Littlewood-Richardson coefficients, the hive model and Horn inequalities

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Contents

- Schur functions and LR-coefficients
- The hive model - vertex and edge labelling
- Horn inequalities
- Factorisation
- Problems of symmetry, conjugacy and complementarity
- Inequalities of the Stembridge and Gutschwager type
- Stretched LR-coefficients and LR-polynomials
- Open problems - positivity of coefficients, degrees and zeros of LR-polynomials

Key sources

- Littlewood-Richardson coefficients
  - D.E. Littlewood and A. Richardson, *Group characters and algebra*, Phil. Trans. Roy. Soc. London A **233** (1934) 99–141
  - D.E. Littlewood, *Theory of Group Characters*, 2nd Ed. Oxford: Clarendon Press, 1940
- The hive model
  - A. Knutson and T. Tao, *The honeycomb model of $GL_n(\mathbb{C})$ tensor products. I: Proof of the saturation conjecture*, J. Amer. Math. Soc. **12** (1999) 1055–1090
  - A.S. Buch, *The saturation conjecture (after A. Knutson and T. Tao). With an appendix by W. Fulton*, Enseign. Math. **46** (2000) 43–60

Key sources

- Horn inequalities
  - A. Horn, *Eigenvalues of sums of Hermitian matrices*, Pacific J. Math. **12** (1962) 225–241
- Puzzles
  - A. Knutson, T. Tao and C. Woodward, *The honeycomb model of $GL_n(\mathbb{C})$ tensor products. II: Puzzles determine facets of the Littlewood-Richardson cone*, Amer. Math. Soc. **17** (2004) 19–48
  - V.I. Danilov and G.A. Koshevoy, *Diskretnaya vognutost' i ermitovy matritsy*, Trudy Math. Inst. Steklova **241** (2003) 68–90, in Russian. Title translation: *Discrete convexity and Hermitian matrices*

Key sources

- Polynomial property of stretched LR-coefficients
  - E. Ehrhart, *Sur les polyedres rationnels homothetiques a n'dimensions*, C. R. Acad. Sci. Paris Ser A **254** (1962) 616-618
  - H. Derksen and J. Weyman, *On the Littlewood-Richardson polynomials*, J. Algebra. **255** (2002) 247–257
  - E. Rassart, *A polynomial property of Littlewood-Richardson coefficients*, J Comb. Theo. Ser. A **107** (2004) 161–179
- Overview
  - W. Fulton, *Eigenvalues, invariant factors, highest weights, and Schubert calculus*, Bull. Amer. Math. Soc. **37** (2000) 209–249

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Schur functions

- Let n be a fixed positive integer and $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ a sequence of indeterminates.
- Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ be a partition of weight $|\lambda|$ and length $\ell(\lambda) \leq n$, so that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0$.
- Then the Schur function $s_\lambda(\mathbf{x})$ is defined by:

$$s_\lambda(\mathbf{x}) = \frac{\left| x_i^{n+\lambda_j-j} \right|_{1\leq i,j\leq n}}{\left| x_i^{n-j} \right|_{1\leq i,j\leq n}}.$$

- The Schur functions form a $\mathbb{Z}$-basis of Λ_n, the ring of polynomial symmetric functions of $x_1, \ldots, x_n$.
- Each Schur function $s_\lambda(\mathbf{x})$ may be interpreted as the character $\mathrm{ch}\, V^\lambda(\mathbf{x})$ of an irrep of $gl(n)$.

LR-coefficients

- Any product of Schur functions can be expressed as a linear sum of Schur functions:

$$s_\lambda(\mathbf{x})\ s_\mu(\mathbf{x}) = \sum_\nu\ c^\nu_{\lambda\mu}\ s_\nu(\mathbf{x})$$

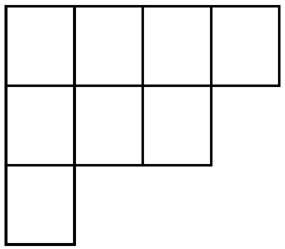
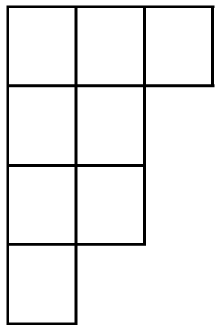
- The coefficients $c^\nu_{\lambda\mu}$ are the multiplicities appearing in the decomposition of the tensor product of irreps of $gl(n)$:

$$V^\lambda \otimes V^\mu = \sum_\nu\ c^\lambda_{\lambda\mu}\ V^\nu$$

- Each Littlewood-Richardson coefficient $c^\nu_{\lambda\mu}$ is a non-negative integer that may be evaluated by means of the Littlewood-Richardson rule

Young diagrams

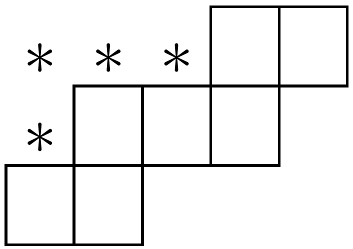
- Each partition λ specifies a Young diagram F^λ consisting of $|\lambda|$ boxes arranged in $\ell(\lambda)$ left adjusted rows of lengths $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_{\ell(\lambda)} > 0$.
- The partition λ' conjugate to λ is such that $F^{\lambda'}$ is obtained from F^λ by interchanging rows and columns.
- Ex: If $\lambda = (4, 3, 1)$ then $|\lambda| = 8$, $\ell(\lambda) = 3$, $\lambda' = (3, 2, 2, 1)$, with

$F^\lambda =$  $\qquad F^{\lambda'} =$ 

Skew Young diagrams

- Given partitions λ and ν such that all boxes of F^λ are contained in F^ν we write $\lambda \subseteq \nu$.
- Removing the boxes of F^λ from F^ν leaves the skew Young diagram $F^{\nu/\lambda}$
- Ex: If $\nu = (5,4,2)$ and $\lambda = (3,1)$ then

$$F^{\nu/\lambda} =$$



- The corresponding skew Schur function is such that

$$s_{\nu/\lambda}(\mathbf{x}) = \sum_{\mu} c^{\nu}_{\lambda\mu}\, s_\mu(\mathbf{x})$$

Littlewood-Richardson rule

- Fill the boxes of the Young diagram F^λ with 0's
- Then fill the boxes of the skew Young diagram $F^{\nu/\lambda}$ with μ_i entries i for $i = 1, 2, \ldots, n$.
- $c_{\lambda\mu}^\nu$ is the number of such diagrams with entries
  - weakly increasing across rows from left to right
  - strictly increasing down columns from top to bottom
  - satisfying the lattice permutation rule -
    i.e. at every stage in the sequence of non-zero entries read from right to left across rows taken in turn from top to bottom $\#1's \geq \#2's \geq \cdots \geq \#n's$

Application of the LR-rule

Ex: $n = 4$, $\lambda = (4,2)$, $\mu = (4,3,2)$, $\nu = (6,5,3,1)$

- The only valid LR-diagrams

$$
\begin{array}{cccccc}
0&0&0&0&1&1\\
0&0&1&1&2&\\
2&2&3&&&\\
3&&&&&
\end{array}
\qquad
\begin{array}{cccccc}
0&0&0&0&1&1\\
0&0&1&2&2&\\
1&2&3&&&\\
3&&&&&
\end{array}
\qquad
\begin{array}{cccccc}
0&0&0&0&1&1\\
0&0&1&2&2&\\
1&1&3&&&\\
3&&&&&
\end{array}
$$

- Some invalid diagrams

$$
\begin{array}{cccccc}
0&0&0&0&1&1\\
0&0&1&2&2&\\
1&\color{red}{3}&\color{red}{2}&&&\\
3&&&&&
\end{array}
\qquad
\begin{array}{cccccc}
0&0&0&0&1&1\\
0&0&1&1&2&\\
\color{red}{2}&\color{blue}{3}&3&&&\\
\color{red}{2}&&&&&
\end{array}
\qquad
\begin{array}{cccccc}
0&0&0&0&1&1\\
0&0&1&1&\color{blue}{3}&\\
2&2&2&&&\\
3&&&&&
\end{array}
$$

- Hence $c^{\nu}_{\lambda\mu} = 3$.

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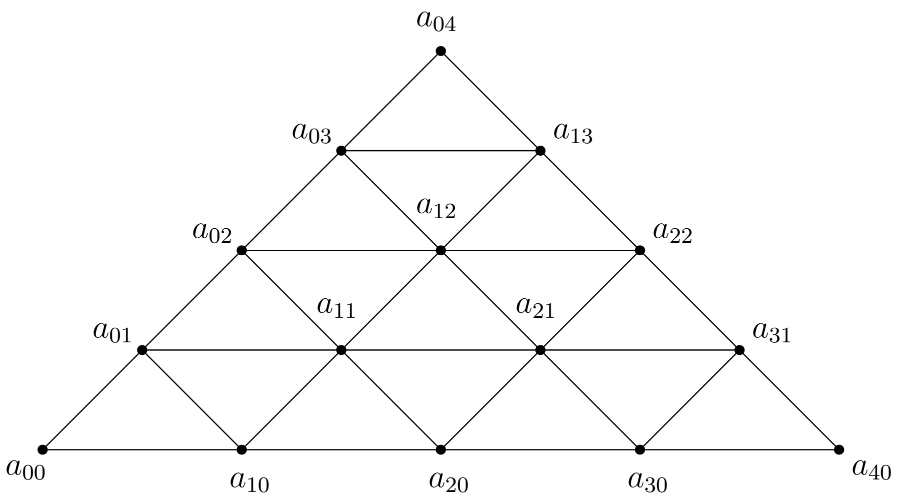
Integer hives

- Knutson & Tao [99], as described by Buch [00]
- An integer n-hive is a triangular graph with vertex labels $a_{ij} \in \mathbb{Z}$ for $0 \leq i, j, i + j \leq n$.

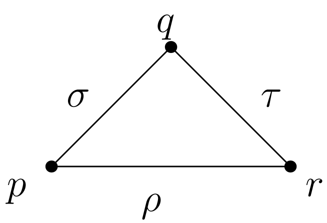
Ex: $n = 4$

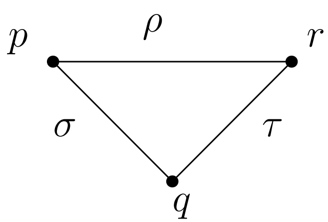


- Vertex labels increase along each edge from left to right

Relation between vertex and edge labels

- Edge labels are the non-negative differences between neighbouring vertex labels
- Vertex and edge labels for the two types of elementary triangle





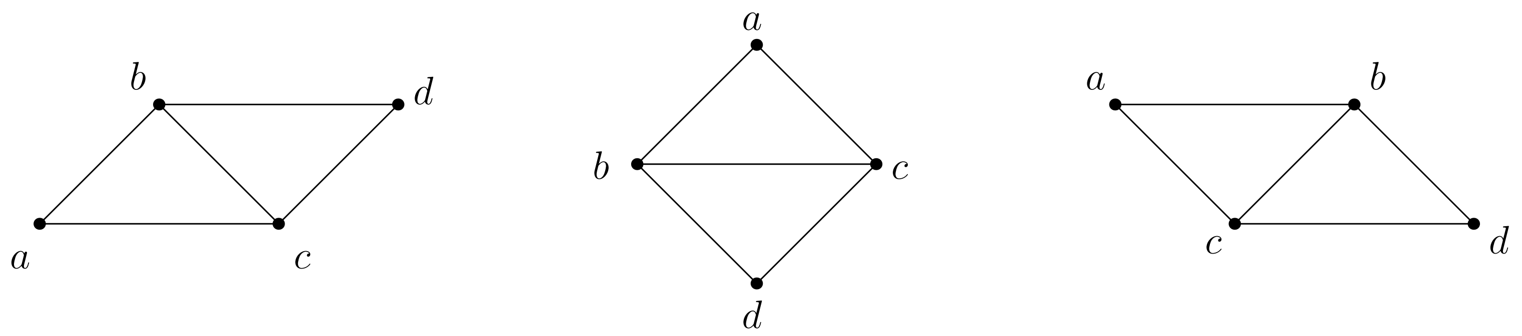
- In each case

$$\sigma = q - p \geq 0, \qquad \tau = r - q \geq 0, \qquad \rho = r - p \geq 0$$

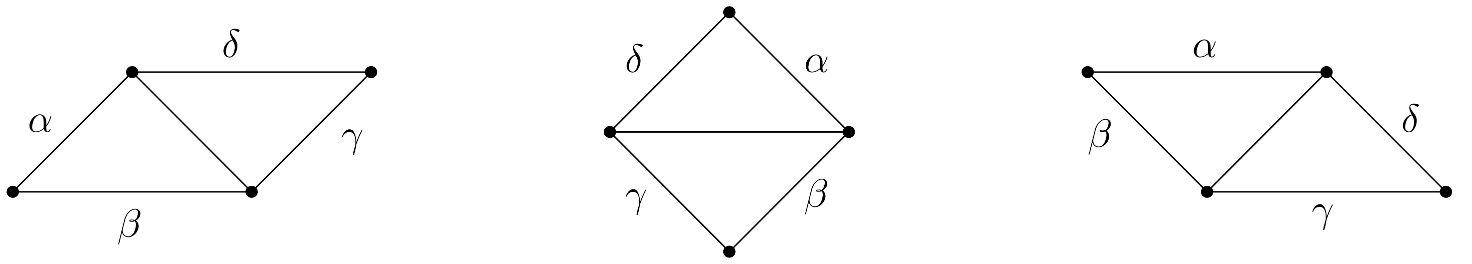
so that automatically we have $\sigma + \tau = \rho$ in any such triangle

Hive conditions

- Three distinct types of rhombi with vertex labels:



- The hive condition for each rhombus: $b + c \geq a + d$
- Three distinct types of rhombi with edge labels:

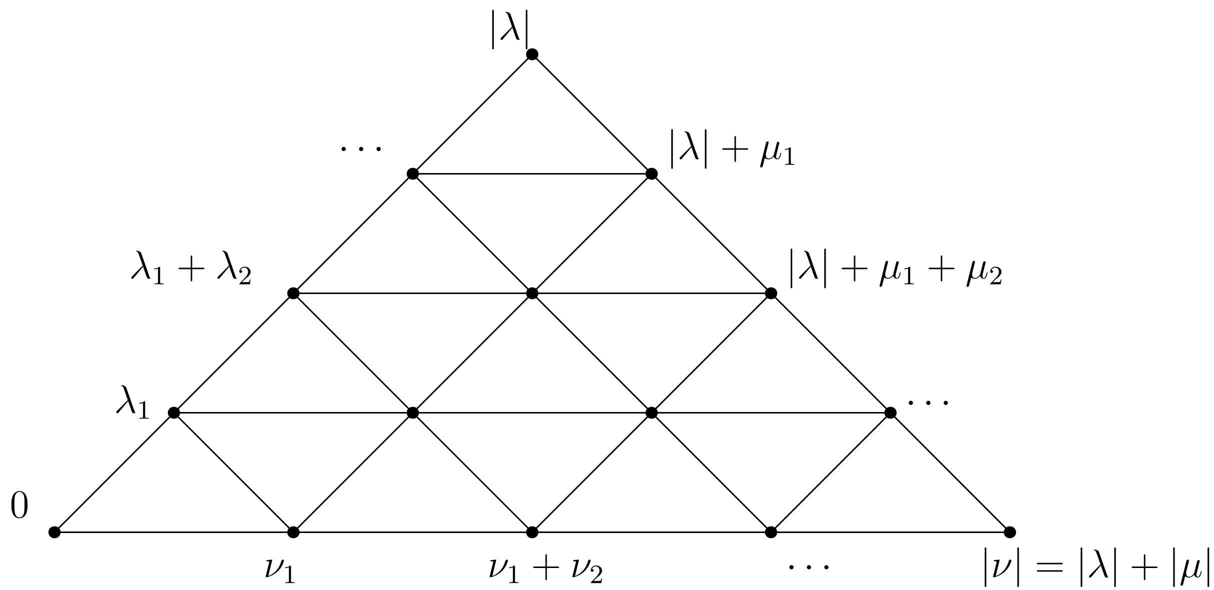


- The hive condition for each rhombus: $\alpha \geq \gamma$ and $\beta \geq \delta$
- Note: The triangle edge condition implies $\alpha + \delta = \beta + \gamma$.

LR-hives vertex labels

Definition An LR-hive is an integer n-hive for which

- all rhombi satisfy the hive conditions;
- boundaries determined by partitions λ, μ, ν with $\ell(\lambda), \ell(\mu), \ell(\nu) \leq n$ and $|\lambda| + |\mu| = |\nu|$;
- boundary vertex labels as shown:



LR-hives edge labels

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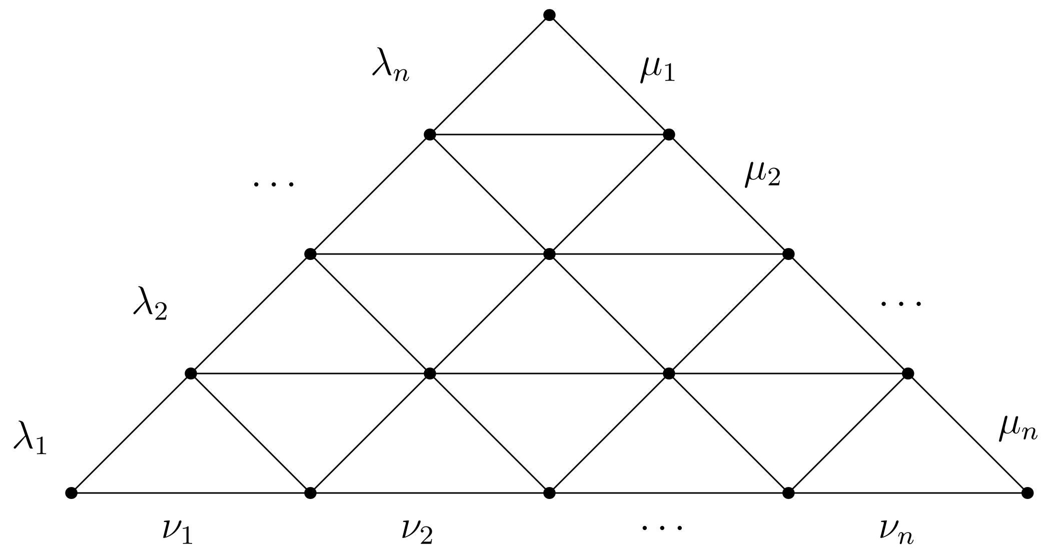

Bijection between LR-diagrams and LR-hives

- Lemma A bijection between LR integer n-hives, H, and LR-diagrams, D, is provided by the formula:

$$a_{ij} = \#\text{ of entries} \leq i \text{ in the first } i+j \text{ rows of } D$$

for the (i,j)th vertex label in H, for all i,j such that $0 \leq i, j, i+j \leq n$.

- Theorem The LR-coefficient $c^{\nu}_{\lambda\mu}$ is the number of LR-hives with boundary labels determined by λ, μ and ν.

- Note: Neither this theorem nor the Littlewood-Richardson rule allows us to see whether or not a given LR-coefficient $c^{\nu}_{\lambda\mu}$ is non-zero.

