Characters of the classical groups: tensor products, branching rules, modification rules and Hopf algebras

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Introduction

Aim: To provide a uniform setting for dealing with:

- characters of representations of the classical groups;
- group-subgroup branching rules;
- decomposition of tensor products;
- modification rules.

Key features: A Schur function approach involving:

- partitions, Young diagrams and their generalisations;
- infinite series of Schur functions;
- the notion of universal characters;
- the Hopf algebra of symmetric functions;
- Jacobi-Trudi and other determinantal expansions.

Origins of the work

Standard texts

- D E Littlewood, *Theory of Group Characters*, 2nd Ed., Clarendon Press, Oxford, 1950.
- I G Macdonald, *Symmetric Functions and Hall Polynmials*, 2nd Ed., Clarendon Press, Oxford, 1995.
- W Fulton and J Harris, *Representation Theory*, Springer-Verlag, New York, 1991.

Further key ideas

- Universal characters: K Koike, I Terada.
- Hopf algebra approach: B Fauser, P D Jarvis.
- Odd symplectic groups: R Proctor.

The classical groups

The general linear group and its subgroups

- Let $M(n, \mathbb{C})$ be the set of $n \times n$ matrices over $\mathbb{C}$.
- Let

$$\begin{aligned}
G_{2k} &= \operatorname{diag}\left[\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right] \\
G_{2k+1} &= \operatorname{diag}\left[\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \end{pmatrix}\right] \\
J_{2k} &= \operatorname{diag}\left[\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \cdots \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right] \\
J_{2k+1} &= \operatorname{diag}\left[\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \cdots \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 0 \end{pmatrix}\right]
\end{aligned}$$

The classical groups

- $GL(n) = \{X \in M(n, \mathbb{C}) \mid \det X \neq 0\}$.
- $O(n) = \{X \in GL(n) \mid X\, G_n\, X^t = G_n\}$ with $G_n^t = G_n$.
- $Sp(n) = \{X \in GL(n) \mid X\, J_n\, X^t = J_n\}$ with $J_n^t = -J_n$.
- $SL(n) = \{X \in GL(n) \mid \det X = 1\}$.
- $SO(2k+1) = \{X \in SL(2k+1) \mid X\, G_{2k+1}\, X^t = G_{2k+1}\}$.
- $Sp(2k) = \{X \in SL(2k) \mid X\, J_{2k}\, X^t = J_{2k}\}$.
- $SO(2k) = \{X \in SL(2k) \mid X\, G_{2k}\, X^t = G_{2k}\}$.
- $Sp(2k+1) = \begin{bmatrix} & & & * \\ & Sp(2k) & & * \\ & & & * \\ 0 & \cdots & 0 & GL(1) \end{bmatrix}$.

Parametrisation of eigenvalues

- $GL(n):\quad x_1, x_2, \ldots, x_n$ with $x_1 x_2 \cdots x_n \neq 0$.
- $SL(n):\quad x_1, x_2, \ldots, x_n$ with $x_1 x_2 \cdots x_n = 1$.
- $SO(2k+1):\quad x_1, x_2, \ldots, x_k, \overline{x}_1, \overline{x}_2, \ldots, \overline{x}_k, 1$.
- $O(2k+1)\backslash SO(2k+1):\quad x_1, x_2, \ldots, x_k, \overline{x}_1, \overline{x}_2, \ldots, \overline{x}_k, -1$.
- $Sp(2k):\quad x_1, x_2, \ldots, x_k, \overline{x}_1, \overline{x}_2, \ldots, \overline{x}_k$.
- $SO(2k):\quad x_1, x_2, \ldots, x_k, \overline{x}_1, \overline{x}_2, \ldots, \overline{x}_k$.
- $O(2k)\backslash SO(2k):\quad x_1, x_2, \ldots, x_{k-1}, \overline{x}_1, \overline{x}_2, \ldots, \overline{x}_{k-1}, 1, -1$.
- $Sp(2k+1):\quad x_1, x_2, \ldots, x_k, \overline{x}_1, \overline{x}_2, \ldots, \overline{x}_k, x_{2k+1}$.

Note $\overline{x}_i = x_i^{-1}$ for all i.

Irreducible representations of $GL(n)$

Covariant tensor irreducible representations

- $V^{\lambda}_{GL(n)}$ where λ is a partition of length $\ell(\lambda) = p \leq n$, so that $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_p, 0, \ldots, 0)$ with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_p > 0$ and $\lambda_i \in \mathbb{N}$ for $1 \leq i \leq p$ and $\lambda_i = 0$ for $p < i \leq n$.

Contravariant tensor irreducible representations

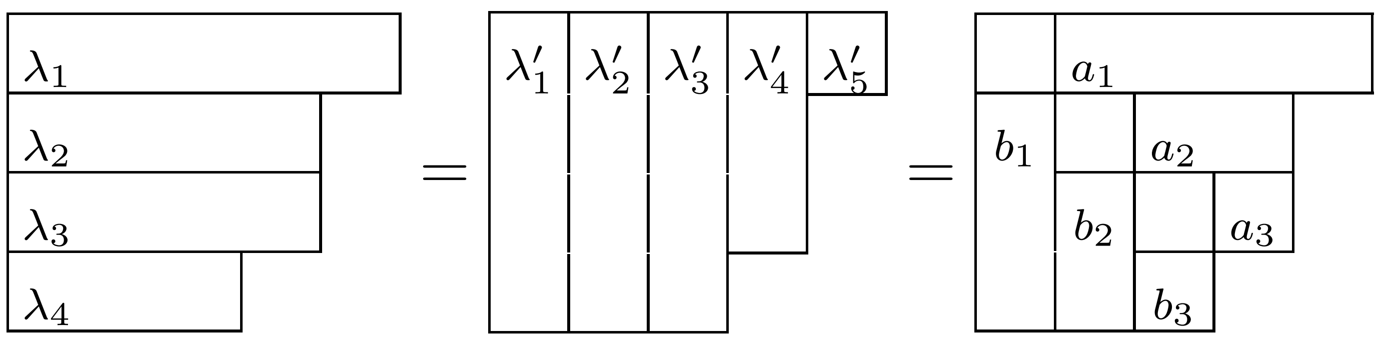
- $V^{\overline{\mu}}_{GL(n)}$ where μ is a partition of length $\ell(\mu) = q \leq n$, and $\overline{\mu} = (0, \ldots, 0, -\mu_q, \ldots, -\mu_2, -\mu_1)$.

Mixed tensor irreducible representations

- $V^{\lambda;\overline{\mu}}_{GL(n)}$ where λ and μ are partitions with $\ell(\lambda) = p$, $\ell(\mu) = q$ and $p + q \leq n$, so that $\lambda;\overline{\mu} = (\lambda_1, \lambda_2, \ldots, \lambda_p, 0, \ldots, 0, -\mu_q, \ldots, -\mu_2, -\mu_1)$.

Partitions and Young diagrams

- Young diagrams F^{λ} consists of $|\lambda|$ boxes arranged in $\ell(\lambda)$ left-adjusted rows of lengths λ_i for $i = 1, 2, \ldots, \ell(\lambda)$.
- Conjugate partition λ' is the partition defined by the column lengths of F^{λ}.
- Frobenius notation $\lambda = \begin{pmatrix} a_1 & a_2 & \cdots & a_r \\ b_1 & b_2 & \cdots & b_r \end{pmatrix}$ if F^{λ} has r boxes on the main diagonal, with arm and leg lengths a_i and b_i, with respect to this diagonal for $i = 1, 2, \ldots, r$.



Frobenius Lemma

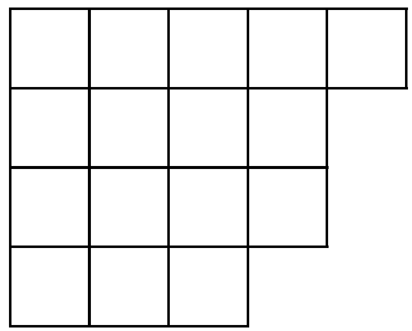
Lemma Let λ be a partition, with conjugate λ'.
For any $m \geq \lambda_1$ and any $n \geq \lambda'_1$ the two sets $\{n - i + 1 + \lambda_i \,|\, 1 \leq i \leq n\}$ and $\{n + j - \lambda'_j \,|\, 1 \leq j \leq m\}$ are disjoint, and their union is $\{1, 2, \ldots, n + m\}$.

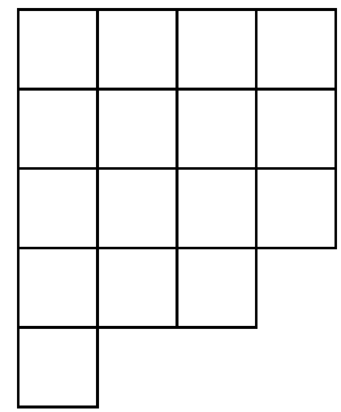
Proof: The cardinalities of the two sets are n and m, and

- $1 \leq 1 + \lambda_n < 2 + \lambda_{n-1} < \cdots < n + \lambda_1 \leq n + m$, since λ is a partition, with $\lambda_n \geq 0$ and $\lambda_1 \leq m$.
- $1 \leq n + 1 - \lambda'_1 < n + 2 - \lambda'_2 < \cdots < n + m - \lambda'_m \leq n + m$, since λ' is a partition, with $\lambda'_1 \leq n$ and $\lambda'_m \geq 0$.
- For $1 \leq i \leq n$ and $1 \leq j \leq m$, we have differences $(n - i + 1 + \lambda_i) - (n + j - \lambda'_j) = h_{ij}$, where $h_{ij} = \lambda_i + \lambda'_j - i - j + 1 \neq 0$ (see later).

Frobenius Lemma illustration

Ex: For $\lambda = (5,4,4,3)$ we have $\lambda' = (4,4,4,3,1)$ as can be seen from the diagrams:

$F^{\lambda} =$ 

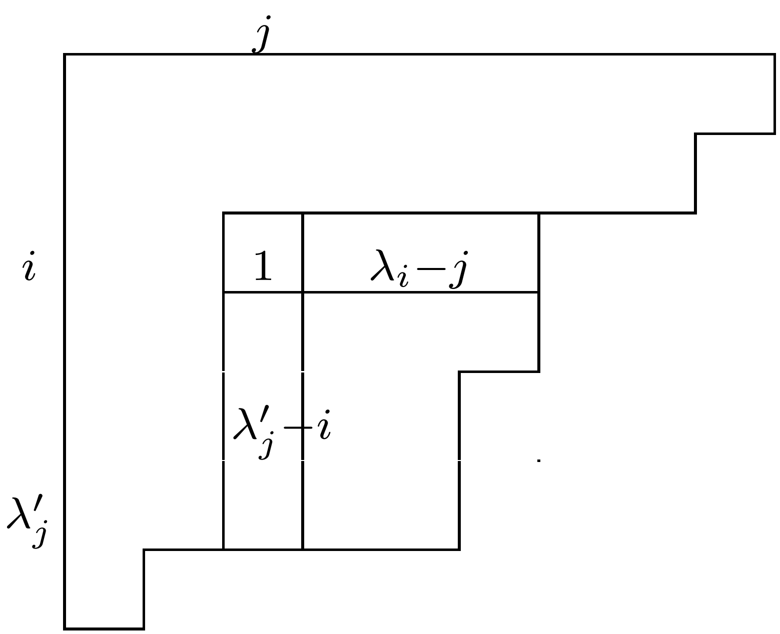
$F^{\lambda'} =$ 

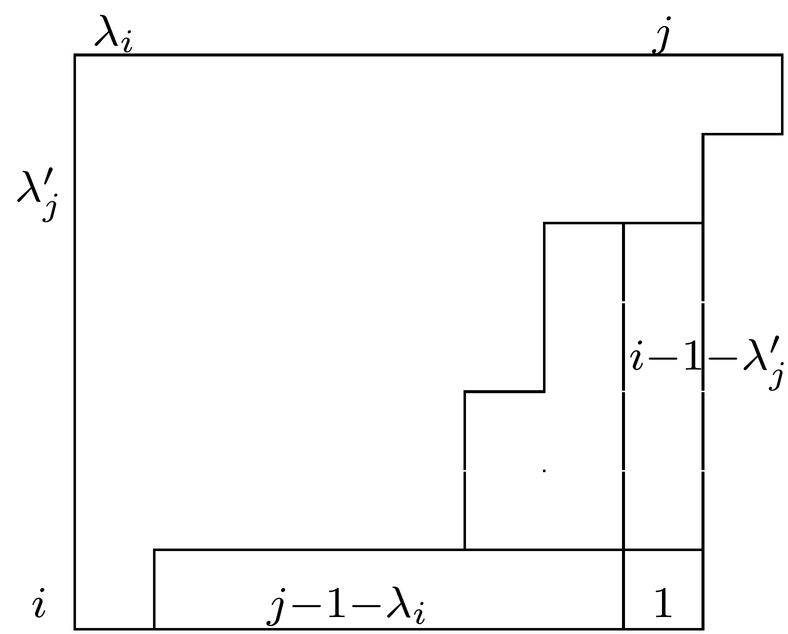
If we set $n = 4$ and $m = 5$, so that $n + m = 9$ we have:

$$\begin{aligned} \{\lambda_i\} &= \{3,4,4,5\} & \{\lambda'_j\} &= \{4,4,4,3,1\} \\ \{n-i+1\} &= \{1,2,3,4\} & \{n+j\} &= \{5,6,7,8,9\} \\ \{n-i+1+\lambda_i\} &= \{4,6,7,9\} & \{n+j-\lambda'_j\} &= \{1,2,3,5,8\} \end{aligned}$$

Proof that $h_{ij} \neq 0$

- $h_{ij} = (\lambda_i - j) + (\lambda'_j - i) + 1 = -(j - 1 - \lambda_i) - (i - 1 - \lambda'_j) - 1.$
- For $(i, j) \in F^\lambda$ we have $h_{ij} > 0$, since h_{ij} is the interior hook length of a box at (i, j) inside F^λ.
- For $(i, j) \notin F^\lambda$ we have $h_{ij} < 0$, since $-h_{ij}$ is the exterior hook length a box would have at (i, j) outside F^λ.





Signature Lemma

Lemma The permutation π_λ has signature $(-1)^{|\lambda|}$ for $\pi_\lambda =$

$$\begin{pmatrix} 1 & 2 & \cdots & n & n+1 & n+2 & \cdots & n+m \\ 1+\lambda_n & 2+\lambda_{n-1} & \cdots & n+\lambda_1 & n+1-\lambda'_1 & n+2-\lambda'_2 & \cdots & n+m-\lambda'_m \end{pmatrix}$$

Proof: The signature is $(-1)^{\#}$, where $\#$ is the number of pairs (k,l) with $k<l$ such that $\pi_\lambda(k) > \pi_\lambda(l)$.

- The first n numbers in the bottom row, and the last m are strictly increasing from left to right.
- For $k = n-i+1$ and $l = n+j$, we have $\pi_\lambda(k) - \pi_\lambda(l) = h_{ij}$ since $\pi_\lambda(k) = (n-i+1+\lambda_i)$ and $\pi_\lambda(l) = (n+j-\lambda'_j)$.
- However, the number of $h_{ij} > 0$ is $|\lambda|$, the number of boxes in F^λ.

