Branching rules, plethysms and Hopf algebras - some surprises

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Introduction

Background: A uniform setting for dealing with:

- characters of classical subgroups of $GL(N)$
- group-subgroup branching rules
- evaluation of tensor products

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Introduction

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Aim: To identify characters of some non-classical subgroups

- by exploiting the Hopf algebra of symmetric functions
- and evaluating plethysms of Schur function series

Outcome:

- new modification, branching and tensor product rules
- identification of non-semisimple non-reductive subgroups
- problems with indecomposability and irreducibility

Symmetric functions

The ring of symmetric functions Λ.

- $\Lambda = \mathbb{Z}[\mathbf{x}]^{S_N}$ polynomial symmetric functions of indeterminates $\mathbf{x} = (x_1, x_2, \ldots, x_N)$.
- $\Lambda = \oplus_n \Lambda_n$ graded by degree.
- Basis of Λ_n provided by Schur functions $s_\lambda(\mathbf{x})$ with partitions λ of weight $|\lambda| = n$ and length $\ell(\lambda) \leq N$.
- Outer product $s_\lambda(\mathbf{x})\, s_\mu(\mathbf{x}) = \sum_\nu \, c^\nu_{\lambda\mu}\, s_\nu(\mathbf{x})$ Littlewood-Richardson coefficients $c^\nu_{\lambda\mu} \in \mathbb{Z}_{\geq 0}$.
- For any countably infinite set of indeterminates $(x_1, x_2, \ldots)$ $s_\lambda(x_1, \ldots, x_N) = s_\lambda(x_1, \ldots, x_N, 0, 0, \ldots)$.
- $c^\nu_{\lambda\mu}$ is independent of N for all $N \geq \ell(\nu)$.

Hopf algebra of symmetric functions

Hopf algebra $(\Lambda, \cdot, \Delta, \iota, \epsilon, \mathsf{S})$.

- Indeterminates $\mathbf{x} = (x_1, x_2, \ldots)$, $\mathbf{y} = (y_1, y_2, \ldots)$.
- Product $\cdot$: $s_\lambda(\mathbf{x}) \cdot s_\mu(\mathbf{x}) = \sum_\nu c^\nu_{\lambda\mu}\, s_\nu(\mathbf{x})$
  Littlewood-Richardson coefficients $c^\nu_{\lambda\mu} \in \mathbb{Z}_{\geq 0}$.
- Identity ι: $s_0(\mathbf{x}) = 1$.
- Coproduct Δ: $s_\nu(\mathbf{x}, \mathbf{y}) = \sum_{\lambda,\mu} c^\nu_{\lambda\mu}\, s_\lambda(\mathbf{x})\, s_\mu(\mathbf{y})$.
- Counit ϵ: $\epsilon(s_\lambda(\mathbf{x})) = \delta_{\lambda 0}$.
- Antipode S: $\mathsf{S}(s_\lambda(\mathbf{x})) = (-1)^{|\lambda|}\, s_{\lambda'}(\mathbf{x})$
  where λ' is the conjugate of λ.

Connection with classical groups

General linear group $GL(N)$

- Covariant tensor irreducible representations $V^{\lambda}_{GL(N)}$ with λ a partition of length $\ell(\lambda) \leq N$.
- Character $\mathrm{ch}\, V^{\lambda}_{GL(N)} = s_{\lambda}(\mathbf{x})$ where $x_1, \ldots, x_N$ are the eigenvalues of $A \in GL(N)$

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- Tensor product representation $V^{\lambda}_{GL(N)} \otimes V^{\mu}_{GL(N)}$.
- Decomposition into irreducibles: $\bigoplus_{\nu} c^{\nu}_{\lambda\mu}\, V^{\nu}_{GL(N)}$.

- Restriction $GL(N+M) \to GL(N) \times GL(M)$.
- Branching rule: $V^{\nu}_{GL(N+M)} \to \bigoplus_{\lambda,\mu} c^{\nu}_{\lambda\mu}\, V^{\lambda}_{GL(N)} \otimes V^{\mu}_{GL(M)}$.

Hopf algebraic properties

Scalar product $(\cdot \,|\, \cdot)$ such that $(s_\lambda \,|\, s_\mu) = \delta_{\lambda\mu}$

- $(s_\nu \,|\, s_\lambda \cdot s_\mu) = (s_\nu \,|\, s_{\lambda \cdot \mu}) = c^\nu_{\lambda\mu} = (s_{\nu/\lambda} \,|\, s_\mu) = (s_\nu / s_\lambda \,|\, s_\mu)$
- $(\Delta(s_\nu) \,|\, s_\lambda \otimes s_\mu) = (s_\nu \,|\, s_\lambda \cdot s_\mu)$ since $\sum_{\sigma,\tau} c^\nu_{\sigma\tau} \left(s_\sigma \otimes s_\tau \,|\, s_\lambda \otimes s_\mu\right) = \sum_{\sigma,\tau} c^\nu_{\sigma\tau} \delta_{\sigma\lambda} \delta_{\tau\mu} = c^\nu_{\lambda\mu}$

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- $(\Delta(s_\nu)\,|\,s_\lambda \otimes s_\mu) = (s_\nu\,|\,s_\lambda \cdot s_\mu)$ since $\sum_{\sigma,\tau} c^\nu_{\sigma\tau}\,(s_\sigma \otimes s_\tau\,|\,s_\lambda \otimes s_\mu) = \sum_{\sigma,\tau} c^\nu_{\sigma\tau}\delta_{\sigma\lambda}\delta_{\tau\mu} = c^\nu_{\lambda\mu}$

Bialgebraic properties

- Commutative: $s_\lambda \cdot s_\mu = s_\mu \cdot s_\lambda$
- Cocommutative: $\Delta(s_\nu) = \sum_\lambda s_\lambda \otimes s_{\nu/\lambda} = \sum_\mu s_{\nu/\mu} \otimes s_\mu$
- Associative: $s_\rho \cdot (s_\sigma \cdot s_\tau) = (s_\rho \cdot s_\sigma) \cdot s_\tau = \sum_{\rho,\sigma,\tau} c^\lambda_{\rho\sigma\tau}\; s_\lambda$
- Coassociative: $(I \otimes \Delta)(\Delta(s_\lambda)) = (\Delta \otimes I)(\Delta(s_\lambda))$ since $\sum_{\rho,\mu} c^\lambda_{\rho\mu}\; s_\rho \otimes \left(\sum_{\sigma,\tau} c^\mu_{\sigma,\tau}\; s_\sigma \otimes s_\tau\right) = \sum_{\rho,\sigma,\tau} c^\lambda_{\rho\sigma\tau}\; s_\lambda$
$\sum_{\nu,\tau} c^\lambda_{\nu\tau}\; \left(\sum_{\rho,\sigma} c^\nu_{\rho,\sigma}\; s_\rho \otimes s_\sigma\right) \otimes s_\tau = \sum_{\rho,\sigma,\tau} c^\lambda_{\rho\sigma\tau}\; s_\lambda$

Hopf algebraic properties

Antipode identity $\quad \cdot\,(\mathsf{S}\otimes I)\,\Delta = \iota\,\epsilon = \cdot\,(I\otimes \mathsf{S})\,\Delta$

- $\cdot\,(\mathsf{S}\otimes I)\,\Delta(s_\lambda) = \sum_\mu \cdot\,(\mathsf{S}\otimes I)\,(s_\mu \otimes s_{\lambda/\mu})$

  $= \sum_\mu \mathsf{S}(s_\mu)\cdot s_{\lambda/\mu} = \sum_\mu (-1)^{|\mu|} s_{\mu'}\cdot s_{\lambda/\mu} = \delta_{\lambda 0}\, s_0$
- $\iota\,\epsilon(s_\lambda) = \iota\,\delta_{\lambda 0} = \delta_{\lambda 0}\, s_0$

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- $\cdot\,(\epsilon\otimes I)\,\Delta(s_\lambda) = \sum_\mu \cdot\,(\epsilon\otimes I)\,(s_\mu \otimes s_{\lambda/\mu})$

  $= \sum_\mu \cdot\,(\delta_{\mu 0}\otimes s_{\lambda/\mu}) = s_\lambda.$
- $I(s_\lambda) = s_\lambda.$

Hopf algebraic properties

Product and coproduct compatibility:

$$\Delta\,(\cdot) = (\cdot \otimes \cdot)\,(I \otimes \mathsf{sw} \otimes I)\,(\Delta \otimes \Delta)$$

