Littlewood-Richardson coefficients, the hive model and Horn inequalities

Ronald C King

School of Mathematics, University of Southampton
Southampton, SO17 1BJ, England

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Contents

- Schur functions and LR-coefficients
- The hive model - vertex and edge labelling
- Horn inequalities
- Factorisation
- Problems of symmetry, conjugacy and complementarity
- Inequalities of the Stembridge and Gutschwager type
- Stretched LR-coefficients and LR-polynomials
- Open problems - positivity of coefficients, degrees and zeros of LR-polynomials
Key sources

- Littlewood-Richardson coefficients

- The hive model
Key sources

- **Horn inequalities**
  

- **Puzzles**
  

Key sources

- Polynomial property of stretched LR-coefficients

- Overview
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Schur functions

Let $n$ be a fixed positive integer and $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ a sequence of indeterminates.

Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ be a partition of weight $|\lambda|$ and length $\ell(\lambda) \leq n$, so that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0$.

Then the Schur function $s_\lambda(\mathbf{x})$ is defined by:

$$s_\lambda(\mathbf{x}) = \left| \frac{x_i^{n+\lambda_j-j}}{x_i^{n-j}} \right|_{1 \leq i,j \leq n}.$$

The Schur functions form a $\mathbb{Z}$-basis of $\Lambda_n$, the ring of polynomial symmetric functions of $x_1, \ldots, x_n$.

Each Schur function $s_\lambda(\mathbf{x})$ may be interpreted as the character $\text{ch} V^\lambda(\mathbf{x})$ of an irrep of $gl(n)$. 
Any product of Schur functions can be expressed as a linear sum of Schur functions:

\[ s_{\lambda}(x) \ s_{\mu}(x) = \sum_{\nu} c_{\lambda\mu}^{\nu} \ s_{\nu}(x) \]

The coefficients \( c_{\lambda\mu}^{\nu} \) are the multiplicities appearing in the decomposition of the tensor product of irreps of \( gl(n) \):

\[ V^{\lambda} \otimes V^{\mu} = \sum_{\nu} c_{\lambda\mu}^{\nu} \ V^{\nu} \]

Each Littlewood-Richardson coefficient \( c_{\lambda\mu}^{\nu} \) is a non-negative integer that may be evaluated by means of the Littlewood-Richardson rule
Young diagrams

- Each partition $\lambda$ specifies a Young diagram $F^\lambda$ consisting of $|\lambda|$ boxes arranged in $\ell(\lambda)$ left adjusted rows of lengths $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_{\ell(\lambda)} > 0$.

- The partition $\lambda'$ conjugate to $\lambda$ is such that $F^{\lambda'}$ is obtained from $F^\lambda$ by interchanging rows and columns.

- Ex: If $\lambda = (4, 3, 1)$ then $|\lambda| = 8$, $\ell(\lambda) = 3$, $\lambda' = (3, 2, 2, 1)$, with

  $F^\lambda =$ \begin{array}{cccc}
  \Box & \Box & \Box & \\
  \Box & \Box & \\
  \Box & \\
\end{array} \quad F^{\lambda'} =$ \begin{array}{cccc}
  \Box & \Box & \Box & \\
  \Box & \Box & \\
\end{array}
Skew Young diagrams

- Given partitions \( \lambda \) and \( \nu \) such that all boxes of \( F^\lambda \) are contained in \( F^\nu \) we write \( \lambda \subseteq \nu \).

- Removing the boxes of \( F^\lambda \) from \( F^\nu \) leaves the skew Young diagram \( F^{\nu/\lambda} \).

- **Ex:** If \( \nu = (5, 4, 2) \) and \( \lambda = (3, 1) \) then

  \[
  F^{\nu/\lambda} = \begin{array}{ccc}
    * & * & * \\
    * & & \\
    & & \\
  \end{array}
  \]

- The corresponding skew Schur function is such that

  \[
  s_{\nu/\lambda}(x) = \sum_{\mu} c_{\lambda \mu}^{\nu} s_{\mu}(x)
  \]
Littlewood-Richardson rule

- Fill the boxes of the Young diagram $F^\lambda$ with 0’s.
- Then fill the boxes of the skew Young diagram $F^{\nu/\lambda}$ with $\mu_i$ entries $i$ for $i = 1, 2, \ldots, n$.
- $c^\nu_{\lambda\mu}$ is the number of such diagrams with entries
  - weakly increasing across rows from left to right
  - strictly increasing down columns from top to bottom
  - satisfying the lattice permutation rule - i.e. at every stage in the sequence of non-zero entries read from right to left across rows taken in turn from top to bottom $\#1’s \geq \#2’s \geq \cdots \geq \#n’s$
Application of the LR-rule

Ex: \( n = 4, \lambda = (4, 2), \mu = (4, 3, 2), \nu = (6, 5, 3, 1) \)

- The only valid LR-diagrams

- Some invalid diagrams

- Hence \( c^\nu_{\lambda\mu} = 3 \).
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Integer hives

- Knutson & Tao [99], as described by Buch [00]

- An integer $n$-hive is a triangular graph with vertex labels $a_{ij} \in \mathbb{Z}$ for $0 \leq i, j, i + j \leq n$.