Signature Lemma illustration

Ex: Let $\lambda = (5,4,4,3)$, so that $\lambda' = (4,4,4,3,1)$. Let $n = 4$ and $m = 5$, so that $n + m = 9$. Then as before:

$$\begin{aligned} \{\lambda_i\} &= \{3,4,4,5\} & \{\lambda'_j\} &= \{4,4,4,3,1\} \\ \{n-i+1\} &= \{1,2,3,4\} & \{n+j\} &= \{5,6,7,8,9\} \\ \{n-i+1+\lambda_i\} &= \{4,6,7,9\} & \{n+j-\lambda'_j\} &= \{1,2,3,5,8\} \end{aligned}$$

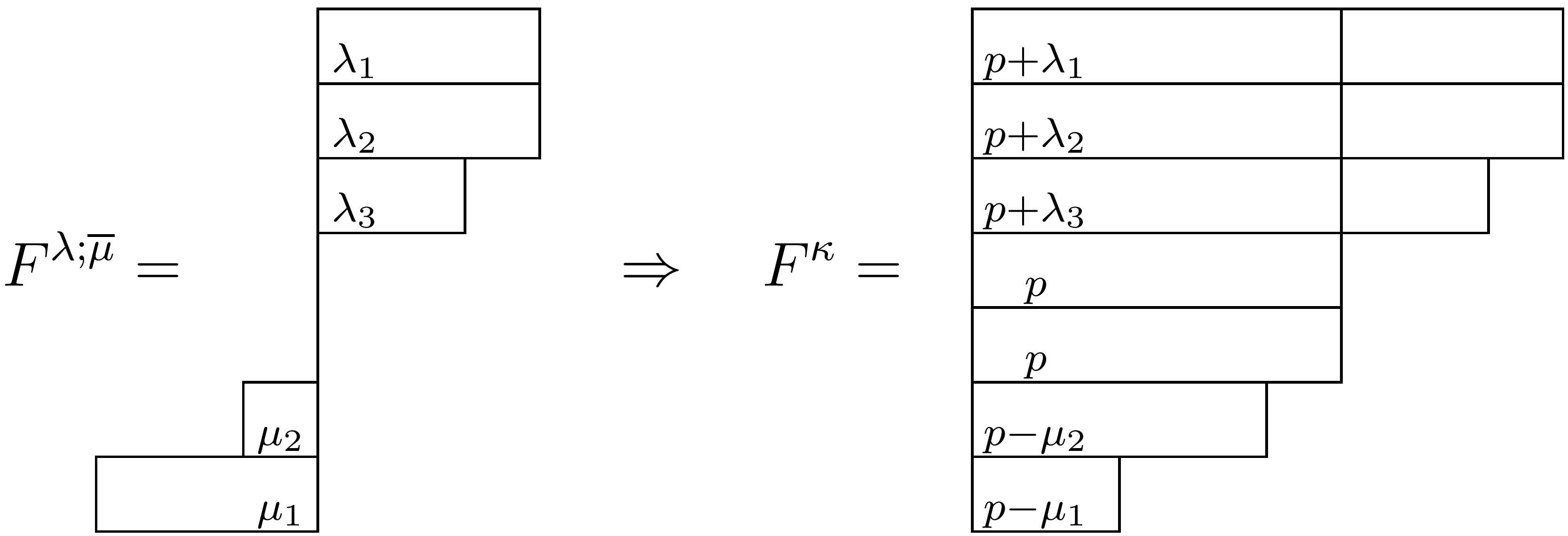
In this case

$$\pi_\lambda = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 4 & 6 & 7 & 9 & 1 & 2 & 3 & 5 & 8 \end{pmatrix}$$

with signature $(-1)^{4+4+4+3+1} = (-1)^{16} = (-1)^{|\lambda|} = +1$.

Generalised partitions and Young diagrams

- Composite Young diagram $F^{\lambda;\overline{\mu}}$ consists of F^{λ} and $F^{\overline{\mu}}$ left and right adjusted to a common vertical line, where $F^{\overline{\mu}}$ is obtained by rotating F^{μ} through π.
- (n,p)-equivalent Young diagram F^{κ} obtained by adding to F^{λ} the complement of $F^{\overline{\mu}}$ in F^{p^n} for any $p \geq \mu_1$.

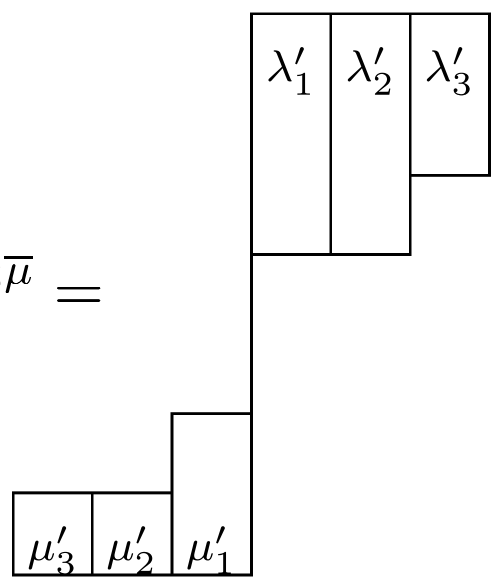


Column structure of (n,p)-equivalence relation

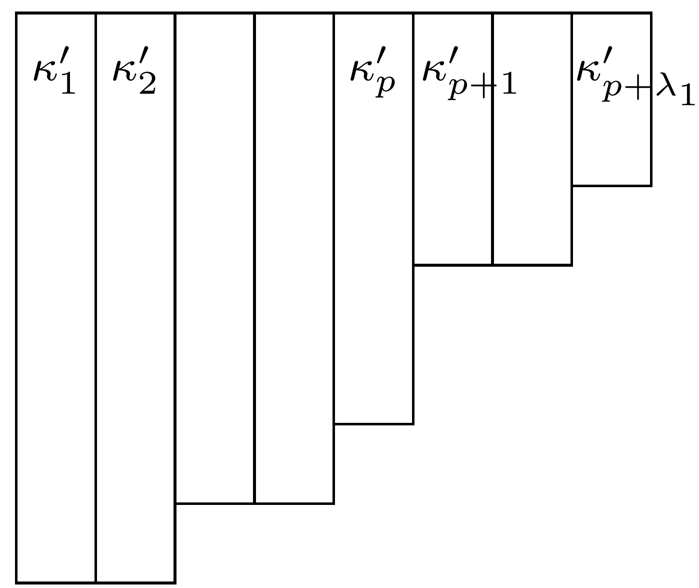
- $F^{\lambda;\overline{\mu}}$ is (n,p)-equivalent to F^{κ} where F^{κ} is obtained by adding to F^{λ} the complement of $F^{\overline{\mu}}$ in F^{p^n} for any $p \geq \mu_1$.
- If κ' is the conjugate of κ then

$$\kappa'_j = \begin{cases} n - \mu'_{p+1-j} & \text{for } 1 \leq j \leq p \\ \lambda'_{j-p} & \text{for } j > p \end{cases}$$

$F^{\lambda;\overline{\mu}} =$



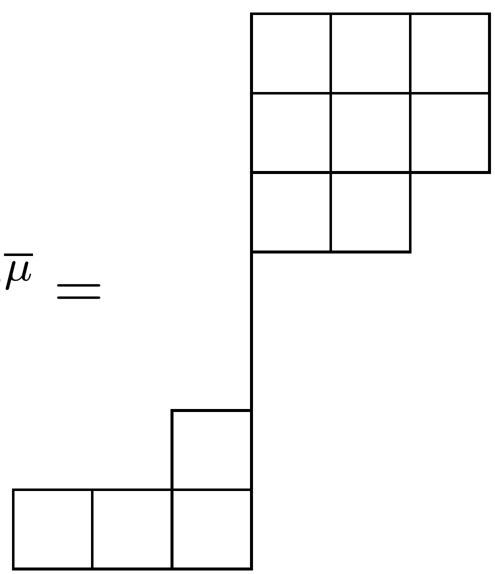
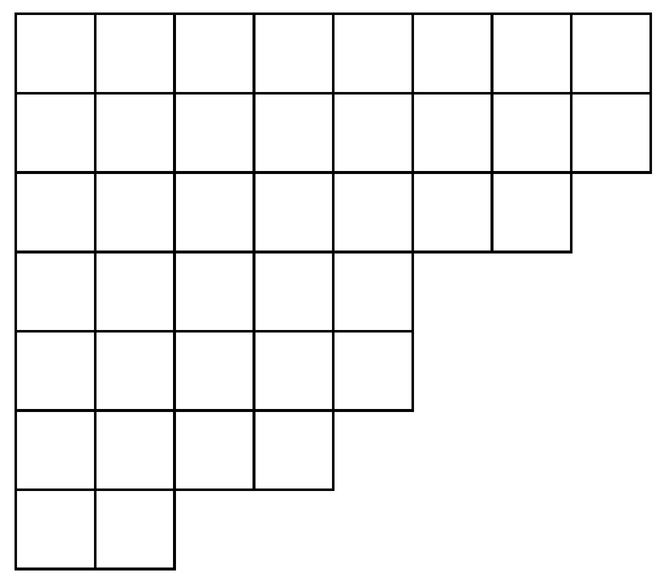
$\Rightarrow \quad F^{\kappa} =$



Generalised partitions and Young diagrams

Ex: Let $n = 7$, $p = 5$, $\lambda = (3,3,2)$, $\mu = (3,1)$.

- $\lambda;\overline{\mu} = (3,3,2,0,0,-1,-3)$.
- $\lambda' = (3,3,1)$, $\mu' = (2,1,1,0,0)$, $n^p - \mu' = (5,6,6,7,7)$.
- $\kappa' = (n-\mu'_5, n-\mu'_4, n-\mu'_3, n-\mu'_2, n-\mu'_1, \lambda'_1, \lambda'_2, \lambda'_3)$
- $\kappa' = (7,7,6,6,5,3,3,1)$.

$F^{\lambda;\overline{\mu}} =$  $\Rightarrow$ $F^{\kappa} =$ 

Irreducible representations of $GL(n)$

- Defining representation: $V = V^1$ with $X \mapsto X$.
- Dual of defining representation: $\overline{V} = V^{\overline{1}}$ with $X \mapsto (X^t)^{-1}$.
- Trivial 1-dimensional representation: V^0 with $X \mapsto 1$.
- Determinantal 1-dimensional representation: $V^\varepsilon = V^{1^n}$ with $X \mapsto \det X$.
- Inverse determinantal representation: $V^{\overline{\varepsilon}} = V^{\overline{1}^n}$ with $X \mapsto (\det X)^{-1}$.
- $V^{\overline{1}^k} = V^{\overline{\varepsilon}} \times V^{1^{n-k}}$.
- $V^{\lambda;\overline{\mu}} = (V^{\overline{\varepsilon}})^{\times m} \times V^\kappa$ where κ is (n,m)-equivalent to $\lambda;\overline{\mu}$.

Characters of irreducible representations of $GL(n$

Weyl's character formula

- Let $X \in GL(n)$ have eigenvalues $(x_1, x_2, \ldots, x_n)$.
- Let $\rho = (n-1, n-2, \ldots, 1, 0)$.
- The character of the irreducible representation $V^{\lambda}_{GL(n}$ is then given by:

$$\mathrm{ch}\, V^{\lambda}_{GL(n)} = s_{\lambda}(x) = \frac{a_{\lambda+\rho}(x)}{a_{\rho}(x)} = \frac{\left| x_i^{\lambda_j+n-j} \right|}{\left| x_i^{n-j} \right|}$$

- The ratio of determinants is the Schur function $s_{\lambda}(x)$.
- The denominator is the Vandermonde determinant:

$$a_{\rho}(x) = \left| x_i^{n-j} \right| = \prod_{1 \le i < j \le n} (x_i - x_j)\,.$$

Characters of irreducible representations of $GL(n$

- In the cases $\lambda = (m)$ and $\lambda = (1^m)$, with $m \geq 0$, the Schur functions $s_\lambda(x)$ coincide with the complete homogeneous and the elementary symmetric functions:

$$
\begin{aligned}
s_m(x) &= h_m(x) = \sum_{1\le i_1\le i_2\le\cdots\le i_m\le n} x_{i_1}x_{i_2}\cdots x_{i_m}\,;\\
s_{1^m}(x) &= e_m(x) = \sum_{1\le i_1< i_2<\cdots< i_m\le n} x_{i_1}x_{i_2}\cdots x_{i_m}\,.
\end{aligned}
$$

- Notice that $e_m(x) = 0$ if $m > n$.
- For $m = 0$ we have $h_0(x) = e_0(x) = s_0(x) = 1$.
- For $m < 0$ we set $h_m(x) = e_m(x) = 0$.

Determinantal expansions

- We then have the Jacobi-Trudi identity:

$$s_\lambda(x) = |\, h_{\lambda_i - i + j}(x) \,| = |\, s_{\lambda_i - i + j}(x) \,| \ ,$$

- and the dual Jacobi-Trudi identity:

$$s_\lambda(x) = s_{(\lambda')'}(x) = \left| e_{\lambda'_j + i - j}(x) \right| = \left| s_{1^{\lambda'_j + i - j}}(x) \right| .$$

- Notice that $s_{1^m}(x) = e_m(x) = 0$ if $m > n$. Since the length, $\ell(\lambda)$ of the partition λ is λ'_1, this implies the existence of the modification rule:

$$s_\lambda(x) = 0 \quad \text{if } \ell(\lambda) > n \, .$$

Determinantal expansions

Ex: For $\lambda = (5,4,4,3)$ we have $\lambda' = (4,4,4,3,1)$.

- Setting $s_m(x) = h_m(x) = \{m\}$ and $s_{1^m}(x) = e_m(x) = \{1^m\}$:

$$s_\lambda(x) = \begin{vmatrix} \{5\} & \{6\} & \{7\} & \{8\} \\ \{3\} & \{4\} & \{5\} & \{6\} \\ \{2\} & \{3\} & \{4\} & \{5\} \\ \{0\} & \{1\} & \{2\} & \{3\} \end{vmatrix} = \begin{vmatrix} \{1^4\} & \{1^3\} & \{1^2\} & \{0\} & - \\ \{1^5\} & \{1^4\} & \{1^3\} & \{1\} & - \\ \{1^6\} & \{1^5\} & \{1^4\} & \{1^2\} & - \\ \{1^7\} & \{1^6\} & \{1^5\} & \{1^3\} & \{0\} \\ \{1^8\} & \{1^7\} & \{1^6\} & \{1^4\} & \{1\} \end{vmatrix}$$

- Note For $n = 3$, the dual J-T identity implies that

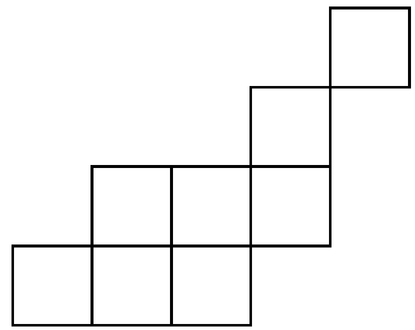
  $s_\lambda(x) = \{5,4,4,3\} = \{4,4,4,3,1\}' = 0$

  since $\{1^m\} = e_m(x_1, x_2, x_3) = 0$ for $m > n = 3$.

Skew Schur functions

- For partitions λ and μ, we write $\mu \subseteq \lambda$ if $\mu_i \leq \lambda_i$ for all i.
- If $\mu \subseteq \lambda$ then the skew Young diagram $F^{\lambda/\mu}$ is defined to be $F^{\lambda} \backslash F^{\mu}$.

Ex: $F^{5443/431} =$



Skew Schur functions: Naegelbach's Formula:

$$s_{\lambda/\mu}(x) = \left| h_{\lambda_i - \mu_j - i + j}(x) \right| = \left| s_{\lambda_i - \mu_j - i + j}(x) \right| ,$$

and Aitken's Theorem:

$$s_{\lambda/\mu}(x) = s_{(\lambda'/\mu')'}(x) = \left| e_{\lambda'_j - \mu'_i + i - j}(x) \right| = \left| s_{1^{\lambda'_j - \mu'_i + i - j}}(x) \right| .$$

Skew Schur functions

Ex: For $\lambda = (5,4,4,3)$ and $\mu = (4,3,1)$, we have:

- $\lambda' = (4,4,4,3,1)$ and $\mu' = (3,2,2,1)$.
- $\lambda - \mu = (1,1,3,3)$ and $\lambda' - \mu' = (1,2,2,2,1)$.
- Diagonal elements determined by $\lambda - \mu$ and $\lambda' - \mu'$.

$$s_{\lambda/\mu}(x) = \begin{vmatrix} \{1\} & \{3\} & \{6\} & \{8\} \\ - & \{1\} & \{4\} & \{6\} \\ - & \{0\} & \{3\} & \{5\} \\ - & - & \{1\} & \{3\} \end{vmatrix} = \begin{vmatrix} \{1\} & \{0\} & - & - & - \\ \{1^3\} & \{1^2\} & \{1\} & - & - \\ \{1^4\} & \{1^3\} & \{1^2\} & \{0\} & - \\ \{1^6\} & \{1^5\} & \{1^4\} & \{1^2\} & - \\ \{1^8\} & \{1^7\} & \{1^6\} & \{1^4\} & \{1\} \end{vmatrix}$$

Note: For $n = 2$, $s_{\lambda/\mu}(x_1, x_2) = \{1\}\{1^2\}\{1^2\}\{1^2\}\{1\} \neq 0$.