- $\Delta\,(\cdot)\,(s_\lambda \otimes s_\mu) = \Delta(s_{\lambda\cdot\mu}) = \sum_\rho s_\rho \otimes s_{(\lambda\cdot\mu)/\rho}$
- $(\cdot \otimes \cdot)\,(I \otimes \mathsf{sw} \otimes I)\,(\Delta \otimes \Delta)(s_\lambda \otimes s_\mu) =$
  $(\cdot \otimes \cdot)\,(I \otimes \mathsf{sw} \otimes I)\,\sum_{\sigma,\tau} s_\sigma \otimes s_{\lambda/\sigma} \otimes s_\tau \otimes s_{\mu/\tau}) =$
  $\sum_{\sigma,\tau} (\cdot \otimes \cdot)\,(s_\sigma \otimes s_\tau \otimes s_{\lambda/\sigma} \otimes s_{\mu/\tau}) = \sum_{\sigma,\tau} \left(s_{\sigma\cdot\tau} \otimes s_{(\lambda/\sigma)\cdot(\mu/\tau)}\right)$

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- $(\cdot \otimes \cdot)\,(I \otimes \mathsf{sw} \otimes I)\,(\Delta \otimes \Delta)(s_\lambda \otimes s_\mu) =$
  $(\cdot \otimes \cdot)\,(I \otimes \mathsf{sw} \otimes I)\,\sum_{\sigma,\tau} \, s_\sigma \otimes s_{\lambda/\sigma} \otimes s_\tau \otimes s_{\mu/\tau}) =$
  $\sum_{\sigma,\tau}\,(\cdot \otimes \cdot)\,(s_\sigma \otimes s_\tau \otimes s_{\lambda/\sigma} \otimes s_{\mu/\tau}) = \sum_{\sigma,\tau}\,(s_{\sigma\cdot\tau} \otimes s_{(\lambda/\sigma)\cdot(\mu/\tau)})$

Implications:

- $s_{(\lambda\cdot\mu)/\rho} = \sum_{\sigma,\tau} \, c^\rho_{\sigma\tau}\, s_{\lambda/\sigma} \cdot s_{\mu/\tau}$
- $\Delta(s_\lambda \cdot s_\mu) = \Delta(s_\lambda) \cdot \Delta(s_\mu)$
- $\Delta(X \cdot Y) = \Delta(X) \cdot \Delta(Y)$ for any $X, Y \in \Lambda$.

Characters of classical groups

Characters of irreps V^λ with λ a partition

$$GL(N): \quad \{\lambda\}(\mathbf{x}) = s_\lambda(\mathbf{x})$$

$$O(N): \quad [\lambda](\mathbf{x}) = s_{\lambda/C}(\mathbf{x}) \quad \text{where} \quad C(\mathbf{x}) = \prod_{i\leq j}(1 - x_i x_j)$$

$$Sp(N): \quad \langle\lambda\rangle(\mathbf{x}) = s_{\lambda/A}(\mathbf{x}) \quad \text{where} \quad A(\mathbf{x}) = \prod_{i<j}(1 - x_i x_j)$$

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Generating functions

$$\prod_{i,a}(1 - x_i y_a)^{-1} = \sum_\lambda \{\lambda\}(\mathbf{x})\ \{\lambda\}(\mathbf{y})$$

$$\prod_{i,a}(1 - x_i y_a)^{-1} \prod_{a\leq b}(1 - y_a y_b) = \sum_\lambda [\lambda](\mathbf{x})\ \{\lambda\}(\mathbf{y})$$

$$\prod_{i,a}(1 - x_i y_a)^{-1} \prod_{a<b}(1 - y_a y_b) = \sum_\lambda \langle\lambda\rangle(\mathbf{x})\ \{\lambda\}(\mathbf{y})$$

Group-subgroup restrictions

Branching rules

- $GL(N) \supset GL(N-1)$: $\{\lambda\} \to \{\lambda/M\}$
- $GL(N) \supset O(N)$: $\{\lambda\} \to [\lambda/D]$
- $GL(N) \supset Sp(N)$: $\{\lambda\} \to \langle\lambda/B\rangle$
  - $M(\mathbf{x}) = \prod_a (1 - x_a)^{-1} = \sum_m s_m(\mathbf{x})$
  - $D(\mathbf{x}) = \prod_{a\leq b} (1 - x_a x_b)^{-1}$ so that $D(\mathbf{x})C(\mathbf{x}) = 1 = s_0(\mathbf{x})$
  - $B(\mathbf{x}) = \prod_{a<b} (1 - x_a x_b)^{-1}$ so that $B(\mathbf{x})A(\mathbf{x}) = 1 = s_0(\mathbf{x})$

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Proof

- $s_\lambda(\mathbf{x}, 1) = \sum_\mu s_{\lambda/\mu}(\mathbf{x}) s_\mu(1) = \sum_m s_{\lambda/m}(\mathbf{x}) = s_{\lambda/M}(\mathbf{x})$
- $s_\lambda(\mathbf{x}) = s_{\lambda/0}(\mathbf{x}) = s_{\lambda/DC}(\mathbf{x}) = s_{(\lambda/D)/C}(\mathbf{x})$
- $s_\lambda(\mathbf{x}) = s_{\lambda/0}(\mathbf{x}) = s_{\lambda/BA}(\mathbf{x}) = s_{(\lambda/B)/A}(\mathbf{x})$

Coproducts of Schur function series

Coproducts

- $\Delta(M) = M \otimes M$
- $\Delta(D) = (D \otimes D) \cdot \Delta''(D)$ with $\Delta''(D) = \sum_\sigma s_\sigma \otimes s_\sigma$
- $\Delta(B) = (B \otimes B) \cdot \Delta''(B)$ with $\Delta''(B) = \sum_\sigma s_\sigma \otimes s_\sigma$

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Proof

- $M(\mathbf{x},\mathbf{y}) = \prod_i (1-x_i)^{-1} \, \prod_a (1-y_a)^{-1} = M(\mathbf{x})\, M(\mathbf{y})$
- $$D(\mathbf{x},\mathbf{y}) = \prod_{i\le j}(1-x_ix_j)^{-1} \prod_{i,a}(1-x_iy_a)^{-1} \prod_{a\le b}(1-y_ay_b)^{-1} = D(\mathbf{x}) \, \sum_\sigma s_\sigma(\mathbf{x})\, s_\sigma(\mathbf{y})\, D(\mathbf{y})$$
- $$B(\mathbf{x},\mathbf{y}) = \prod_{i< j}(1-x_ix_j)^{-1} \prod_{i,a}(1-x_iy_a)^{-1} \prod_{a< b}(1-y_ay_b)^{-1} = B(\mathbf{x}) \, \sum_\sigma s_\sigma(\mathbf{x})\, s_\sigma(\mathbf{y})\, B(\mathbf{y})$$

Tensor product rule for $GL(N-1)$

Under restriction from $GL(N)$ to $GL(N-1)$:

- $\{\lambda\}_N = \{\lambda/M\}_{N-1}$ and $\{\lambda\}_{N-1} = \{\lambda/L\}_N$ with $L = M^{-1}$.
- $(\{\lambda\}_{N-1} \cdot \{\mu\}_{N-1} \mid \{\nu\}_{N-1}) = c^{\nu}_{\lambda\mu}$.

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- $(\{\lambda\}_{N-1} \cdot \{\mu\}_{N-1} \mid \{\nu\}_{N-1}) = c^{\nu}_{\lambda\mu}$.

Proof Note that $\{\lambda\}_{N-1} \cdot \{\mu\}_{N-1} = \{((\lambda/L) \cdot (\mu/L))/M\}_{N-1}$

Hence $(\{\lambda\}_{N-1} \cdot \{\mu\}_{N-1} \mid \{\nu\}_{N-1})$

$$= (\{((\lambda/L) \cdot (\mu/L))/M\} \mid \{\nu\})$$
$$= (\{(\lambda/L) \cdot (\mu/L)\} \mid M \cdot \{\nu\})$$
$$= (\{\lambda/L\} \otimes \{\mu/L\} \mid \Delta(M \cdot \{\nu\}))$$
$$= (\{\lambda/L\} \otimes \{\mu/L\} \mid (M \otimes M) \cdot \Delta(\{\nu\}))$$
$$= (\{\lambda/LM\} \otimes \{\mu/LM\} \mid \Delta(\{\nu\}))$$
$$= (\{\lambda\} \otimes \{\mu\} \mid \Delta(\{\nu\})) = (\{(\lambda\} \cdot \{\mu\} \mid \{\nu\}) = c^{\nu}_{\lambda\mu}$$

Tensor products of irreps

Tensor product rule for $O(N)$: $[\lambda] \cdot [\mu] = \sum_\sigma [(\lambda/\sigma) \cdot (\mu/\sigma)]$

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Proof Note that $[\lambda] \cdot [\mu] = [((\lambda/C) \cdot (\mu/C))/D]$.