Example of bijection

Ex: $n = 4,\ \lambda = (753),\ \mu = (742),\ \nu = (9964)$

a_{ij}:

$$
\begin{array}{ccccccccc}
 & & & & 15 & & & & \\
 & & & 15 & & 22 & & & \\
 & & 12 & & 21 & & 26 & & \\
 & 7 & & 16 & & 24 & & 28 & \\
0 & & 9 & & 18 & & 24 & & 28
\end{array}
\iff
\begin{array}{|c|c|c|c|c|c|c|c|c|}
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
\hline
0 & 0 & 0 & 0 & 0 & 1 & 1 & 2 & 2 \\
\hline
0 & 0 & 0 & 1 & 1 & 2 \\
\cline{1-6}
1 & 2 & 3 & 3 \\
\cline{1-4}
\end{array}
$$

Example of bijection

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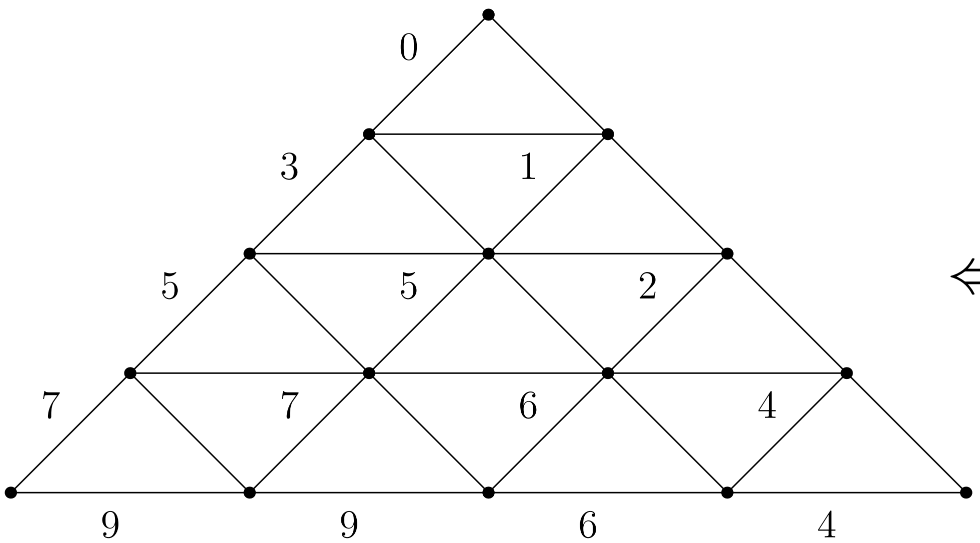
a_{ij}:



$\Longleftrightarrow$

0	0	0	0	0	0	0	1	1
0	0	0	0	0	1	1	2	2
0	0	0	1	1	2			
1	2	3	3					

Note The given edge labels are sufficient to fix all edge labels using the triangle condition

LR-hives showing that $c^{9964}_{753,742} = 6$

$$\begin{array}{ccccccccc} &&&&15&&&& \\ &&&15&&22&&& \\ &&12&&21&&26&& \\ &7&&16&&24&&28& \\ 0&&9&&18&&24&&28 \end{array} \qquad \begin{array}{ccccccccc} &&&&15&&&& \\ &&&15&&22&&& \\ &&12&&21&&26&& \\ &7&&16&&23&&28& \\ 0&&9&&18&&24&&28 \end{array} \qquad \begin{array}{ccccccccc} &&&&15&&&& \\ &&&15&&22&&& \\ &&12&&20&&26&& \\ &7&&16&&24&&28& \\ 0&&9&&18&&24&&28 \end{array}$$

$$\begin{array}{ccccccccc} &&&&15&&&& \\ &&&15&&22&&& \\ &&12&&20&&26&& \\ &7&&16&&23&&28& \\ 0&&9&&18&&24&&28 \end{array} \qquad \begin{array}{ccccccccc} &&&&15&&&& \\ &&&15&&22&&& \\ &&12&&20&&26&& \\ &7&&16&&22&&28& \\ 0&&9&&18&&24&&28 \end{array} \qquad \begin{array}{ccccccccc} &&&&15&&&& \\ &&&15&&22&&& \\ &&12&&19&&26&& \\ &7&&16&&23&&28& \\ 0&&9&&18&&24&&28 \end{array}$$

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Non-zero conditions

- We know that $c^{\nu}_{\lambda\mu}$ is the number of LR-hives with boundary labels determined by λ, μ and ν
- For given λ, μ and ν we would like some way of determining if $c^{\nu}_{\lambda\mu}$ is non-zero
- Horn [62] defined a set of inequalities and conjectured that they gave necessary and sufficient conditions for the solution of a problem later realised to be equivalent to that of LR-coefficients being non-zero
- The validity of Horn's conjecture was proved by the efforts of Klyachko [98], Knutson and Tao [99], Belkale [01], and Knutson, Tao and Woodward [04]
- For a comprehensive review see Fulton [00]

Partial sums

- Let $N = \{1, 2, \ldots, n\}$, then for fixed r, with $1 \leq r \leq n$, let $I = \{i_1, i_2, \ldots, i_r\} \subseteq N$ and $\overline{I} = N \backslash I$.
- For any partition λ let:
$$ps\,(\lambda)_I = \lambda_{i_1} + \lambda_{i_2} + \cdots + \lambda_{i_r}.$$
- If $i_1 < i_2 < \cdots < i_r$ then let
$$part\,(I) = (i_r - r, \ldots, i_2 - 2, i_1 - 1).$$
- Let T_r^n be the set of triples (I, J, K) with $I, J, K \subset N$ and $\#I = \#J = \#K = r$ with $c_{part\,(I) part\,(J)}^{part\,(K)} > 0$.
- Let R_r^n be the set of triples (I, J, K) with $I, J, K \subset N$ and $\#I = \#J = \#K = r$ with $c_{part\,(I) part\,(J)}^{part\,(K)} = 1$.

Non-zero conditions

Theorem: The LR-coefficient $c^{\nu}_{\lambda\mu}$ is non-zero if and only if

$$|\nu| = |\lambda| + |\mu|$$

and Horn's inequalities,

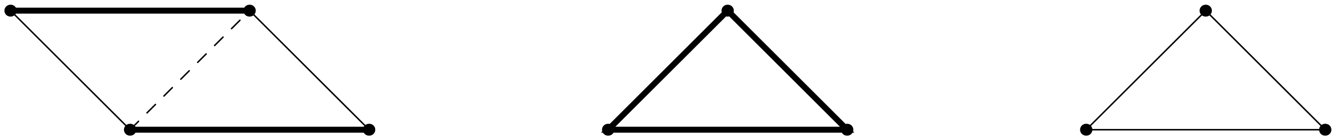
$$ps\,(\nu)_K \leq ps\,(\lambda)_I + ps\,(\mu)_J\,,$$

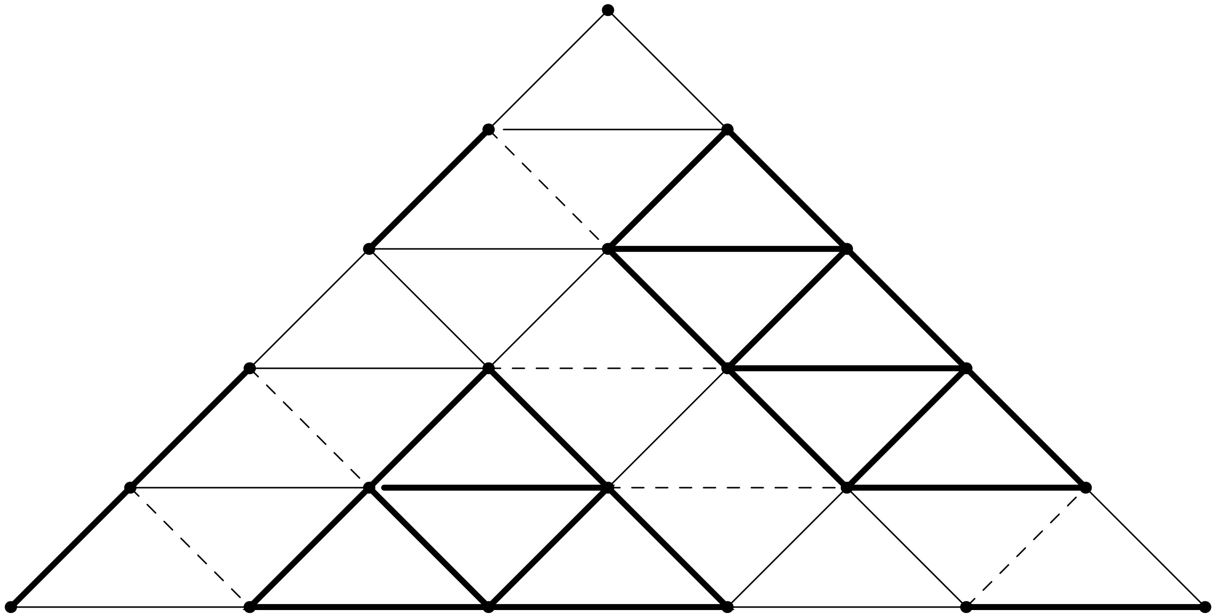
are satisfied for all $r = 1, 2, \ldots, n-1$ and all $(I, J, K) \in T^n_r$

Note: Not all of Horn's inequalities are essential. Horn's essential inequalities are those for which $(I, J, K) \in R^n_r$ - but where do they come from?

Puzzles - Knutson, Tao & Woodward [04]

Definition A puzzle is a diagram on a triangular lattice in which edges are distinguished so that it is composed of copies of the following pieces oriented in any way so as to fit:





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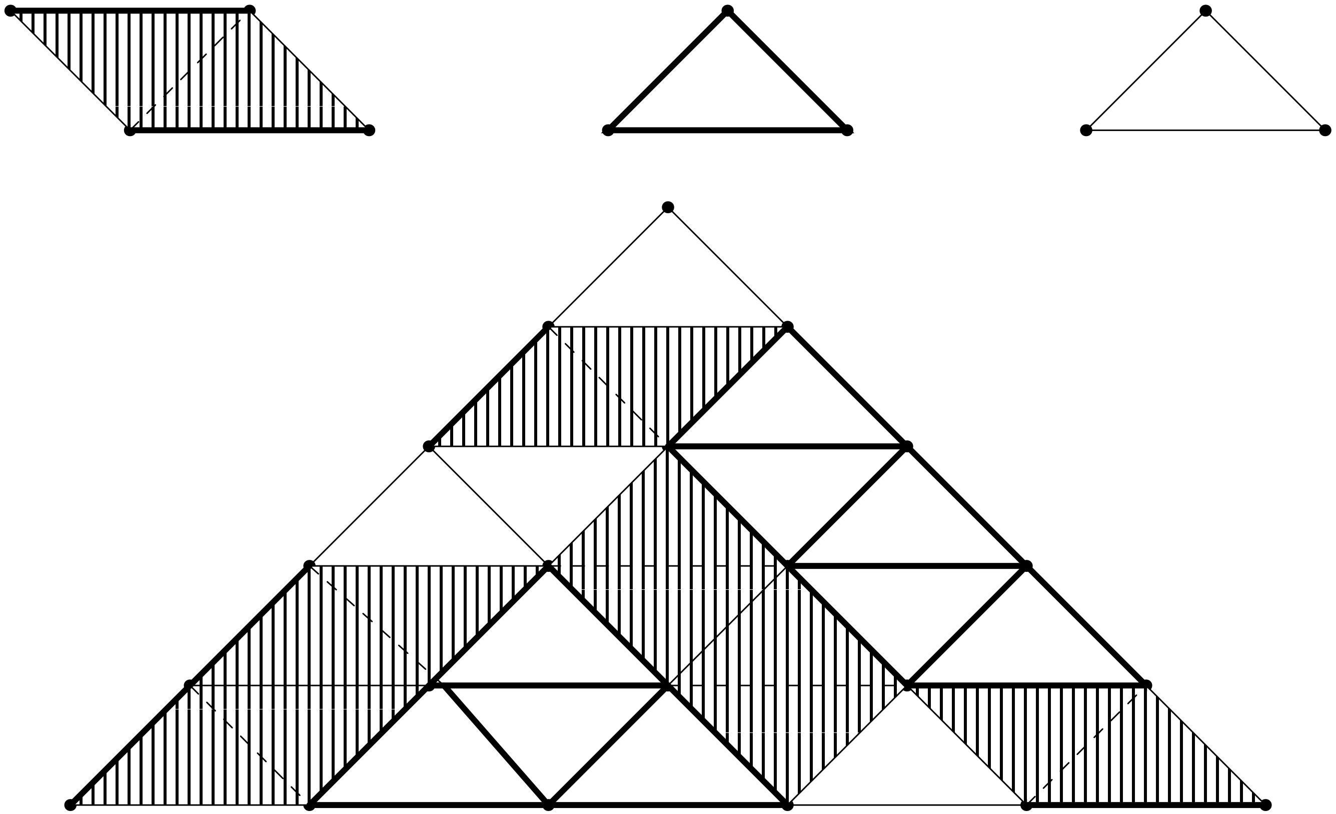

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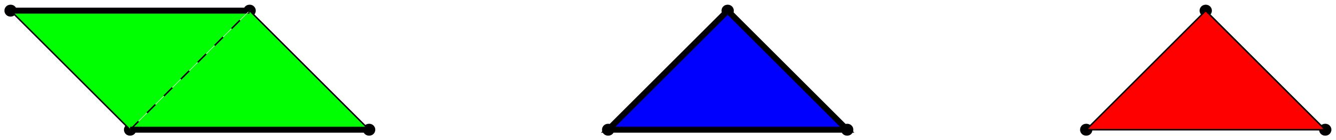
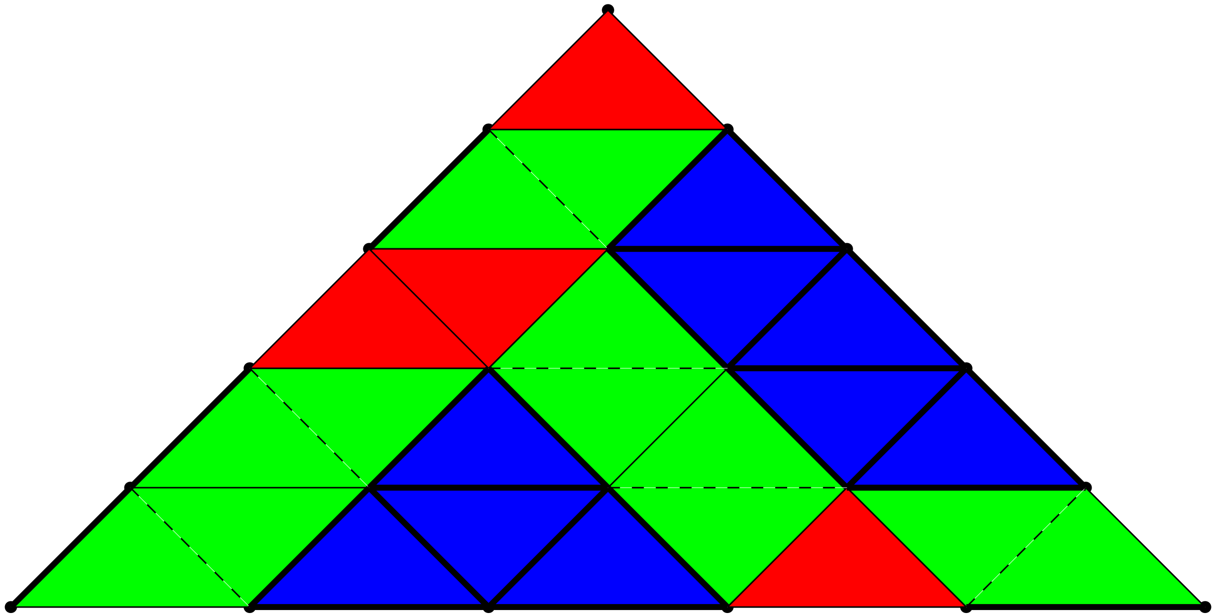
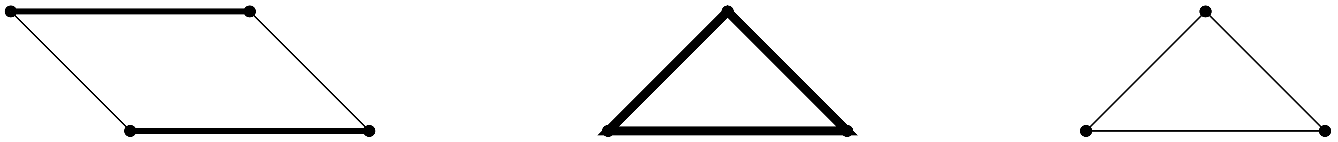
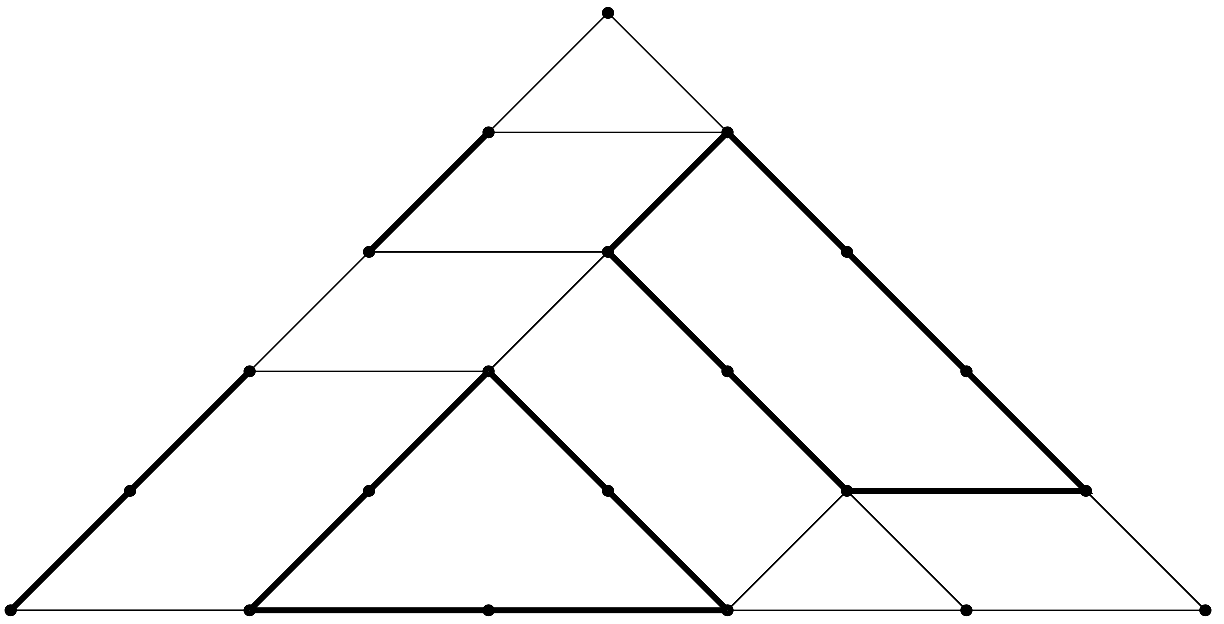