Characters of mixed tensor representations of $GL(n$

- Let $x = (x_1, x_2, \dots, x_n)$ and $y = (y_1, y_2, \dots, y_m)$.
- Let $\overline{x} = (\overline{x}_1, \overline{x}_2, \dots, \overline{x}_n)$ with $\overline{x}_k = x_k^{-1}$ for $k = 1, 2, \dots, n$.
- Let λ and μ be partitions with $\lambda_1 \leq q$ and $\mu_1 \leq p$.

Definition $$s_{\lambda;\overline{\mu}}(x;y) = \left| \, s_{1^{\mu'_{p-j+1}-i+j}}(y) \,\vdots\, s_{1^{\lambda'_{j-p}-j+i}}(x) \right| ,$$

with $1 \leq i, j \leq p+q$, and the partition $\vdots$ occurring between the pth and $(p+1)$th columns.

Proposition Let $X \in GL(n)$ have eigenvalues $(x_1, x_2, \dots, x_n)$. Then the character of the irreducible representation $V^{\lambda;\overline{\mu}}_{GL(n}$ is given by:

$$\mathrm{ch}\, V^{\lambda;\overline{\mu}}_{GL(n)} = s_{\lambda;\overline{\mu}}(x;\overline{x}) \, .$$

Relations between representations of $GL(n)$

Recall our claim: $V^{\overline{1}^k}_{GL(n)} = V^{\overline{\varepsilon}}_{GL(n)} \times V^{1^{n-k}}_{GL(n)}$

Justification by means of character identities

- As usual, consider $X \in GL(n)$ with eigenvalues $x = (x_1, x_2, \ldots, x_n)$.
- Let $\overline{x} = (\overline{x}_1, \overline{x}_2, \ldots, \overline{x}_n)$ with $\overline{x}_k = x_k^{-1}$ for $k = 1, 2, \ldots, n$.

$$\mathrm{ch}\, V^{\overline{1}^k}_{GL(n)} = s_{1^k}(\overline{x}) = e_k(\overline{x}) = \sum_{1 \leq i_1 < \cdots < i_k \leq n} \overline{x}_{i_1} \cdots \overline{x}_{i_k}$$

$$= (\overline{x}_1 \cdots \overline{x}_n) \sum_{1 \leq j_1 < \cdots < j_{n-k} \leq n} x_{j_1} \cdots x_{j_{n-k}} = e_n(\overline{x})\, e_{n-k}(x)$$

$$= \det(X)^{-1}\, s_{1^{n-k}}(x) = \mathrm{ch}\, V^{\overline{\varepsilon}}_{GL(n)}\ \mathrm{ch}\, V^{1^{n-k}}_{GL(n)}\,.$$

Relations between representations of $GL(n)$

Recall our claim: $V^{\lambda;\overline{\mu}}_{GL(n)} = (V^{\overline{\varepsilon}}_{GL(n)})^{\times p} \times V^{\kappa}_{GL(n)}$ where κ is (n,p)-equivalent to $\lambda;\overline{\mu}$ with $p \geq \mu_1$.

Justification by means of character identities

$$\mathrm{ch}\, V^{\lambda;\overline{\mu}}_{GL(n)} = s_{\lambda;\overline{\mu}}(x;\overline{x}) = \left| s_{1^{\mu'_{p-j+1}-i+j}}(\overline{x}) \,\vdots\, s_{1^{\lambda'_{j-p}-j+i}}(x) \right|$$

$$= \left| e_{\mu'_{p-j+1}-i+j}(\overline{x}) \,\vdots\, e_{\lambda'_{j-p}-j+i}(x) \right|$$

$$= (e_n(\overline{x}))^p \, \left| e_{n-\mu'_{p-j+1}+i-j}(x) \,\vdots\, e_{\lambda'_{j-p}-j+i}(x) \right|$$

$$= (e_n(\overline{x}))^p \, \left| e_{\kappa'_j+i-j}(x) \,\vdots\, e_{\kappa'_j-j+i}(x) \right| = (\mathrm{ch}\, V^{\overline{\varepsilon}}_{GL(n)})^p \; \mathrm{ch}\, V^{\kappa}_{GL(n)} \, .$$

Schur function expansion of $s_{\lambda;\overline{\mu}}(x;y)$

Proposition: $s_{\lambda;\overline{\mu}}(x;y) = \sum_\sigma (-1)^{|\sigma|}\, s_{\lambda/\sigma}(x)\, s_{\mu/\sigma'}(y)\,.$

Proof: With $P=\{1,2,\ldots,p\}$ and $Q^p=\{p{+}1,p{+}2,\ldots,p{+}q\}$, let $R=P\cup Q^p$ and let π be a permutation such that:

- $P_\pi \cup Q^p_\pi = R$ and $P_\pi \cap Q^p_\pi = \emptyset$;
- $P_\pi = \{\pi(1),\pi(2),\ldots,\pi(p)\} = \{1+\sigma_p, 2+\sigma_{p-1},\ldots,p+\sigma_1\}$;
- $Q^p_\pi = \{\pi(p+1),\pi(p+2),\ldots,\pi(p+q)\}$
  $= \{p+1-\sigma'_1, p+2-\sigma'_2,\ldots,p+q-\sigma'_q\}$.
- Let the elements of P_π and Q^p_π increase from left to right.
- Then the Frobenius and Signature Lemmas imply that:
  σ is a partition, with conjugate σ';
  $\pi=\pi_\sigma$ with signature $(-1)^{|\sigma|}$.

Proof continued

Setting $Q = Q^0 = \{1, 2, \ldots, q\}$, the Laplace expansion of the determinant defining $s_{\lambda;\overline{\mu}}(x;y)$ gives:

$$s_{\lambda;\overline{\mu}}(x;y) = \left| \, s_{1^{\mu'_{p-j+1}-i+j}}(y) \,\vdots\, s_{1^{\lambda'_{j-p}-j+i}}(x) \, \right|_{i,j\in R}$$

$$= \sum_{\sigma} (-1)^{|\sigma|} \, \left| \, s_{1^{\mu'_{p-j+1}-i+j}}(y) \, \right|_{i\in P_\pi, j\in P} \, \left| \, s_{1^{\lambda'_{j-p}-j+i}}(x) \, \right|_{i\in Q^p_\pi, j\in Q^p}$$

$$= \sum_{\sigma} (-1)^{|\sigma|} \, \left| \, s_{1^{\mu'_k-(p-l+1+\sigma_l)+(p-k+1)}}(y) \, \right|_{k,l\in P} \, \left| \, s_{1^{\lambda'_k-(p+k)+(p+l-\sigma'_l)}}(x) \, \right|_{k,l\in}$$

$$= \sum_{\sigma} (-1)^{|\sigma|} \, \left| \, s_{1^{\mu'_k-\sigma_l-k+l}}(y) \, \right|_{k,l\in P} \, \left| \, s_{1^{\lambda'_k-\sigma'_l-k+l}}(x) \, \right|_{k,l\in Q}$$

$$= \sum_{\sigma} (-1)^{|\sigma|} \, s_{\lambda/\sigma}(x) \, s_{\mu/\sigma'}(y) \, .$$

Example with $\lambda = (2,1)$ and $\mu = (3)$

Since $s_{21;\overline{3}}(x;y) = \sum_\sigma (-1)^{|\sigma|}\, s_{21/\sigma}(x)\, s_{3/\sigma'}(y)$ we have

$$
\begin{aligned}
s_{21;\overline{3}}(x;y) &= s_{21}(x)\, s_3(y) && = s_{21}(x)\, s_3(y) \\
&\quad - s_{21/1}(x)\, s_{3/1}(y) && \quad - (s_2(x) + s_{1^2}(x))\, s_2(y) \\
&\quad + s_{21/2}(x)\, s_{3/1^2}(y) && \quad - \\
&\quad + s_{21/1^2}(x)\, s_{3/2}(y) && \quad + s_1(x)\, s_1(y) \\
&\quad - s_{21/3}(x)\, s_{3/1^3}(y) && \quad - \\
&\quad - s_{21/21}(x)\, s_{3/21}(y) && \quad - \\
&\quad - s_{21/1^3}(x)\, s_{3/3}(y) && \quad - \\
&\quad + \cdots && \quad -
\end{aligned}
$$

$$= s_{21}(x)\, s_3(y) - s_2(x)\, s_2(y) - s_{1^2}(x)\, s_2(y) + s_1(x)\, s_1(y)\,.$$

Symmetric functions

The ring of symmetric functions $\Lambda^{(n)}$.

- $\Lambda^{(n)} = \mathbb{Z}[x]^{S_n}$ polynomial symmetric functions of indeterminates $x = (x_1, x_2, \dots, x_n)$.
- $\Lambda^{(n)} = \bigoplus_d \Lambda_d^{(n)}$ graded by degree.
- Basis of $\Lambda_d^{(n)}$ provided by Schur functions $s_\lambda(x)$ with partitions λ of weight $|\lambda| = d$ and length $\ell(\lambda) \leq n$.
- Outer product $s_\lambda(x)\, s_\mu(x) = \sum_\nu c_{\lambda\mu}^\nu\, s_\nu(x)$.
- The Littlewood-Richardson coefficients $c_{\lambda\mu}^\nu$ are non-negative integers for all λ, μ, ν.
- They are independent of n, and may be evaluated by means of the Littlewood-Richardson rule.

The ring of symmetric functions Λ

Universal Schur functions

- Let $x_1, x_2, \ldots$ be a sequence of countably many independent variables.
- Then for all partitions λ of weight $|\lambda| = d$ with d finite, there exists a "universal" Schur function $s_\lambda(x)$ of $x = (x_1, x_2, \ldots)$ such that for all finite n we have $s_\lambda(x_1, x_2, \ldots, x_n, 0, 0, \ldots) = s_\lambda(x_1, x_2, \ldots, x_n) \in \Lambda_d^{(n)}$.

The ring Λ

- Λ is the ring generated by $s_\lambda(x)$ for all partitions λ.
- $s_\lambda(x)$ for all partitions λ provides an integral basis of Λ.
- $s_\lambda(x)\ s_\mu(x) = \sum_\nu\ c_{\lambda\mu}^\nu\ s_\nu(x)$ and $s_{\nu/\lambda}(x) = \sum_\mu\ c_{\lambda\mu}^\nu\ s_\mu(x)$.

The Littlewood-Richardson rule

Theorem $s_\lambda(x)\, s_\mu(x) = \sum_\nu\ c^\nu_{\lambda\mu}\ s_\nu(x)$

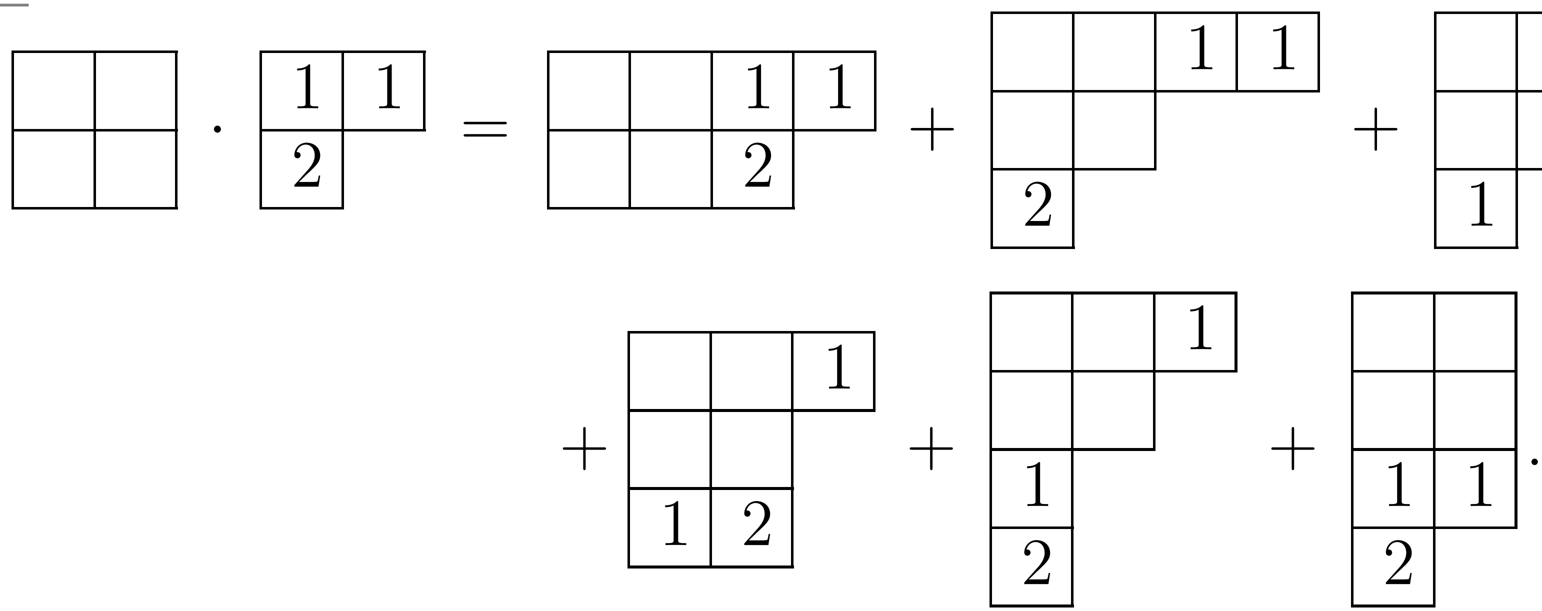
and $s_{\nu/\lambda}(x) = \sum_\mu\ c^\nu_{\lambda\mu}\ s_\mu(x)$,

where $c^\nu_{\lambda\mu}$ is the number of tableaux obtained by adding to F^λ the boxes of F^μ, with entries equal to their row number in F^μ, to give a diagram of shape F^ν, in which the entries:

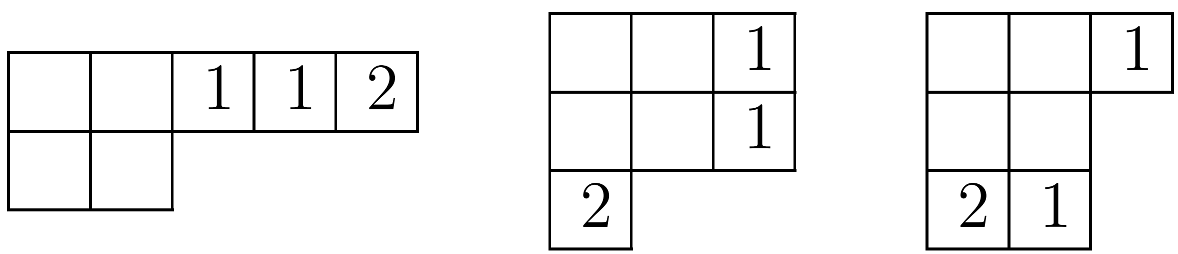
- increase weakly across rows and strictly down columns;
- and $\#i \geq \#(i+1)$ for all i when read from right to left across rows from top to bottom.