**Ex:** $n = 4$

- Vertex labels increase along each edge from left to right
Relation between vertex and edge labels

- Edge labels are the non-negative differences between neighbouring vertex labels.
- Vertex and edge labels for the two types of elementary triangle:

  ![Triangle Diagram]

  - In each case:
    \[ \sigma = q - p \geq 0, \quad \tau = r - q \geq 0, \quad \rho = r - p \geq 0 \]

  so that automatically we have \( \sigma + \tau = \rho \) in any such triangle.
Hive conditions

- Three distinct types of rhombi with vertex labels:

  ![Rhombus with vertex labels](image1)

  ![Rhombus with vertex labels](image2)

  ![Rhombus with vertex labels](image3)

  The hive condition for each rhombus: \( b + c \geq a + d \)

- Three distinct types of rhombi with edge labels:

  ![Rhombus with edge labels](image4)

  ![Rhombus with edge labels](image5)

  ![Rhombus with edge labels](image6)

  The hive condition for each rhombus: \( \alpha \geq \gamma \) and \( \beta \geq \delta \)

- Note: The triangle edge condition implies \( \alpha + \delta = \beta + \gamma \)
**LR-hives vertex labels**

**Definition** An LR-hive is an integer \( n \)-hive for which

- all rhombi satisfy the hive conditions;
- boundaries determined by partitions \( \lambda, \mu, \nu \) with \( \ell(\lambda), \ell(\mu), \ell(\nu) \leq n \) and \( |\lambda| + |\mu| = |\nu| \);
- boundary vertex labels as shown:
LR-hives edge labels

**Definition** An LR-hive is an integer $n$-hive for which

- all rhombi satisfy the hive conditions;
- boundaries determined by partitions $\lambda, \mu, \nu$ with
  $\ell(\lambda), \ell(\mu), \ell(\nu) \leq n$ and $|\lambda| + |\mu| = |\nu|$;
- boundary edge labels as shown:
Bijection between LR-diagrams and LR-hives

**Lemma** A bijection between LR integer $n$-hives, $H$, and LR-diagrams, $D$, is provided by the formula:

$$a_{ij} = \# \text{ of entries } \leq i \text{ in the first } i + j \text{ rows of } D$$

for the $(i, j)$th vertex label in $H$, for all $i, j$ such that $0 \leq i, j, i+j \leq n$.

**Theorem** The LR-coefficient $c_{\lambda\mu}^{\nu}$ is the number of LR-hives with boundary labels determined by $\lambda, \mu$ and $\nu$.

**Note:** Neither this theorem nor the Littlewood-Richardson rule allows us to see whether or not a given LR-coefficient $c_{\lambda\mu}^{\nu}$ is non-zero.
Example of bijection

Ex: $n = 4, \lambda = (753), \mu = (742), \nu = (9964)$

$a_{ij}$:

\[
\begin{array}{cccc}
15 & 15 & 22 & \\
12 & 21 & 26 & \\
7 & 16 & 24 & 28 \\
0 & 9 & 18 & 24 & 28
\end{array}
\]

\[
\begin{array}{cccc}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 1 \\
1 & 2 & 3 & 3
\end{array}
\]
Example of bijection

Ex: \( n = 4, \ \lambda = (753), \ \mu = (742), \ \nu = (9964) \)

\( a_{ij}: \)

Note The given edge labels are sufficient to fix all edge labels using the triangle condition
LR-hives showing that \( c_{753,742}^{9964} = 6 \)

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Non-zero conditions

- We know that $c_{\lambda\mu}^\nu$ is the number of LR-hives with boundary labels determined by $\lambda$, $\mu$ and $\nu$.

- For given $\lambda$, $\mu$ and $\nu$ we would like some way of determining if $c_{\lambda\mu}^\nu$ is non-zero.

- Horn [62] defined a set of inequalities and conjectured that they gave necessary and sufficient conditions for the solution of a problem later realised to be equivalent to that of LR-coefficients being non-zero.

- The validity of Horn’s conjecture was proved by the efforts of Klyachko [98], Knutson and Tao [99], Belkale [01], and Knutson, Tao and Woodward [04].

- For a comprehensive review see Fulton [00].
Partial sums

- Let $N = \{1, 2, \ldots, n\}$, then for fixed $r$, with $1 \leq r \leq n$, let $I = \{i_1, i_2, \ldots, i_r\} \subseteq N$ and $\bar{I} = N \setminus I$.