Hence $([\lambda] \cdot [\mu] \mid [\nu])$

$$= (\{((\lambda/C) \cdot (\mu/C))/D\} \mid \{\nu\})$$
$$= (\{(\lambda/C) \cdot (\mu/C)\} \mid D \cdot \{\nu\})$$
$$= (\{\lambda/C\} \otimes \{\mu/C\} \mid \Delta(D \cdot \{\nu\}))$$
$$= (\{\lambda/C\} \otimes \{\mu/C\} \mid (D \otimes D) \cdot \Delta''(D) \cdot \Delta(\{\nu\}))$$
$$= (\{(\lambda/CD\} \otimes \{\mu/CD\} \mid \Delta''(D) \cdot \Delta(\{\nu\}))$$
$$= (\{\lambda\} \otimes \{\mu\} \mid \sum_\sigma \{\sigma\} \otimes \{\sigma\} \cdot \Delta(\{\nu\}))$$
$$= \sum_\sigma (\{\lambda/\sigma\} \otimes \{\mu/\sigma\} \mid \Delta(\{\nu\}))$$
$$= \sum_\sigma (\{\lambda/\sigma\} \cdot \{\mu/\sigma\} \mid \{\nu\})$$

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Tensor product rule for $Sp(N)$: $\langle\lambda\rangle \cdot \langle\mu\rangle = \sum_\sigma \langle(\lambda/\sigma)\cdot(\mu/\sigma)\rangle$

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Tensor product rule for $Sp(N)$: $\langle\lambda\rangle \cdot \langle\mu\rangle = \sum_\sigma \langle(\lambda/\sigma)\cdot(\mu/\sigma)\rangle$

Proof Note that $\langle\lambda\rangle \cdot \langle\mu\rangle = \langle((\lambda/A)\cdot(\mu/A))/B\rangle$.

Hence $(\langle\lambda\rangle \cdot \langle\mu\rangle \mid \langle\nu\rangle)$

$$
\begin{aligned}
&= (\{((\lambda/A)\cdot(\mu/A))/B\} \mid \{\nu\}) \\
&= (\{(\lambda/A)\cdot(\mu/A)\} \mid B \cdot \{\nu\}) \\
&= (\{\lambda/A\} \otimes \{\mu/A\} \mid \Delta(B \cdot \{\nu\})) \\
&= (\{\lambda/A\} \otimes \{\mu/A\} \mid (B \otimes B) \cdot \Delta''(B) \cdot \Delta(\{\nu\})) \\
&= (\{(\lambda/AB\} \otimes \{\mu/AB\} \mid \Delta''(B) \cdot \Delta(\{\nu\})) \\
&= (\{\lambda\} \otimes \{\mu\} \mid \textstyle\sum_\sigma \{\sigma\} \otimes \{\sigma\} \cdot \Delta(\{\nu\})) \\
&= \textstyle\sum_\sigma (\{\lambda/\sigma\} \otimes \{\mu/\sigma\} \mid \Delta(\{\nu\})) \\
&= \textstyle\sum_\sigma (\{\lambda/\sigma\} \cdot \{\mu/\sigma\} \mid \{\nu\})
\end{aligned}
$$

Tensorial approach

Basic subgroup invariants

- $GL(N-1) = \{A \in GL(N) \,|\, A_i^a\, v_a = v_i\}$
- $O(N) = \{A \in GL(N) \,|\, A_i^a\, A_j^b\, g_{ab} = g_{ij}$ with $g_{ij} = g_{ji}\}$
- $Sp(N) = \{A \in GL(N) \,|\, A_i^a\, A_j^b\, f_{ab} = f_{ij}$ with $f_{ij} = -f_{ji}\}$

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Polynomial invariants

- $M(\mathbf{x}) = \prod_a (1 - x_a)^{-1} = \sum_m \; s_m(\mathbf{x}) = \sum_m \; s_m(s_1(\mathbf{x}))$
- $D(\mathbf{x}) = \prod_{a \leq b} (1 - x_a x_b)^{-1} = \sum_m \; s_m(s_2(\mathbf{x}))$
- $B(\mathbf{x}) = \prod_{a < b} (1 - x_a x_b)^{-1} = \sum_m \; s_m(s_{1^2}(\mathbf{x}))$

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Plethysms $s_\lambda(s_\pi(\mathbf{x})) = s_\lambda(\mathbf{y})$ with $\mathbf{y} = (y_1, y_2 \ldots)$
where $s_\pi(\mathbf{x}) = \sum_k y_k$ is the monomial expansion of $s_\pi(\mathbf{x})$.

Generalisation to other subgroups

Invariant $\eta_{ij\cdots k}$ of rank n and symmetry π

- π is a partition of n with corresponding Young symmetrizer Y^π an idempotent in the algebra of S_n.
- S_n acts naturally by permuting the n indices of $\eta_{ij\cdots k}$.
- $Y^\pi : \ \eta_{ij\cdots k} \mapsto \eta_{ij\cdots k}$.

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Subgroup $H_\pi(N)$ of $GL(N)$ leaving η invariant

- $H_\pi(N) = \{A \in GL(N) \,|\, A_i^a\, A_j^b \cdots A_k^c\, \eta_{ab\cdots c} = \eta_{ij\cdots k}\}$

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Polynomial invariants

- $M_\pi(\mathbf{x}) = M(s_\pi(\mathbf{x})) = \sum_m s_m(s_\pi(\mathbf{x})) = \prod_k (1 - y_k)^{-1}$
- Ex: $M_1(\mathbf{x}) = M(\mathbf{x}), \quad M_2(\mathbf{x}) = D(\mathbf{x}), \quad M_{1^2}(\mathbf{x}) = B(\mathbf{x})$.

Branching rule and tensor products

Branching rule

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Under restriction from $GL(N)$ to $H_\pi(N)$:

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Tensor product rule for $H_\pi(N)$

- $[\![\lambda]\!] \cdot [\![\mu]\!]$

$$= \sum_{\sigma(\xi,\zeta,k)} [\![\{\lambda / \prod_{\xi,\zeta<\pi} \prod_{k=1}^{c^\pi_{\xi\zeta}} \xi \otimes_{pl} \sigma(\xi,\zeta,k)\}$$

$$\cdot \{\mu / \prod_{\xi,\zeta<\pi} \prod_{k=1}^{c^\pi_{\xi\zeta}} \zeta \otimes_{pl} \sigma(\xi,\zeta,k)\}]\!].$$

where for each ξ, ζ and k the summation is over all $\sigma(\xi,\zeta,k)$.

Proof of tensor product rule

Notation for plethysms

- $s_\lambda(s_\pi) = \{\pi\}^{\otimes_{pl}\{\lambda\}} = \{\pi\} \otimes_{pl} \{\lambda\} = \{\pi \otimes_{pl} \lambda\} = s_{\pi \otimes_{pl} \lambda}$

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Some properties of plethysms with M

- $\Delta(\{\pi\} \otimes_{pl} M) = (\Delta(\{\pi\}) \otimes_{pl} M$ since both $= M(s_\pi(\mathbf{x}, \mathbf{y}))$
- $(\{\lambda\} + \{\mu\}) \otimes_{pl} M = (\{\lambda\} \otimes_{pl} M) \cdot (\{\mu\} \otimes_{pl} M)$
- $(\{\lambda\} \otimes \{\mu\}) \otimes_{pl} M = \sum_\sigma (\{\lambda\} \otimes_{pl} \{\sigma\}) \otimes (\{\mu\} \otimes_{pl} \{\sigma\})$

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Proofs: Let $s_\lambda = \sum_i y_i$ and $s_\mu = \sum_a z_a$

- $M(s_\lambda + s_\mu) = \prod_i (1 - y_i)^{-1} \prod_a (1 - z_a)^{-1} = M(s_\lambda)\, M(s_\mu)$
- $M(s_\lambda \otimes s_\mu) = \prod_{i,a} (1 - y_i z_a)^{-1} = \sum_\sigma s_\sigma \otimes s_\sigma$
- Note: $M(s_\lambda \otimes 1) = \prod_i (1 - y_i\, 1)^{-1} = M(s_\lambda) \otimes 1$.