Proof: Result stated by Littlewood and Richardson, incomplete "proof" by Robinson, finally proved by Thomas, Macdonald, ...

Product with $\lambda = (2,2)$ and $\mu = (2,1)$



Excluded:



$\dots$

Hence, denoting $s_\lambda(x)$ by $\{\lambda\}$ for all λ, we obtain:

$$\{22\}\,\{21\} = \{42\} + \{421\} + \{331\} + \{322\} + \{3211\} + \{2221\}.$$

Quotient with $\nu = (4,3,1)$ and $\lambda = (2,1)$

$$\begin{array}{|c|c|c|c|}\hline * & * & & \\ \hline * & & \\ \cline{1-3} \\ \cline{1-1}\end{array} = \begin{array}{|c|c|c|c|}\hline * & * & 1 & 1 \\ \hline * & 1 & 2 \\ \cline{1-3} 1 \\ \cline{1-1}\end{array} + \begin{array}{|c|c|c|c|}\hline * & * & 1 & 1 \\ \hline * & 1 & 2 \\ \cline{1-3} 2 \\ \cline{1-1}\end{array} + \begin{array}{|c|c|c|c|}\hline * & * & 1 & 1 \\ \hline * & 2 & 2 \\ \cline{1-3} 1 \\ \cline{1-1}\end{array}$$

$$+ \begin{array}{|c|c|c|c|}\hline * & * & 1 & 1 \\ \hline * & 1 & 2 \\ \cline{1-3} 3 \\ \cline{1-1}\end{array} + \begin{array}{|c|c|c|c|}\hline * & * & 1 & 1 \\ \hline * & 2 & 2 \\ \cline{1-3} 3 \\ \cline{1-1}\end{array}$$

Excluded: $\begin{array}{|c|c|c|c|}\hline * & * & 1 & 1 \\ \hline * & 1 & 1 \\ \cline{1-3} 1 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|c|c|}\hline * & * & 1 & 1 \\ \hline * & 2 & 1 \\ \cline{1-3} 2 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|c|c|}\hline * & * & 1 & 1 \\ \hline * & 1 & 3 \\ \cline{1-3} 2 \\ \cline{1-1}\end{array} \quad \cdots$

Hence, denoting $s_{\nu/\lambda}(x)$ by $\{\nu/\lambda\}$, we obtain:

$$\{431/21\} = \{41\} + 2\,\{32\} + \{311\} + \{221\}$$

Outer products and skew products

- Outer product: $s_{\lambda\cdot\mu}(x) = s_\lambda(x)\, s_\mu(x) = \sum_\nu c^\nu_{\lambda\mu}\, s_\nu(x)$.
- Quotient or skew product: $s_{\nu/\lambda}(x) = \sum_\mu c^\nu_{\lambda\mu}\, s_\mu(x)$.
- Note: $s_{(\lambda\cdot\mu)\cdot\nu}(x) = s_{\lambda\cdot(\mu\cdot\nu)}(x)$ and $s_{(\lambda/\mu)/\nu}(x) = s_{\lambda/(\mu\cdot\nu)}(x)$.

Lemma: The outer and skew products are related by:

$$\sum_\mu (-1)^{|\mu|} s_{(\nu/\mu)\cdot\mu'} = \delta_{\nu,0}\, s_0.$$

Cauchy's formulae For all $x = (x_1, x_2, \ldots)$ and $y = (y_1, y_2, \ldots)$:

$$J(x,y) = \prod_{i,a} (1 - x_i y_a)^{-1} = \sum_\lambda s_\lambda(x)\, s_\lambda(y);$$

$$I(x,y) = \prod_{i,a} (1 - x_i y_a) = \sum_\lambda (-1)^{|\lambda|}\, s_\lambda(x)\, s_{\lambda'}(y).$$

Proof of the outer-skew product relation

$$\begin{aligned} J(x,y)\,I(x,y) &= \sum_\lambda s_\lambda(x)\,s_\lambda(y)\,\sum_\mu (-1)^{|\mu|}\,s_\mu(x)\,s_{\mu'}(y) \\ &= \sum_{\lambda,\mu,\nu} c^\nu_{\lambda\mu}\,s_\nu(x)\,(-1)^{|\mu|}\,s_\lambda(y)\,s_{\mu'}(y) \\ &= \sum_{\mu,\nu} s_\nu(x)\,(-1)^{|\mu|}\,s_{\nu/\mu}(y)\,s_{\mu'}(y)\,. \\ J(x,y)\,I(x,y) &= \prod_{i,a}(1-x_iy_a)^{-1}(1-x_iy_a) = 1 \\ &= s_0(x)\,s_0(y) = \sum_\nu s_\nu(x)\,\delta_{\nu,0}\,s_0(y)\,. \end{aligned}$$

Comparing coefficients of $s_\nu(x)$, we have

$$\sum_\mu (-1)^{|\mu|}\,s_{\nu/\mu}(y)\,s_{\mu'}(y) \;=\; \delta_{\nu,0}\,s_0(y).$$

Illustration of the outer-skew product relation

Relation: $$\sum_\mu (-1)^{|\mu|}\, s_{\nu/\mu}(x)\, s_{\mu'}(x) \;=\; \delta_{\nu,0}\; s_0(x).$$

Ex: Setting $\nu = 2$ in the left hand side gives:

$$\sum_\mu (-1)^{|\mu|}\, \{2/\mu\}(x)\, \{\mu'\}(x)$$

$$= \{2/0\}(x)\,\{0\}(x) - \{2/1\}(x)\,\{1\}(x) + \{2/2\}(x)\,\{1^2\}(x)$$

$$= \{2\}(x)\,\{0\}(x) - \{1\}(x)\,\{1\}(x) + \{0\}(x)\,\{1^2\}(x)$$

$$= \{2\}(x) - (\{2\}(x) + \{1^2\}(x)) + \{1^2\}(x) = 0\,,$$

as required, since $\nu \neq 0$.

Hopf algebra of symmetric functions

Hopf algebra: $Symm = (\Lambda, \cdot, \Delta, \iota, \epsilon, \mathsf{S})$.

- Indeterminates: $x = (x_1, x_2, \ldots)$, $y = (y_1, y_2, \ldots)$.
- Product $\cdot$: $s_\lambda(x) \cdot s_\mu(x) = \sum_\nu c^\nu_{\lambda\mu} \, s_\nu(x)$.
- Identity ι: $s_0(x) = 1$.
- Coproduct Δ: $s_\nu(x, y) = \sum_{\lambda,\mu} c^\nu_{\lambda\mu} \, s_\lambda(x) \, s_\mu(y)$.
- Counit ϵ: $\epsilon(s_\lambda(x)) = \delta_{\lambda 0}$.
- Antipode S: $\mathsf{S}(s_\lambda(x)) = (-1)^{|\lambda|} \, s_{\lambda'}(x)$

where the $c^\nu_{\lambda\mu}$ are Littlewood-Richardson coefficients and λ' is the conjugate of λ.

Properties of $Symm$

Self duality: Scalar product $(\cdot \mid \cdot)$ such that $(s_\lambda \mid s_\mu) = \delta_{\lambda\mu}$.

- $(s_\nu \mid s_\lambda \cdot s_\mu) = (s_\nu \mid s_{\lambda\,\cdot\,\mu}) = c^\nu_{\lambda\mu} = (s_{\nu/\lambda} \mid s_\mu) = (s_\nu / s_\lambda \mid s_\mu)$.
- $(\Delta(s_\nu) \mid s_\lambda \otimes s_\mu) = (s_\nu \mid s_\lambda \cdot s_\mu)$

  since $\displaystyle \sum_{\sigma,\tau} c^\nu_{\sigma\tau}\,(s_\sigma \otimes s_\tau \mid s_\lambda \otimes s_\mu) = \sum_{\sigma,\tau} c^\nu_{\sigma\tau}\,\delta_{\sigma\lambda}\,\delta_{\tau\mu} = c^\nu_{\lambda\mu}$.

Commutativity:

- Commutative: $s_\lambda \cdot s_\mu = s_\mu \cdot s_\lambda$. This implies $c^\nu_{\lambda\mu} = c^\nu_{\mu\lambda}$.
- Cocommutative: $\Delta(s_\nu) = \sum_\lambda s_\lambda \otimes s_{\nu/\lambda} = \sum_\lambda s_{\nu/\lambda} \otimes s_\lambda$

  since

  $$\Delta(s_\nu) = \sum_{\lambda\mu} c^\nu_{\lambda\mu}\, s_\lambda \otimes s_\mu = \sum_{\mu\lambda} c^\nu_{\mu\lambda}\, s_\mu \otimes s_\lambda = \sum_{\lambda\mu} c^\nu_{\lambda\mu}\, s_\mu \otimes s_\lambda.$$

Properties of $Symm$

Associativity:

- Associative: $s_\rho \cdot (s_\sigma \cdot s_\tau) = (s_\rho \cdot s_\sigma) \cdot s_\tau$.

Setting both sides equal to $\sum_\lambda c^\lambda_{\rho\sigma\tau}\, s_\lambda$, this implies that $c^\lambda_{\rho\sigma\tau} = \sum_\mu c^\lambda_{\rho\mu}\, c^\mu_{\sigma\tau} = \sum_\nu c^\nu_{\rho\sigma} c^\lambda_{\nu\tau}$.

- Coassociative: $(I \otimes \Delta)(\Delta(s_\lambda)) = (\Delta \otimes I)(\Delta(s_\lambda))$.

This follows from the identities:

$$\sum_{\rho,\mu} c^\lambda_{\rho\mu}\, s_\rho \otimes \Big(\sum_{\sigma,\tau} c^\mu_{\sigma\tau}\, s_\sigma \otimes s_\tau\Big) = \sum_{\rho,\sigma,\tau} c^\lambda_{\rho\sigma\tau}\, s_\lambda\,;$$

$$\sum_{\nu,\tau} c^\lambda_{\nu\tau}\, \Big(\sum_{\rho,\sigma} c^\nu_{\rho\sigma}\, s_\rho \otimes s_\sigma\Big) \otimes s_\tau = \sum_{\rho,\sigma,\tau} c^\lambda_{\rho\sigma\tau}\, s_\lambda\,.$$

Properties of $Symm$

Bilinearity: The scalar product $(\cdot\,|\,\cdot)$ is linear in each argument. For any two Schur function series $X = \sum_\sigma a_\sigma s_\sigma$ and $Y = \sum_\tau b_\tau s_\tau$ with $a_\sigma, b_\tau \in \mathbb{Z}$ for all σ, we have:

- $$(X\,|\,Y) = \sum_{\sigma,\tau} a_\sigma b_\tau (s_\sigma\,|\,s_\tau) = \sum_{\sigma,\tau} a_\sigma b_\tau \delta_{\sigma,\tau} = \sum_\sigma a_\sigma b_\sigma.$$
- $$(s_\lambda\,|\,X) = \sum_\sigma a_\sigma (s_\lambda\,|\,s_\sigma) = \sum_\sigma a_\sigma \delta_{\lambda,\sigma} = a_\lambda.$$
- $X \cdot s_\mu = \sum_\sigma a_\sigma (s_\sigma \cdot s_\mu)$ and $s_\lambda / X = \sum_\sigma a_\sigma (s_\lambda / s_\sigma)$.
- $$(s_\lambda\,|\,X \cdot s_\mu) = \sum_\sigma a_\sigma (s_\lambda\,|\,s_\sigma \cdot s_\mu) = \sum_\sigma a_\sigma (s_\lambda / s_\sigma\,|\,s_\mu) = (s_\lambda / X\,|\,s_\mu).$$

General Hopf algebraic properties

Antipode identity $\quad \cdot\,(\mathsf{S}\otimes I)\,\Delta = \iota\,\epsilon = \cdot\,(I\otimes\mathsf{S})\,\Delta$

- $\cdot\,(\mathsf{S}\otimes I)\,\Delta(s_\lambda) = \sum_\mu \cdot\,(\mathsf{S}\otimes I)\,(s_\mu\otimes s_{\lambda/\mu})$

  $= \sum_\mu \mathsf{S}(s_\mu)\cdot s_{\lambda/\mu} = \sum_\mu (-1)^{|\mu|} s_{\mu'}\cdot s_{\lambda/\mu} = \delta_{\lambda 0}\, s_0$
- $\iota\,\epsilon(s_\lambda) = \iota\,\delta_{\lambda 0} = \delta_{\lambda 0}\, s_0$

Counitarity $\quad \cdot\,(\epsilon\otimes I)\,\Delta = I = \cdot\,(I\otimes\epsilon)\,\Delta$

- $\cdot\,(\epsilon\otimes I)\,\Delta(s_\lambda) = \sum_\mu \cdot\,(\epsilon\otimes I)\,(s_\mu\otimes s_{\lambda/\mu})$

  $= \sum_\mu \cdot\,(\delta_{\mu 0}\otimes s_{\lambda/\mu}) = 1\cdot s_\lambda = s_\lambda$.
- $I(s_\lambda) = s_\lambda$.

General Hopf algebraic properties

Product and coproduct compatibility: $\Delta\,(\cdot) = (\cdot \otimes \cdot)\,(\Delta \otimes \Delta)$

- $\Delta\,(\cdot)\,(s_\lambda \otimes s_\mu) = \Delta(s_{\lambda\cdot\mu}) = \sum_\rho\, s_\rho \otimes s_{(\lambda\cdot\mu)/\rho}$
- $(\cdot \otimes \cdot)\,(\Delta \otimes \Delta)(s_\lambda \otimes s_\mu)$
  $= (\cdot \otimes \cdot)\, \sum_{\sigma,\tau} \left(s_\sigma \otimes s_{\lambda/\sigma}\right) \otimes \left(s_\tau \otimes s_{\mu/\tau}\right)$
  $= \sum_{\sigma,\tau} \left(s_\sigma \cdot s_\tau\right) \otimes \left(s_{(\lambda/\sigma)} \cdot s_{(\mu/\tau)}\right)$
  $= \sum_{\rho,\sigma,\tau}\, c^\rho_{\sigma\tau}\, s_\rho \otimes \left(s_{\lambda/\sigma} \cdot s_{\mu/\tau}\right)$.

Implications:

- $s_{(\lambda\cdot\mu)/\rho} = \sum_{\sigma,\tau}\, c^\rho_{\sigma\tau}\, s_{\lambda/\sigma} \cdot s_{\mu/\tau}$
- $\Delta(s_\lambda \cdot s_\mu) = \Delta(s_\lambda) \cdot \Delta(s_\mu)$
- $\Delta(X \cdot Y) = \Delta(X) \cdot \Delta(Y)$ for any $X, Y \in \Lambda$.

Tensorial approach to subgroups of $GL(n)$

Basic subgroup invariants

- $GL(n-1) = \{X \in GL(n) \,|\, X_i^a\, v_a = v_i\}$
- $O(n) = \{X \in GL(n) \,|\, X_i^a\, X_j^b\, G_{ab} = G_{ij}$ with $G_{ij} = G_{ji}\}$
- $Sp(n) = \{X \in GL(n) \,|\, X_i^a\, X_j^b\, J_{ab} = J_{ij}$ with $J_{ij} = -J_{ji}\}$

Generating functions for polynomial invariants

- $M(x) = \prod_i (1 - x_i)^{-1} = \sum_m \, s_m(x) = \sum_m \, s_m(s_1(x))$
- $D(x) = \prod_{i \le j} (1 - x_i x_j)^{-1} = \sum_m \, s_m(s_2(x))$
- $B(x) = \prod_{i < j} (1 - x_i x_j)^{-1} = \sum_m \, s_m(s_{1^2}(x))$

Plethysms $\quad s_\lambda(s_\pi(x)) = s_\lambda(y)$ with $y = (y_1, y_2 \ldots)$
where $s_\pi(x) = \sum_k \, y_k$ is the monomial expansion of $s_\pi(x)$.