- For any partition $\lambda$ let:
  $$ps(\lambda)_I = \lambda_{i_1} + \lambda_{i_2} + \cdots + \lambda_{i_r}.$$

- If $i_1 < i_2 < \cdots < i_r$ then let
  $$\text{part}(I) = (i_r - r, \ldots, i_2 - 2, i_1 - 1).$$

- Let $T^n_r$ be the set of triples $(I, J, K)$ with $I, J, K \subset N$ and
  $\#I = \#J = \#K = r$ with $c_{\text{part}(K)}^{\text{part}(I) \text{part}(J)} > 0$.

- Let $R^n_r$ be the set of triples $(I, J, K)$ with $I, J, K \subset N$ and
  $\#I = \#J = \#K = r$ with $c_{\text{part}(K)}^{\text{part}(I) \text{part}(J)} = 1$. 
Non-zero conditions

**Theorem:** The LR-coefficient $c_{\lambda \mu}^\nu$ is non-zero if and only if

$$|\nu| = |\lambda| + |\mu|$$

and Horn’s inequalities,

$$ps (\nu)_K \leq ps (\lambda)_I + ps (\mu)_J,$$

are satisfied for all $r = 1, 2, \ldots, n - 1$ and all $(I, J, K) \in T^n_r$

**Note:** Not all of Horn’s inequalities are essential. Horn’s **essential** inequalities are those for which $(I, J, K) \in R^n_r$ - but where do they come from?
Definition A puzzle is a diagram on a triangular lattice in which edges are distinguished so that it is composed of copies of the following pieces oriented in any way so as to fit:
Puzzles

**Definition** A puzzle is a diagram on a triangular lattice in which edges are distinguished so that it is composed of copies of the following pieces oriented in any way so as to fit:
Definition A puzzle is a diagram on a triangular lattice in which edges are distinguished so that it is composed of copies of the following pieces oriented in any way so as to fit:
Definition A hive plan is made up of corridors, dark rooms and light rooms obtained by deleting interior edges of a puzzle:
Hive plan or labyrinth

**Definition** A hive plan is made up of shaded corridors, dark rooms and light rooms obtained by deleting interior edges of a puzzle:
Hive plan or labyrinth

**Definition** A hive plan is made up of **corridors**, **blue rooms** and **red rooms** obtained by deleting interior edges of a puzzle:
Link between puzzles and Horn triples

- \((I, J, K)\) is Horn triple if it specifies the positions of the thick edges on the boundary of any puzzle. It is essential if the puzzle with these boundary thick edges is unique.

- For \(I = (1, 2, 4)\), \(J = (2, 3, 4)\) and \(K = (2, 3, 5)\) we have:
Each Horn triple defines an inequality

\[ \nu_2 + \nu_3 + \nu_5 \leq (\nu_2 + \nu_3) + \gamma_4 = (\alpha_1 + \alpha_2 + \beta_1 + \beta_2) + \gamma_4 \]
\[ \leq \lambda_1 + \alpha_2 + \beta_1 + \beta_2 + \gamma_4 \leq \lambda_1 + \lambda_2 + \beta_1 + \beta_2 + \gamma_4 \]
\[ \leq \lambda_1 + \lambda_2 + \beta_3 + \beta_2 + \gamma_4 \leq \lambda_1 + \lambda_2 + (\beta_3 + \beta_4 + \gamma_4) \]
\[ = \lambda_1 + \lambda_2 + (\alpha_4 + \mu_2 + \mu_3 + \mu_4) \leq \lambda_1 + \lambda_2 + \lambda_4 + \mu_2 + \mu_3 + \mu_4 \]

That is \( ps (\nu)_K \leq ps (\lambda)_I + ps (\mu)_J \).
The same procedure applied to thin-edge inequalities gives
\[ \nu_1 + \nu_4 \geq \gamma_1 + \nu_4 = \gamma_1 + (\alpha_5 + \beta_5) \]
\[ \geq \gamma_1 + \alpha_3 + \beta_5 \geq (\gamma_1 + \alpha_3) + \mu_5 = (\lambda_3 + \gamma_2) + \mu_5 \]
\[ \geq \lambda_3 + \gamma_3 + \mu_5 = \lambda_3 + \lambda_5 + \mu_1 + \mu_5 \]
That is \[ ps(\nu)_K \geq ps(\lambda)_T + ps(\mu)_J \]
Significance of inequalities derived from puzzles

Each inequality \( ps (\nu)_K \leq ps (\lambda)_I + ps (\mu)_J \) derived from a puzzle must be satisfied if \( c_{\nu\lambda\mu} \) is to be non-zero.

To show that a puzzle triple \((I, J, K)\) is a Horn triple, and the inequality a Horn inequality, we must make a connection between puzzles with thick boundary edges specified by \((I, J, K)\) and \(c_{\text{part}(K)}\) \(c_{\text{part}(I), \text{part}(J)}\).

First note that the passage from \(I\) to \(\text{part}(I)\) is easy.