Hive plan or labyrinth

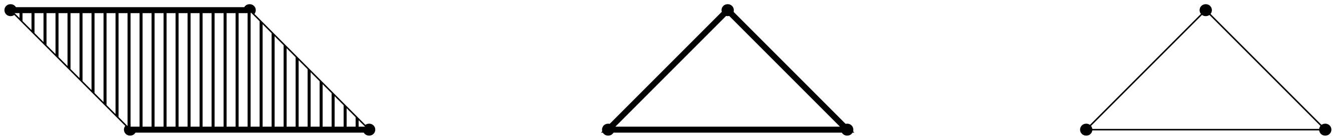
Definition A hive plan is made up of corridors, dark rooms and light rooms obtained by deleting interior edges of a puzzle:

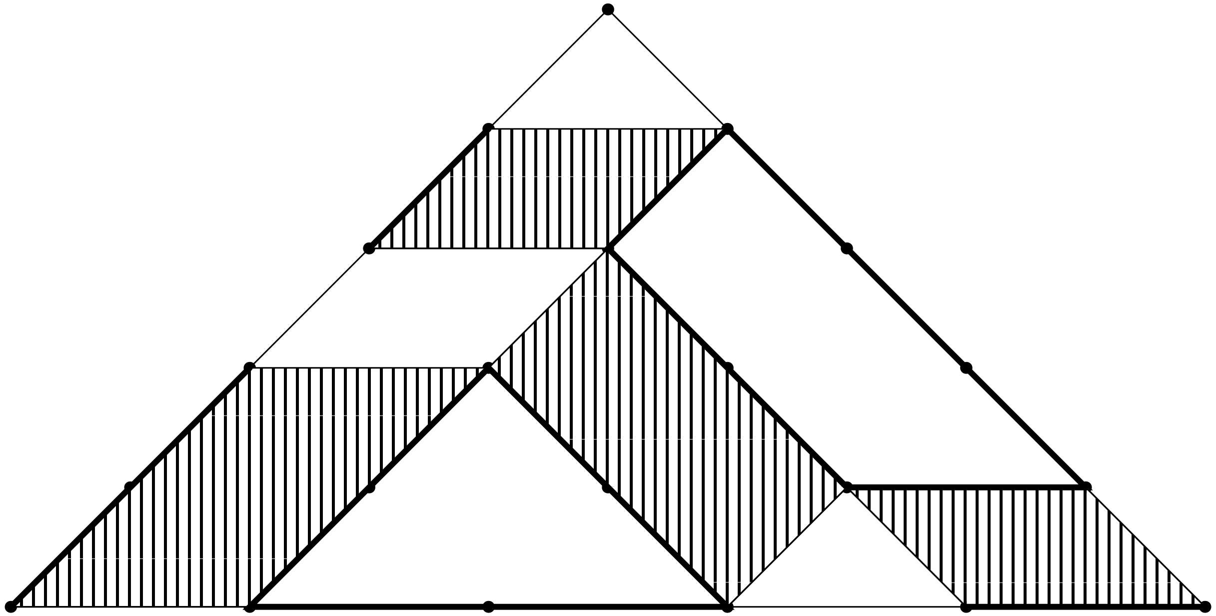




Hive plan or labyrinth

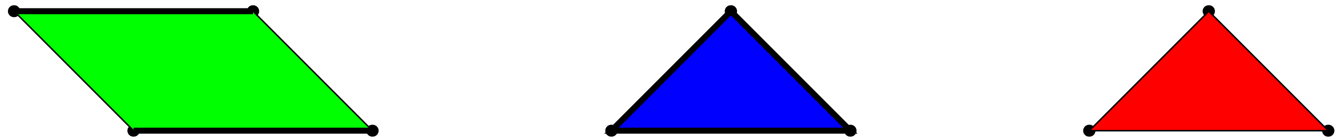
Definition A hive plan is made up of shaded corridors, dark rooms and light rooms obtained by deleting interior edges of a puzzle:

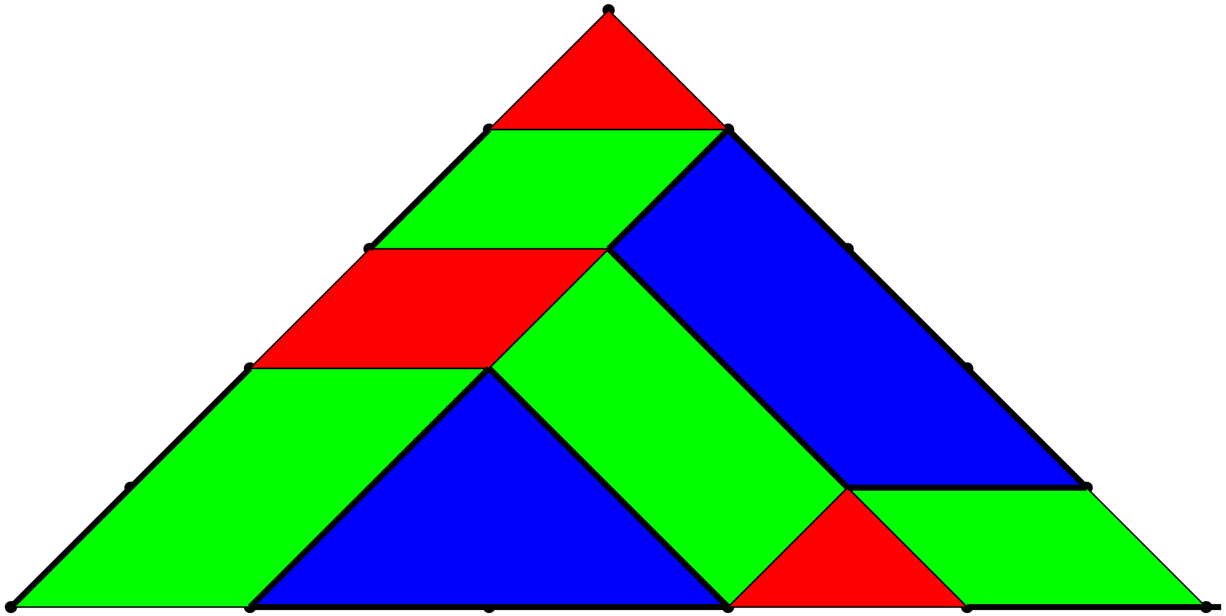




Hive plan or labyrinth

Definition A hive plan is made up of corridors, blue rooms and red rooms obtained by deleting interior edges of a puzzle:

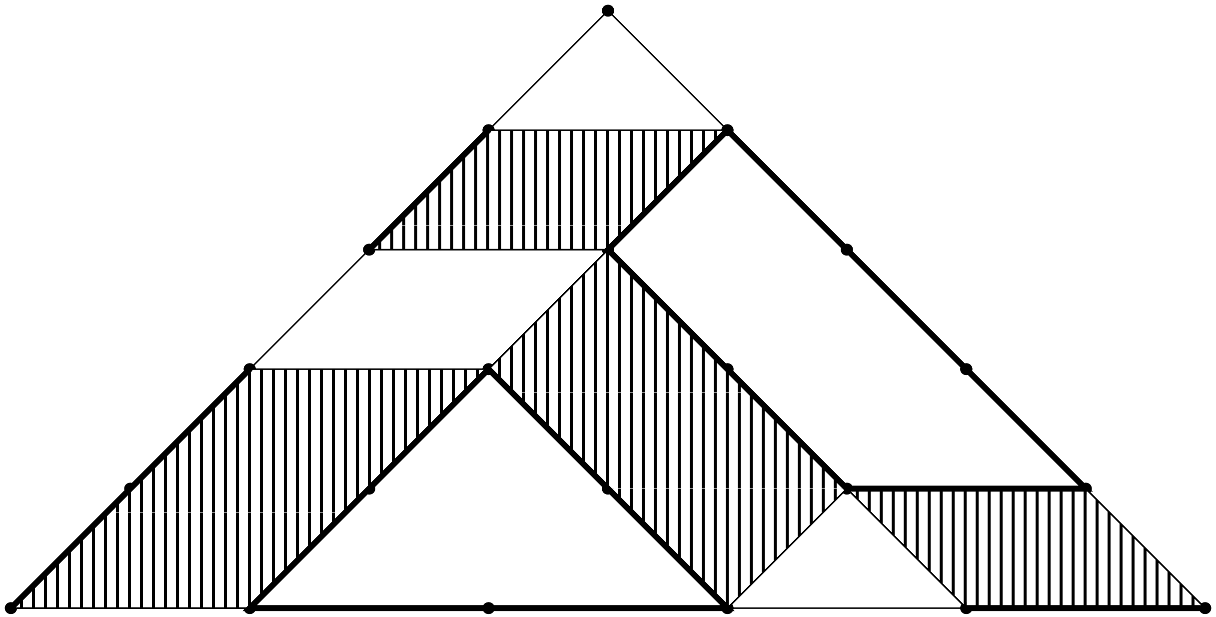




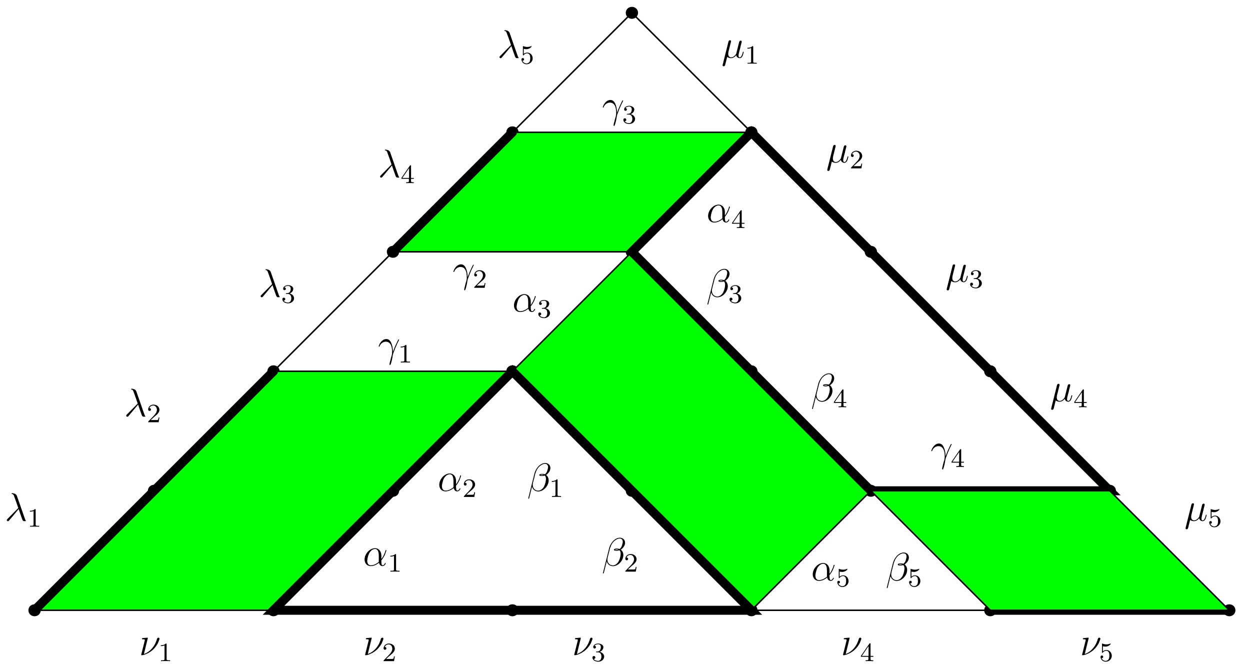
Link between puzzles and Horn triples

- (I, J, K) is Horn triple if it specifies the positions of the thick edges on the boundary of any puzzle. It is essential if the puzzle with these boundary thick edges is unique.

- For $I = (1, 2, 4)$, $J = (2, 3, 4)$ and $K = (2, 3, 5)$ we have:



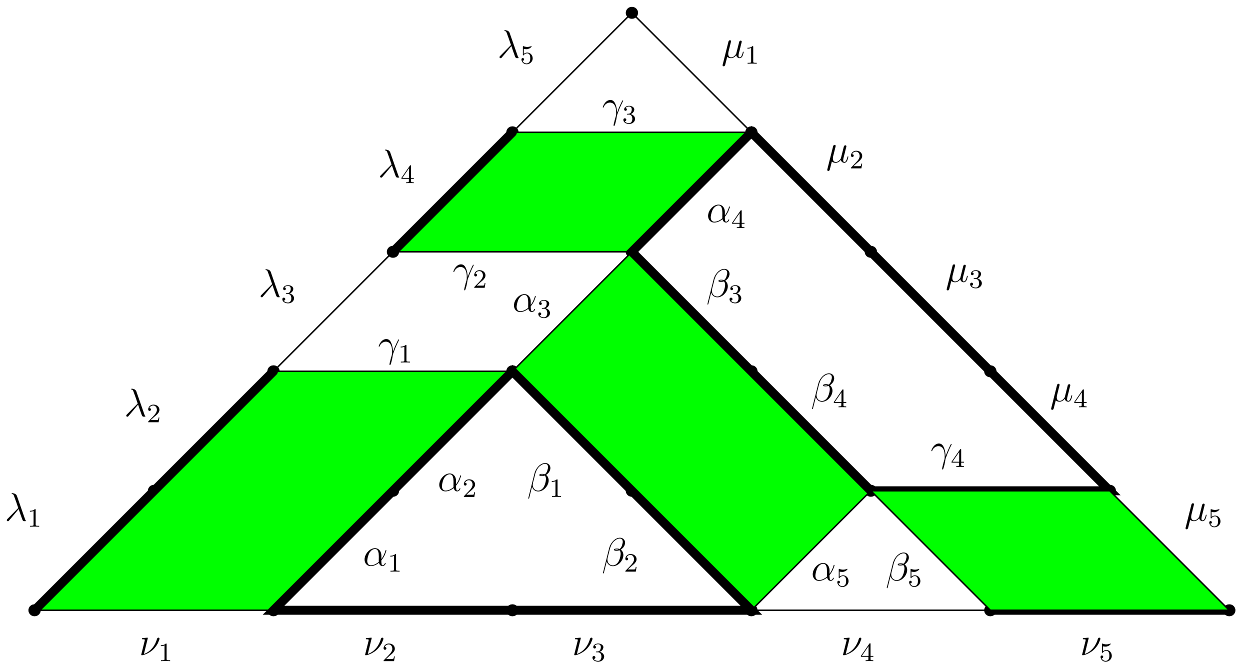
Each Horn triple defines an inequality



$$\nu_2 + \nu_3 + \nu_5 \leq (\nu_2 + \nu_3) + \gamma_4 = (\alpha_1 + \alpha_2 + \beta_1 + \beta_2) + \gamma_4$$
$$\leq \lambda_1 + \alpha_2 + \beta_1 + \beta_2 + \gamma_4 \leq \lambda_1 + \lambda_2 + \beta_1 + \beta_2 + \gamma_4$$
$$\leq \lambda_1 + \lambda_2 + \beta_3 + \beta_2 + \gamma_4 \leq \lambda_1 + \lambda_2 + (\beta_3 + \beta_4 + \gamma_4)$$
$$= \lambda_1 + \lambda_2 + (\alpha_4 + \mu_2 + \mu_3 + \mu_4) \leq \lambda_1 + \lambda_2 + \lambda_4 + \mu_2 + \mu_3 + \mu_4$$

That is $\quad ps\,(\nu)_K \leq ps\,(\lambda)_I + ps\,(\mu)_J$.

Complementary Horn inequalities



The same procedure applied to thin-edge inequalities gives

$$\nu_1 + \nu_4 \geq \gamma_1 + \nu_4 = \gamma_1 + (\alpha_5 + \beta_5)$$

$$\geq \gamma_1 + \alpha_3 + \beta_5 \geq (\gamma_1 + \alpha_3) + \mu_5 = (\lambda_3 + \gamma_2) + \mu_5$$

$$\geq \lambda_3 + \gamma_3 + \mu_5 = \lambda_3 + \lambda_5 + \mu_1 + \mu_5$$

That is $\quad ps\,(\nu)_{\overline{K}} \geq ps\,(\lambda)_{\overline{I}} + ps\,(\mu)_{\overline{J}}$

Significance of inequalities derived from puzzles

- Each inequality $ps\,(\nu)_K \leq ps\,(\lambda)_I + ps\,(\mu)_J$ derived from a puzzle must be satisfied if $c_{\lambda\mu}^{\nu}$ is to be non-zero
- To show that a puzzle triple (I, J, K) is a Horn triple, and the inequality a Horn inequality, we must make a connection between puzzles with thick boundary edges specified by (I, J, K) and $c_{part\,(I), part\,(J)}^{part\,(K)}$
- First note that the passage from I to $part\,(I)$ is easy

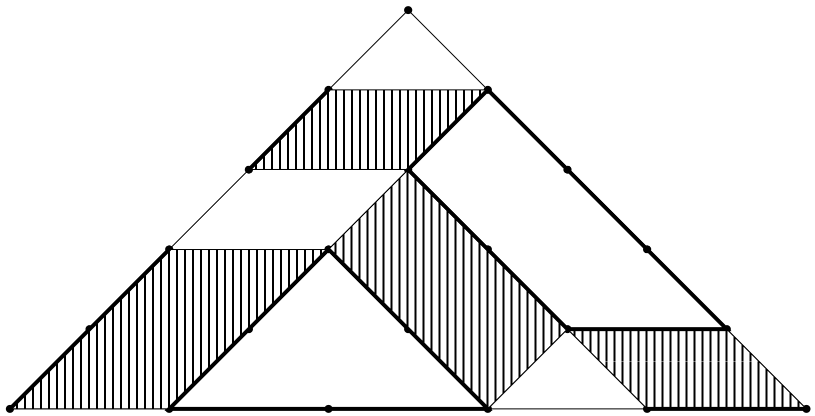
Ex If $I = \{1, 3, 6, 8\}$
then $(1, 3, 6, 8) - (1, 2, 3, 4) = (0, 1, 3, 4)$
so that $part\,(I) = (4, 3, 1)$

How may puzzles are there?

