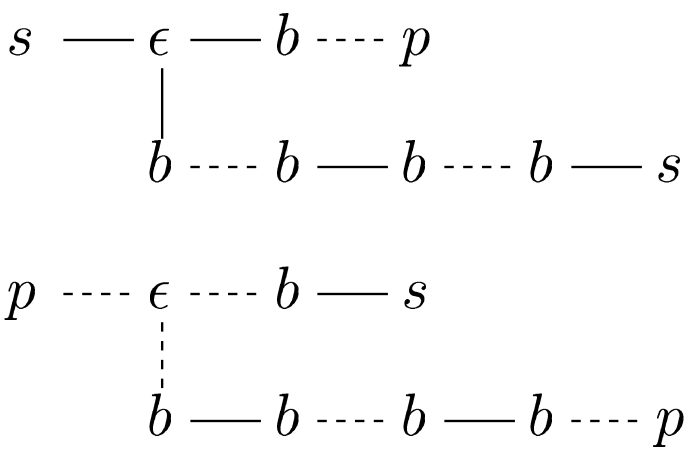
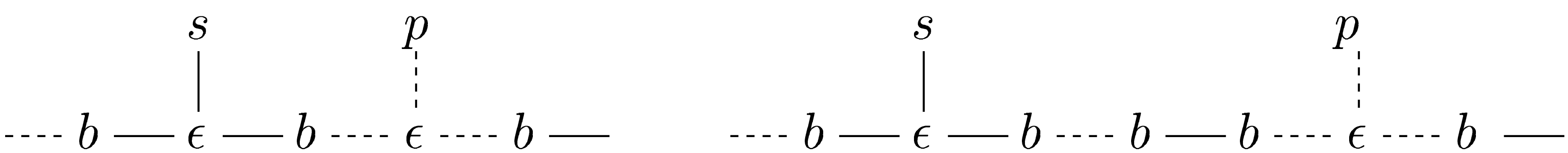


- Two different ϵs connected by one or more bs.



Syzygies I

- Elimination of U_2^2

$$U_2 * U_2 - 2 * X_1 * K_9 - 2 * K_8 * X_2 + 2 * K_8 * K_9 * K_2$$
$$+K_6 * K_6 * K_7 - K_6 * K_6 * K_2^2 - 2 * K_3 * K_4 * K_5^2$$
$$+2 * U_3 * K_5 * K_6 = 0.$$

- Elimination of U_3^2

$$U_3 * U_3 + 8 * K_6 * U_2 - 4 * K_6 * K_6 * K_2 - 4 * K_8 * K_9$$
$$+4 * K_3 * K_9 * K_2 + 4 * K_8 * K_4 * K_2 + 2 * K_3 * K_4 * K_7$$
$$-2 * K_3 * K_4 * K_2^2 - 4 * X_1 * K_4 - 4 * K_3 * X_2 = 0.$$

- There exists no similar expansion for $U_4 := U_2 * U_3$, but U_4^2 can be eliminated using the above.

Syzygies II

- Elimination of all $U_i * V_j$. Example:

$$2 * U_2 * V_3 - W_1 * K_6 * K_7 + W_1 * K_6 * K_2^2 - 2 * V_4 * K_9$$
$$-2 * K_5 * V_1 * K_8 + 4 * K_5 * V_2 * K_6 - 2 * K_5 * W_3 * K_3 = 0.$$

- Elimination of all $U_i * W_j$. Example:

$$U_3 * W_2 + 2 * V_1 * K_6 * K_2 - 2 * V_2 * K_9 + 4 * W_3 * K_6$$
$$-2 * W_4 * K_3 - 2 * K_5 * W_1 * K_4 = 0.$$

- Similarly, all V_i^2, W_j^2 and $V_i * W_j$ may be eliminated.

Syzygies III

- Elimination of $W_k * X_1$ and $V_k * X_2$. Examples:

$$W_1 * X_1 - W_1 * K_8 * K_2 - V_3 * K_8 + V_4 * K_6 - K_5 * V_2 * K_3 = 0.$$

$$V_1 * X_2 - V_1 * K_9 * K_2 - W_3 * K_9 + W_4 * K_6 - K_5 * W_2 * K_4 = 0.$$

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- Elimination of $X_1 * X_2$:

$$2 * X_1 * X_2 - 2 * X_1 * K_9 * K_2 - 2 * K_8 * X_2 * K_2$$
$$+ K_8 * K_9 * K_7 + K_8 * K_9 * K_2^2 - 2 * U_2 * K_6 * K_7$$
$$+ 2 * U_2 * K_6 * K_2^2 + K_6 * K_6 * K_7 * K_2 - K_6 * K_6 * K_2^3$$
$$- 2 * U_4 * K_5 + 2 * U_3 * K_5 * K_6 * K_2 - 2 * K_6 * K_6 * K_5^2$$
$$+ 2 * K_8 * K_4 * K_5^2 + 2 * K_3 * K_9 * K_5^2 - 2 * K_3 * K_4 * K_5^2 * K_2 = 0.$$