Proof of tensor product rule

Proper cut coproduct definitions

- $\Delta(s_\pi) = s_\pi \otimes 1 + 1 \otimes s_\pi + \Delta'(s_\pi)$
- $\Delta(M_\pi) = (M_\pi \otimes M_\pi) \cdot \Delta''(M_\pi)$

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Relation between cut coproducts

- $\Delta''(M_\pi) = M(\Delta'(s_\pi))$

Proof:

$$\Delta(M_\pi) = \Delta(M(s_\pi)) = M(\Delta(s_\pi)) = M(s_\pi \otimes 1 + 1 \otimes s_\pi + \Delta'(s_\pi))$$
$$= M(s_\pi \otimes 1) \cdot M(1 \otimes s_\pi) \cdot M(\Delta'(s_\pi))$$
$$= (M(s_\pi) \otimes 1) \cdot (1 \otimes M(s_\pi)) \cdot M(\Delta'(s_\pi))$$
$$= (M(s_\pi) \otimes M(s_\pi)) \cdot M(\Delta'(s_\pi))$$
$$= (M_\pi \otimes M_\pi) \cdot M(\Delta'(s_\pi))$$

Proof of tensor product rule

Proof Note that $[\![\lambda]\!] \cdot [\![\mu]\!] = [\![((\lambda/L_\pi) \cdot (\mu/L_\pi))/M_\pi]\!]$

Hence $([\![\lambda]\!] \cdot [\![\mu]\!] \,|\, [\![\nu]\!])$

$$= (\{(\lambda/L_\pi) \cdot (\mu/L_\pi))/M_\pi\} \,|\, \{\nu\})$$

$$= \cdots$$

$$= (\{\lambda\} \otimes \{\mu\} \mid \Delta''(M_\pi) \;\cdot\; \Delta(\{\nu\}))$$

$$= (\{\lambda\} \otimes \{\mu\} \mid M(\Delta'(s_\pi)) \;\cdot\; \Delta(\{\nu\}))$$

Proof of tensor product rule

Proof Note that $[\![\lambda]\!] \cdot [\![\mu]\!] = [\![((\lambda/L_\pi) \cdot (\mu/L_\pi))/M_\pi]\!]$

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$= \cdots$

$= (\{\lambda\} \otimes \{\mu\} \,|\, \Delta''(M_\pi) \cdot \Delta(\{\nu\}))$

$= (\{\lambda\} \otimes \{\mu\} \,|\, M(\Delta'(s_\pi)) \cdot \Delta(\{\nu\}))$

Now use

- $\Delta'(s_\pi) = \sum_{\xi,\zeta<\pi} c^\pi_{\xi\zeta}\, s_\xi \otimes s_\zeta = \sum_{\xi,\zeta<\pi} \sum_{k=1}^{c^\pi_{\xi\zeta}} s_\xi \otimes s_\zeta$
- $M(\Delta'(s_\pi)) = \prod_{\xi,\zeta<\pi} \prod_{k=1}^{c^\pi_{\xi\zeta}} M(s_\xi \otimes s_\zeta)$
- $M(s_\xi \otimes s_\zeta) = \sum_\sigma s_\sigma(s_\xi) \otimes s_\sigma(s_\zeta) = \sum_\sigma \{\xi \otimes_{pl} \sigma\} \otimes \{\zeta \otimes_{pl} \sigma\}$.

Examples: plethysms of the series M

π	$M_\pi = M(s_\pi)$ up to weight 6
1	$0 + 1 + 2 + 3 + 4 + 5 + 6 + \cdots$
2	$0 + 2 + 4 + 2^2 + 6 + 42 + 2^3 + \cdots$
1^2	$0 + 1^2 + 2^2 + 1^4 + 3^2 + 2^2 1^2 + 1^6 \cdots$
3	$0 + 3 + 6 + 42 + \cdots$
21	$0 + 21 + 42 + 321 + 31^3 + 2^3 + \cdots$
1^3	$0 + 1^3 + 2^3 + 21^4 + \cdots$

Examples: branching rules

Restrictions from $GL(N)$ to $H_\pi(N)$: $\{\lambda\} \to [\![\lambda/M_\pi]\!]$

π	1	2	1^2	3	21	1^3
λ	$GL(N-1)$	$O(N)$	$Sp(N)$	$H_3(N)$	$H_{21}(N)$	$H_{1^3}(N)$
0	0	0	0	0	0	0
1	$1+0$	1	1	1	1	1
2	$2+1+0$	$2+0$	2	2	2	2
1^2	1^2+1	1^2	1^2+0	1^2	1^2	1^2
3	$3+2+1+0$	$3+1$	3	$3+0$	3	3
21	$21+2+1^2+1$	$21+1$	$21+1$	21	$21+0$	21
1^3	1^3+1^2	1^3	1^3+1	1^3	1^3	1^3+0

Examples: coproducts of s_π

π	$\Delta(s_\pi)$ and $\Delta'(s_\pi)$
1	$1,0+0,1$
2	$2,0+0,2+1,1$
1^2	$1^2,0+0,1^2+1,1$
3	$3,0+0,3+2,1+1,2$
21	$21,0+0,21+2,1+1^2,1+1,2+1,1^2$
1^3	$1^3,0+0,1^3+1^2,2+1,1^2$

Examples: proper cut coproducts of M_π

π	$\Delta''(M_\pi) = M(\Delta'(s_\pi))$
1	$0 \otimes 0$
2	$\sum_\sigma \sigma \otimes \sigma$
1^2	$\sum_\sigma \sigma \otimes \sigma$
3	$\sum_{\sigma,\tau} (\sigma(2) \cdot \tau) \otimes (\sigma \cdot \tau(2))$
21	$\sum_{\sigma,\tau,\phi,\psi} (\sigma(2) \cdot \tau(1^2) \cdot \phi \cdot \psi) \otimes (\sigma \cdot \tau \cdot \phi(2) \cdot \psi(1^2))$
1^3	$\sum_{\sigma,\tau} (\sigma(1^2) \cdot \tau) \otimes (\sigma \cdot \tau(1^2))$

Examples: tensor products for classical groups

$$GL(N) \quad \{2^2\} \cdot \{21\}$$

$$= \{43\} + \{421\} + \{3^2 1\} + \{32^2\} + \{321^2\} + \{2^3 1\}$$

$$O(N) \quad [2^2] \cdot [21]$$

$$= [43] + [421] + [3^2 1] + [32^2] + [321^2] + [2^3 1]$$

$$+[41] + 2[32] + 2[31^2] + 2[2^2 1] + [21^3]$$

$$+[3] + 2[21] + [1^3] + [1]$$

$$Sp(N) \quad \langle 2^2 \rangle \cdot \langle 21 \rangle$$

$$= \langle 43 \rangle + \langle 421 \rangle + \langle 3^2 1 \rangle + \langle 32^2 \rangle + \langle 321^2 \rangle + \langle 2^3 1 \rangle$$

$$+\langle 41 \rangle + 2\langle 32 \rangle + 2\langle 31^2 \rangle + 2\langle 2^2 1 \rangle + \langle 21^3 \rangle$$

$$+\langle 3 \rangle + 2\langle 21 \rangle + \langle 1^3 \rangle + \langle 1 \rangle$$

Examples: tensor products for $H_\pi(N)$

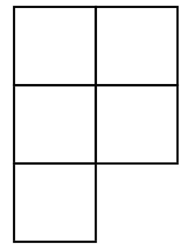
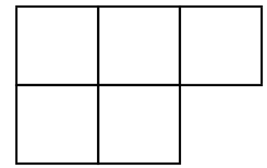
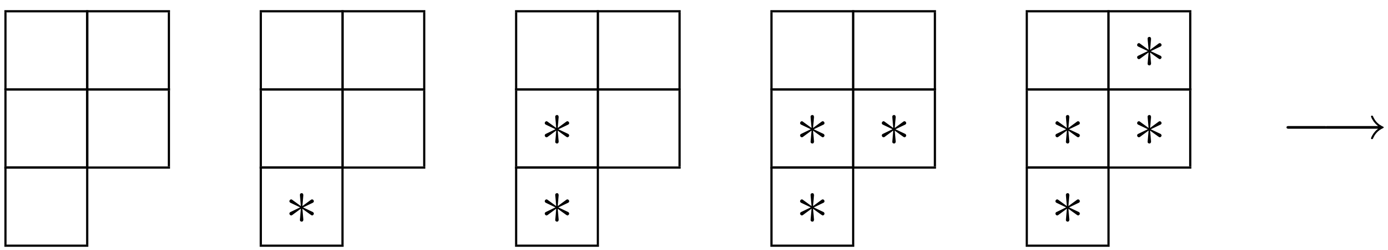
Evaluation of $[\![2^2]\!] \cdot [\![21]\!]$