Theorem The number of puzzles with the positions of the thick edges on the boundary specified by (I, J, K) is given by $c_{part\,(I),part\,(J)}^{part\,(K)}$

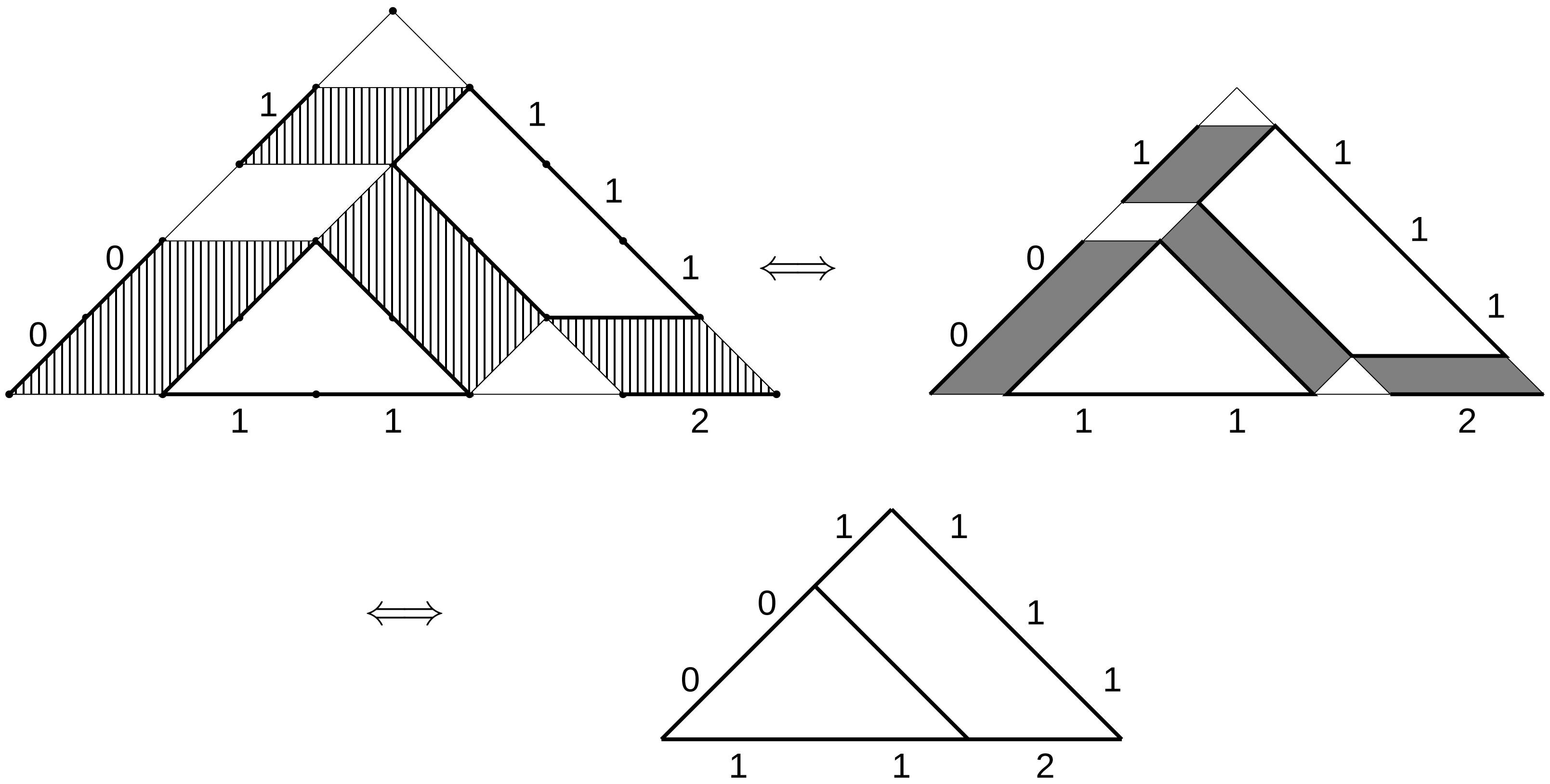
Ex: $n = 5$, $r = 2$, $I = (1, 2, 4)$, $J = (2, 3, 4)$, $K = (2, 3, 5)$

- In this case $c_{part\,(I),part\,(J)}^{part\,(K)} = c_{1,111}^{311} = 1$
- and there exists just the one puzzle identified earlier



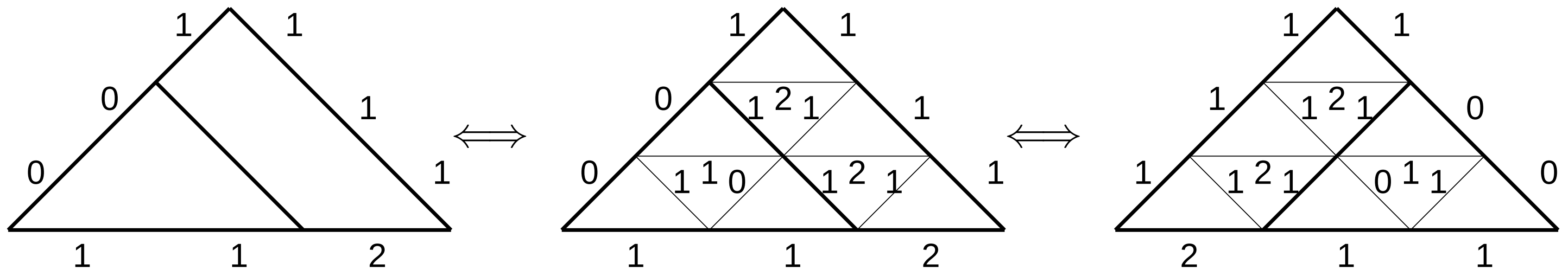
A map from puzzles to hives

- Let the thick edges of the puzzle be specified by (I, J, K)
- For $M = I, J, K$ in turn, label each thick boundary edge of the puzzle by the corresponding row part of $part\,(M)$
- Scale the length of all thin edges by t and let $t \to 0$



A map from puzzles to hives contd

- In each thick edged room set all parallel edge labels equal using the triangle condition wherever required
- Reflect the resulting diagram in its vertical axis of symmetry to obtain a hive



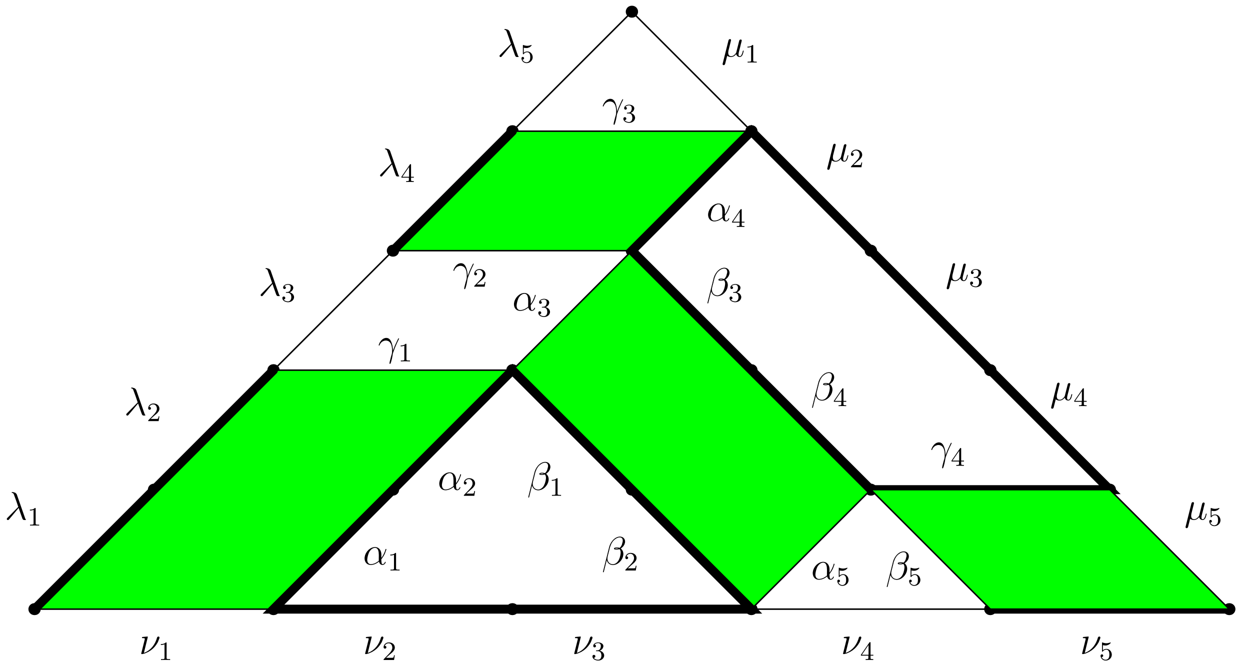
Lemma This map provides a bijection between puzzles with thick edges specified by (I, J, K) and LR-hives with boundary specified by $part\,(J), part\,(I), part\,(K)$

Corollaries

- Each triple (I, J, K) is a Horn triple if and only if it specifies the thick boundary edges of a puzzle
- The corresponding inequality $ps\,(\nu)_K \leq ps\,(\lambda)_I + ps\,(\mu)_J$ defined by the puzzle is a Horn inequality
- The number of puzzles specified by (I, J, K) is equal to $c^{part\,(K)}_{part\,(J), part\,(I)} = c^{part\,(K)}_{part\,(I), part\,(J)}$
- The puzzle is said to be rigid if it is unique, that is $c^{part\,(K)}_{part\,(J), part\,(I)} = 1$, and the corresponding Horn inequality is essential
- The LR-coefficient $c^{\nu}_{\lambda\mu}$ is non-zero if and only if every essential Horn inequality is satisfied

Contents

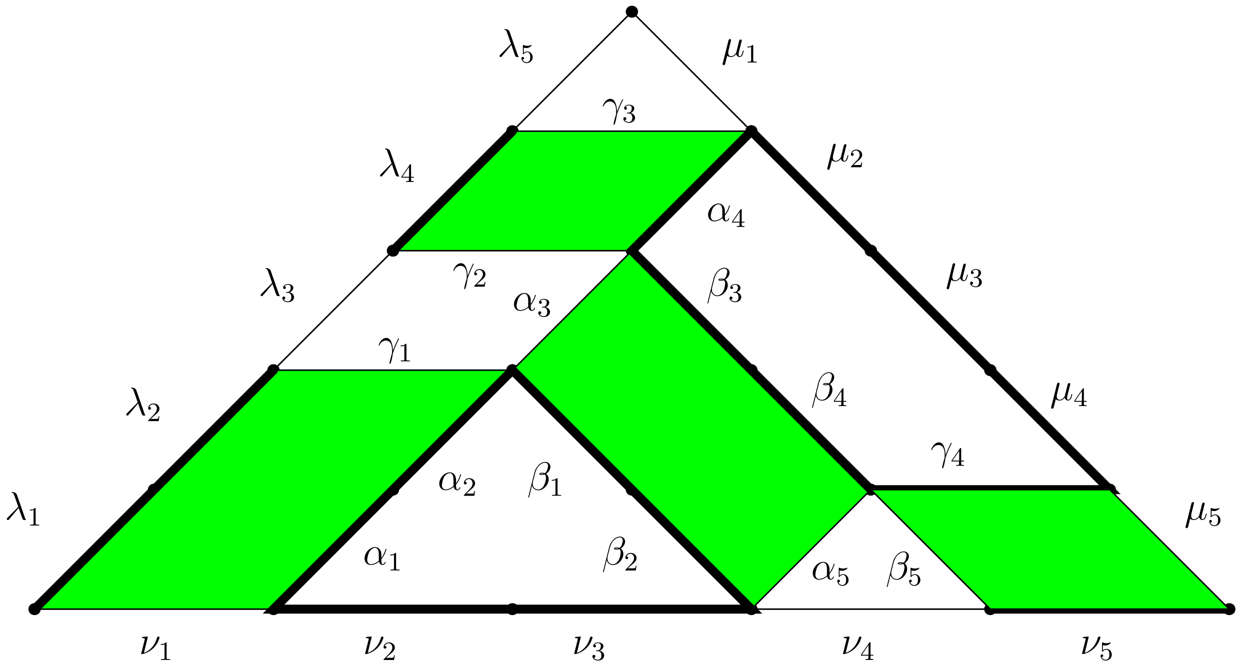
Consequences of any Horn equality



All sequences of inequalities become equalities.

- $\nu_2+\nu_3+\nu_5 = (\nu_2+\nu_3)+\gamma_4 = (\alpha_1+\alpha_2+\beta_1+\beta_2)+\gamma_4$
  $= \lambda_1+\alpha_2+\beta_1+\beta_2+\gamma_4 = \lambda_1+\lambda_2+\beta_1+\beta_2+\gamma_4$
  $= \lambda_1+\lambda_2+\beta_3+\beta_2+\gamma_4 = \lambda_1+\lambda_2+(\beta_3+\beta_4+\gamma_4)$
  $= \lambda_1+\lambda_2+(\alpha_4+\mu_2+\mu_3+\mu_4) = \lambda_1+\lambda_2+\lambda_4+\mu_2+\mu_3+\mu_4$.

Edge label equalities



- $\nu_1 + \nu_4 = \gamma_1 + \nu_4 = \gamma_1 + (\alpha_5 + \beta_5)$
  $= \gamma_1 + \alpha_3 + \beta_5 = (\gamma_1 + \alpha_3) + \mu_5 = (\lambda_3 + \gamma_2) + \mu_5$
  $= \lambda_3 + \gamma_3 + \mu_5 = \lambda_3 + \lambda_5 + \mu_1 + \mu_5$
- These equalities imply: $\nu_5 = \gamma_4$, $\alpha_1 = \lambda_1$, $\alpha_2 = \lambda_2$, $\beta_1 = \beta_3$, $\beta_2 = \beta_4$, $\alpha_4 = \lambda_4$, $\beta_5 = \mu_5$, $\nu_1 = \gamma_1$, $\gamma_2 = \gamma_3$.

Factorisation of LR-hives

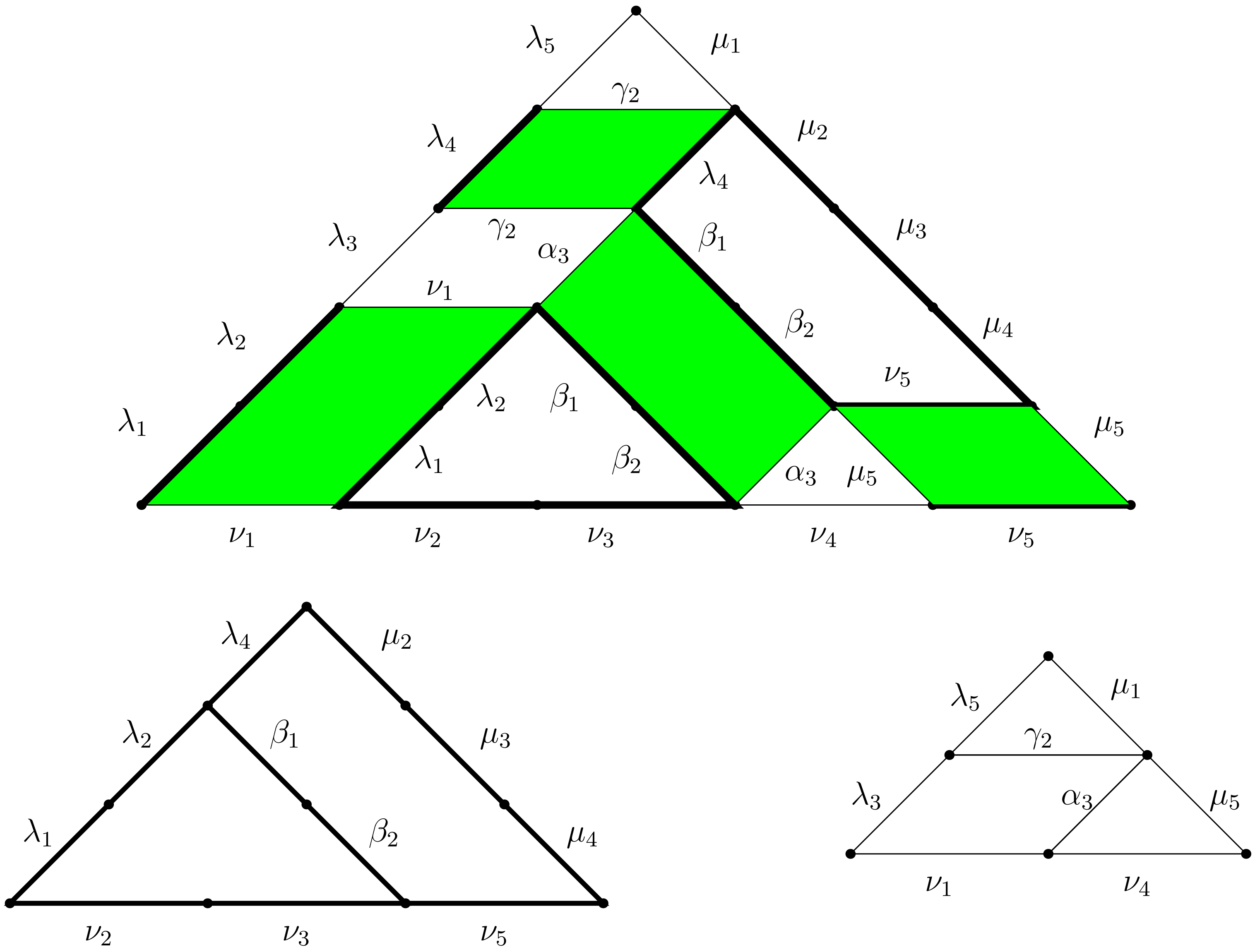
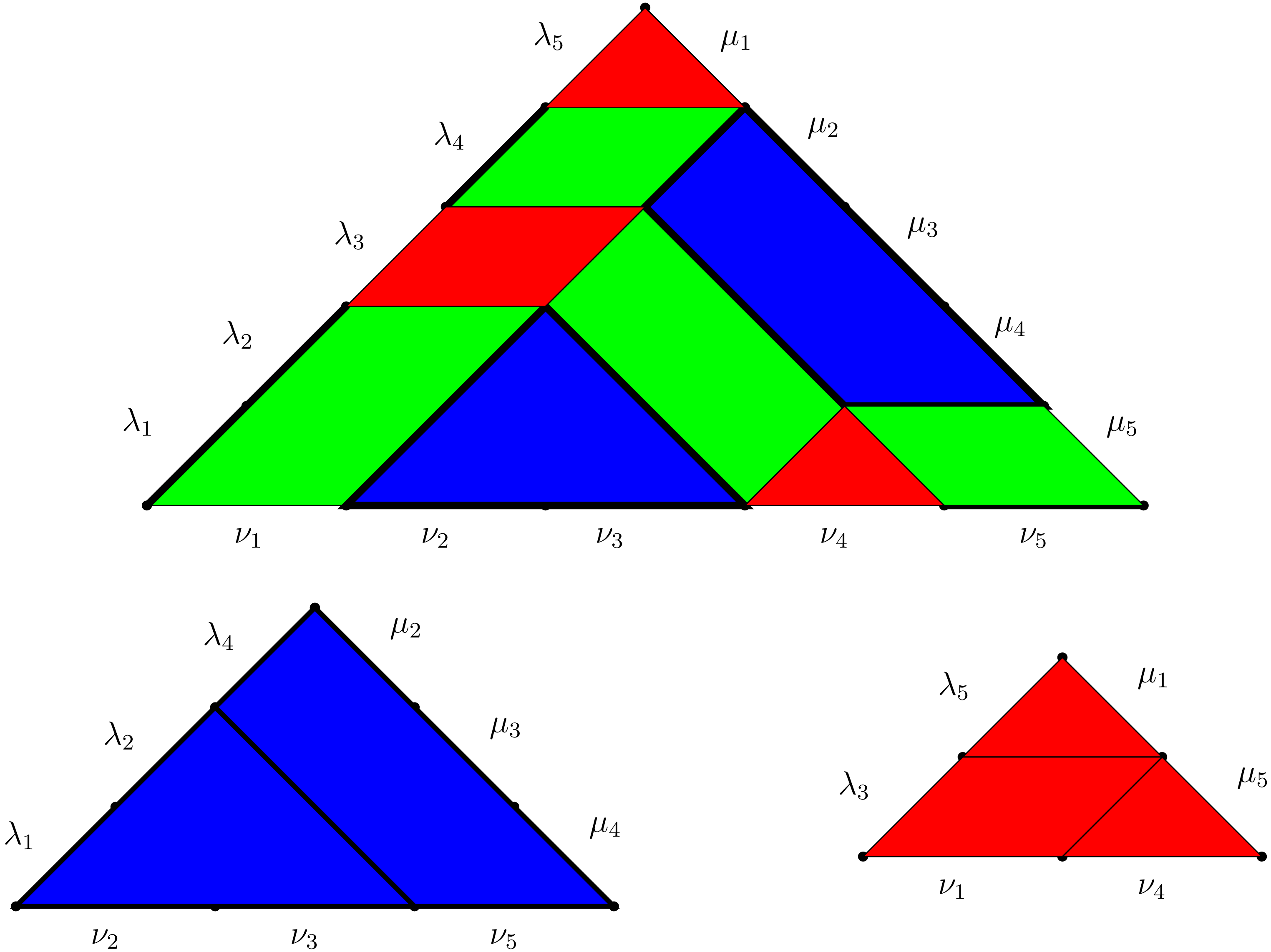


Illustration of H_n and subhives H_r,H_{n-r}



LR-coefficient factorisation

- Lemma In the case of any Horn equality and a corresponding puzzle, the deletion of redundant corridors from any LR-hive H_n gives a pair of LR-subhives H_r and H_{n-r}.
- Lemma In the case of any essential Horn equality, this map from the LR-hives H_n to pairs of LR-hives H_r and H_{n-r} is a bijection.
- Theorem If an essential Horn inequality is saturated then $c^{\nu}_{\lambda\mu}$ factorises.
- Definition If all essential Horn inequalities are strict $c^{\nu}_{\lambda\mu}$ is said to be primitive.