Infinite series of Schur functions

- $L(x) = \prod_i (1 - x_i) = \sum_m (-1)^m \{1^m\}(x)$
- $M(x) = \prod_i (1 - x_i)^{-1} = \sum_m \{m\}(x)$
- $A(x) = \prod_{i<j} (1 - x_i x_j) = \sum_{\alpha \in \mathcal{A}} (-1)^{|\alpha|/2} \{\alpha\}(x)$
- $B(x) = \prod_{i<j} (1 - x_i x_j)^{-1} = \sum_{\beta \in \mathcal{B}} \{\beta\}(x)$
- $C(x) = \prod_{i \leq j} (1 - x_i x_j) = \sum_{\gamma \in \mathcal{C}} (-1)^{|\gamma|/2} \{\gamma\}(x)$
- $D(x) = \prod_{i \leq j} (1 - x_i x_j)^{-1} = \sum_{\delta \in \mathcal{D}} \{\delta\}(x)$

  where
  - $L(x)\, M(x) = A(x)\, B(x) = C(x)\, D(x) = 1$
  - $B(x)\, D(x) = \prod_{i,j} (1 - x_i x_j)^{-1} = \sum_\sigma \{\sigma\}(x)\ \{\sigma\}(x)$
  - $A(x)\, C(x) = \prod_{i,j} (1 - x_i x_j) = \sum_\sigma (-1)^{|\sigma|}\ \{\sigma\}(x)\ \{\sigma'\}(x)$

Expansions of infinite series

- $A = \{0\} - \{1^2\} + \{21^2\} - \{31^3\} - \{2^3\} + \cdots$
- $B = \{0\} + \{1^2\} + \{2^2\} + \{1^4\} + \{3^2\} + \{2^2 1^2\} + \{1^6\} + \cdots$
- $C = \{0\} - \{2\} + \{31\} - \{41^2\} - \{3^2\} + \cdots$
- $D = \{0\} + \{2\} + \{4\} + \{2^2\} + \{6\} + \{42\} + \{2^3\} + \cdots$

More generally

- $\alpha \in \mathcal{A} \quad \Leftrightarrow \quad \alpha = \begin{pmatrix} a_1 & a_2 & \cdots & a_r \\ a_1+1 & a_2+1 & \cdots & a_r+1 \end{pmatrix}$
- $\beta \in \mathcal{B} \quad \Leftrightarrow \quad \beta'_k$ is even for all k
- $\gamma \in \mathcal{C} \quad \Leftrightarrow \quad \gamma = \begin{pmatrix} a_1+1 & a_2+1 & \cdots & a_r+1 \\ a_1 & a_2 & \cdots & a_r \end{pmatrix}$
- $\delta \in \mathcal{D} \quad \Leftrightarrow \quad \delta_k$ is even for all k

Branching rules and universal characters

Branching rules: On restriction from $GL(n)$ to a classical subgroup the irrep V^λ decomposes in accordance with the following branching rules:

- $GL(n) \supset GL(n-1)$: $\{\lambda\}_n \to \{\lambda/M\}_{n-1}$
- $GL(n) \supset O(n)$: $\{\lambda\} \to [\lambda/D]$
- $GL(n) \supset Sp(n)$: $\{\lambda\} \to \langle\lambda/B\rangle$

Subgroup characters This implies that the characters of the subgroups are given by:

- $GL(n-1))$: $\{\lambda\}_{n-1} = \{\lambda/M^{-1}\}_n = \{\lambda/L\}_n$
- $O(n)$: $[\lambda] = \{\lambda/D^{-1}\} = \{\lambda/C\}$
- $Sp(n)$: $\langle\lambda\rangle = \{\lambda/B^{-1}\} = \{\lambda/A\}$

Branching rules examples

- $GL(n) \supset GL(n-1): \quad \{\lambda\} \to \{\lambda/M\}$
  - $\{4\} \to \{4\} + \{3\} + \{2\} + \{1\} + \{0\}$
  - $\{1^4\} \to \{1^4\} + \{1^3\}$
  - $\{2^2 1^2\} \to \{2^2 1^2\} + \{2^2 1\} + \{21^3\} + \{21^2\}$
- $GL(n) \supset O(n): \quad \{\lambda\} \to [\lambda/D]$
  - $\{4\} \to [4] + [2] + [0]$
  - $\{1^4\} \to [1^4]$
  - $\{2^2 1^2\} \to [2^2 1^2] + [21^2] + [1^2]$
- $GL(n) \supset Sp(n): \quad \{\lambda\} \to \langle\lambda/B\rangle$
  - $\{4\} \to \langle 4\rangle$
  - $\{1^4\} \to \langle 1^4\rangle + \langle 1^2\rangle + \langle 0\rangle$
  - $\{2^2 1^2\} \to \langle 2^2 1^2\rangle + \langle 2^2\rangle + \langle 21^2\rangle + \langle 1^4\rangle + 2\,\langle 1^2\rangle + \langle 0\rangle$

Universal characters of classical groups

Character generating functions

$$\prod_{i,a}(1-x_iy_a)^{-1} = \sum_\lambda \{\lambda\}(x)\ \{\lambda\}(y)$$

$$\prod_{i,a}(1-x_iy_a)^{-1}\prod_{a\le b}(1-y_ay_b) = \sum_\lambda [\lambda](x)\ \{\lambda\}(y)$$

$$\prod_{i,a}(1-x_iy_a)^{-1}\prod_{a<b}(1-y_ay_b) = \sum_\lambda \langle\lambda\rangle(x)\ \{\lambda\}(y)$$

Proposition The characters of V^λ are given by:

$GL(n):\quad \{\lambda\}(x) = s_\lambda(x)$

$O(n):\quad [\lambda](x) = s_{\lambda/C}(x)$ where $C(x) = \prod_{i\le j}(1-x_ix_j)$

$Sp(n):\quad \langle\lambda\rangle(x) = s_{\lambda/A}(x)$ where $A(x) = \prod_{i<j}(1-x_ix_j)$

Character formulae for $O(n)$ and $Sp(n)$

For $O(n)$ the required characters $[\lambda](x)$ are defined by $\prod_{i,a}(1-x_iy_a)^{-1}\,\prod_{a\leq b}(1-y_ay_b)=\sum_\lambda[\lambda](x)\,\{\lambda\}(y)$. Hence

$$
\begin{aligned}
[\lambda](x) &= \Big(\prod_{i,a}(1-x_iy_a)^{-1}\,\prod_{a\leq b}(1-y_ay_b)\;\Big|\;\{\lambda\}(y)\Big)\\
&= \Big(\sum_\sigma\{\sigma\}(x)\,\{\sigma\}(y)\,C(y)\;\Big|\;\{\lambda\}(y)\Big)\\
&= \sum_\sigma\{\sigma\}(x)\,\Big(\{\sigma\}(y)\cdot C(y)\;\Big|\;\{\lambda\}(y)\Big)\\
&= \sum_\sigma\{\sigma\}(x)\,\Big(\{\sigma\}(y)\;\Big|\;\{\lambda\}(y)/C(y)\Big)=s_{\lambda/C}(x).
\end{aligned}
$$

For $Sp(n)$ simply replace $a\leq b$ by $a<b$, and $C(y)$ by $A(y)$.
This gives: $\langle\lambda\rangle(x)=s_{\lambda/A}(x)$.

Coproducts of Schur function series

Coproducts

- $\Delta(M) = M \otimes M$
- $\Delta(D) = (D \otimes D) \cdot \Delta''(D)$ with $\Delta''(D) = \sum_\sigma s_\sigma \otimes s_\sigma$
- $\Delta(B) = (B \otimes B) \cdot \Delta''(B)$ with $\Delta''(B) = \sum_\sigma s_\sigma \otimes s_\sigma$

Proof

- $M(x,y) = \prod_i (1-x_i)^{-1} \; \prod_a (1-y_a)^{-1} = M(x)\, M(y)$
- $$D(x,y) = \prod_{i\leq j}(1-x_ix_j)^{-1} \prod_{i,a}(1-x_iy_a)^{-1} \prod_{a\leq b}(1-y_ay_b)^{-1} = D(x) \; \sum_\sigma s_\sigma(x)\, s_\sigma(y) \; D(y)$$
- $$B(x,y) = \prod_{i<j}(1-x_ix_j)^{-1} \prod_{i,a}(1-x_iy_a)^{-1} \prod_{a<b}(1-y_ay_b)^{-1} = B(x) \; \sum_\sigma s_\sigma(x)\, s_\sigma(y) \; B(y)$$

Subgroups $H(n)$ of $GL(n)$

Characters

- Let $H(n)$ be a subgroup of $GL(n)$.
- Let $X \in H(n) \subset GL(n)$ have eigenvalues $x = (x_1, x_2, \ldots)$.
- Let there exist irreps V^λ of $H(n)$ specified by partitions λ.
- Let the character of V^λ be $[\![\lambda]\!](x)$.

Assumption: Let the embedding $H(n) \subset GL(n)$ be such that there exists two series of Schur functions $S(x)$ and $T(x)$ with the properties:

- $[\![\lambda]\!](x) = \{\lambda/S\}(x)$;
- $S(x)\, T(x) = 1$;

Branching rules and tensor products

Branching rule: Under restriction from $GL(n)$ to $H(n)$:

$$\{\lambda\} \to [\![\lambda/S]\!].$$

Note: It follows that for all $X \in H(n) \subset GL(n)$:

$$\{\lambda\} = [\![\lambda/S]\!] \quad \text{and conversely} \quad [\![\lambda]\!] = \{\lambda/T\}.$$

Coproduct

Let $\Delta(S) = (S \otimes S) \cdot \Delta''(S)$

with $\Delta''(S) = \sum_{\sigma,\tau} b^S_{\sigma\tau}\ \{\sigma\} \otimes \{\tau\}$.

Proposition: Tensor product rule for $H(n)$:

$$[\![\lambda]\!] \cdot [\![\mu]\!] = \sum_{\sigma,\tau} b^S_{\sigma\tau}\ [\![(\lambda/\sigma) \cdot (\mu/\tau)]\!].$$

Proof of tensor product rule

Proof Note that

$$[\![\lambda]\!] \cdot [\![\mu]\!] = \{\lambda/T\} \cdot \{\mu/T\} = [\![((\lambda/T)\cdot(\mu/T))/S]\!].$$

This implies that the multiplicity of $[\![\nu]\!]$ in $[\![\lambda]\!] \cdot [\![\mu]\!]$ is the same as the multiplicity of $\{\nu\}$ in $\{((\lambda/T)\cdot(\mu/T))/S\}$, that is:

$$\begin{aligned}
&(\{((\lambda/T)\cdot(\mu/T))/S\} \mid \{\nu\}) \\
&\quad = (\{(\lambda/T)\cdot(\mu/T)\} \mid S \cdot \{\nu\}) \\
&\quad = (\{\lambda/T\} \otimes \{\mu/T\} \mid \Delta(S \cdot \{\nu\})) \\
&\quad = (\{\lambda/T\} \otimes \{\mu/T\} \mid (S \otimes S) \cdot \Delta''(S) \cdot \Delta(\{\nu\})) \\
&\quad = (\{\lambda/TS\} \otimes \{\mu/TS\} \mid \Delta''(S) \cdot \Delta(\{\nu\})) \\
&\quad = (\{\lambda\} \otimes \{\mu\} \mid \textstyle\sum_{\sigma,\tau} b^S_{\sigma\tau} \{\sigma\} \otimes \{\tau\} \cdot \Delta(\{\nu\})) \\
&\quad = \textstyle\sum_{\sigma,\tau} b^S_{\sigma\tau} (\{\lambda/\sigma\} \otimes \{\mu/\tau\} \mid \Delta(\{\nu\})) \\
&\quad = \textstyle\sum_{\sigma,\tau} b^S_{\sigma\tau} (\{\lambda/\sigma\} \cdot \{\mu/\sigma\} \mid \{\nu\})
\end{aligned}$$

Tensor products of irreps

Tensor product rule for $GL(n-1)$:

$$\{\lambda\}_{n-1} \cdot \{\mu\}_{n-1} = \sum_{\nu} c^{\nu}_{\lambda\mu} \, \{\nu\}_{n-1}.$$

Proof Under restriction from $GL(n)$ to $GL(n-1)$:

- $\{\lambda\}_n = \{\lambda/M\}_{n-1}$ and $\{\lambda\}_{n-1} = \{\lambda/L\}_n$ with $L\,M = 1$.
- $\Delta(M) = M \otimes M$ so that $\Delta''(M) = 1 \otimes 1 = \{0\} \otimes \{0\}$.
- Thus $b^{M}_{\sigma\tau} = \delta_{\sigma,0}\,\delta_{\tau,0}$.

Hence $\{\lambda\}_{n-1} \cdot \{\mu\}_{n-1} = \sum_{\sigma,\tau} b^{M}_{\sigma\tau} \, \{(\lambda/\sigma)\cdot(\mu/\tau)\}_{n-1}$

$$= \{\lambda \cdot \mu\}_{n-1} = \sum_{\nu} c^{\nu}_{\lambda\mu} \, \{\nu\}_{n-1}.$$

Note: This just confirms that tensor products in $GL(n)$ are stable with respect to n.

Tensor products of irreps

Tensor product rule for $O(n)$:

$$[\lambda] \cdot [\mu] = \sum_\sigma [(\lambda/\sigma) \cdot (\mu/\sigma)].$$

Proof Under restriction from $GL(n)$ to $O(n)$:

- $\{\lambda\} = [\lambda/D]$ and $[\lambda] = \{\lambda/C\}$ with $C\,D = 1$.
- $\Delta(D) = (D \otimes D) \cdot \Delta''(D)$ with $\Delta''(D) = \sum_\sigma \{\sigma\} \otimes \{\sigma\}$.
- Thus $b^D_{\sigma\tau} = \delta_{\sigma,\tau}$.

Hence $$[\lambda] \cdot [\mu] = \sum_{\sigma,\tau} b^D_{\sigma\tau}\, [(\lambda/\sigma) \cdot (\mu/\tau)] = \sum_\sigma [(\lambda/\sigma) \cdot (\mu/\sigma)].$$

Tensor products of irreps

Tensor product rule for $Sp(n)$:

$$\langle\lambda\rangle \cdot \langle\mu\rangle = \textstyle\sum_\sigma \langle(\lambda/\sigma)\cdot(\mu/\sigma)\rangle.$$

Proof Under restriction from $GL(n)$ to $Sp(n)$:

- $\{\lambda\} = \langle\lambda/B\rangle$ and $\langle\lambda\rangle = \{\lambda/A\}$ with $A\,B = 1$.
- $\Delta(B) = (B \otimes B)\cdot\Delta''(B)$ with $\Delta''(B) = \sum_\sigma \{\sigma\}\otimes\{\sigma\}$.
- Thus $b^B_{\sigma\tau} = \delta_{\sigma,\tau}$.

Hence $\langle\lambda\rangle\cdot\langle\mu\rangle = \sum_{\sigma,\tau} b^B_{\sigma\tau}\,\langle(\lambda/\sigma)\cdot(\mu/\tau)\rangle = \sum_\sigma \langle(\lambda/\sigma)\cdot(\mu/\sigma)\rangle$.

Note: The tensor product rules for $O(n)$ and $Sp(n)$ are identical!