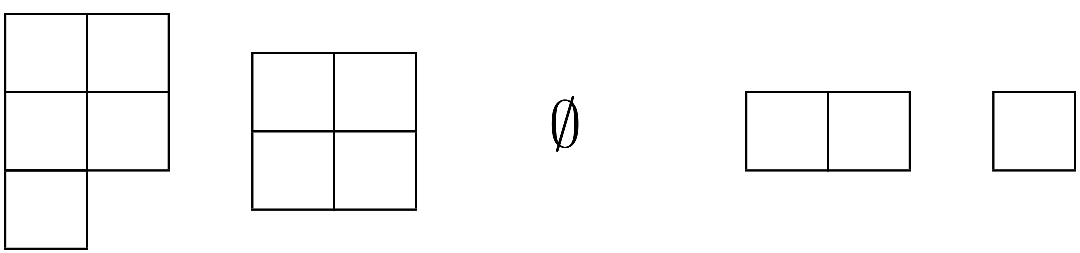
$$H_3(N) \quad [\![43]\!] + [\![421]\!] + [\![3^21]\!] + [\![32^2]\!] + [\![321^2]\!] + [\![2^31]\!]$$
$$+[\![4]\!] + 3[\![31]\!] + 2[\![2^2]\!] + 2[\![21^2]\!] + 2[\![1]\!]$$

$$H_{21}(N) \quad [\![43]\!] + [\![421]\!] + [\![3^21]\!] + [\![32^2]\!] + [\![321^2]\!] + [\![2^31]\!]$$
$$+[\![4]\!] + 5[\![31]\!] + 4[\![2^2]\!] + 5[\![21^2]\!] + [\![1^4]\!] + 6[\![1]\!]$$

$$H_{1^3}(N) \quad [\![43]\!] + [\![421]\!] + [\![3^21]\!] + [\![32^2]\!] + [\![321^2]\!] + [\![2^31]\!]$$
$$+2[\![31]\!] + 2[\![2^2]\!] + 3[\![21^2]\!] + [\![1^4]\!] + 2[\![1]\!]$$

Young diagrams and boundary strip removal

- Each partition λ define a Young diagram F^{λ}
- Transposition gives $F^{\lambda'}$ where λ' is the conjugate of λ
- Ex: $\lambda = (2^2 1)$, $\lambda' = (32)$ $F^{2^2 1} =$  $F^{32} =$ 
- $F^{\lambda - h}$ is obtained from F^{λ} by removing strip of length h starting at foot of 1st column extending over c columns.
- 

- 

where $\emptyset$ indicates $F^{\lambda - h}$ not regular.

Modification rules

Standard characters

- $GL(N)$: $\{\lambda\}$ with $\lambda'_1 \leq N$.
- $O(N)$: $[\lambda]$ with $\lambda'_1 + \lambda'_2 \leq N$.
- $Sp(N)$: $\langle\lambda\rangle$ with $2\lambda'_1 \leq N$.

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- $Sp(N)$: $\langle\lambda\rangle$ with $2\lambda'_1 \leq N$.

Modification rules for non-standard characters

- $GL(N)$: $\{\lambda\} = 0$ if $\lambda'_1 > N$.
- $O(N)$: $[\lambda] = (-1)^{c-1}[\lambda - h]^*$ if $h = 2\lambda'_1 - N > 0$.
- $Sp(N)$: $\langle\lambda\rangle = (-1)^c\langle\lambda - h\rangle$ if $h = 2\lambda'_1 - N - 2 \geq 0$.

with $[\lambda - h]$ and $\langle\lambda - h\rangle$ both $= 0$ if $F^{\lambda-h}$ is not regular

Application of modification rules for $GL(N)$

$GL(N)$ modifications: $\{\lambda\} = 0$ if $\ell(\lambda) > N$.

- Examples:
  - $N < 4$: $\{321^2\} = \{2^31\} = 0$
  - $N < 3$: $\{421\} = \{3^21\} = \{32^2\} = 0$

Application of modification rules for $GL(N)$

$GL(N)$ modifications: $\{\lambda\} = 0$ if $\ell(\lambda) > N$.

- Examples:
  - $N < 4$: $\{321^2\} = \{2^31\} = 0$
  - $N < 3$: $\{421\} = \{3^21\} = \{32^2\} = 0$
- For $GL(N)$ the product $\{2^2\} \cdot \{21\}$ modifies to give

$$N \geq 4 \quad \{43\} + \{421\} + \{3^21\} + \{32^2\} + \{321^2\} + \{2^31\}$$

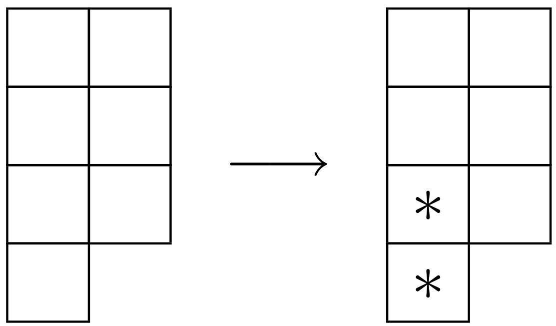
$$N = 3 \quad \{43\} + \{421\} + \{3^21\} + \{32^2\}$$

$$N = 2 \quad \{43\}$$

Examples of modifications for $O(N)$

$O(N)$ modifications: $[\lambda] = (-1)^{c-1}[\lambda - h]^*$ with $h = 2\ell(\lambda) - N$

- $N = 6 \quad \lambda = (2^3 1)$, $h = 2$



$\Rightarrow \quad [2^3 1] = 0$

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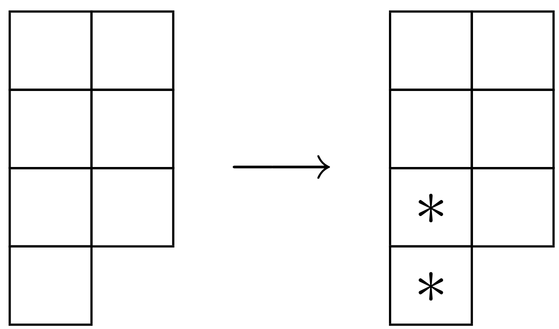
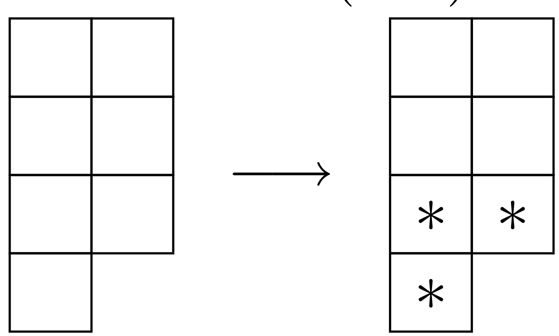
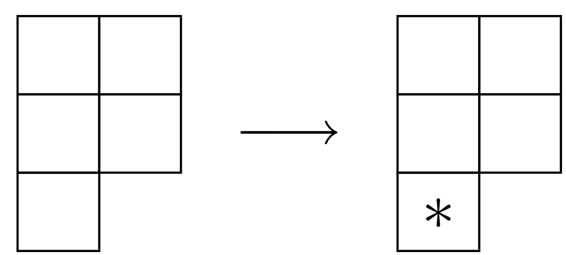


$\Rightarrow \quad [2^3 1] = 0$

- $N = 5 \quad \lambda = (2^3 1)$, $h = 3$ and $\lambda = (2^2 1)$, $h = 1$



and



$\Rightarrow \quad [2^3 1] = -[2^2]^*$ and $[2^2 1] = [2^2]^*$ $\Rightarrow \quad [2^3 1] = -[2^2 1]$.

$O(N)$ tensor product $[2^2] \cdot [21]$

$$N \geq 7 \quad [43] + [421] + [3^21] + [32^2] + [321^2] + [2^31]$$
$$+[41] + 2[32] + 2[31^2] + 2[2^21] + [21^3]$$
$$+[3] + 2[21] + [1^3] + [1]$$

$$N = 6 \quad [43] + [421] + [3^21] + [32^2] + [321^2]$$
$$+[41] + 2[32] + 2[31^2] + 2[2^21] + [21^3]$$
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$$N = 5 \quad [43] + [421] + [3^21] + [41] + 2[32] + 2[31^2]$$
$$+[2^21] + [21^3] + [3] + 2[21] + [1^3] + [1]$$

$$N = 4 \quad [43] + [41] + [32] + [31^2] + [3] + [21] + [1^3] + [1]$$

Examples of modifications for $Sp(N)$

$Sp(N)$: $\langle\lambda\rangle = (-1)^c\langle\lambda - h\rangle$ with $h = 2\ell(\lambda) - N - 2$

- Ex: In $Sp(6)$ $\langle 321^2\rangle = \langle 2^3 1\rangle = \langle 21^3\rangle = 0$ since $h = 0$ and $c = 1$ (strip of length 0 removed from the 1st column)

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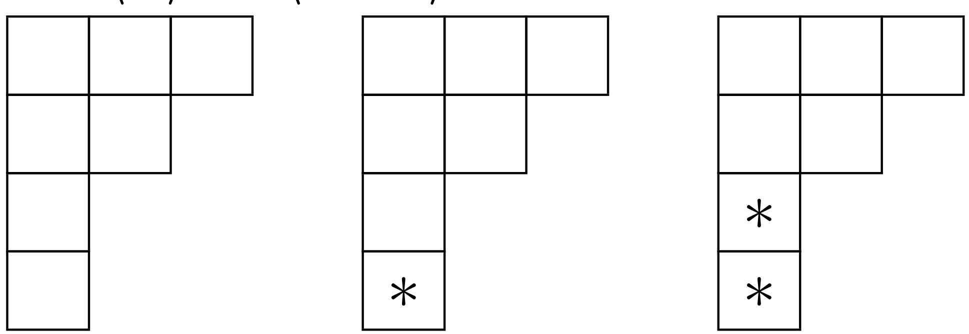
- Ex: $\langle\lambda\rangle = \langle 321^2\rangle$, $h = 6 - N$.