LR factorisation example

Ex: $n = 5$, $r = 3$, $n - r = 2$:

- $\lambda = (9, 7, 6, 2, 0)$, $\mu = (13, 5, 3, 1, 0)$, $\nu = (14, 12, 11, 5, 4)$.
- $I = \{1, 2, 4\}$, $J = \{2, 3, 4\}$, $K = \{2, 3, 5\}$.
- $\lambda_I = (9, 7, 2)$, $\mu_J = (5, 3, 1)$, $\nu_K = (12, 11, 4)$
- $\lambda_{\overline{I}} = (6, 0)$, $\mu_{\overline{J}} = (13, 0)$, $\nu_{\overline{K}} = (14, 5)$
- $ps\,(\nu)_K = 27 = 18 + 9 = ps\,(\lambda)_I + ps\,(\mu)_J$

Hence

$$c^{(14,12,11,5,4)}_{(9,7,6,2,0),(13,5,3,1,0)} = c^{(12,11,4)}_{(9,7,2),(5,3,1)}\; c^{(14,5)}_{(6,0),(13,0)} = 2 \cdot 1 = 2$$

Note: This is an example of the reduction of an LR-coefficient, since $c^{(14,12,11,5,4)}_{(9,7,6,2,0),(13,5,3,1,0)} = c^{(12,11,4)}_{(9,7,2),(5,3,1)} = 2$

Proof of factorisation

To be shown:

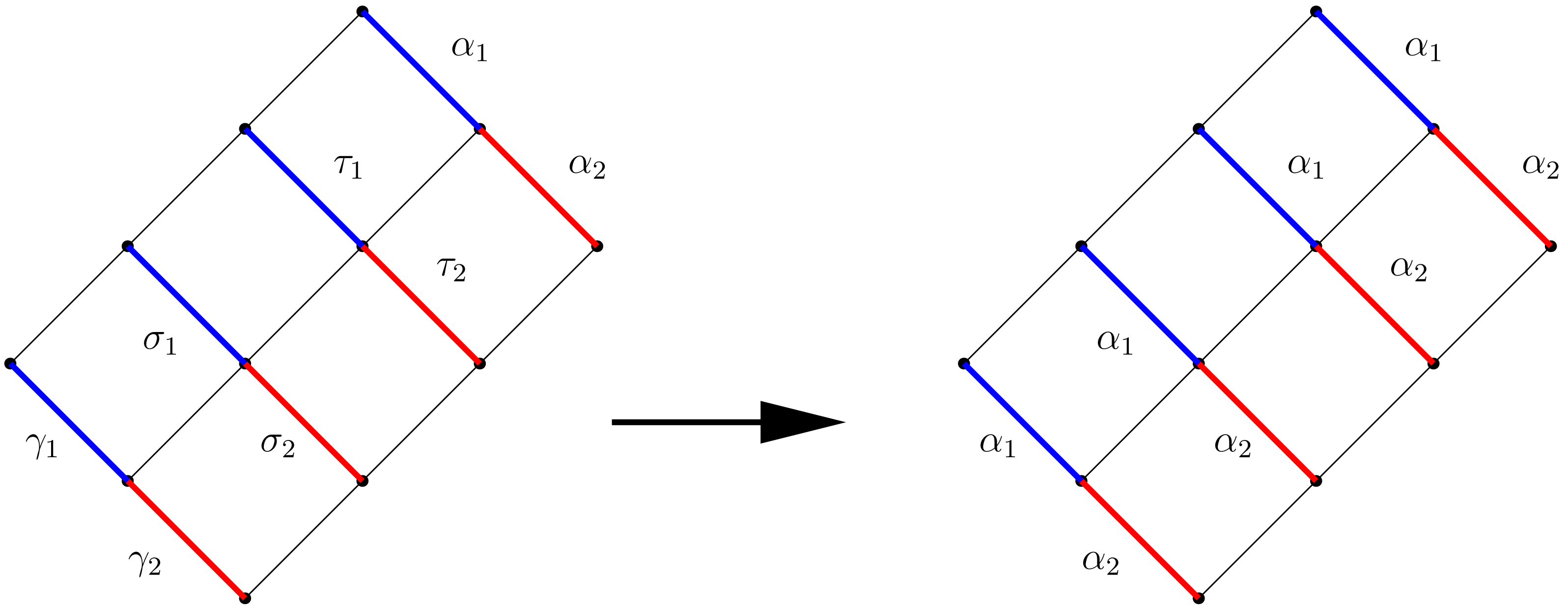
- the corridors R_n of H_n are redundant;
- the dark rooms constitute an LR-hive H_r;
- the light rooms constitute an LR-hive H_{n-r};
- any LR hives H_r,H_{n-r} joined by R_n gives an LR-hive H_n.

To be checked that the Horn equality implies:

- all corridor edge labels fixed;
- LR hive conditions for any rhombus split by corridor;
- LR hive conditions across corridor/dark room boundary;
- LR hive conditions across corridor/light room boundary.

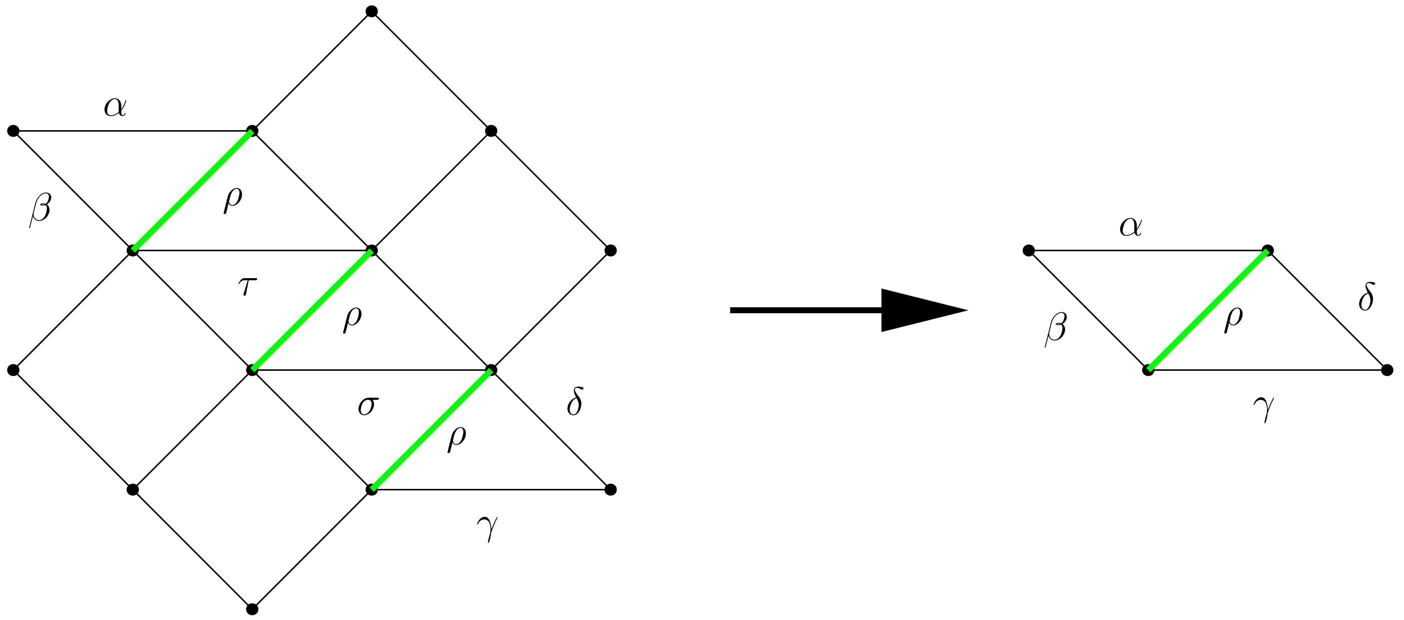
Corridor edges fixed

- Hive conditions: $\gamma_1 \le \sigma_1 \le \tau_1 \le \alpha_1, \quad \gamma_2 \le \sigma_2 \le \tau_2 \le \alpha_2$.
- Horn equality: $\gamma_1 + \gamma_2 = \alpha_1 + \alpha_2$.
- Implies: $\gamma_1 = \alpha_1$ and $\gamma_2 = \alpha_2$.
- Implies: $\gamma_1 = \sigma_1 = \tau_1 = \alpha_1$ and $\gamma_2 = \sigma_2 = \tau_2 = \alpha_2$.



Deletion of corridor

- Initial hive conditions: $\gamma \leq \sigma, \ \sigma \leq \tau, \ \tau \leq \alpha$.
- Implies final hive condition: $\gamma \leq \alpha$.



- Horn equality $\alpha - \beta = \rho = \gamma - \delta$ implies $\alpha + \delta = \beta + \gamma$.

Paths - gentle and good

- Path: a continuous sequence of connected corridor walls with dark rooms, thick-edged 0-regions, on the right and light rooms, thin-edged 1-regions, on the left.
- Gentle path: at each vertex the deviation is 0 or $\pm\pi/3$.
- Gentle loop: a gentle path that forms a closed interior loop.
- An edge is good if it forms the short diagonal of a rhombus satisfying the hive condition, otherwise it is bad.
- Good path: gentle path along which all the edges are good, ie with the hive condition satisfied across each edge.

Good paths and factorisation

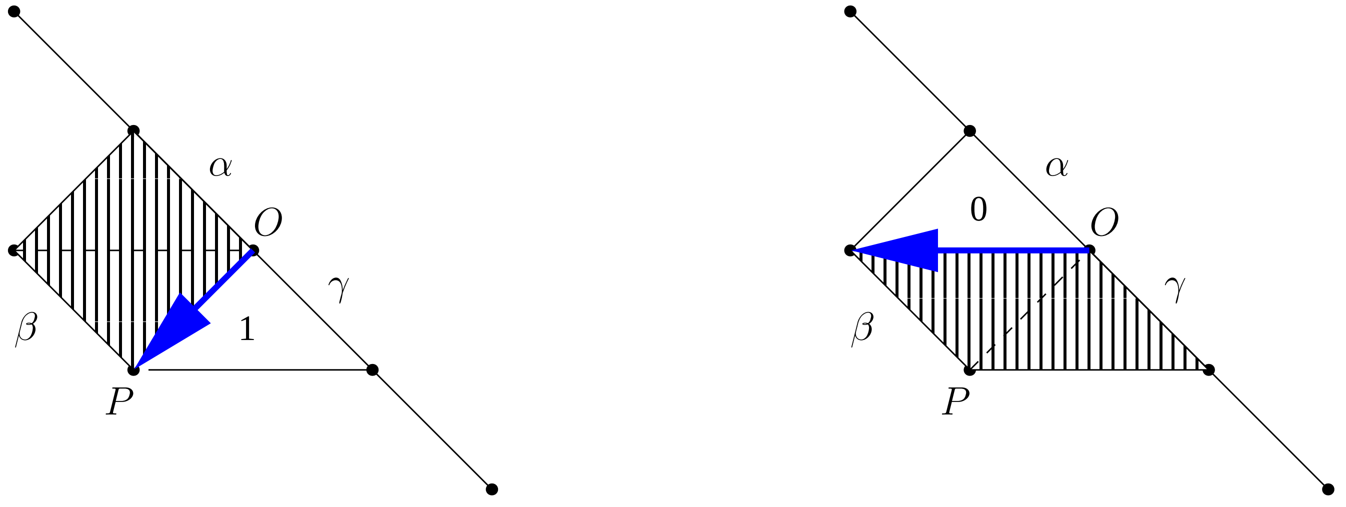
Observations

- When subdividing two LR subhives, H_r and H_{n-r}, and inserting corridors to create a hive H_n, this hive will be an LR hive if and only if each edge of every internal corridor wall of the corresponding hive plan is good.
- Let the boundary labels λ, μ, ν of the hive H_n be such that for a given puzzle specified by (I, J, K) the corresponding Horn inequality is saturated, ie. $ps\,(\nu)_K = ps\,(\lambda)_I + ps\,(\mu)_J$. If in the corresponding hive plan each edge of every internal corridor wall lies on a good path, then the LR-coefficient $c^{\nu}_{\lambda\mu}$ factorises.

Start of good paths

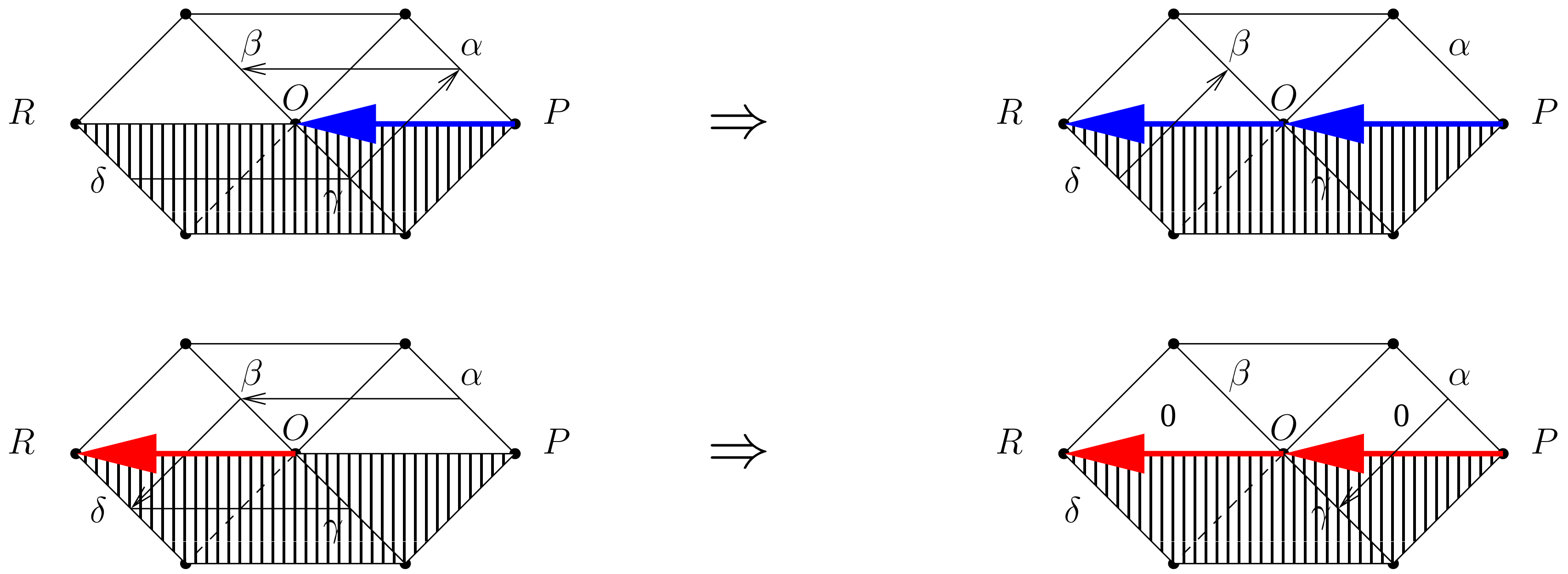
Lemma The first edge of any path starting from any boundary is good

- Each boundary has edges specified by a partition: $\alpha \geq \gamma$.
- Horn equality applied to corridors: $\beta = \alpha$ or $\beta = \gamma$.
- Hence $\beta = \alpha \geq \gamma$ or $\beta = \gamma \leq \alpha$ so that in both cases OP is good.