**Ex**  If \(I = \{1, 3, 6, 8\}\),
then \((1, 3, 6, 8) - (1, 2, 3, 4) = (0, 1, 3, 4)\)
so that \(\text{part}(I) = (4, 3, 1)\).
How may puzzles are there?

**Theorem**  The number of puzzles with the positions of the thick edges on the boundary specified by \((I, J, K)\) is given by

\[
\binom{\text{part}(K)}{\text{part}(I), \text{part}(J)}
\]

**Ex:** \(n = 5, \ r = 2, \ I = (1, 2, 4), \ J = (2, 3, 4), \ K = (2, 3, 5)\)

- In this case \(\binom{\text{part}(K)}{\text{part}(I), \text{part}(J)} = \binom{311}{1,111} = 1\)

- and there exists just the one puzzle identified earlier
A map from puzzles to hives

Let the thick edges of the puzzle be specified by \((I, J, K)\).

For \(M = I, J, K\) in turn, label each thick boundary edge of the puzzle by the corresponding row part of \(\text{part}(M)\).

Scale the length of all thin edges by \(t\) and let \(t \to 0\).
A map from puzzles to hives contd

- In each thick edged room set all parallel edge labels equal using the triangle condition wherever required.
- Reflect the resulting diagram in its vertical axis of symmetry to obtain a hive.

Lemma  This map provides a bijection between puzzles with thick edges specified by \((I, J, K)\) and LR-hives with boundary specified by \(\text{part}(J), \text{part}(I), \text{part}(K)\)
Corollaries

- Each triple \((I, J, K)\) is a Horn triple if and only if it specifies the thick boundary edges of a puzzle.
- The corresponding inequality \(p_s(\nu)_K \leq p_s(\lambda)_I + p_s(\mu)_J\) defined by the puzzle is a Horn inequality.
- The number of puzzles specified by \((I, J, K)\) is equal to \(c_{part(K)}\).
- The puzzle is said to be **rigid** if it is unique, that is \(c_{part(K)} = 1\), and the corresponding Horn inequality is **essential**.
- The LR-coefficient \(c^\nu_{\lambda\mu}\) is non-zero if and only if every essential Horn inequality is satisfied.
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Consequences of any Horn equality

All sequences of inequalities become equalities.

\[
\nu_2 + \nu_3 + \nu_5 = (\nu_2 + \nu_3) + \gamma_4 = (\alpha_1 + \alpha_2 + \beta_1 + \beta_2) + \gamma_4 \\
= \lambda_1 + \alpha_2 + \beta_1 + \beta_2 + \gamma_4 = \lambda_1 + \lambda_2 + \beta_1 + \beta_2 + \gamma_4 \\
= \lambda_1 + \lambda_2 + \beta_3 + \beta_2 + \gamma_4 = \lambda_1 + \lambda_2 + (\beta_3 + \beta_4 + \gamma_4) \\
= \lambda_1 + \lambda_2 + (\alpha_4 + \mu_2 + \mu_3 + \mu_4) = \lambda_1 + \lambda_2 + \lambda_4 + \mu_2 + \mu_3 + \mu_4.
\]
\[ \nu_1 + \nu_4 = \gamma_1 + \nu_4 = \gamma_1 + (\alpha_5 + \beta_5) \]
\[ = \gamma_1 + \alpha_3 + \beta_5 = (\gamma_1 + \alpha_3) + \mu_5 = (\lambda_3 + \gamma_2) + \mu_5 \]
\[ = \lambda_3 + \gamma_3 + \mu_5 = \lambda_3 + \lambda_5 + \mu_1 + \mu_5 \]

These equalities imply:
\[ \nu_5 = \gamma_4, \quad \alpha_1 = \lambda_1, \quad \alpha_2 = \lambda_2, \]
\[ \beta_1 = \beta_3, \quad \beta_2 = \beta_4, \quad \alpha_4 = \lambda_4, \quad \beta_5 = \mu_5, \quad \nu_1 = \gamma_1, \quad \gamma_2 = \gamma_3. \]
Factorisation of LR-hives
Illustration of $H_n$ and subhives $H_r, H_{n-r}$
LR-coefficient factorisation

Lemma  In the case of any Horn equality and a corresponding puzzle, the deletion of redundant corridors from any LR-hive $H_n$ gives a pair of LR-subhives $H_r$ and $H_{n-r}$.

Lemma  In the case of any essential Horn equality, this map from the LR-hives $H_n$ to pairs of LR-hives $H_r$ and $H_{n-r}$ is a bijection.

Theorem  If an essential Horn inequality is saturated then $c_{\lambda\mu}^\nu$ factorises.