  - $N = 6: \quad \langle 321^2\rangle = -\langle 321^2\rangle = 0$
  - $N = 5: \quad \langle 321^2\rangle = -\langle 321\rangle$
  - $N = 4: \quad \langle 321^2\rangle = -\langle 32\rangle$

$Sp(N)$ **tensor product** $\langle 2^2 \rangle \cdot \langle 21 \rangle$

$$N \geq 7 \quad \langle 43\rangle + \langle 421\rangle + \langle 3^21\rangle + \langle 32^2\rangle + \langle 321^2\rangle + \langle 2^31\rangle$$
$$+\langle 41\rangle + 2\langle 32\rangle + 2\langle 31^2\rangle + 2\langle 2^21\rangle + \langle 21^3\rangle$$
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$$N = 6 \quad \langle 43\rangle + \langle 421\rangle + \langle 3^21\rangle + \langle 32^2\rangle$$
$$+\langle 41\rangle + 2\langle 32\rangle + 2\langle 31^2\rangle + 2\langle 2^21\rangle$$
$$+\langle 3\rangle + 2\langle 21\rangle + \langle 1^3\rangle + \langle 1\rangle$$

$$N = 5 \quad \langle 43\rangle + \langle 421\rangle + \langle 3^21\rangle + \langle 32^2\rangle - \langle 321\rangle - \langle 2^3\rangle$$
$$+\langle 41\rangle + 2\langle 32\rangle + 2\langle 31^2\rangle + 2\langle 2^21\rangle - \langle 21^2\rangle$$
$$+\langle 3\rangle + 2\langle 21\rangle + \langle 1^3\rangle + \langle 1\rangle$$

$$N = 4 \quad \langle 43\rangle + \langle 41\rangle + \langle 32\rangle + \langle 3\rangle + \langle 21\rangle + \langle 1\rangle$$

Odd symplectic groups

Identification of group $H_{1^2}(2k+1) = Sp(2k+1)$

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- Canonical form of second rank antisymmetric tensor η_{ij}

$$\eta \;=\; \mathrm{diag}\left[\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \;\cdots\; \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \;\begin{pmatrix} 0 \end{pmatrix}\right]$$

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- $Sp(2k+1)$ neither semisimple nor reductive (Proctor)

$$Sp(2k+1) \quad = \quad \begin{bmatrix} & & & * \\ & Sp(2k) & & * \\ & & & * \\ 0 & \cdots & 0 & GL(1) \end{bmatrix}$$

Odd symplectic groups

Identification of characters of $H_{1^2}(2k+1) = Sp(2k+1)$

- The defining representation is indecomposable but reducible: $\langle 1\rangle = \mathrm{ch}\, V^{\langle 1\rangle}_{Sp(2k+1)} = \mathrm{ch}\, V^{\langle 1\rangle}_{Sp(2k)} + \mathrm{ch}\, V^{\{1\}}_{GL(1)}$
- Each $\langle\lambda\rangle$ is the character of an indecomposable representation (Proctor)

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- Each $\langle\lambda\rangle$ is the character of an indecomposable representation (Proctor)

Tensor products are not fully reducible

- Ex: In $Sp(3)$ $V^{\langle 2\rangle}_{Sp(3)} \otimes V^{\langle 1^2\rangle}_{Sp(3)}$ is not fully reducible.
- $\langle 2\rangle \cdot \langle 1^2\rangle = \langle 31\rangle + \langle 21^2\rangle + \langle 2\rangle + \langle 1^2\rangle = \langle 31\rangle - \langle 21\rangle + \langle 2\rangle + \langle 1^2\rangle$
- $V^{\langle 31\rangle}_{Sp(3)} + V^{\langle 2\rangle}_{Sp(3)} + V^{\langle 1^2\rangle}_{Sp(3)}$ must contain all the irreducible components of $V^{\langle 21\rangle}_{Sp(3)}$.

The group $H_{1^3}(3)$

Identification of group $H_{1^3}(3)$

- Canonical form of third rank antisymmetric invariant in 3 dimensions: $\eta_{ijk} = \epsilon_{ijk}$.
- For $A \in H_{1^3}(3)$ we have $A: \; \epsilon_{ijk} \mapsto A_i^p \, A_j^q \, A_k^r \, \epsilon_{pqr} \; = \; \epsilon_{ijk}$
- In 3-dimensions this implies and is implied by $\det A = 1$.
- Hence $H_{1^3}(3) = SL(3) = \{A \in GL(3) \,|\, \det A = 1\}$.

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- Hence $H_{1^3}(3) = SL(3) = \{A \in GL(3) \,|\, \det A = 1\}$.

Identification of characters of $H_{1^3}(3)$

- $SL(3)$ is semisimple
- All its representations are fully reducible
- In particular each $V_{SL(3)}^{[\![\lambda]\!]}$ is irreducible with $[\![\lambda]\!] = \mathrm{ch}\, V_{SL(3)}^{[\![\lambda]\!]}$

Branching rule for $GL(3) \supset H_{1^3}(3)$

$$\{\lambda\} \to [\![\lambda/M_{1^3}]\!] = [\![\lambda/(0 + 1^3 + 2^3 + 21^4 + 3^3 + \cdots)]\!]$$

Branching rule for $GL(3) \supset H_{1^3}(3)$

$$\{\lambda\} \to [\![\lambda/M_{1^3}]\!] = [\![\lambda/(0 + 1^3 + 2^3 + 21^4 + 3^3 + \cdots)]\!]$$

$\{\lambda\}_{\rm dim}$	$[\![\lambda/M_{1^3}]\!]_{\rm dim}$	$\{\lambda\}_{\rm dim}$	$[\![\lambda/M_{1^3}]\!]_{\rm dim}$
$\{0\}_1$	$[\![0]\!]_1$	$\{21^3\}_0$	$[\![21^3]\!]_{-9} + [\![2]\!]_6 + [\![1^2]\!]_3$
$\{1\}_3$	$[\![1]\!]_3$	$\{2^2\}_6$	$[\![2^2]\!]_6$
$\{1^2\}_3$	$[\![1^2]\!]_3$	$\{2^21\}_3$	$[\![2^21]\!]_0 + [\![1^2]\!]_3$
$\{1^3\}_1$	$[\![1^3]\!]_0 + [\![0]\!]_1$	$\{2^21^2\}_0$	$[\![2^21^2]\!]_{-8} + [\![21]\!]_8 + [\![1^3]\!]_0$
$\{1^4\}_0$	$[\![1^4]\!]_{-3} + [\![1]\!]_3$	$\{2^3\}_1$	$[\![2^3]\!]_0 + [\![1^3]\!]_0 + [\![0]\!]_1$
$\{2\}_6$	$[\![2]\!]_6$	$\{2^31\}_0$	$[\![2^31]\!]_0 + [\![21^2]\!]_0 + [\![1^4]\!]_{-3} + [\![1]\!]_3$
$\{21\}_8$	$[\![21]\!]_8$	$\{2^4\}_0$	$[\![2^4]\!]_3 + [\![21^3]\!]_{-9} + [\![2]\!]_6$
$\{21^2\}_3$	$[\![21^2]\!]_0 + [\![1]\!]_3$		

Modification rules and revised branchings

$\ell(\lambda) = 3$	$\ell(\lambda) = 4$
$[\![1^3]\!]_0 = 0$	$[\![1^4]\!]_{-3} = -[\![1]\!]_3$
$[\![21^2]\!]_0 = 0$	$[\![21^3]\!]_{-9} = -[\![2]\!]_6 - [\![1^2]\!]_3$
$[\![2^21]\!]_0 = 0$	$[\![2^21^2]\!]_{-8} = -[\![21]\!]_8$
$[\![2^3]\!]_0 = 0$	$[\![2^31]\!]_0 = 0, \quad [\![2^4]\!]_3 = [\![1^2]\!]_3$

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$[\![2^21]\!]_0 = 0$	$[\![2^21^2]\!]_{-8} = -[\![21]\!]_8$
$[\![2^3]\!]_0 = 0$	$[\![2^31]\!]_0 = 0, \quad [\![2^4]\!]_3 = [\![1^2]\!]_3$