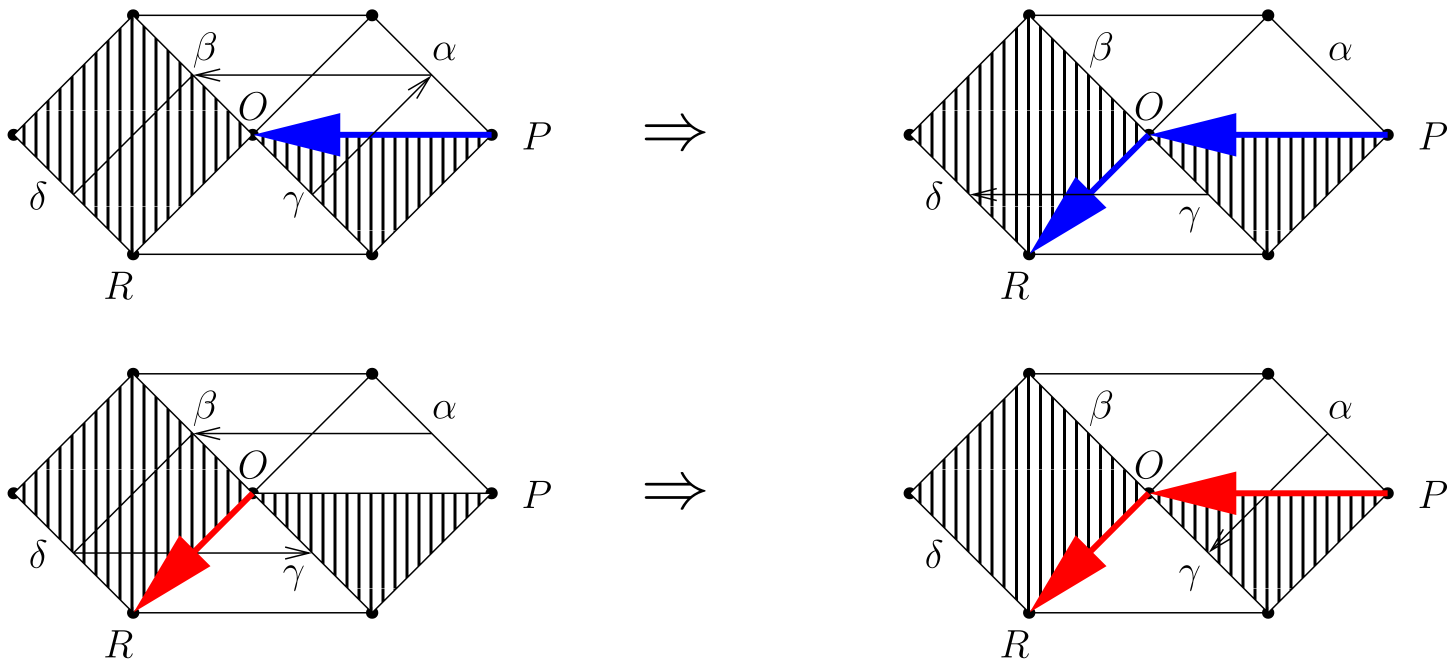
Path along an interior corridor wall

- Initial hive conditions: $\alpha \le \beta$
- Horn equality applied to corridors: $\delta = \gamma$.
- PO good $\Rightarrow \gamma \le \alpha \Rightarrow \delta = \gamma \le \alpha \le \beta \Rightarrow OR$ good
- OR bad $\Rightarrow \delta > \beta \Rightarrow \gamma = \delta > \beta \ge \alpha \Rightarrow PO$ bad



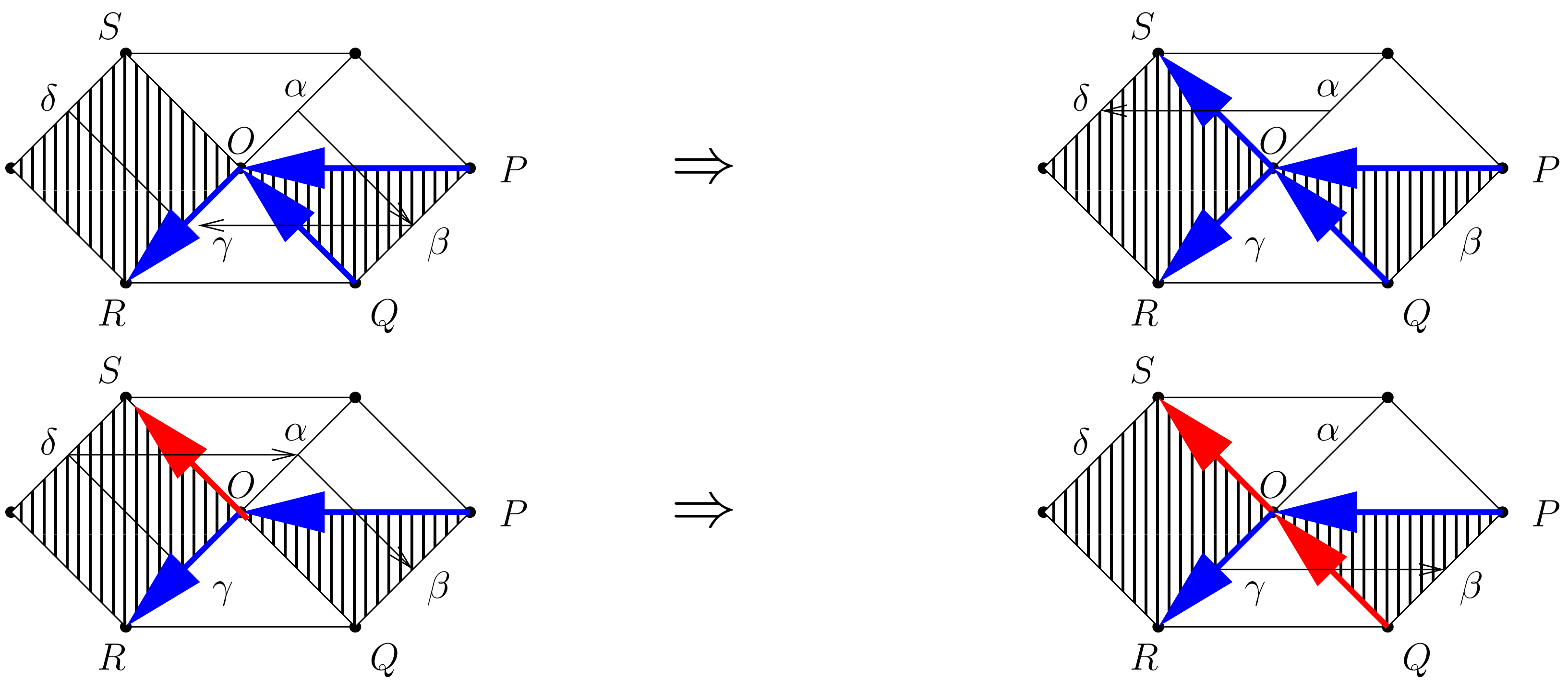
Path reaching a vertex

- Initial hive condition: $\alpha \leq \beta$; Horn equality $\delta = \beta$
- PO good $\Rightarrow \gamma \leq \alpha \Rightarrow \gamma \leq \alpha \leq \beta = \delta \Rightarrow OR$ good
- OR bad $\Rightarrow \gamma > \delta \Rightarrow \gamma > \delta = \beta \geq \alpha \Rightarrow PO$ bad

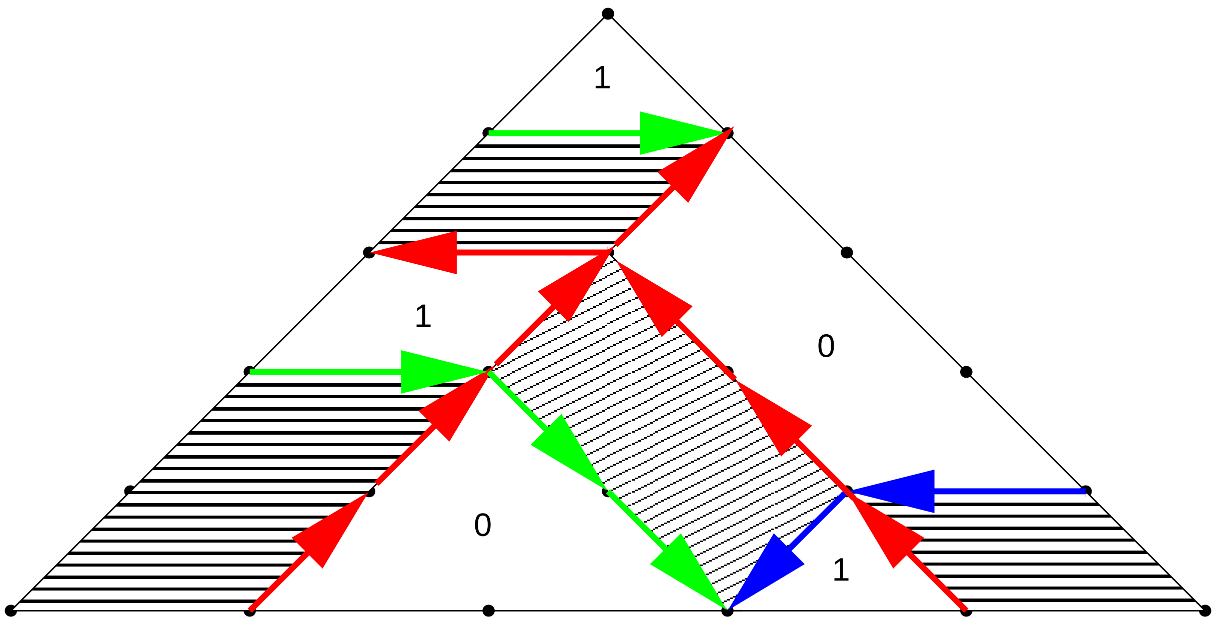


Intersecting paths

- Assume PO good , so that $\beta \geq \alpha$.
- Horn equality $\gamma = \delta$
- QO good $\Rightarrow \gamma \geq \beta \Rightarrow \delta = \gamma \geq \beta \geq \alpha \Rightarrow OS$ good
- OS bad $\Rightarrow \delta < \alpha \Rightarrow \gamma = \gamma = \delta < \alpha \leq \beta \Rightarrow QO$ bad



Union of good paths



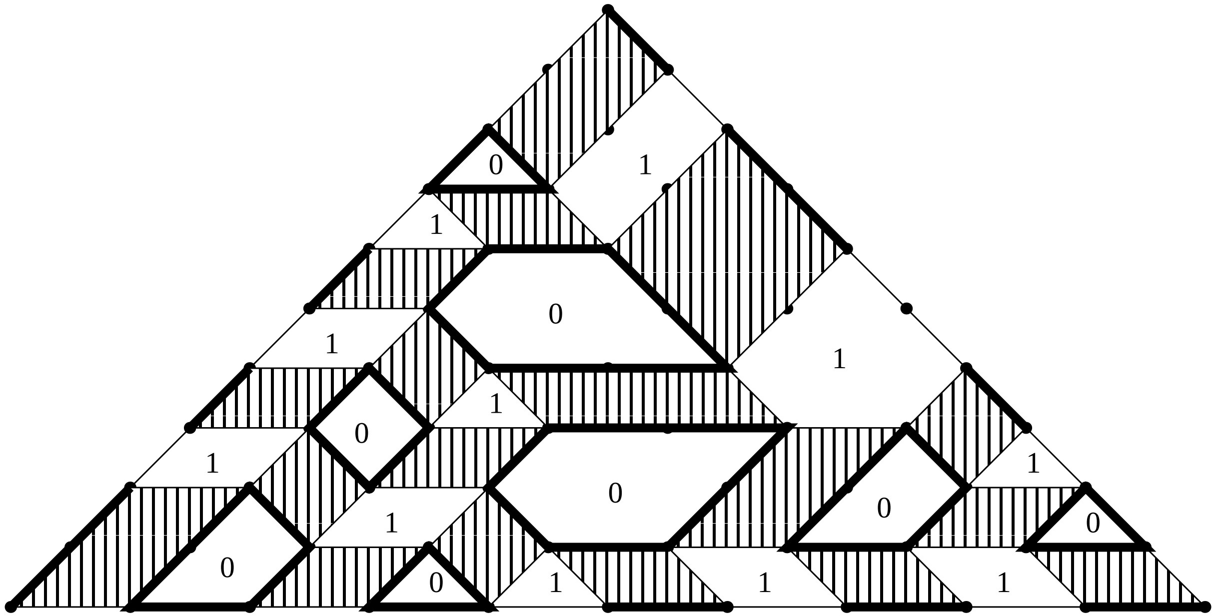
- Good paths cover all interior corridor walls
- All LR hive conditions satisfied
- Hence we have factorisation

Bad edges and gentle loops

- Good paths may not cover all interior corridor edges
- If there exists a bad edge then its predecessor on some gentle path must also be bad
- A reverse gentle path of bad edges cannot reach the boundary, since all gentle paths start from the boundary with a good edge
- A reverse gentle path of bad edges must therefore continue indefinitely
- Since there are only a finite number of edges, it follows that there may only be bad edges if there exists a gentle loop

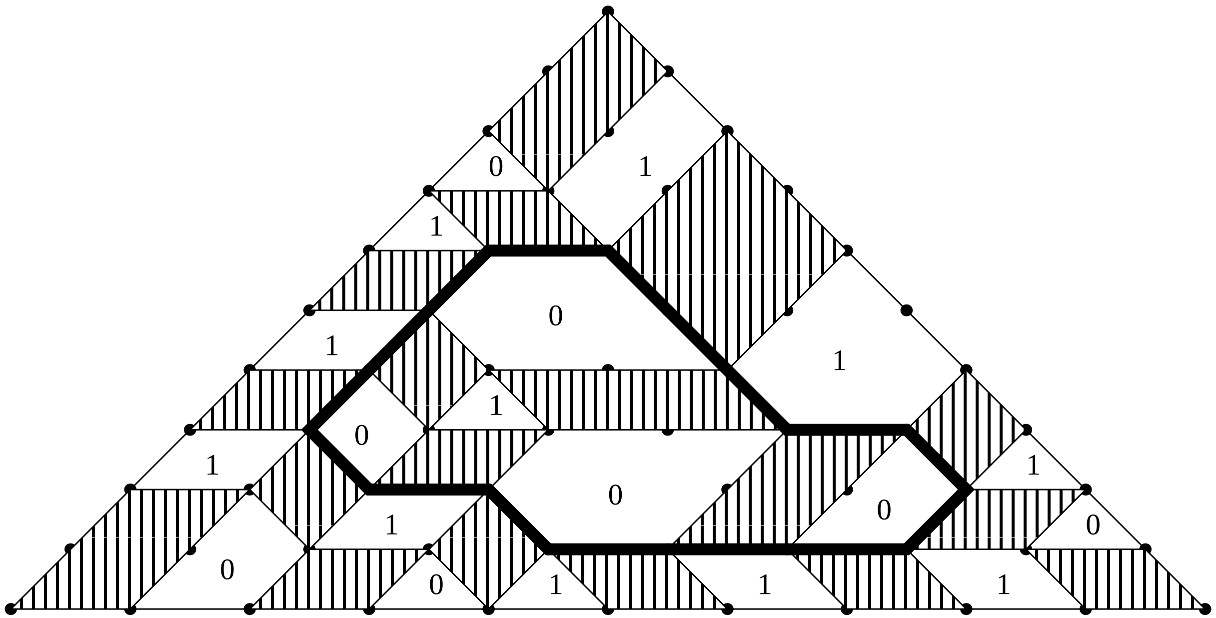
Example exhibiting a gentle loop

- $n = 10$, $r = 5$.
- $I = (1, 2, 4, 6, 8)$, $J = (1, 3, 4, 7, 9)$, $K = (2, 4, 6, 8, 10)$.



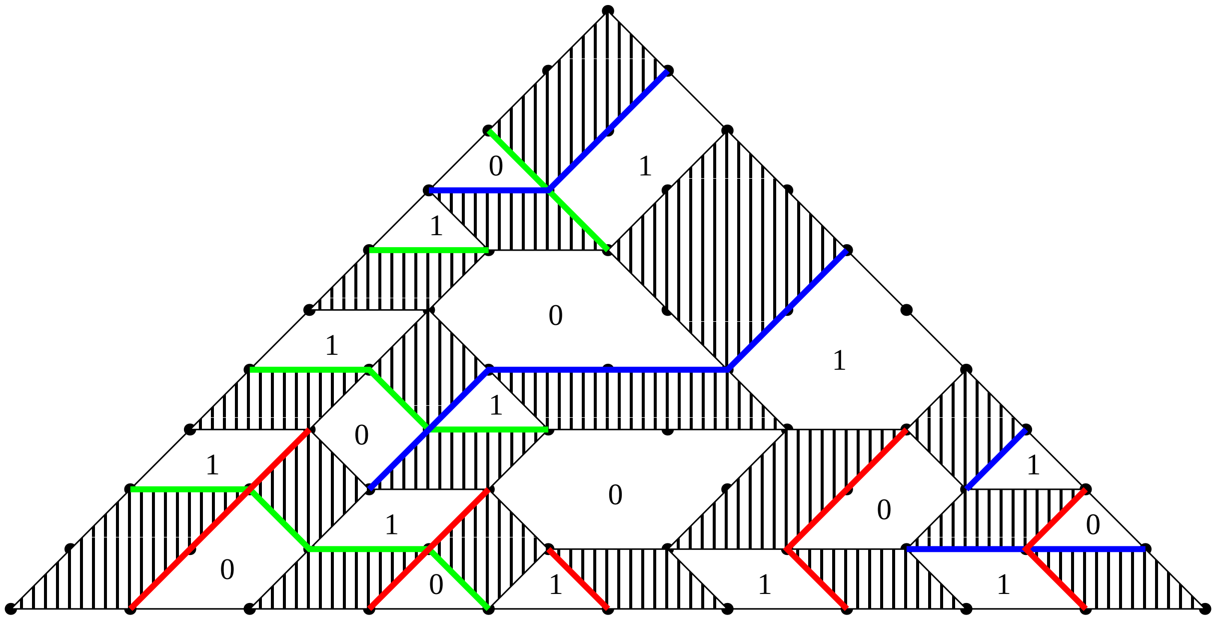
Example of a gentle loop

- Gentle loop is any closed interior gentle path.



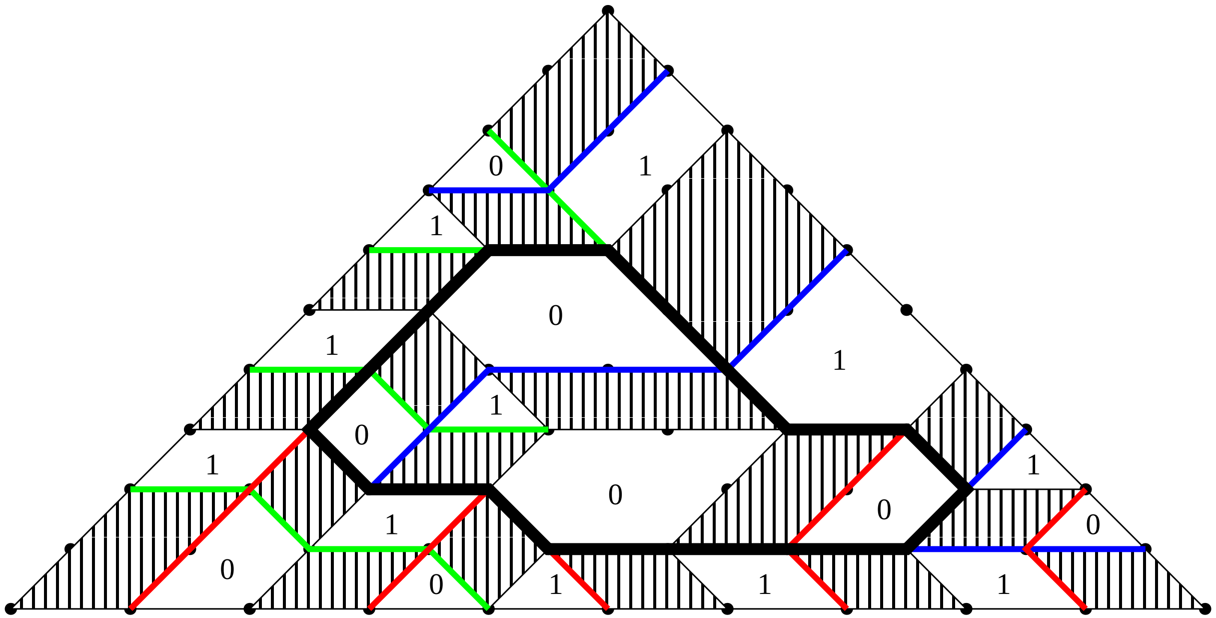
Good paths

- Good paths do not include all interior corridor walls.