Definition  If all essential Horn inequalities are strict $c_{\lambda\mu}^\nu$ is said to be primitive.
LR factorisation example

Ex: \( n = 5, r = 3, n - r = 2 \):

- \( \lambda = (9, 7, 6, 2, 0), \mu = (13, 5, 3, 1, 0), \nu = (14, 12, 11, 5, 4) \).
- \( I = \{1, 2, 4\}, J = \{2, 3, 4\}, K = \{2, 3, 5\} \).
- \( \lambda_I = (9, 7, 2), \mu_J = (5, 3, 1), \nu_K = (12, 11, 4) \)
- \( \lambda_I = (6, 0), \mu_J = (13, 0), \nu_K = (14, 5) \)
- \( ps(\nu)_K = 27 = 18 + 9 = ps(\lambda)_I + ps(\mu)_J \)

Hence

\[
C_{(14,12,11,5,4)}^{(9,7,6,2,0),(13,5,3,1,0)} = C_{(12,11,4)}^{(9,7,2),(5,3,1)} \cdot C_{(14,5)}^{(6,0),(13,0)} = 2 \cdot 1 = 2
\]

Note: This is an example of the reduction of an LR-coefficient, since

\[
C_{(14,12,11,5,4)}^{(9,7,6,2,0),(13,5,3,1,0)} = C_{(12,11,4)}^{(9,7,2),(5,3,1)} = 2
\]
Proof of factorisation

To be shown:

- the corridors $R_n$ of $H_n$ are redundant;
- the dark rooms constitute an LR-hive $H_r$;
- the light rooms constitute an LR-hive $H_{n-r}$;
- any LR hives $H_r, H_{n-r}$ joined by $R_n$ gives an LR-hive $H_n$.

To be checked that the Horn equality implies:

- all corridor edge labels fixed;
- LR hive conditions for any rhombus split by corridor;
- LR hive conditions across corridor/dark room boundary;
- LR hive conditions across corridor/light room boundary.
Corridor edges fixed

- Hive conditions: \( \gamma_1 \leq \sigma_1 \leq \tau_1 \leq \alpha_1, \quad \gamma_2 \leq \sigma_2 \leq \tau_2 \leq \alpha_2 \).
- Horn equality: \( \gamma_1 + \gamma_2 = \alpha_1 + \alpha_2 \).
- Implies: \( \gamma_1 = \alpha_1 \) and \( \gamma_2 = \alpha_2 \).
- Implies: \( \gamma_1 = \sigma_1 = \tau_1 = \alpha_1 \) and \( \gamma_2 = \sigma_2 = \tau_2 = \alpha_2 \).
Deletion of corridor

- Initial hive conditions: $\gamma \leq \sigma$, $\sigma \leq \tau$, $\tau \leq \alpha$.
- Implies final hive condition: $\gamma \leq \alpha$.

Horn equality $\alpha - \beta = \rho = \gamma - \delta$ implies $\alpha + \delta = \beta + \gamma$. 
Paths - gentle and good

- **Path**: a continuous sequence of connected corridor walls with dark rooms, thick-edged 0-regions, on the right and light rooms, thin-edged 1-regions, on the left.

- **Gentle path**: at each vertex the deviation is 0 or $\pm \pi/3$.

- **Gentle loop**: a gentle path that forms a closed interior loop.

An edge is **good** if it forms the short diagonal of a rhombus satisfying the hive condition, otherwise it is **bad**.

- **Good path**: gentle path along which all the edges are good, ie with the hive condition satisfied across each edge.
Good paths and factorisation

Observations

- When subdividing two LR subhives, $H_r$ and $H_{n-r}$, and inserting corridors to create a hive $H_n$, this hive will be an LR hive if and only if each edge of every internal corridor wall of the corresponding hive plan is good.

- Let the boundary labels $\lambda, \mu, \nu$ of the hive $H_n$ be such that for a given puzzle specified by $(I, J, K)$ the corresponding Horn inequality is saturated, i.e. $ps(\nu)_K = ps(\lambda)_I + ps(\mu)_J$. If in the corresponding hive plan each edge of every internal corridor wall lies on a good path, then the LR-coefficient $c_{\lambda\mu}^{\nu}$ factorises.
**Lemma**  The first edge of any path starting from any boundary is **good**

- Each boundary has edges specified by a partition: \( \alpha \geq \gamma \).
- Horn equality applied to corridors: \( \beta = \alpha \) or \( \beta = \gamma \).
- Hence \( \beta = \alpha \geq \gamma \) or \( \beta = \gamma \leq \alpha \) so that in both cases \( OP \) is **good**.
Path along an interior corridor wall

- Initial hive conditions: $\alpha \leq \beta$
- Horn equality applied to corridors: $\delta = \gamma$.
- $PO$ good $\Rightarrow \gamma \leq \alpha \Rightarrow \delta = \gamma \leq \alpha \leq \beta \Rightarrow OR$ good
- $OR$ bad $\Rightarrow \delta > \beta \Rightarrow \gamma = \delta > \beta \geq \alpha \Rightarrow PO$ bad
Path reaching a vertex

- Initial hive condition: $\alpha \leq \beta$; Horn equality $\delta = \beta$
- $PO$ good $\Rightarrow \gamma \leq \alpha \Rightarrow \gamma \leq \alpha \leq \beta = \delta \Rightarrow OR$ good
- $OR$ bad $\Rightarrow \gamma > \delta \Rightarrow \gamma > \delta = \beta \geq \alpha \Rightarrow PO$ bad
Intersecting paths

Assume \( PO \) good, so that \( \beta \geq \alpha \).