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- K_k^m cannot be eliminated for any finite m
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- $U_k * X_1$ cannot be eliminated
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Generating function for invariants

- Definitions

$$D := \prod_{k=1}^{9} (1 - K_k),$$

$$U := 1 + \sum_{i=2}^{4} U_i, \quad V := \sum_{i=1}^{4} V_i, \quad W := \sum_{i=1}^{4} W_i,$$

$$F := \frac{1}{D} \left(U + \frac{U\,X_1}{1 - X_1} + \frac{U\,X_2}{1 - X_2} + \frac{V}{1 - X_1} + \frac{W}{1 - X_2} \right)$$

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- Theorem Every linearly independent invariant appears once and once only in the formal expansion of F.

Algebraic identity linking X_1 and X_2

- The syzygies imply an identity linking just X_1, X_2 and K_k for $k = 1, 2, \ldots, 9$. It is about 10 pages long and is of degree 48. Its dependence on X_1, X_2 is illustrated by setting $K_k = z^{\deg(K_k)}$ for $k = 1, 2, \ldots, 9$:

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$$
\begin{aligned}
&16 * X_1^4 * X_2^4 + 8832 * z^{36} * X_1 * X_2 - 7040 * X_1^2 * X_2 * z^{30} \\
&-2112 * z^{30} * X_2^3 + 144 * z^{24} * X_1^4 - 2112 * z^{30} * X_1^3 - 1088 * z^{42} * X \\
&+1984 * X_1^2 * X_2^2 * z^{24} + 128 * z^{24} * X_2^3 * X_1 - 1088 * z^{42} * X_2 \\
&+144 * z^{24} * X_2^4 + 896 * X_1^3 * z^{18} * X_2^2 + 6496 * z^{36} * X_2^2 \\
&-7040 * z^{30} * X_2^2 * X_1 - 32 * z^{12} * X_1^4 * X_2^2 - 64 * X_1^4 * X_2^3 * z^6 \\
&+192 * z^{18} * X_1^4 * X_2 + 128 * X_1^3 * X_2 * z^{24} - 32 * z^{12} * X_2^4 * X_1^2 \\
&+896 * X_1^2 * X_2^3 * z^{18} - 6512 * z^{48} + 192 * z^{18} * X_2^4 * X_1 \\
&-384 * X_1^3 * X_2^3 * z^{12} + 6496 * z^{36} * X_1^2 - 64 * z^6 * X_2^4 * X_1^3 = 0.
\end{aligned}
$$

Implications

- X_1, X_2 cannot both be primary (algebraically independent) invariants.
- In our ring of invariants we require arbitrarily large powers of both X_1 and X_2.
- Define $K_{10} = X_1 + X_2$ and $J_1 = X_1 - X_2$
- Then K_k for $k = 1, 2, \ldots, 10$ are algebraically independent.

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- Define $K_{10} = X_1 + X_2$ and $J_1 = X_1 - X_2$
- Then K_k for $k = 1, 2, \ldots, 10$ are algebraically independent.
- Moreover $J_1^2 = K_{10}^2 - 4 * X_1 * X_2$ with $X_1 * X_2$ known to be linear in X_1, X_2, U_2, U_3, U_4.
- Conclude that J_1^2 is linear in J_1, U_2, U_3, U_4, with coefficients polynomial in $K_1, K_2, \ldots, K_{10}$, and can be eliminated.

Primary and secondary invariants

- As the Molien function suggested there are

  10 primary and 15 secondary invariants.

- 10 primaries: $K_1, K_2, \ldots, K_9$ and $K_{10} = X_1 + X_2$.

- 15 secondaries:

$$
\begin{array}{lll}
J_1 = X_1 - X_2 & J_4 = J_1\, J_2 & J_7 = J_1\, J_2\, J_3 \\
J_2 = U_2 & J_5 = J_1\, J_3 & J_{7+k} = V_k \quad \text{for } k = 1, 2, 3, 4 \\
J_3 = U_3 & J_6 = J_2\, J_3 & J_{11+k} = W_k \quad \text{for } k = 1, 2, 3, 4
\end{array}
$$

Conclusion

- Setting $J_0 = 1$, the ring $\mathcal{R}$ of invariants takes the Cohen-Macauley form:

$$\mathcal{R} = \oplus_{m=0}^{15} J_m \, \mathbb{C}[K_1, K_2, \ldots, K_{10}]$$

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- with 10 primary invariants K_k of degrees $1, 2, 2, 2, 3, 3, 4, 4, 4, 6$, and 15 secondary invariants J_m of degrees $4, 5, 6, 6, 6, 7, 7, 8, 8, 9, 9, 9, 10, 11, 15$.