Obstruction to good paths

- Good paths do not include all interior corridor walls.

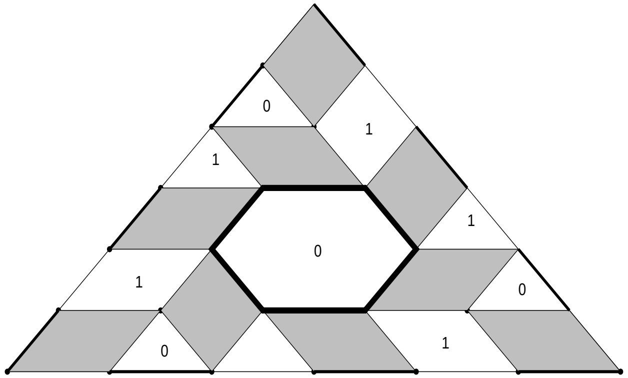


Example exhibiting a gentle loop

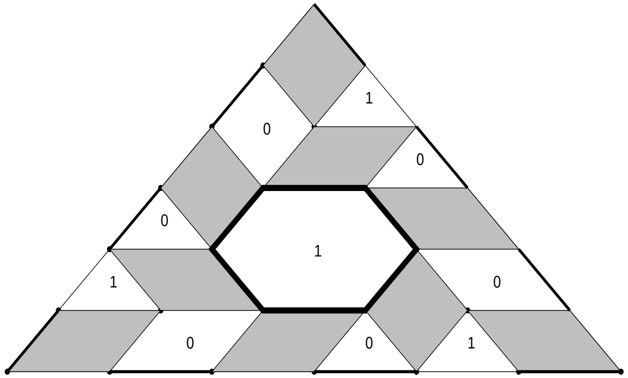
Ex: $n = 6,\ r = 3,\ I = J = (1,3,5)$ and $K = (2,4,6)$.

There exist two puzzles, each exhibiting a gentle loop

$P1$:



$P2$:



- The corresponding inessential saturated Horn inequality
$$\nu_2 + \nu_4 + \nu_6 = \lambda_1 + \lambda_3 + \lambda_5 + \mu_1 + \mu_3 + \mu_5$$
is satisfied by $\lambda = \mu = (221100)$ and $\nu = (332211)$

- There can be no corresponding factorisation since

$$c^{\nu}_{\lambda\mu} = c^{332211}_{221100,221100} = 3 \neq 2 \cdot 2 = c^{321}_{210,210}\ c^{321}_{210,210} = c^{\nu_K}_{\lambda_I \mu_J}\ c^{\nu_{\overline{K}}}_{\lambda_{\overline{I}} \mu_{\overline{J}}}$$

Rigid puzzles and gentle loops

Theorem [KTW 04] The hive plan of a puzzle has no gentle loops if only if the puzzle is rigid

- A puzzle is rigid if and only if the corresponding Horn inequality is essential
- All gentle paths in the hive plan of a rigid puzzle are good, ie have no bad edges
- This implies that in the case of any essential Horn equality, the map from the LR-hives H_n to pairs of LR-hives H_r and H_{n-r} is a bijection
- Hence we have proved: [KTT 08]
  Theorem If an essential Horn inequality is saturated then $c^{\nu}_{\lambda\mu}$ factorises.

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Some relations between LR-coefficients

Symmetries

- $c^{\nu}_{\mu\,\lambda} = c^{\nu}_{\lambda\,\mu}$
- $c^{\nu'}_{\lambda'\,\mu'} = c^{\nu}_{\lambda\,\mu}$
- $c^{\tilde{\mu}}_{\tilde{\nu}\,\tilde{\lambda}} = c^{\nu}_{\lambda\,\mu}$
    - where λ' is the conjugate of λ
    - and $\tilde{\lambda} = (m - \lambda_n, \ldots, m - \lambda_2, m - \lambda_1)$

Inequalities

- $c^{\nu+(1^c)}_{\lambda+(1^a),\,\mu+(1^b)} \geq c^{\nu}_{\lambda\,\mu}$
- $c^{\nu\cup(c)}_{\lambda\cup(a),\,\mu\cup(b)} \geq c^{\nu}_{\lambda\,\mu}$
    - for all non-negative integers a, b, c, with $c = a + b$
    - where $\lambda + (1^a) = (\lambda_1 + 1, \ldots, \lambda_a + 1, \lambda_{a+1}, \ldots, \lambda_n)$
    - and $\lambda \cup (a) = (\lambda_1, \ldots, \lambda_k, a, \lambda_{k+1}, \ldots, \lambda_n)$ with $\lambda_k \geq a > \lambda_{k+1}$

Hive based proofs of the symmetry relations

Commutativity $\quad c^{\nu}_{\mu\,\lambda} = c^{\nu}_{\lambda\,\mu}$

Proof: It remains an open problem to find a hive-based proof.

- We would like a bijection between the LR-hives for $c^{\nu}_{\mu\,\lambda}$ and those for $c^{\nu}_{\lambda\,\mu}$
- The result is by no means obvious from the Littlewood-Richardson rule.
- Considerable effort has gone into devising combinatorial proofs
- See for example the work of Benkart, Sotille & Stroomer [96] and of Azenhas [99]

Hive based proofs of the symmetry relations

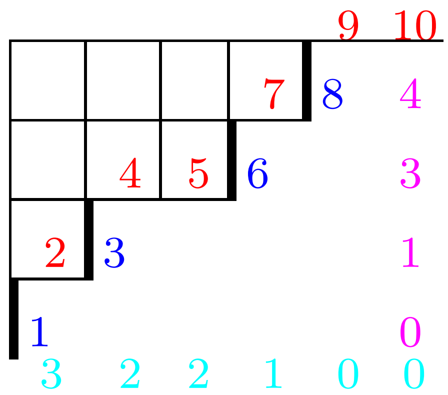
Conjugacy $\quad c^{\nu'}_{\lambda'\,\mu'} = c^{\nu}_{\lambda\,\mu}$

- Let $N = \{1, 2, \ldots, n\} = M \cup \overline{M}$ with $M \cap \overline{M} = \emptyset$
- Note that if $\zeta = part\,(M)$ then $\zeta' = part\,(\overline{M}_n)$ where $\overline{M}_n = \{n+1-i \,|\, i \in \overline{M}\}$

Ex: If $n = 10$ and $M = \{1, 3, 6, 8\}$ then $part\,(M) = (4, 3, 1)$ while $\overline{M} = \{2, 4, 5, 7, 9, 10\}$ and $\overline{M}_n = \{1, 2, 4, 6, 7, 9\}$ so that $part\,(\overline{M}) = (3, 2, 2, 1)$. Diagrammatically:

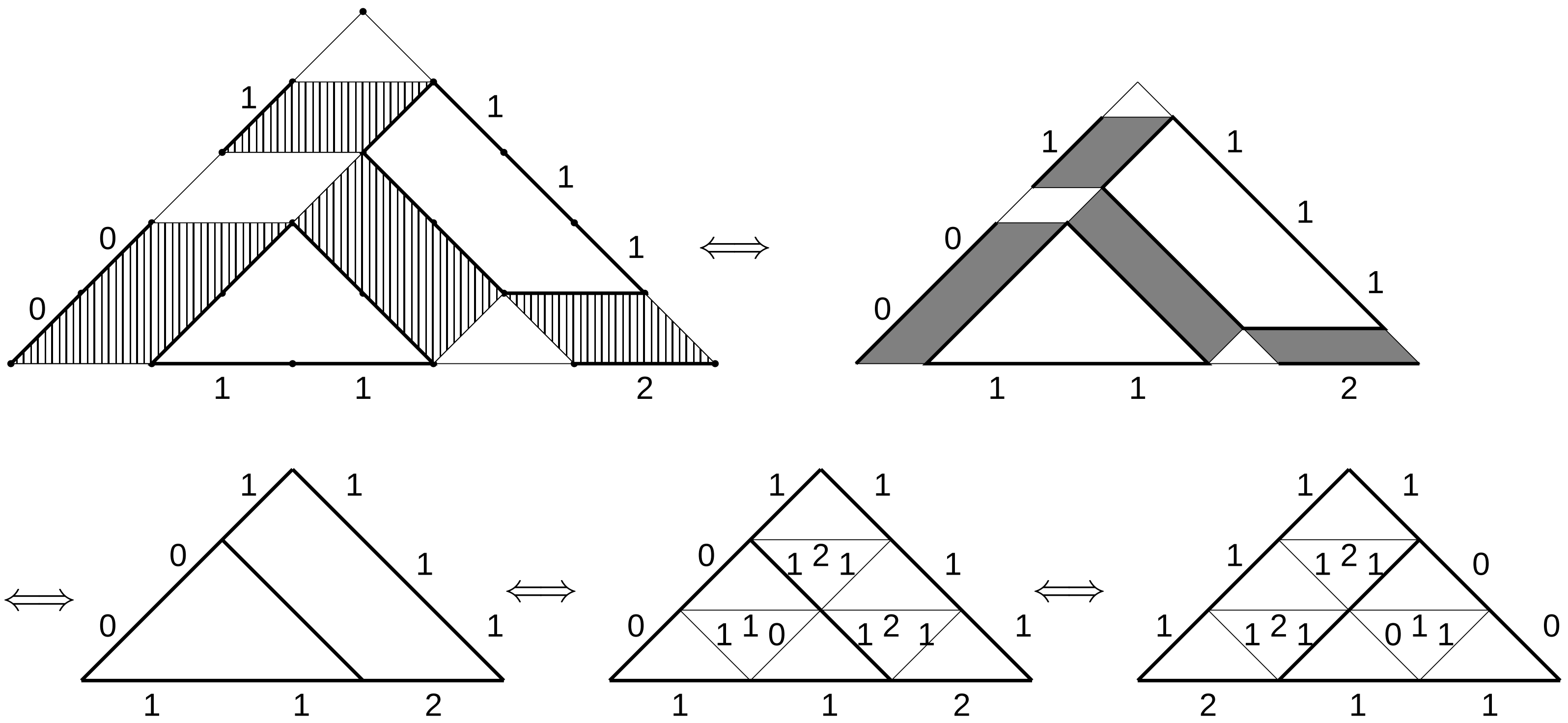
1	2	3	4	5	6	7	8	9	10
0	3	1	2	2	3	1	4	0	0

$\Longleftrightarrow$



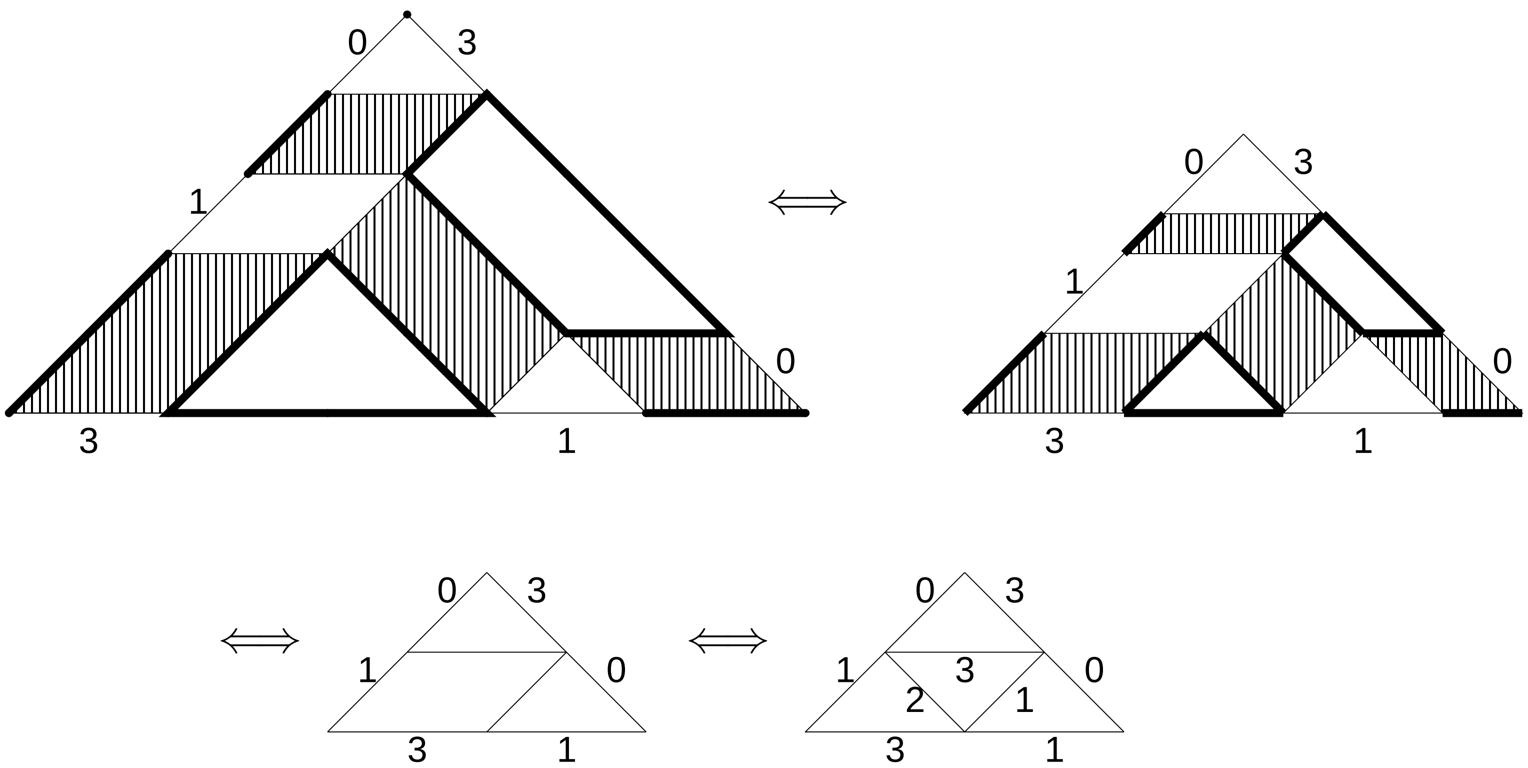
Proof of conjugacy contd.

- Recall that the number of n-puzzles with thick edges specified by I, J, K is $c^{part\,(K)}_{part\,(J),part\,(I)} = c^{part\,(K)}_{part\,(I),part\,(J)}$
- This was proved by scaling all thick edges by t and allowing $t \to 0$



Proof of conjugacy contd.

- It remains to show that the number of n-puzzles with thin edges specified by $\overline{I}, \overline{J}, \overline{K}$ is $c^{part\,(\overline{K}_n)}_{part\,(\overline{I}_n), part\,(\overline{J}_n)}$
- This is proved by scaling all thin edges by t and allowing $t \to 0$

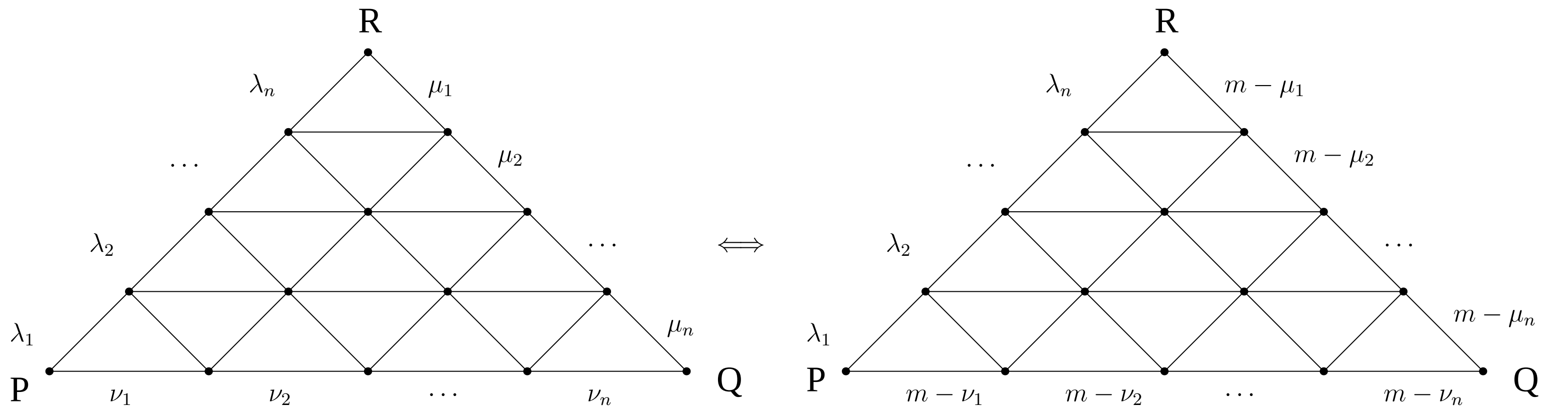


Hive based proofs of the symmetry relations

Complementarity $c_{\tilde{\nu}\,\tilde{\lambda}}^{\tilde{\mu}} = c_{\lambda\,\mu}^{\nu}$

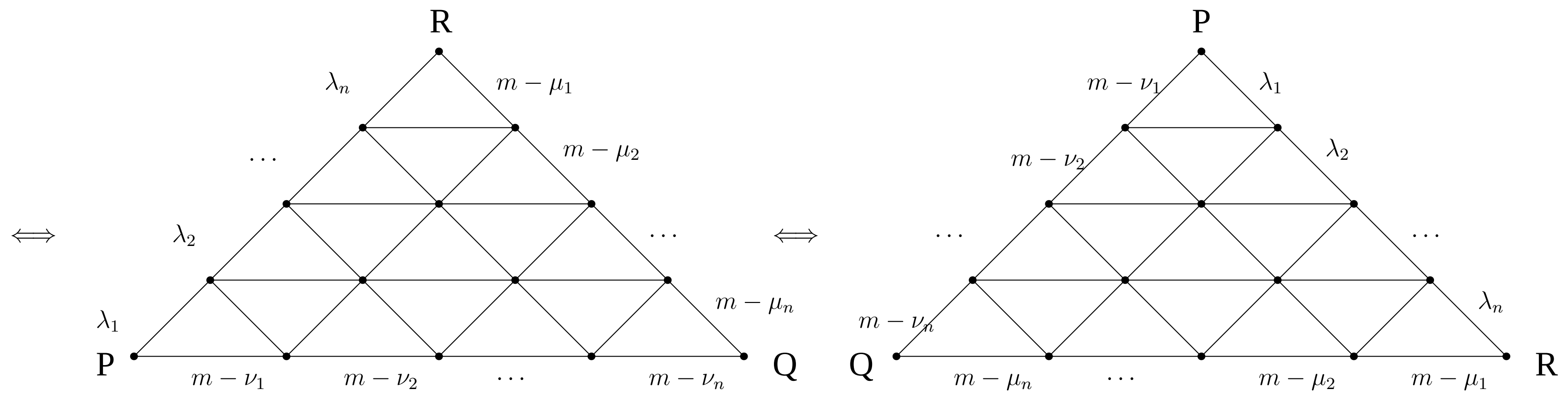
Proof Consider any LR-hive for $c_{\lambda\,\mu}^{\nu}$. Then

- For all edges parallel to the NE,SE,WE boundaries replace the edge labels α,β,γ by α,$m-\beta$,$m-\gamma$
- This breaks the triangle and the rhombus hive conditions and the result is not an LR-hive



Complementarity contd.