Horn equality \( \gamma = \delta \)

\( QO \) good \( \Rightarrow \gamma \geq \beta \Rightarrow \delta = \gamma \geq \beta \geq \alpha \Rightarrow OS \) good

\( OS \) bad \( \Rightarrow \delta < \alpha \Rightarrow \gamma = \gamma = \delta < \alpha \leq \beta \Rightarrow QO \) bad
Good paths cover all interior corridor walls

All LR hive conditions satisfied

Hence we have factorisation
Bad edges and gentle loops

- Good paths may not cover all interior corridor edges
- If there exists a bad edge then its predecessor on some gentle path must also be bad
- A reverse gentle path of bad edges cannot reach the boundary, since all gentle paths start from the boundary with a good edge
- A reverse gentle path of bad edges must therefore continue indefinitely
- Since there are only a finite number of edges, it follows that there may only be bad edges if there exists a gentle loop
Example exhibiting a gentle loop

- \( n = 10, r = 5. \)
- \( I = (1, 2, 4, 6, 8), J = (1, 3, 4, 7, 9), K = (2, 4, 6, 8, 10). \)
Example of a gentle loop

Gentle loop is any closed interior gentle path.
Good paths do not include all interior corridor walls.
Obstruction to good paths

Good paths do not include all interior corridor walls.
Example exhibiting a gentle loop

Ex: \( n = 6, \ r = 3, \ I = J = (1, 3, 5) \) and \( K = (2, 4, 6) \).

There exist two puzzles, each exhibiting a gentle loop

\[
P_1: \begin{array}{cccccc}
0 & 1 & 0 & 1 & 0 & 1 \\
1 & 0 & 1 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 & 0 & 1 \\
\end{array}
\]

\[
P_2: \begin{array}{cccccc}
0 & 0 & 0 & 0 & 1 & 1 \\
1 & 0 & 1 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 & 0 & 1 \\
\end{array}
\]

The corresponding inessential saturated Horn inequality

\[
\nu_2 + \nu_4 + \nu_6 = \lambda_1 + \lambda_3 + \lambda_5 + \mu_1 + \mu_3 + \mu_5
\]

is satisfied by \( \lambda = \mu = (221100) \) and \( \nu = (332211) \)

There can be no corresponding factorisation since

\[
C^\nu_{\lambda\mu} = C^{332211}_{221100, 221100} = 3 \neq 2 \cdot 2 = C^{321}_{210, 210} C^{321}_{210, 210} = C^{\nu K}_{\lambda I \mu} C^{\nu K}_{\lambda J \mu J}
\]
Rigid puzzles and gentle loops

Theorem [KTW 04] The hive plan of a puzzle has no gentle loops if only if the puzzle is rigid

- A puzzle is rigid if and only if the corresponding Horn inequality is essential
- All gentle paths in the hive plan of a rigid puzzle are good, i.e., have no bad edges
- This implies that in the case of any essential Horn equality, the map from the LR-hives $H_n$ to pairs of LR-hives $H_r$ and $H_{n-r}$ is a bijection
- Hence we have proved: [KTT 08] Theorem If an essential Horn inequality is saturated then $C^\nu_{\lambda\mu}$ factorises.
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Some relations between LR-coefficients

Symmetries

\[ c^{\nu}_{\mu \lambda} = c^{\nu}_{\lambda \mu} \]
\[ c^{\nu'}_{\lambda' \mu'} = c^{\nu}_{\lambda \mu} \]
\[ c^{\tilde{\mu}}_{\tilde{\nu} \tilde{\lambda}} = c^{\nu}_{\lambda \mu} \]
\[ \text{where } \lambda' \text{ is the conjugate of } \lambda \]
\[ \text{and } \tilde{\lambda} = (m - \lambda_n, \ldots, m - \lambda_2, m - \lambda_1) \]

Inequalities

\[ c^{\nu + (1^c)}_{\lambda + (1^a), \mu + (1^b)} \geq c^{\nu}_{\lambda \mu} \]
\[ c^{\nu \cup (c)}_{\lambda \cup (a), \mu \cup (b)} \geq c^{\nu}_{\lambda \mu} \]

\[ \text{for all non-negative integers } a, b, c, \text{ with } c = a + b \]
\[ \text{where } \lambda + (1^a) = (\lambda_1 + 1, \ldots, \lambda_a + 1, \lambda_{a+1}, \ldots, \lambda_n) \]
\[ \text{and } \lambda \cup (a) = (\lambda_1, \ldots, \lambda_k, a, \lambda_{k+1}, \ldots, \lambda_n) \]
\[ \text{with } \lambda_k \geq a > \lambda_{k+1} \]
Hive based proofs of the symmetry relations

Commutativity

\[ c^\nu_\mu \lambda = c^\nu_\lambda \mu \]

Proof: It remains an open problem to find a hive-based proof.