$\{\lambda\}$	$[\![\lambda/M_{1^3}]\!]$	$\{\lambda\}$	$[\![\lambda/M_{1^3}]\!]$	$\{\lambda\}$	$[\![\lambda/M_{1^3}]\!]$
$\{0\}_1$	$[\![0]\!]_1$	$\{2\}_6$	$[\![2]\!]_6$	$\{2^2\}_6$	$[\![2^2]\!]_6$
$\{1\}_3$	$[\![1]\!]_3$	$\{21\}_8$	$[\![21]\!]_8$	$\{2^21\}_3$	$[\![1^2]\!]_3$
$\{1^2\}_3$	$[\![1^2]\!]_3$	$\{21^2\}_3$	$[\![1]\!]_3$	$\{2^3\}_1$	$[\![0]\!]_1$
$\{1^3\}_1$	$[\![0]\!]_1$				

Modification rules and revised branchings

$[\![\lambda_1,\lambda_2,\lambda_3,\lambda_4]\!]$ with $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \lambda_4 \geq 0$
$[\![\lambda_1,\lambda_2,\lambda_3,0]\!] = 0$ if $\lambda_3 \geq 1$
$[\![\lambda_1,\lambda_1,1,1]\!] = -[\![\lambda_1,\lambda_1-1]\!]$
$[\![\lambda_1,\lambda_2,1,1]\!] = -[\![\lambda_1,\lambda_2-1]\!] - [\![\lambda_1-1,\lambda_2]\!] = 0$ if $\lambda_1 > \lambda_2$
$[\![\lambda_1,\lambda_2,2,1]\!] = 0$
$[\![\lambda_1,\lambda_2,2,2]\!] = [\![\lambda_1-1,\lambda_2-1]\!]$
$[\![\lambda_1,\lambda_2,\lambda_3,\lambda_4]\!] = 0$ if $\lambda_3 \geq 3$

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$[\![\lambda_1,\lambda_2,\lambda_3,\lambda_4]\!] = 0$ if $\lambda_3 \geq 3$

$\{\lambda\}$	$[\![\lambda/M_{1^3}]\!]$	
$\{\lambda_1,\lambda_2\}$	$[\![\lambda_1,\lambda_2]\!]$	for $\lambda_1 \geq \lambda_2 \geq 0$
$\{\lambda_1,\lambda_2,\lambda_3\}$	$[\![\lambda_1-\lambda_3,\lambda_2-\lambda_3]\!]$	for $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq 0$

Tensor products for $H_{1^3}(3)$

General

rule: $[\![\lambda]\!] \cdot [\![\mu]\!] = \sum_{\sigma,\tau} [\![(\lambda/(\sigma \cdot \tau(1^2))) \cdot (\mu/(\sigma(1^2) \cdot \tau))]\!]$

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Example: Evaluation of $[\![2^2]\!] \cdot [\![21]\!]$ in $H_{1^3}(3)$

$$H_{1^3}(N) \quad [\![43]\!] + [\![421]\!] + [\![3^21]\!] + [\![32^2]\!] + [\![321^2]\!] + [\![2^31]\!] + 2[\![31]\!] + 2[\![2^2]\!] + 3[\![21^2]\!] + [\![1^4]\!] + 2[\![1]\!]$$

$$H_{1^3}(3) \quad [\![43]\!] + [\![31]\!] + [\![2^2]\!] + [\![1]\!]$$

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$$H_{1^3}(N) \quad [\![43]\!] + [\![421]\!] + [\![3^21]\!] + [\![32^2]\!] + [\![321^2]\!] + [\![2^31]\!]$$
$$+ 2[\![31]\!] + 2[\![2^2]\!] + 3[\![21^2]\!] + [\![1^4]\!] + 2[\![1]\!]$$
$$H_{1^3}(3) \quad [\![43]\!] + [\![31]\!] + [\![2^2]\!] + [\![1]\!]$$

Check: Direct evaluation of $\{2^2\} \cdot \{21\}$ in $SL(3)$

$$SL(N) \quad \{43\} + \{421\} + \{3^21\} + \{32^2\} + \{321^2\} + \{2^31\}$$
$$SL(3) \quad \{43\} + \{31\} + \{2^2\} + \{1\}$$

The group $H_{1^3}(4)$

- Canonical form of third rank antisymmetric invariant in 4 dimensions: $\eta_{ijk} = \begin{cases} \epsilon_{ijk} & \text{if } \; i,j,k \in \{1,2,3\} \\ 0 & \text{otherwise} \end{cases}$
- For $A \in H_{1^3}(4)$ we have $A: \; \eta_{ijk} \mapsto A_i^p \, A_j^q \, A_k^r \, \epsilon_{pqr} \; = \; \eta_{ijk}$
- $\Rightarrow \quad A = \begin{bmatrix} B & D \\ 0 & C \end{bmatrix}$ where B is 3×3 and C is 1×1,

  with $\det B = 1$, D arbitrary and $\det C \neq 0$.
- Hence $H_{1^3}(4) = \begin{bmatrix} SL(3) & * \\ 0 & GL(1) \end{bmatrix}$

The group $H_{1^3}(4)$

- $H_{1^3}(4)$ is neither semisimple nor reductive
- In general its representations are not fully reducible

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$\{\lambda\}_{\mathrm{dim}}$	$[\![\lambda/M_{1^3}]\!]_{\mathrm{dim}}$	$\{\lambda\}_{\mathrm{dim}}$	$[\![\lambda/M_{1^3}]\!]_{\mathrm{dim}}$
$\{0\}_1$	$[\![0]\!]_1$	$\{21^3\}_4$	$[\![21^3]\!]_{-12} + [\![2]\!]_{10} + [\![1^2]\!]_6$
$\{1\}_4$	$[\![1]\!]_4$	$\{2^2\}_{20}$	$[\![2^2]\!]_{20}$
$\{1^2\}_6$	$[\![1^2]\!]_6$	$\{2^21\}_{20}$	$[\![2^21]\!]_{14} + [\![1^2]\!]_6$
$\{1^3\}_4$	$[\![1^3]\!]_3 + [\![0]\!]_1$	$\{2^21^2\}_6$	$[\![2^21^2]\!]_{-17} + [\![21]\!]_{20} + [\![1^3]\!]_3$
$\{1^4\}_1$	$[\![1^4]\!]_{-3} + [\![1]\!]_4$	$\{2^3\}_{10}$	$[\![2^3]\!]_6 + [\![1^3]\!]_3 + [\![0]\!]_1$
$\{2\}_{10}$	$[\![2]\!]_{10}$	$\{2^31\}_4$	$[\![2^31]\!]_{-8} + [\![21^2]\!]_{1^2} + [\![1^4]\!]_{-3} + [\![1]\!]_4$
$\{21\}_{20}$	$[\![21]\!]_{20}$	$\{2^4\}_1$	$[\![2^4]\!]_3 + [\![21^3]\!]_{-12} + [\![2]\!]_{10}$
$\{21^2\}_{15}$	$[\![21^2]\!]_{1^2} + [\![1]\!]_4$		

Modification rules for $H_{1^3}(4)$

In $GL(4)$:

- $\{\lambda_1, \lambda_2, \lambda_3, \lambda_4\} = \epsilon^{\lambda_4}\,\{\lambda_1 - \lambda_4, \lambda_2 - \lambda_4, \lambda_3 - \lambda_4, 0\}$
- $\epsilon = \{1^4\}$ is the character of the determinant rep of $GL(4)$.
- Ex. $\{21^3\} = \epsilon\,\{1\}$

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- Ex. $\{21^3\} = \epsilon\{1\}$

Under restriction from $GL(4)$ to $H_{1^3}(4)$:

- $\{1\}_4 \mapsto [\![1]\!]_4$
- $\{1^4\}_1 = \epsilon\{0\}_1 \mapsto \epsilon[\![0]\!]_1$
- $\{21^3\}_4 \mapsto [\![21^3]\!]_{-12} + [\![2]\!]_{10} + [\![1^2]\!]_6$

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- $\{1^4\}_1 = \epsilon\{0\}_1 \mapsto \epsilon[\![0]\!]_1$
- $\{21^3\}_4 \mapsto [\![21^3]\!]_{-12} + [\![2]\!]_{10} + [\![1^2]\!]_6$

Hence in $H_{1^3}(4)$ we have the modification rule:

- $[\![21^3]\!]_{-12} = \epsilon[\![1]\!]_4 - [\![2]\!]_{10} - [\![1^2]\!]_6$

Tensor products for $H_{1^3}(4)$

General rule for $H_{1^3}(N)$:

$$[\![\lambda]\!] \cdot [\![\mu]\!] = \sum_{\sigma,\tau} [\![(\lambda/(\sigma \cdot \tau(1^2))) \cdot (\mu/(\sigma(1^2) \cdot \tau))]\!]$$