Examples: tensor products for classical groups

$$GL(n) \quad \{2^2\} \cdot \{21\}$$

$$= \{43\} + \{421\} + \{3^21\} + \{32^2\} + \{321^2\} + \{2^31\}$$

$$O(n) \quad [2^2] \cdot [21]$$

$$= [43] + [421] + [3^21] + [32^2] + [321^2] + [2^31]$$

$$+[41] + 2[32] + 2[31^2] + 2[2^21] + [21^3]$$

$$+[3] + 2[21] + [1^3] + [1]$$

$$Sp(n) \quad \langle 2^2 \rangle \cdot \langle 21 \rangle$$

$$= \langle 43 \rangle + \langle 421 \rangle + \langle 3^21 \rangle + \langle 32^2 \rangle + \langle 321^2 \rangle + \langle 2^31 \rangle$$

$$+\langle 41 \rangle + 2\langle 32 \rangle + 2\langle 31^2 \rangle + 2\langle 2^21 \rangle + \langle 21^3 \rangle$$

$$+\langle 3 \rangle + 2\langle 21 \rangle + \langle 1^3 \rangle + \langle 1 \rangle$$

Relations between $s_{\lambda\,;\,\overline{\mu}}(x;y)$ and $s_\lambda(x)\,s_\mu(y)$

Proposition:
$$s_{\lambda\,;\,\overline{\mu}}(x;y)=\sum_{\sigma}(-1)^{|\sigma|}\,s_{\lambda/\sigma}(x)\;s_{\mu/\sigma'}(y);$$
$$s_\lambda(x)\;s_\mu(y)=\sum_{\sigma}s_{\lambda/\sigma\,;\,\overline{\mu/\sigma}}(x;y).$$

Proof: The first formula has already been proved. Used in the second, together with the outer-skew product relation, it gives:

$$\sum_{\sigma}s_{\lambda/\sigma\,;\,\overline{\mu/\sigma}}(x;y)=\sum_{\sigma,\tau}(-1)^{|\tau|}\,s_{(\lambda/\sigma)/\tau}(x)\;s_{(\mu/\sigma)/\tau'}(y)$$
$$=\sum_{\sigma,\tau}(-1)^{|\tau|}\,s_{\lambda/(\sigma\cdot\tau)}(x)\;s_{\mu/(\sigma\cdot\tau')}(y)=\sum_{\sigma,\tau,\nu}(-1)^{|\tau|}\,c^{\nu}_{\sigma\tau}\,s_{\lambda/\nu}(x)\;s_{\mu/(\sigma\cdot\tau')}(y$$
$$=\sum_{\tau,\nu}s_{\lambda/\nu}(x)\;(-1)^{|\tau|}\,s_{\mu/((\nu/\tau)\cdot\tau')}(y)=\sum_{\nu}s_{\lambda/\nu}(x)\;\delta_{\nu,0}\,s_{\mu/0}(y)$$
$$=s_\lambda(x)\;s_\mu(y)\,.$$

Mixed tensor universal characters of GL(n)

Characters $GL(n): \quad \mathrm{ch}\, V^{\lambda;\overline{\mu}}_{GL(n)} = s_{\lambda;\overline{\mu}}(x;\overline{x})$.

- $s_{\lambda;\overline{\mu}}(x;y) = \sum_\sigma (-1)^{|\sigma|} s_{\lambda/\sigma}(x)\, s_{\mu/\sigma'}(y)$.
- $s_\lambda(x)\, s_\mu(y) = \sum_\sigma s_{\lambda/\sigma;\overline{\mu/\sigma}}(x;y)$.

Cauchy's formulae

- $I(x,y) = \prod_{i,a} (1 - x_i y_a) = \sum_\sigma (-1)^{|\sigma|} s_\sigma(x)\, s_{\sigma'}(y)$.
- $J(x,y) = \prod_{i,a} (1 - x_i y_a)^{-1} = \sum_\sigma s_\sigma(x)\, s_\sigma(y)$.

Symbolically For all $x = (x_1, x_2, \ldots)$ and $y = (y_1, y_2, \ldots)$

- $\{\lambda;\overline{\mu}\}(x;y) := s_{\lambda;\overline{\mu}}(x;y)$ and $\{\lambda \times \mu\}(x;y) := s_\lambda(x)\, s_\mu(y)$.
- $\{\lambda;\overline{\mu}\}(x;y) = \{(\lambda \times \mu)/I\}(x;y)$.
- $\{\lambda \times \mu\}(x;y) = \{(\lambda;\overline{\mu})/J\}(x;y)$.

Mixed tensor products

Formal structure:

- $\{\kappa;\overline{\lambda}\}(x;y)=\{(\kappa\times\lambda)/I\}(x;y)$.
- $\{\kappa\times\lambda\}(x;y)=\{(\kappa;\overline{\lambda})/J\}(x;y)$.

Coproduct

Let $\Delta(J)=(J(x,y)\otimes J(u,v))\cdot\Delta''(J)(x,y,u,v)$

with $\Delta''(J)(x,y,u,v)$
$$=\sum_{\sigma,\tau,\eta,\zeta} b^J_{\sigma\tau\eta\zeta}\,(\{\sigma\}(x)\ \{\tau\}(y))\otimes(\{\eta\}(u)\{\zeta\}(v)).$$

Proposition: Tensor product rule:

$$\{\kappa;\overline{\lambda}\}\cdot\{\mu;\overline{\nu}\}=\sum_{\sigma,\tau,\eta,\zeta} b^J_{\sigma\tau\eta\zeta}\,\{(\kappa/\sigma)\cdot(\mu/\eta)\,;\,\overline{(\lambda/\tau)\cdot(\nu/\zeta)}\}$$

Coproducts of the Cauchy kernel

Lemma

$\Delta(J) = (J \otimes J) \cdot \Delta''(J)$ with $\Delta''(J) = \sum_{\sigma,\tau} (s_\sigma \times s_\tau) \otimes (s_\tau \times s_\sigma)$

Proof For $J(x, y) = \prod_{i,a} (1 - x_i\, y_a)^{-1}$, under the coproduct maps $x \to (x, u)$ and $y \to (y, v)$ we have

$$\Delta(J(x, y)) = J(x, u, y, v)$$

$$= \prod_{i,a} (1 - x_i\, y_a)^{-1} \prod_{i,b} (1 - x_i\, v_b)^{-1} \prod_{j,a} (1 - u_j\, y_a)^{-1} \prod_{j,b} (1 - u_j\, v_b)^{-1}$$

$$= J(x, y) \sum_\sigma s_\sigma(x)\, s_\sigma(v) \sum_\tau s_\tau(u)\, s_\tau(y)\; J(u, v).$$

$$= J(x, y)\; J(u, v) \sum_{\sigma,\tau} s_\sigma(x)\, s_\tau(y)\, s_\tau(u)\, s_\sigma(v).$$

Mixed tensor product rule for $GL(n)$

Proposition $\{\kappa;\overline{\lambda}\}\cdot\{\mu;\overline{\nu}\} = \sum_{\sigma,\tau}\{(\kappa/\sigma)\cdot(\mu/\tau);\overline{(\lambda/\tau)\cdot(\nu/\sigma)}\}$

Proof

- $\{\kappa;\overline{\lambda}\} = \{(\kappa\times\lambda)/I\}$ and $\{\kappa\times\lambda\} = \{(\kappa;\overline{\lambda})/J\}$ with $I\,J = 1$.
- $\Delta(J) = (J\otimes J)\cdot\Delta''(J)$ with $\Delta''(J) = \sum_{\sigma,\tau}\,(s_\sigma\times s_\tau)\otimes(s_\tau\times s_\sigma)$.
- Thus $b^J_{\sigma\tau\eta\zeta} = \delta_{\eta,\tau}\;\delta_{\zeta,\sigma}$.

Hence

$$\{\kappa;\overline{\lambda}\}\cdot\{\mu;\overline{\nu}\} = \sum_{\sigma,\tau,\eta,\zeta} b^J_{\sigma\tau\eta\zeta}\,\{(\kappa/\sigma)\cdot(\mu/\eta)\,;\,\overline{(\lambda/\tau)\cdot(\nu/\zeta)}\}$$

$$= \sum_{\sigma,\tau}\{(\kappa/\sigma)\cdot(\mu/\tau);\overline{(\lambda/\tau)\cdot(\nu/\sigma)}\}$$

Example of mixed tensor product

In general $\{\kappa;\overline{\lambda}\}\cdot\{\mu;\overline{\nu}\} = \sum_{\sigma,\tau}\{(\kappa/\sigma)\cdot(\mu/\tau);\overline{(\lambda/\tau)\cdot(\nu/\sigma)}\}$.

Ex Let $\{\kappa;\overline{\lambda}\} = \{2;\overline{1}\}$ and $\{\mu;\overline{\nu}\} = \{1^2;\overline{1}\}$.

$$
\begin{aligned}
&\{2;\overline{1}\}\cdot\{1^2;\overline{1}\} = \\
&= \{(2\cdot 1^2;\overline{1\cdot 1}\} + \{(2/1\cdot 1^2;\overline{1\cdot 1/1}\} \\
&\qquad +\{(2\cdot 1^2/1;\overline{1/1\cdot 1}\} + \{(2/1\cdot 1^2/1;\overline{1/1\cdot 1/1}\} \\
&= \{(2\cdot 1^2;\overline{1\cdot 1}\} + \{(1\cdot 1^2;\overline{1\cdot 0}\} \\
&\qquad +\{(2\cdot 1;\overline{0\cdot 1}\} + \{(1\cdot 1;\overline{0\cdot 0}\} \\
&= \{31;\overline{2}\} + \{31;\overline{1^2}\} + \{21^2;\overline{2}\} + \{21^2;\overline{1^2}\} \\
&\qquad +\{3;\overline{1}\} + 2\{21;\overline{1}\} + \{1^3;\overline{1}\} + \{2;0\} + \{1^2;0\}
\end{aligned}
$$

Subgroups $H(n)$ of $GL(n)$ for finite n

Each $X \in H(n) \subset GL(n)$ has eigenvalues:

- $SO(2k+1): \quad x_1, x_2, \ldots, x_k, \overline{x}_1, \overline{x}_2, \ldots, \overline{x}_k, 1$.
- $O(2k+1)\backslash SO(2k+1): \quad x_1, x_2, \ldots, x_k, \overline{x}_1, \overline{x}_2, \ldots, \overline{x}_k, -1$.
- $Sp(2k): \quad x_1, x_2, \ldots, x_k, \overline{x}_1, \overline{x}_2, \ldots, \overline{x}_k$.
- $SO(2k): \quad x_1, x_2, \ldots, x_k, \overline{x}_1, \overline{x}_2, \ldots, \overline{x}_k$.
- $O(2k)\backslash SO(2k): \quad x_1, x_2, \ldots, x_{k-1}, \overline{x}_1, \overline{x}_2, \ldots, \overline{x}_{k-1}, 1, -1$.
- $SSp(2k+1): \quad x_1, x_2, \ldots, x_k, \overline{x}_1, \overline{x}_2, \ldots, \overline{x}_k, 1$.
  - As usual $\overline{x}_i = x_i^{-1}$ for all i.
  - If x_k is an eigenvalue of $X \in H(n)$ then so is $\overline{x}_k = x_k^{-1}$.
  - In each case $\det X = \det X^{-1} = \pm 1$.

Character identities for $GL(n)$ restricted to $H(n)$

- For all $X \in GL(n)$ with eigenvalues $x = (x_1, x_2, \ldots, x_n)$:

$$\mathrm{ch}\, V^{\lambda}_{GL(n)} = s_\lambda(x) = \{\lambda\} \text{ and } \mathrm{ch}\, V^{\overline{\lambda}}_{GL(n)} = s_\lambda(\overline{x}) = \{\overline{\lambda}\}.$$

- When restricted to $X \in H(n) \subset GL(n)$:

$$\overline{x} = (\overline{x}_1, \overline{x}_2, \ldots, \overline{x}_n) = (x_1, x_2, \ldots, x_n) = x.$$

- Hence under this restriction:

$$\{\overline{\lambda}\} = s_\lambda(\overline{x}) = s_\lambda(x) = \{\lambda\}.$$

- In particular $\{\overline{1}^k\} = \{1^k\}$ for all k.
- Recall: $\mathrm{ch}\, V^{\overline{1}^k}_{GL(n)} = \det X^{-1}\; chV^{1^{n-k}}_{GL(n)}$.
- Under restriction, and letting * denote multiplication by $\det X = \pm 1$, this gives: $\{1^k\} = \det X\; \{1^{n-k}\} = \{1^{n-k}\}^*$.

Character identities for $O(n)$

- $\{\lambda\} = [\lambda/D] = [\lambda/(0 + 2 + 4 + 2^2 + 6 + 42 + 2^3 + \cdots)]$.
- $[\lambda] = \{\lambda/C\} = \{\lambda/(0 - 2 + 31 - 41^2 - 3^2 + \cdots)\}$.
- $\{1^k\} = [1^k]$. Hence $[1^k] = \{1^k\} = \{1^{n-k}\}^* = [1^{n-k}]^*$.

Lemma For $O(n)$ we have the elementary modification rule: $[1^p] = [1^{p-h}]^*$ where $h = 2p - n$.

- For $n = 6, p = 5$ we have $h = 4$ so that $[1^5] = [1]^*$.
- For $n = 6, p = 3$ we have $h = 0$ so that $[1^3] = [1^3]^*$.
- For $n = 6, p = 7$ we have $h = -8$ so that $[1^7] = [1^{-1}]^* = 0$.

Note: $[1^7] = \{1^7\} = 0$ by the $GL(6)$ modification rule, and $[1^3] = [1^3]^*$ is said to be self-associate.

Character identities for $Sp(n)$

- $\{\lambda\} = \langle\lambda/B\rangle = \langle\lambda/(0 + 1^2 + 2^2 + 1^4 + 3^2 + 2^2 1^2 + 1^6)\rangle$.
- $\langle\lambda\rangle = \{\lambda/A\} = \{\lambda/(0 - 1^2 + 21^2 - 31^3 - 2^3 + \cdots)\}$.
- $\{1^k\} = \langle 1^k\rangle + \langle 1^{k-2}\rangle + \cdots$ and $\langle 1^k\rangle = \{1^k\} - \{1^{k-2}\}$.
- Hence $\langle 1^k\rangle = \{1^{n-k}\}^* - \{1^{n-k+2}\}^* = -\langle 1^{n-k+2}\rangle^*$.

Lemma For $Sp(n)$ we have the elementary modification rule: $\langle 1^p\rangle = -\langle 1^{p-h}\rangle^*$ where $h = 2p - n - 2$.

- For $n = 6, p = 5$ we have $h = 2$ so that $\langle 1^5\rangle = -\langle 1^3\rangle^*$.
- For $n = 6, p = 4$ we have $h = 0$ so that $\langle 1^4\rangle = -\langle 1^4\rangle^*$.

Note: For $Sp(2k)$ we have $\det X = 1$ so that $\langle 1^p\rangle^* = \langle 1^p\rangle$.
Hence for $n = 6, p = 4$ we have $\langle 1^4\rangle = -\langle 1^4\rangle = 0$.