- We would like a bijection between the LR-hives for \( c^\nu_\mu \lambda \) and those for \( c^\nu_\lambda \mu \).

- The result is by no means obvious from the Littlewood-Richardson rule.

- Considerable effort has gone into devising combinatorial proofs.

- See for example the work of Benkart, Sotille & Stroomer [96] and of Azenhas [99].
Hive based proofs of the symmetry relations

Conjugacy \( c_{\lambda' \mu'} = c_{\lambda \mu} \)

- Let \( N = \{1, 2, \ldots, n\} = M \cup \overline{M} \) with \( M \cap \overline{M} = \emptyset \)
- Note that if \( \zeta = \text{part}(M) \) then \( \zeta' = \text{part}(\overline{M}_n) \)
  where \( \overline{M}_n = \{n + 1 - i \mid i \in M\} \)

Ex: If \( n = 10 \) and \( M = \{1, 3, 6, 8\} \) then \( \text{part}(M) = (4, 3, 1) \)
while \( \overline{M} = \{2, 4, 5, 7, 9, 10\} \) and \( \overline{M}_n = \{1, 2, 4, 6, 7, 9\} \) so that 
\( \text{part}(\overline{M}) = (3, 2, 2, 1) \). Diagrammatically:
Proof of conjugacy contd.

Recall that the number of $n$-puzzles with thick edges specified by $I, J, K$ is

$$c_{\text{part}(K), \text{part}(J), \text{part}(I)} = c_{\text{part}(K), \text{part}(I), \text{part}(J)}$$

This was proved by scaling all thick edges by $t$ and allowing $t \to 0$
Proof of conjugacy contd.

It remains to show that the number of \( n \)-puzzles with thin edges specified by \( \overline{I}, \overline{J}, \overline{K} \) is

\[
\text{part}(\overline{K}_n) = \text{part}(\overline{I}_n) + \text{part}(\overline{J}_n)
\]

This is proved by scaling all thin edges by \( t \) and allowing \( t \to 0 \).
Hive based proofs of the symmetry relations

Complementarity  \( \tilde{C}_{\tilde{\nu} \tilde{\lambda}} = C_{\lambda \mu} \)

Proof  Consider any LR-hive for \( C_{\lambda \mu} \). Then

- For all edges parallel to the NE, SE, WE boundaries replace the edge labels \( \alpha, \beta, \gamma \) by \( \alpha, m - \beta, m - \gamma \)
- This breaks the triangle and the rhombus hive conditions and the result is not an LR-hive
Complementarity contd.

- Rotate clockwise through $\pi/3$
- The result is an LR-hive for $c_{\tilde{\nu} \tilde{\lambda}}$

Lemma  The above sequence of two maps provides a bijection between the LR-hives of $c^{\nu}_{\lambda \mu}$ and those of $c^{\tilde{\mu}}_{\tilde{\nu} \tilde{\lambda}}$
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Column insertion

Inequality \( c_{\lambda+(1^a),\mu+(1^b)}^{\nu+(1^c)} \geq c_{\lambda\mu}^{\nu} \) for all \( a, b, c \), with \( c = a + b \)

Proof Since \( c_{1^a,1^b}^{1^c} = 1 \) if \( c = a + b \), there exist a unique LR-hive:

Here each red, magenta and blue edge is labelled 1 and all other edges are labelled 0.
Column insertion

The elementary rhombi take one or other of the following forms

However, we can now interpret our multi-coloured LR-hive as being an LR-hive for $c_{\lambda \mu}^{\nu}$ whose red, magenta and blue edge labels have all been increased by 1, with no change to the uncoloured edge labels.
Column insertion

This yields an LR-five for \( c^{\nu+(1^c)}_{\lambda+(1^a), \mu+(1^b)} \) as can be seen by checking the hive conditions in each of the elementary rhombi

Hence the number of \( c^{\nu+(1^c)}_{\lambda+(1^a), \mu+(1^b)} \) LR-hives is at least as many as the number of \( c^\nu_{\lambda, \mu} \) LR-hives
Row insertion

Inequality\[ c_{\lambda \cup (a), \mu \cup (b)}^{\nu} \geq c_{\lambda \mu}^{\nu} \quad \text{with} \quad c = a + b \]

- Each LR-hive for \( c_{\lambda \mu}^{\nu} \) may be divided into regions as shown on the left below.