- Rotate clockwise through $\pi/3$
- The result is an LR-hive for $c^{\tilde{\mu}}_{\tilde{\nu}\,\tilde{\lambda}}$



Lemma The above sequence of two maps provides a bijection between the LR-hives of $c^{\nu}_{\lambda\,\mu}$ and those of $c^{\tilde{\mu}}_{\tilde{\nu}\,\tilde{\lambda}}$

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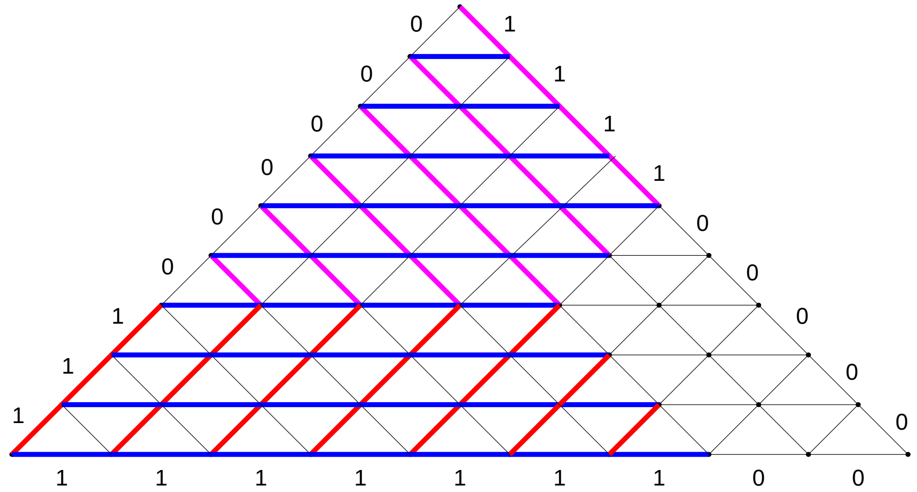
Column insertion

Inequality $c^{\nu+(1^c)}_{\lambda+(1^a),\,\mu+(1^b)} \geq c^{\nu}_{\lambda\,\mu}$ for all a, b, c, with $c = a + b$

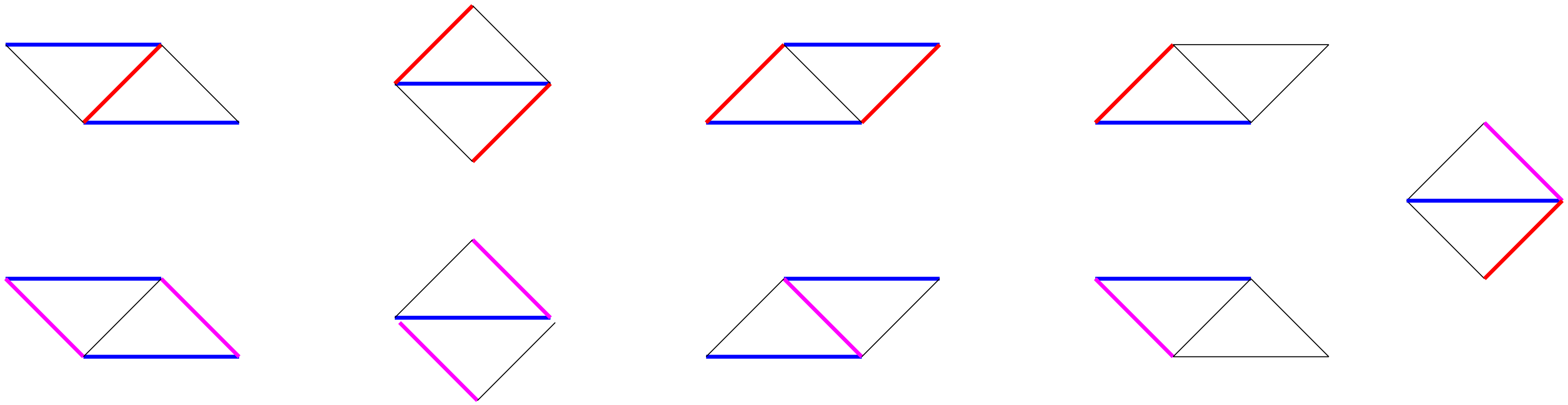
Proof Since $c^{1^c}_{1^a,\,1^b} = 1$ if $c = a + b$, there exist a unique LR-hive:



- Here each red, magenta and blue edge is labelled 1 and all other edges are labelled 0

Column insertion

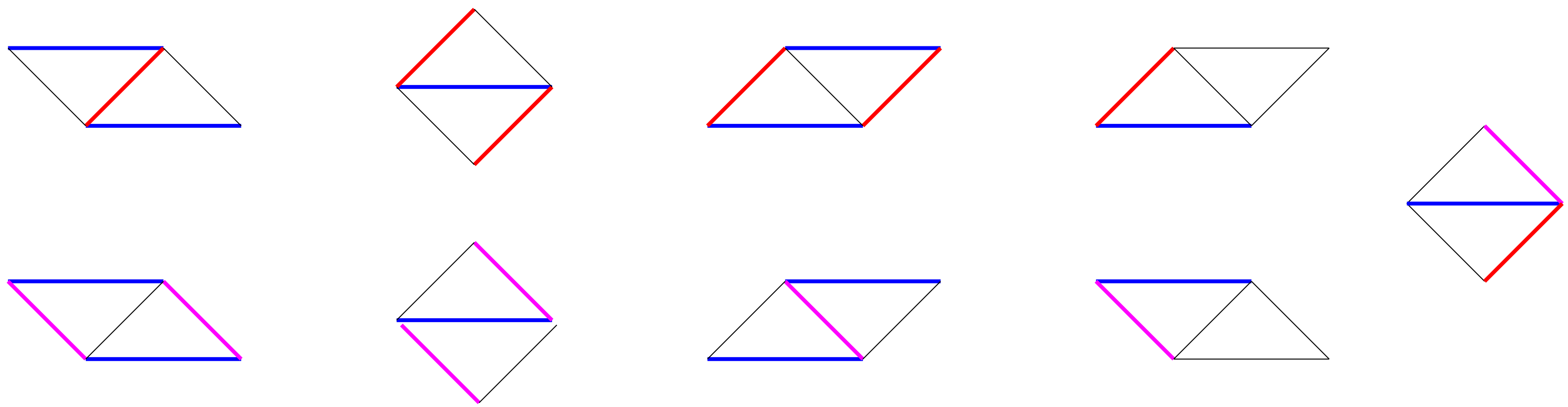
- The elementary rhombi take one or other of the following forms



- However, we can now interpret our multi-coloured LR-hive as being an LR-hive for $c_{\lambda\,\mu}^{\nu}$ whose red, magenta and blue edge labels have all been increased by 1, with no change to the uncoloured edge labels

Column insertion

- This yields an LR-five for $c^{\nu+(1^c)}_{\lambda+(1^a),\,\mu+(1^b)}$ as can be seen by checking the hive conditions in each of the elementary rhombi

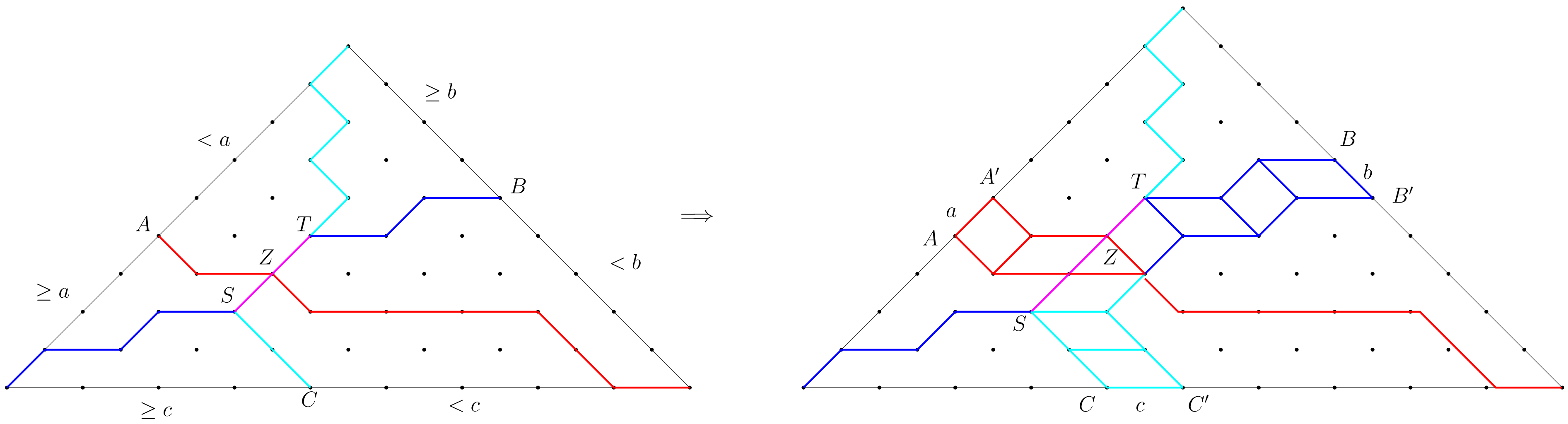


- Hence the number of $c^{\nu+(1^c)}_{\lambda+(1^a),\,\mu+(1^b)}$ LR-hives is at least as many as the number of $c^{\nu}_{\lambda\,\mu}$ LR-hives

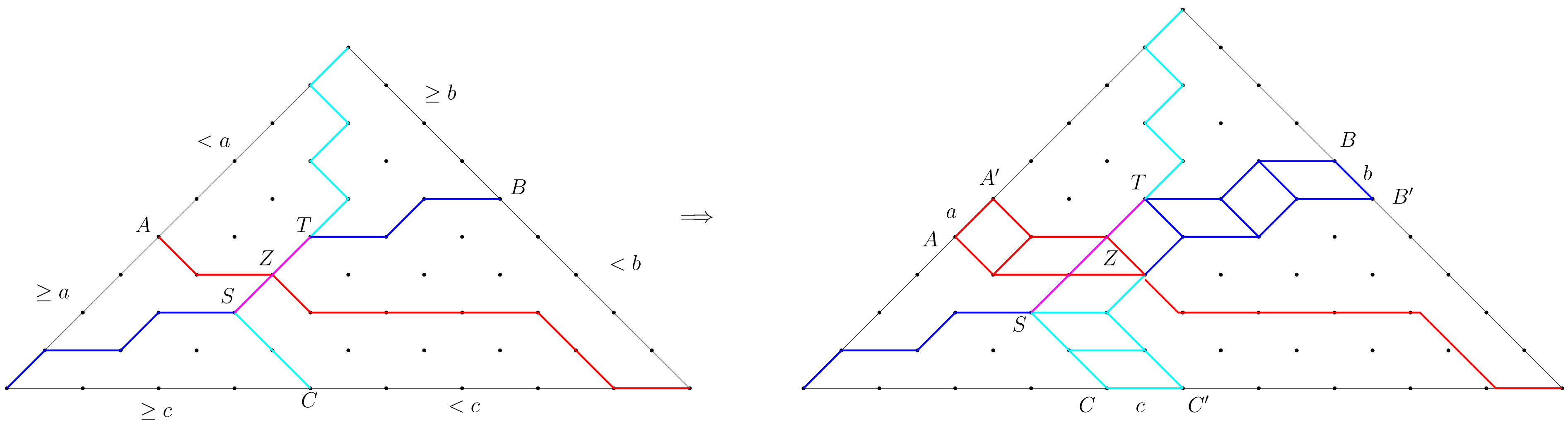
Row insertion

Inequality $c^{\nu\cup(c)}_{\lambda\cup(a),\,\mu\cup(b)} \geq c^{\nu}_{\lambda\,\mu}$ with $c = a + b$

- Each LR-hive for $c^{\nu}_{\lambda\,\mu}$ may be divided into regions as shown on the left below
- Redundant corridors are then inserted as shown on the right to give an LR-hive for $c^{\nu\cup(c)}_{\lambda\cup(a),\,\mu\cup(b)}$

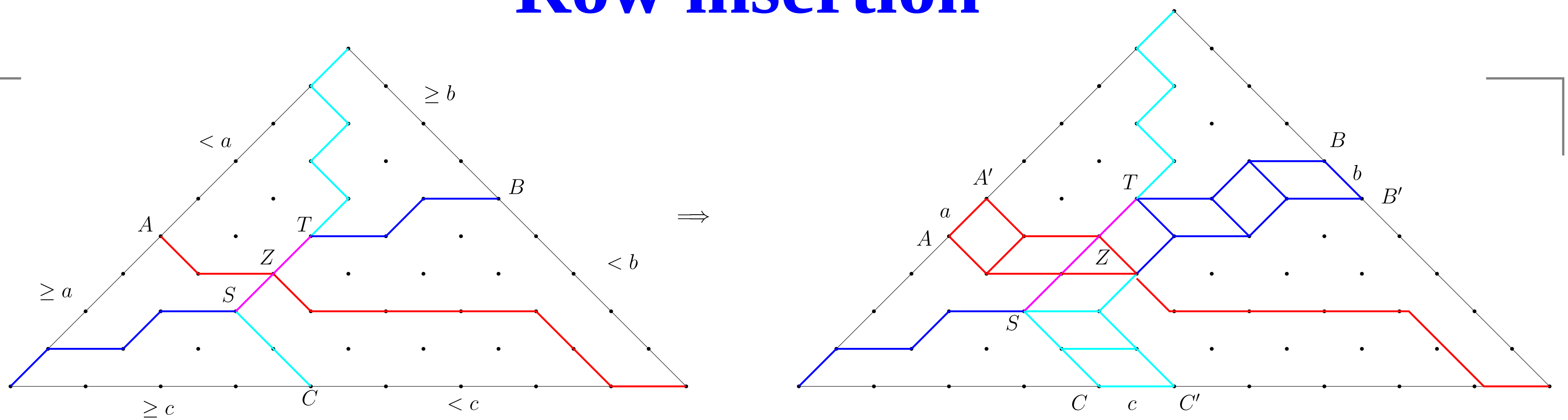


Row insertion



- The red line divides the hive into those regions for which NW edge labels are either $\leq a$ or $> a$
- The blue line divides the hive into those regions for which SW edge labels are either $\leq b$ or $> b$
- The cyan line divides the hive into those regions for which WE edge labels are either $\leq c$ or $> c$

Row insertion



- These lines always intersect at a single point Z
- Edges labelled a, b, c are inserted on the boundary
- Each edge along the first part of the red, blue and cyan lines is replaced by an appropriate redundant rhombus
- The point Z is replaced by a triangle with edge labels a, b, c, and all triangle and rhombus hive conditions are satisfied

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Stretched LR coefficients

- Littlewood-Richardson coefficient $c^{\nu}_{\lambda\mu}$
- Partition $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$ stretching parameter $t \in \mathbb{N}$
- Stretched partition $t\lambda = (t\lambda_1, t\lambda_2, \dots, t\lambda_n)$
- Stretched Littlewood-Richardson coefficient $c^{t\nu}_{t\lambda,t\mu}$

- Ex: $n = 3$, $\lambda = (2,1,0)$, $\mu = (3,2,0)$, $\nu = (4,3,1)$
  - $t = 1$: $c^{431}_{21,32} = 2$
  - $t = 2$: $c^{862}_{42,64} = 3$
  - $t = 3$: $c^{12\,93}_{63,94} = 4$
  - $\dots$
  - suggests $c^{t\nu}_{t\lambda,t\mu} = t + 1$.

LR coefficients and polynomials

Ex: Let $c^{\nu}_{421,532} = c$ and $c^{t\nu}_{t(421),t(532)} = P(t)$.

$$
\begin{array}{lll}
c = 1 & \nu = (953) & P(t) = 1 \\
c = 2 & \nu = (9431) & P(t) = (t+1) \\
c = 3 & \nu = (8441) & P(t) = (t+1)(t+2)/2 \\
c = 4 & \nu = (8531) & P(t) = (t+1)(t+2)(t+3)/6 \\
c = 4 & \nu = (7442) & P(t) = (t+1)^2 \\
c = 5 & \nu = (7541) & P(t) = (t+1)(t+2)(2t+3)/6 \\
c = 6 & \nu = (7532) & P(t) = (t+1)^2(t+2)/2 \\
c = 7 & \nu = (74321) & P(t) = (t+1)(t+2)(t^2+3t+6)/6
\end{array}
$$

Generating function for LR-polynomials

Ex: Let $F(z) = G(z)/(1-z)^{d+1} = \sum_{t=0}^{\infty} P(t)\, z^t$.

$$
\begin{array}{llll}
c = 1 & \nu = (953) & d = 1 & G(z) = 1 \\
c = 2 & \nu = (9431) & d = 2 & G(z) = 1 \\
c = 3 & \nu = (8441) & d = 3 & G(z) = 1 \\
c = 4 & \nu = (8531) & d = 4 & G(z) = 1 \\
c = 4 & \nu = (7442) & d = 3 & G(z) = 1 + z \\
c = 5 & \nu = (7541) & d = 4 & G(z) = 1 + z \\
c = 6 & \nu = (7532) & d = 4 & G(z) = 1 + 2z \\
c = 7 & \nu = (74321) & d = 5 & G(z) = 1 + 2z + z^2
\end{array}
$$

Further example

Ex: $n = 7, \ \lambda = (433210), \ \mu = (432210), \ \nu = (7444321)$.

- LR coefficient $c^{\nu}_{\lambda\mu} = 13$
- LR polynomial

$$
\begin{aligned}
c^{t\nu}_{t\lambda.t\mu} &= 1/10080 \\
&\times (t+1)(t+2)(t+3)(t+4)(t+5) \\
&\times (5t+21)(t^2+2t+4)
\end{aligned}
$$

- where $10080 = 5!\ 84$
- $d = 8$ and $G(z) = 1 + 4z + 12z^2 + 3z^3$

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Polynomial behaviour

Theorem For all λ, μ, ν such that $c^{\nu}_{\lambda\mu} > 0$ there exists

- a polynomial $P^{\nu}_{\lambda\mu}(t)$ in t with $P^{\nu}_{\lambda\mu}(0) = 1$
- such that $P^{\nu}_{\lambda\mu}(t) = c^{t\nu}_{t\lambda,t\mu}$ for all positive integers t.

Conjectures

- coefficients in $P^{\nu}_{\lambda\mu}(t)$ are all rational and non-negative.
- coefficients in $G(z)$ are all positive integers.

Problems

- predict degree of polynomial
- explain origin of factors of form $(t+1)(t+2)\cdots(t+m)$
- prove (if true) and account for positivity of coefficients