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Example of product evaluated in $H_{1^3}(N)$:

$$[\![2]\!] \cdot [\![1^3]\!] = [\![31^2]\!] + [\![21^3]\!] + [\![2]\!] + [\![1^2]\!]$$

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Example of product evaluated in $H_{1^3}(4)$:

$$\begin{aligned} [\![2]\!]_{10} \cdot [\![1^3]\!]_3 &= [\![31^2]\!]_{26} + [\![21^3]\!]_{-12} + [\![2]\!]_{10} + [\![1^2]\!]_6 \\ &= [\![31^2]\!]_{26} + (\epsilon\,[\![1]\!]_4 - [\![2^2]\!]_{10} - [\![1]\!]_6) + [\![2^2]\!]_{10} + [\![1]\!]_6 \\ &= [\![31^2]\!]_{26} + \epsilon\,[\![1]\!]_4 \end{aligned}$$

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$$[\![\lambda]\!] \cdot [\![\mu]\!] = \sum_{\sigma,\tau} [\![(\lambda/(\sigma\cdot\tau(1^2)))\cdot(\mu/(\sigma(1^2)\cdot\tau))]\!]$$

Example of product evaluated in $H_{1^3}(N)$:

$$[\![2]\!] \cdot [\![1^3]\!] = [\![31^2]\!] + [\![21^3]\!] + [\![2]\!] + [\![1^2]\!]$$

Example of product evaluated in $H_{1^3}(4)$:

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Note: In general $[\![\lambda]\!]$ is the character of an indecomposable but reducible representation.

Variety of groups $H_{1^3}(N)$

Canonical forms for η_{ijk} of symmetry $\pi = 1^3$

- Let $p, q, r, s, t, \ldots$ be linearly independent vectors.
- Let $[pqr]$ be the tri-vector with components $p_{[i}q_j r_{k]}$ where $[\cdots]$ indicates antisymmetrisation of indices.
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- Canonical forms for η (Gurevich)

N	η	N	η
3	$[pqr]$	6	$[pqr]$
4	$[pqr]$		$[pqr]+[pst]$
5	$[pqr]$		$[pqr]+[stu]$
	$[pqr]+[pst]$		$[pqr]+[pst]+[qsu]$

Variety of groups $H_{1^3}(N)$

Candidate groups: $\begin{bmatrix} A & * \\ 0 & B \end{bmatrix}$ with $*$ arbitrary

N	η	$A * B$
3	$[pqr]$	$SL(3)$
4	$[pqr]$	$SL(3) * GL(1)$
5	$[pqr]$	$SL(3) * GL(2)$
	$[pqr] + [pst]$	??
6	$[pqr]$	$SL(3) * GL(3)$
	$[pqr] + [pst]$	$?? * GL(1)$
	$[pqr] + [stu]$	$SL(3) * SL(3)$
	$[pqr] + [pst] + [qsu]$	??

The case $\pi = 3$

Implications of invariance of η_{ijk}

- Third rank symmetric invariant in 3 dimensions:

$$\eta_{ijk} = \delta_{ijk} = \begin{cases} 1 & \text{if } i = j = k \\ 0 & \text{otherwise} \end{cases}$$

- For $A \in H_3(N)$ we have $A : \delta_{ijk} \mapsto A_i^p \, A_j^q \, A_k^r \, \delta_{pqr} \; = \; \delta_{ijk}$
  - $\sum_{p=1}^{N} (A_i^p)^3 = 1$
  - $\sum_{p=1}^{N} (A_i^p)^2 A_j^p = 0$ for $i \neq j$
  - $\sum_{p=1}^{N} A_i^p \, A_j^p \, A_k^p = 0$ for $i \neq j \neq k \neq i$.
- For $N \leq 3$ we find $A_i^p \in \mathbb{Z}_3 = \{1, \omega, \omega^2\}$ with $\omega = e^{i2\pi/3}$ for all i, p.

The groups $H_3(N)$ and their characters

N	$H_3(N)$
2	$\mathbb{Z}_3 \wr S_2$
3	$\mathbb{Z}_3 \wr S_3$
4	$?? \supseteq \mathbb{Z}_3 \wr S_4$
N	$?? \supseteq \mathbb{Z}_3 \wr S_N$

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N	$?? \supseteq \mathbb{Z}_3 \wr S_N$

Branching rule for $GL(N) \supset H_3(N)$

- $\{\lambda\} \to [\![\lambda/M_3]\!] = [\![\lambda/(0+3+6+42+\cdots)]\!]$

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Tensor product rule for $H_3(N)$

- $[\![\lambda]\!] \cdot [\![\mu]\!] = \sum_{\sigma,\tau} [\![(\lambda/(\sigma \cdot \tau(2))) \cdot (\mu/(\sigma(2) \cdot \tau))]\!]$
  $= [\![(\lambda \cdot \mu]\!] + [\![(\lambda/1) \cdot (\mu/2)]\!] + [\![(\lambda/2) \cdot (\mu/1]\!] + \cdots$

Examples in the case $H_3(3)$

Typical restrictions from $GL(3)$ to $H_3(3)$

$\{\lambda\}_{\mathrm{dim}}$	$[\![\lambda/M_3]\!]_{\mathrm{dim}}$
$\{5\}_{21}$	$[\![5]\!]_{15} + [\![2]\!]_6$
$\{41\}_{24}$	$[\![41]\!]_{15} + [\![2]\!]_6 + [\![1^2]\!]_3$
$\{32\}_{15}$	$[\![32]\!]_9 + [\![2]\!]_6$

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Typical tensor product decomposition for $H_3(3)$

$$[\![3]\!]_9 \cdot [\![2]\!]_6 = [\![5]\!]_{15} + [\![41]\!]_{15} + [\![32]\!]_9 + 2[\![2]\!]_6 + [\![1^2]\!]_3$$

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Problems:

- $H_3(3) = \mathbb{Z}_3 \wr S_3$ is finite
- All its representations are fully reducible
- The dimensions of the irreducible reps are all 1 or 2

More than one invariant

Subgroup $H_{\pi,\rho}(N) \supset GL(N)$ leaving invariant both $\eta_{ij\ldots}$ and $\zeta_{ij\ldots}$ of symmetry π and ρ.

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- Example: $\pi = 2$ and $\rho = 1^N$
- $\eta = g_{ij}$ symmetric, rank 2 and
  $\zeta = \epsilon_{ij\ldots k}$ antisymmetric, rank N
- $H_{2,1^N}(N) = SO(N)$

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Subgroup $H_{\pi,\rho}(N) \supset GL(N)$ leaving invariant both $\eta_{ij\dots}$ and $\zeta_{ij\dots}$ of symmetry π and ρ.

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  $\zeta = \epsilon_{ij\dots k}$ antisymmetric, rank N
- $H_{2,1^N}(N) = SO(N)$

Characters

- Branching rule $\{\lambda\} \to [\lambda/M_2]$
- Tensor product $[\lambda] \cdot [\mu] = \sum_\sigma [(\lambda/\sigma) \cdot (\mu/\sigma)]$
- Additional modification rule $[\lambda] = [\lambda]_+ + [\lambda]_-$
  if $N = 2k$ and $\ell(\lambda) = k$

Subgroup $H_{2,3}(N)$ of $GL(N)$

Invariants: $\eta_{ij} = \delta_{ij}$, $\zeta_{ijk} = \delta_{ijk}$ of symmetry $\pi = 2$, $\rho = 3$

- Further invariants

  $\delta_k = \sum_{i,j} \delta_{ij}\, \delta_{ijk}$ and $\delta_{ijkl} = \sum_m \delta_{ijm}\, \delta_{mkl}$, etc.

- $H_{2,3}(N) = S_N$ (Littlewood)

Subgroup $H_{2,3}(N)$ of $GL(N)$

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- $H_{2,3}(N) = S_N$ (Littlewood)

Characters of S_N in reduced notation

- Branching rule for $GL(N) \supset S_N$

  $\{\lambda\} \to \sum_{\sigma,\tau,\dots} \langle(\lambda/(M \cdot M_2 \cdot \sigma(2) \cdot M_3 \cdot \tau(3) \cdots) \cdot (\sigma \cdot \tau \cdots)\rangle$

- Tensor product rule for S_N

  $\langle\lambda\rangle \cdot \langle\mu\rangle = \sum_{\nu,\sigma,\tau,} \langle(\lambda/(\nu \cdot \sigma)) \cdot (\mu/(\nu \cdot \tau)) \cdot (\sigma \circ \tau)\rangle$

- Modification rule for S_N:

  $\langle\lambda\rangle = (-1)^c \langle(\lambda' - h)'\rangle$ with $h = |\lambda| + \lambda_1 - N - 1$

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