Determinantal identities

Jacobi-Trudi determinantal identities

- $\{\lambda\} = |\,\{\lambda_i - i + j\}\,|$
- $[\lambda] = |\,[\lambda_i - i + j] + (1-\delta_{i0})[\lambda_i - i - j + 2]\,|$
- $\langle\lambda\rangle = |\,\langle\lambda_i - i + j\rangle + (1-\delta_{i0})\langle\lambda_i - i - j + 2\rangle\,|$

Dual Jacobi-Trudi determinantal identities

- $\{\lambda\} = \left|\,\{1^{\lambda'_j + i - j}\}\,\right|$
- $[\lambda] = \left|\,[1^{\lambda'_j + i - j}] + (1-\delta_{i0})[1^{\lambda'_j - i - j + 2}]\,\right|$
- $\langle\lambda\rangle = \left|\,\langle 1^{\lambda'_j + i - j}\rangle + (1-\delta_{i0})\langle 1^{\lambda'_j - i - j + 2}\rangle\,\right|$

Giambelli determinantal identities

- $\{\lambda\} = \left|\,\left\{\begin{matrix} a_i \\ b_j \end{matrix}\right\}\,\right| \qquad [\lambda] = \left|\,\left[\begin{matrix} a_i \\ b_j \end{matrix}\right]\,\right| \qquad \langle\lambda\rangle = \left|\,\left\langle\begin{matrix} a_i \\ b_j \end{matrix}\right\rangle\,\right|$

Dual Jacobi-Trudi identity for $O(n)$

- In general $[\lambda] = \left| [1^{\lambda'_j+i-j}] + (1-\delta_{i0})[1^{\lambda'_j-i-j+2}] \right|$.
- For $\lambda = (3,2,1,1)$ we have $\lambda' = (4,2,1)$ and in this case:

$$[321^2] = \begin{vmatrix} [1^4] & [1] & [1^{-1}] \\ [1^5]+[1^3] & [1^2]+[1^0] & [1^0]+[1^{-2}] \\ [1^6]+[1^2] & [1^3]+[1^{-1}] & [1^1]+[1^{-3}] \end{vmatrix}$$

$$= \begin{vmatrix} [1^4] & [1] & - \\ [1^5]+[1^3] & [1^2]+[0] & [0] \\ [1^6]+[1^2] & [1^3] & [1] \end{vmatrix}.$$

$O(n)$ modifications example

- For $n=7$ we have $[1^4]=[1^3]^*$, $[1^5]=[1^2]^*$ and $[1^6]=[1^1]^*$ so that $[321^2]=[321]^*$ since

$$\begin{vmatrix} [1^4] & [1] & - \\ [1^5]+[1^3] & [1^2]+[0] & [0] \\ [1^6]+[1^2] & [1^3] & [1] \end{vmatrix} = \begin{vmatrix} [1^3]^* & [1] & - \\ [1^2]^*+[1^4]^* & [1^2]+[0] & [0] \\ [1^1]^*+[1^5]^* & [1^3] & [1] \end{vmatrix}.$$

- For $n=6$ we have $[1^3]=[1^3]^*$, $[1^4]=[1^2]^*$, $[1^5]=[1]^*$ and $[1^6]=[0]^*$ so that $[321^2]=[32]^*$ since

$$\begin{vmatrix} [1^4] & [1] & - \\ [1^5]+[1^3] & [1^2]+[0] & [0] \\ [1^6]+[1^2] & [1^3] & [1] \end{vmatrix} = \begin{vmatrix} [1^2]^* & [1] & - \\ [1]^*+[1^3]^* & [1^2]+[0] & [0] \\ [0]^*+[1^4]^* & [1^3] & [1] \end{vmatrix}.$$

$O(n)$ modifications example

- For $n = 5$ we have $[1^3] = [1^2]^*$, $[1^4] = [1]^*$, $[1^5] = [0]^*$ and $[1^6] = \{1^6\} = 0$ so that $[321^2] = 0$ since

$$\begin{vmatrix} [1^4] & [1] & - \\ [1^5]+[1^3] & [1^2]+[0] & [0] \\ [1^6]+[1^2] & [1^3] & [1] \end{vmatrix} = \begin{vmatrix} [1]^* & [1] & - \\ [0]^*+[1^2]^* & [1^2]+[0] & [0] \\ [1^3]^* & [1^3] & [1] \end{vmatrix}.$$

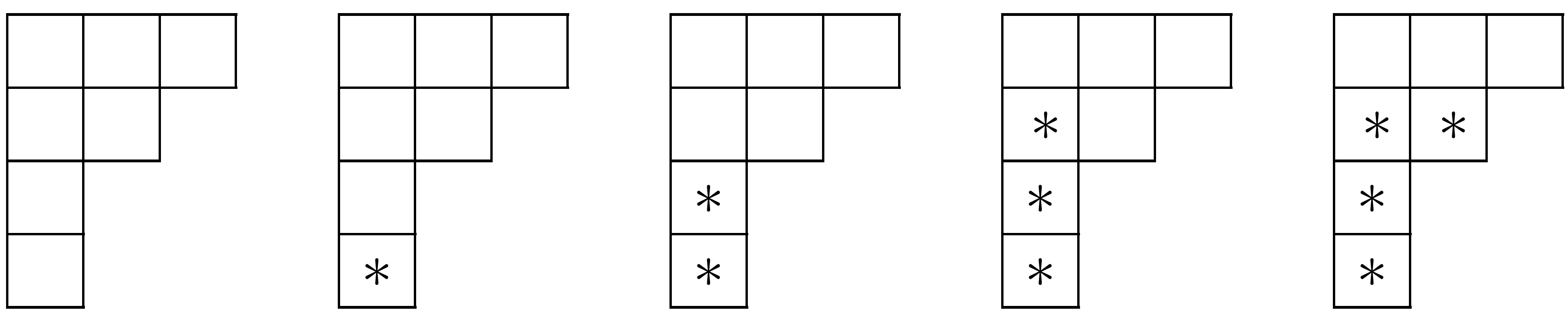
- For $n = 4$ we have $[1^2] = [1^2]^*$, $[1^3] = [1]^*$, $[1^4] = [0]^*$, $[1^5] = [1^6] = 0$ so that $[321^2] = -[3]^*$ since

$$\begin{vmatrix} [1^4] & [1] & - \\ [1^5]+[1^3] & [1^2]+[0] & [0] \\ [1^6]+[1^2] & [1^3] & [1] \end{vmatrix} = \begin{vmatrix} [0]^* & [1] & - \\ [1]^* & [1^2]+[0] & [0] \\ [1^2]^* & [1^3] & [1] \end{vmatrix}.$$

Summary of $O(n)$ modification example

- For $O(n)$: $[321^2]$ standard character for all $n \geq 8$.
- For $O(7)$: $[321^2] = [321]^*$.
- For $O(6)$: $[321^2] = [32]^*$.
- For $O(5)$: $[321^2] = 0$.
- For $O(4)$: $[321^2] = -[3]^*$.

Note: The corresponding modifications to Young diagrams take the form:



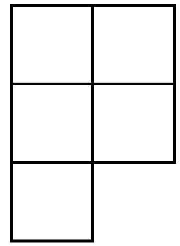
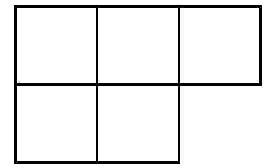
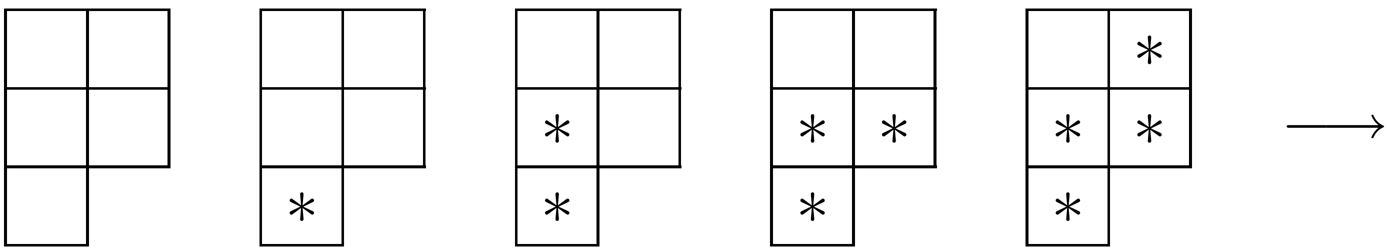
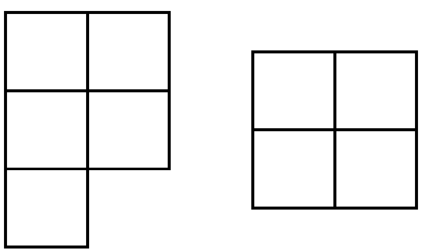
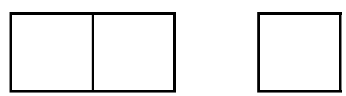
Structure of $O(n)$ modification rule

Summary of procedure

- Let λ, with conjugate λ' have length $\ell(\lambda) = \lambda'_1 = p$.
- λ' determines diagonal entries of dual J-T determinant.
- Modify each term in 1st column using $[1^k] = [1^{n-k}]^*$
- Diagonal term modification: $[1^p] = [1^{p-h}]^*$ with $h = 2p - n$.
- Either result is 0 if 1st column is identical with any other.
- Or permute columns until in form of dual J-T determinant specified by some σ'

Conclude: Either $[\lambda] = 0$, or $[\lambda] = \pm[\sigma]$ with the sign determined by the required permutation of columns.

Young diagrams and boundary strip removal

- Each partition λ define a Young diagram F^{λ}
- Transposition gives $F^{\lambda'}$ where λ' is the conjugate of λ
- Ex: $\lambda = (2^2 1)$, $\lambda' = (32)$ $F^{2^2 1} =$  $F^{32} =$ 
- $F^{\lambda - h}$ is obtained from F^{λ} by removing strip of length h starting at foot of 1st column extending over c columns.
- 
-  $\emptyset$  where $\emptyset$ indicates $F^{\lambda - h}$ not regular.

Modification rules

Standard characters

- $GL(n)$: $\{\lambda\}$ with $\lambda'_1 \leq n$.
- $O(n)$: $[\lambda]$ with $\lambda'_1 + \lambda'_2 \leq n$.
- $Sp(n)$: $\langle\lambda\rangle$ with $2\lambda'_1 \leq n$.

Modification rules for non-standard characters

- $GL(n)$: $\{\lambda\} = 0$ if $\lambda'_1 > n$.
- $O(n)$: $[\lambda] = (-1)^{c-1}[\lambda - h]^*$ if $h = 2\lambda'_1 - n > 0$.
- $Sp(n)$: $\langle\lambda\rangle = (-1)^c\langle\lambda - h\rangle^*$ if $h = 2\lambda'_1 - n - 2 \geq 0$.

with $[\lambda - h]$ and $\langle\lambda - h\rangle$ both $= 0$ if $F^{\lambda - h}$ is not regular

Application of modification rules for $GL(n)$

$GL(n)$ modifications: $\{\lambda\} = 0$ if $\ell(\lambda) > n$.

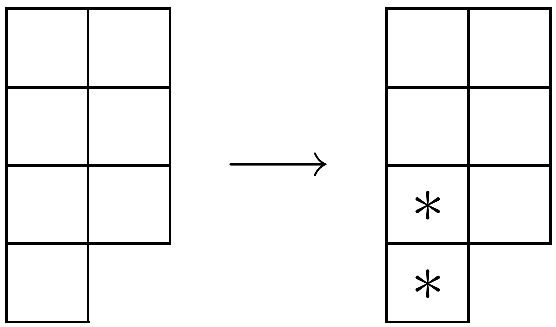
- Examples:
  - $n < 4$: $\{321^2\} = \{2^31\} = 0$
  - $n < 3$: $\{421\} = \{3^21\} = \{32^2\} = 0$
- For $GL(n)$ the product $\{2^2\} \cdot \{21\}$ modifies to give

$$
\begin{array}{ll}
n \geq 4 & \{43\} + \{421\} + \{3^21\} + \{32^2\} + \{321^2\} + \{2^31\} \\
n = 3 & \{43\} + \{421\} + \{3^21\} + \{32^2\} \\
n = 2 & \{43\}
\end{array}
$$

Examples of modifications for $O(n)$

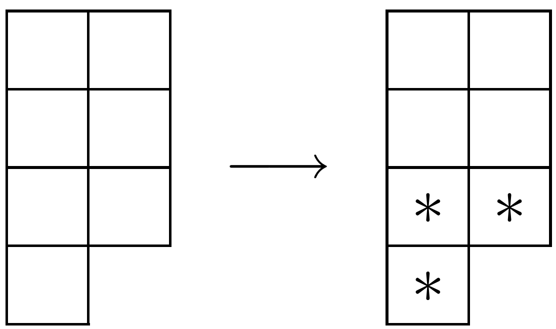
$O(n)$ modifications: $[\lambda] = (-1)^{c-1}[\lambda - h]^*$ with $h = 2\ell(\lambda) - n$

- $n = 6 \quad \lambda = (2^3 1)$, $h = 2$

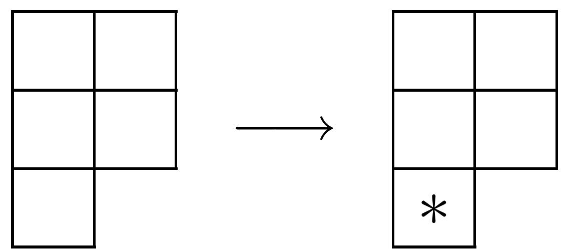


$\Rightarrow \quad [2^3 1] = 0$

- $n = 5 \quad \lambda = (2^3 1)$, $h = 3$ and $\lambda = (2^2 1)$, $h = 1$



and



$\Rightarrow \quad [2^3 1] = -[2^2]^*$ and $[2^2 1] = [2^2]^* \Rightarrow \quad [2^3 1] = -[2^2 1]$.

$O(n)$ **tensor product** $[2^2] \cdot [21]$

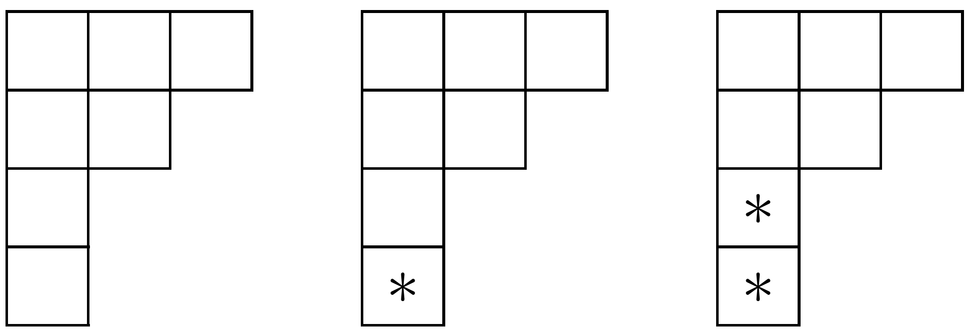
$$n \geq 7 \quad [43] + [421] + [3^21] + [32^2] + [321^2] + [2^31]$$
$$+[41] + 2[32] + 2[31^2] + 2[2^21] + [21^3]$$
$$+[3] + 2[21] + [1^3] + [1]$$
$$n = 6 \quad [43] + [421] + [3^21] + [32^2] + [321^2]$$
$$+[41] + 2[32] + 2[31^2] + 2[2^21] + [21^3]$$
$$+[3] + 2[21] + [1^3] + [1]$$
$$n = 5 \quad [43] + [421] + [3^21] + [41] + 2[32] + 2[31^2]$$
$$+[2^21] + [21^3] + [3] + 2[21] + [1^3] + [1]$$
$$n = 4 \quad [43] + [41] + [32] + [31^2] + [3] + [21] + [1^3] + [1]$$

Examples of modifications for $Sp(n)$

$Sp(n)$: $\langle\lambda\rangle = (-1)^c\langle\lambda - h\rangle^*$ with $h = 2\ell(\lambda) - n - 2$

- Ex: In $Sp(6)$ $\langle 321^2\rangle = \langle 2^3 1\rangle = \langle 21^3\rangle = 0$ since $h = 0$ and $c = 1$ (strip of length 0 removed from the 1st column)

- Ex: $\langle\lambda\rangle = \langle 321^2\rangle$, $h = 6 - n$.