- Redundant corridors are then inserted as shown on the right to give an LR-hive for \( c_{\lambda \cup (a), \mu \cup (b)}^{\nu} \).
Row insertion

- The red line divides the hive into those regions for which NW edge labels are either $\leq a$ or $> a$.
- The blue line divides the hive into those regions for which SW edge labels are either $\leq b$ or $> b$.
- The cyan line divides the hive into those regions for which WE edge labels are either $\leq c$ or $> c$. 
Row insertion

- These lines always intersect at a single point $Z$
- Edges labelled $a, b, c$ are inserted on the boundary
- Each edge along the first part of the red, blue and cyan lines is replaced by an appropriate redundant rhombus
- The point $Z$ is replaced by a triangle with edge labels $a, b, c$, and all triangle and rhombus hive conditions are satisfied
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Stretched LR coefficients

- Littlewood-Richardson coefficient $c^\nu_{\lambda\mu}$
- Partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ stretching parameter $t \in \mathbb{N}$
- Stretched partition $t\lambda = (t\lambda_1, t\lambda_2, \ldots, t\lambda_n)$
- Stretched Littlewood-Richardson coefficient $c^{t\nu}_{t\lambda,t\mu}$

**Ex:** $n = 3$, $\lambda = (2, 1, 0)$, $\mu = (3, 2, 0)$, $\nu = (4, 3, 1)$

- $t = 1$: $c^{431}_{21,32} = 2$
- $t = 2$: $c^{862}_{42,64} = 3$
- $t = 3$: $c^{1293}_{63,94} = 4$
- $\ldots$

suggests $c^{t\nu}_{t\lambda,t\mu} = t + 1$. 
**LR coefficients and polynomials**

**Ex:** Let $c^\nu_{421,532} = c$ and $c_{t(421),t(532)}^{t\nu} = P(t)$.

- $c = 1 \quad \nu = (953) \quad P(t) = 1$
- $c = 2 \quad \nu = (9431) \quad P(t) = (t + 1)$
- $c = 3 \quad \nu = (8441) \quad P(t) = (t + 1)(t + 2)/2$
- $c = 4 \quad \nu = (8531) \quad P(t) = (t + 1)(t + 2)(t + 3)/6$
- $c = 4 \quad \nu = (7442) \quad P(t) = (t + 1)^2$
- $c = 5 \quad \nu = (7541) \quad P(t) = (t + 1)(t + 2)(2t + 3)/6$
- $c = 6 \quad \nu = (7532) \quad P(t) = (t + 1)^2(t + 2)/2$
- $c = 7 \quad \nu = (74321) \quad P(t) = (t + 1)(t + 2)(t^2 + 3t + 6)/6$
Generating function for LR-polynomials

Ex: Let \( F(z) = G(z)/(1 - z)^{d+1} = \sum_{t=0}^{\infty} P(t) z^t \).

\[
\begin{align*}
    &c = 1 \quad \nu = (953) \quad d = 1 \quad G(z) = 1 \\
    &c = 2 \quad \nu = (9431) \quad d = 2 \quad G(z) = 1 \\
    &c = 3 \quad \nu = (8441) \quad d = 3 \quad G(z) = 1 \\
    &c = 4 \quad \nu = (8531) \quad d = 4 \quad G(z) = 1 \\
    &c = 4 \quad \nu = (7442) \quad d = 3 \quad G(z) = 1 + z \\
    &c = 5 \quad \nu = (7541) \quad d = 4 \quad G(z) = 1 + z \\
    &c = 6 \quad \nu = (7532) \quad d = 4 \quad G(z) = 1 + 2z \\
    &c = 7 \quad \nu = (74321) \quad d = 5 \quad G(z) = 1 + 2z + z^2
\end{align*}
\]
Further example

Ex: $n = 7$, $\lambda = (433210)$, $\mu = (432210)$, $\nu = (7444321)$.

- LR coefficient $c_{\lambda\mu}^{\nu} = 13$
- LR polynomial

$$c_{t\lambda.t\mu}^{tv} = \frac{1}{10080} \times (t + 1)(t + 2)(t + 3)(t + 4)(t + 5) \times (5t + 21)(t^2 + 2t + 4)$$

- where $10080 = 5! \times 84$
- $d = 8$ and $G(z) = 1 + 4z + 12z^2 + 3z^3$
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Polynomial behaviour

Theorem  For all $\lambda, \mu, \nu$ such that $c_{\lambda \mu}^{\nu} > 0$ there exists

- a polynomial $P_{\lambda \mu}^{\nu}(t)$ in $t$ with $P_{\lambda \mu}^{\nu}(0) = 1$
- such that $P_{\lambda \mu}^{\nu}(t) = c_{t \lambda, t \mu}^{t \nu}$ for all positive integers $t$.

Conjectures

- coefficients in $P_{\lambda \mu}^{\nu}(t)$ are all rational and non-negative.
- coefficients in $G(z)$ are all positive integers.

Problems

- predict degree of polynomial
- explain origin of factors of form $(t + 1)(t + 2) \cdots (t + m)$
- prove (if true) and account for positivity of coefficients