  - $n = 6 : \quad \langle 321^2\rangle = -\langle 321^2\rangle^* = -\langle 321^2\rangle = 0$
  - $n = 5 : \quad \langle 321^2\rangle = -\langle 321\rangle^*$
  - $n = 4 : \quad \langle 321^2\rangle = -\langle 32\rangle^* = -\langle 32\rangle$

$Sp(n)$ **tensor product** $\langle 2^2 \rangle \cdot \langle 21 \rangle$

$$
\begin{aligned}
n \geq 7 \quad & \langle 43 \rangle + \langle 421 \rangle + \langle 3^2 1 \rangle + \langle 32^2 \rangle + \langle 321^2 \rangle + \langle 2^3 1 \rangle \\
& + \langle 41 \rangle + 2\langle 32 \rangle + 2\langle 31^2 \rangle + 2\langle 2^2 1 \rangle + \langle 21^3 \rangle \\
& + \langle 3 \rangle + 2\langle 21 \rangle + \langle 1^3 \rangle + \langle 1 \rangle \\
n = 6 \quad & \langle 43 \rangle + \langle 421 \rangle + \langle 3^2 1 \rangle + \langle 32^2 \rangle \\
& + \langle 41 \rangle + 2\langle 32 \rangle + 2\langle 31^2 \rangle + 2\langle 2^2 1 \rangle \\
& + \langle 3 \rangle + 2\langle 21 \rangle + \langle 1^3 \rangle + \langle 1 \rangle \\
n = 5 \quad & \langle 43 \rangle + \langle 421 \rangle + \langle 3^2 1 \rangle + \langle 32^2 \rangle - \langle 321 \rangle - \langle 2^3 \rangle \\
& + \langle 41 \rangle + 2\langle 32 \rangle + 2\langle 31^2 \rangle + 2\langle 2^2 1 \rangle - \langle 21^2 \rangle \\
& + \langle 3 \rangle + 2\langle 21 \rangle + \langle 1^3 \rangle + \langle 1 \rangle \\
n = 4 \quad & \langle 43 \rangle + \langle 41 \rangle + \langle 32 \rangle + \langle 3 \rangle + \langle 21 \rangle + \langle 1 \rangle
\end{aligned}
$$

Odd symplectic groups

Identification of group $H_{1^2}(2k+1) = Sp(2k+1)$

- Canonical form of second rank antisymmetric tensor η_{ij}

$$\eta \;=\; \mathrm{diag}\left[\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \cdots \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 0 \end{pmatrix}\right]$$

- $Sp(2k+1)$ neither semisimple nor reductive (Proctor)

$$Sp(2k+1) \;=\; \begin{bmatrix} & & & * \\ & Sp(2k) & & * \\ & & & * \\ 0 & \cdots & 0 & GL(1) \end{bmatrix}$$

Odd symplectic groups

Identification of characters of $H_{1^2}(2k+1) = Sp(2k+1)$

- The defining representation is indecomposable but reducible: $\langle 1\rangle = \mathrm{ch}\, V^{\langle 1\rangle}_{Sp(2k+1)} = \mathrm{ch}\, V^{\langle 1\rangle}_{Sp(2k)} + \mathrm{ch}\, V^{\{1\}}_{GL(1)}$
- Each $\langle\lambda\rangle$ is the character of an indecomposable representation (Proctor)

Tensor products are not fully reducible

- Ex: In $Sp(3)$ $V^{\langle 2\rangle}_{Sp(3)} \otimes V^{\langle 1^2\rangle}_{Sp(3)}$ is not fully reducible.
- $\langle 2\rangle \cdot \langle 1^2\rangle = \langle 31\rangle + \langle 21^2\rangle + \langle 2\rangle + \langle 1^2\rangle = \langle 31\rangle - \langle 21\rangle + \langle 2\rangle + \langle 1^2\rangle$
- $V^{\langle 31\rangle}_{Sp(3)} + V^{\langle 2\rangle}_{Sp(3)} + V^{\langle 1^2\rangle}_{Sp(3)}$ must contain all the irreducible components of $V^{\langle 21\rangle}_{Sp(3)}$.

Mixed tensor characters of $GL(n)$

Definition $s_{\lambda;\overline{\mu}}(x;\overline{x}) = \left| s_{1^{\mu'_{p-j+1}-i+j}}(\overline{x}) \vdots s_{1^{\lambda'_{j-p}-j+i}}(x) \right|$

- Thus $\{\lambda;\overline{\mu}\} = \left| \{\overline{1}^{\mu'_{p-j+1}-i+j}\} \vdots \{1^{\lambda'_{j-p}-j+i}\} \right|$
- Ex: For $\lambda = (322)$ and $\mu = (211)$,
  we have $\lambda' = (331)$, $\mu' = (31)$, so that

$$\{332;\overline{112}\} = \begin{vmatrix} \{\overline{1}\} & \{\overline{1}^4\} & \{1\} & \{0\} & - \\ \{\overline{0}\} & \{\overline{1}^3\} & \{1^2\} & \{1\} & - \\ - & \{\overline{1}^2\} & \{1^3\} & \{1^2\} & - \\ - & \{\overline{1}\} & \{1^4\} & \{1^3\} & \{0\} \\ - & \{\overline{0}\} & \{1^5\} & \{1^4\} & \{1\} \end{vmatrix}$$

Modifications of mixed tensor characters of $GL(n)$

Elementary modifications

- Let $\varepsilon = \det X$ and $\overline{\varepsilon} = \det X^{-1}$ so that $\varepsilon\,\overline{\varepsilon} = 1$.
- $\{\overline{1}^k\} = \overline{\varepsilon}\,\{1^{n-k}\}$ and $\{1^k\} = \varepsilon\,\{\overline{1}^{n-k}\}$

Ex: For $n=4$, applying these to columns 2 and 3 of our previous example, we find $\{322;\overline{112}\} = \{32;\overline{2}\}$

$$\begin{vmatrix} \{\overline{1}\} & - & \{\overline{1}^2\} & \{0\} & - \\ \{\overline{0}\} & \{0\} & \{\overline{1}\} & \{1\} & - \\ - & \{1\} & \{\overline{0}\} & \{1^2\} & - \\ - & \{1^2\} & - & \{1^3\} & \{0\} \\ - & \{1^3\} & - & \{1^4\} & \{1\} \end{vmatrix} = \begin{vmatrix} \{\overline{1}\} & \{\overline{1}^2\} & \{0\} & - & - \\ \{\overline{0}\} & \{\overline{1}\} & \{1\} & \{0\} & - \\ - & \{\overline{0}\} & \{1^2\} & \{1\} & - \\ - & - & \{1^3\} & \{1^2\} & \{0\} \\ - & - & \{1^4\} & \{1^3\} & \{1\} \end{vmatrix}$$

Structure of mixed tensor $GL(n)$ modification rul

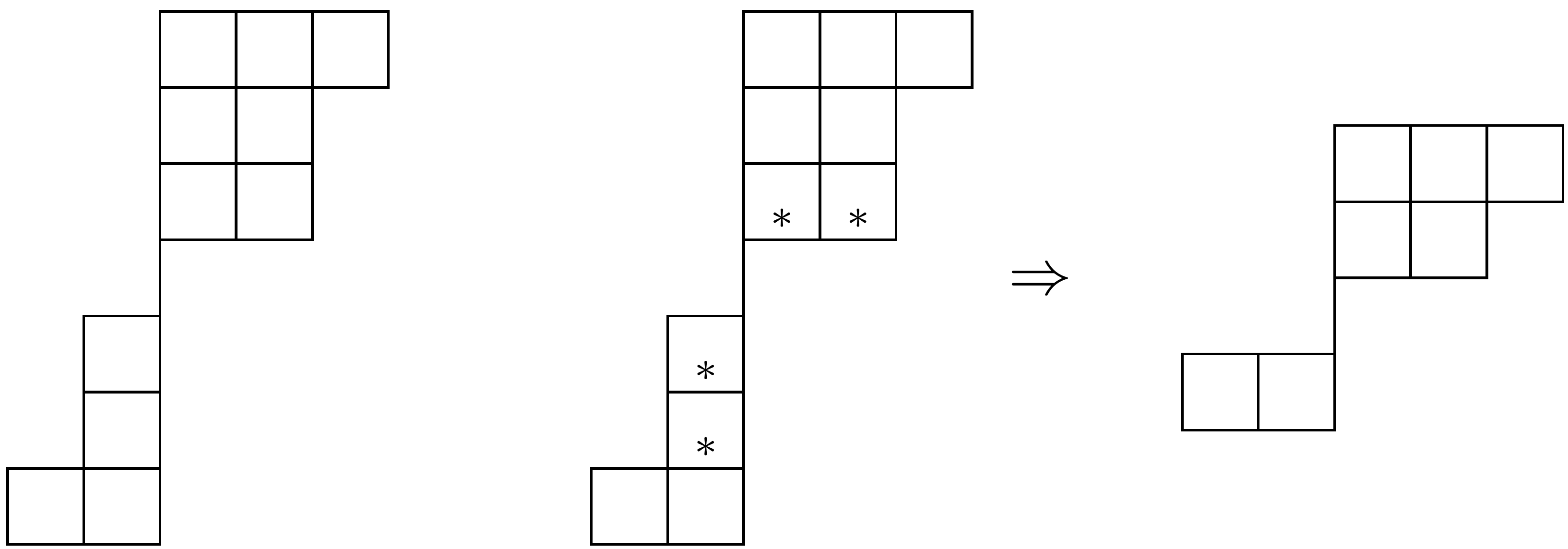
Summary of procedure

- Let λ and μ have lengths $\ell(\lambda) = p$ and $\ell(\mu) = q$.
- The conjugates λ' and μ' determine diagonal entries.
- Modify each term in two neighbouring columns using $\{\overline{1}^k\} = \overline{\varepsilon}\{1^{n-k}\}$ and $\{1^k\} = \varepsilon\{\overline{1}^{n-k}\}$ with $\overline{\varepsilon}\,\varepsilon = 1$.
- Interchange the two columns, so that the original diagonal entries $\{1^p\}$ and $\{\overline{1}^q\}$ map to $\{1^{p-h}\}$ and $\{\overline{1}^{q-h}\}$ with $h = p + q - n - 1$.
- Either $\{\lambda; \overline{\mu}\} = 0$ if any two column are identical.
- Or $\{\lambda; \overline{\mu}\} = \pm\{\sigma; \overline{\tau}\}$ with the sign determined by the required permutation of columns.

Summary of $GL(n)$ modification example

- For $GL(7)$: $\{322;\overline{112}\}$ standard character for all $n \geq 6$
- For $GL(4)$: $\{322;\overline{112}\} = \{32;\overline{2}\}$
- For $GL(n)$: $\{322;\overline{112}\} = 0$ for $n = 5, 3, 2, 1$.

Note: The corresponding modifications to Young diagrams take the form:



Full $GL(n)$ modification rule

Standard characters

- $GL(n)$: $\{\lambda\}$ with $\lambda'_1 \leq n$.
- $GL(n)$: $\{\lambda;\overline{\mu}\}$ with $\lambda'_1 + \mu'_1 \leq n$.

Modification rules for non-standard characters

- $GL(n)$: $\{\lambda\} = 0$ if $\lambda'_1 > n$.
- $GL(n)$: $\{\lambda;\overline{\mu}\} = (-1)^{c+d+1}\{\lambda - h;\overline{\mu - h}\}$ if $h \geq 0$
  - with $h = \lambda'_1 + \mu'_1 - n - 1 \geq 0$,
  - $c = \#$ columns covered by strip removal in $F^{\lambda-h}$,
  - $d = \#$ columns covered by strip removal in $F^{\mu-h}$.
  - $\{\lambda - h;\overline{\mu - h}\} = 0$ if either $F^{\lambda-h}$ or $F^{\mu-h}$ is not regular.

$GL(n)$ **tensor product** $\{2;\overline{1}\} \cdot \{1^2;\overline{1}\}$

$$n \geq 5 \quad \{31;\overline{2}\} + \{31;\overline{1^2}\} + \{21^2;\overline{2}\} + \{21^2;\overline{1^2}\}$$
$$+\{3;\overline{1}\} + 2\{21;\overline{1}\} + \{1^3;\overline{1}\} + \{2;0\} + \{1^2;0\}$$
$$n = 4 \quad \{31;\overline{2}\} + \{31;\overline{1^2}\} + \{21^2;\overline{2}\}$$
$$+\{3;\overline{1}\} + 2\{21;\overline{1}\} + \{1^3;\overline{1}\} + \{2;0\} + \{1^2;0\}$$
$$n = 3 \quad \{31;\overline{2}\} + \{3;\overline{1}\} + \{21;\overline{1}\} + \{2;0\} + \{1^2;0\}$$

where we have used the modifications:

- For $n = 4$: $\{21^2;\overline{1^2}\} = 0$.
- For $n = 3$: $\{21^2;\overline{1^2}\} = -\{21;\overline{1}\}$
  and $\{31;\overline{1^2}\} = \{21^2;\overline{2}\} = \{1^3;\overline{1}\} = 0$.

Generalisation to other subgroups

Invariant $\eta_{ij\cdots k}$ of rank r and symmetry π

- π is a partition of r with corresponding Young symmetrizer Y^π an idempotent in the algebra of S_r.
- S_r acts naturally by permuting the r indices of $\eta_{ij\cdots k}$.
- $Y^\pi : \ \eta_{ij\cdots k} \mapsto \eta_{ij\cdots k}$.

Subgroup $H_\pi(n)$ of $GL(n)$ leaving η invariant

- $H_\pi(n) = \{A \in GL(n) \,|\, A_i^a\, A_j^b \cdots A_k^c\, \eta_{ab\cdots c} = \eta_{ij\cdots k}\}$

Classical examples

- $\pi = (1)$: $\quad H_1(n) = GL(n-1)$.
- $\pi = (2)$: $\quad H_2(n) = O(n)$.
- $\pi = (1^2)$: $\quad H_{1^2}(n) = Sp(n)$.

The group $H_{1^3}(3)$

Identification of group $H_{1^3}(3)$

- Canonical form of third rank antisymmetric invariant in 3 dimensions: $\eta_{ijk} = \epsilon_{ijk}$.
- For $A \in H_{1^3}(3)$ we have $A: \ \epsilon_{ijk} \mapsto A_i^p \, A_j^q \, A_k^r \, \epsilon_{pqr} \ = \ \epsilon_{ijk}$
- In 3-dimensions this implies and is implied by $\det A = 1$.
- Hence $H_{1^3}(3) = SL(3) = \{A \in GL(3) \mid \det A = 1\}$.

Identification of characters of $H_{1^3}(3)$

- $SL(3)$ is semisimple
- All its representations are fully reducible
- In particular each $V_{SL(3)}^{[\![\lambda]\!]}$ is irreducible with $[\![\lambda]\!] = \mathrm{ch}\, V_{SL(3)}^{[\![\lambda]\!]}$

Branching rule for $GL(3) \supset H_{1^3}(3)$

$$\{\lambda\} \to [\![\lambda/M_{1^3}]\!] = [\![\lambda/(0 + 1^3 + 2^3 + 21^4 + 3^3 + \cdots)]\!]$$

$\{\lambda\}_{\rm dim}$	$[\![\lambda/M_{1^3}]\!]_{\rm dim}$	$\{\lambda\}_{\rm dim}$	$[\![\lambda/M_{1^3}]\!]_{\rm dim}$
$\{0\}_1$	$[\![0]\!]_1$	$\{21^3\}_0$	$[\![21^3]\!]_{-9} + [\![2]\!]_6 + [\![1^2]\!]_3$
$\{1\}_3$	$[\![1]\!]_3$	$\{2^2\}_6$	$[\![2^2]\!]_6$
$\{1^2\}_3$	$[\![1^2]\!]_3$	$\{2^21\}_3$	$[\![2^21]\!]_0 + [\![1^2]\!]_3$
$\{1^3\}_1$	$[\![1^3]\!]_0 + [\![0]\!]_1$	$\{2^21^2\}_0$	$[\![2^21^2]\!]_{-8} + [\![21]\!]_8 + [\![1^3]\!]_0$
$\{1^4\}_0$	$[\![1^4]\!]_{-3} + [\![1]\!]_3$	$\{2^3\}_1$	$[\![2^3]\!]_0 + [\![1^3]\!]_0 + [\![0]\!]_1$
$\{2\}_6$	$[\![2]\!]_6$	$\{2^31\}_0$	$[\![2^31]\!]_0 + [\![21^2]\!]_0 + [\![1^4]\!]_{-3} + [\![1]\!]_3$
$\{21\}_8$	$[\![21]\!]_8$	$\{2^4\}_0$	$[\![2^4]\!]_3 + [\![21^3]\!]_{-9} + [\![2]\!]_6$
$\{21^2\}_3$	$[\![21^2]\!]_0 + [\![1]\!]_3$		

Modification rules and revised branchings

$\ell(\lambda) = 3$	$\ell(\lambda) = 4$
$[\![1^3]\!]_0 = 0$	$[\![1^4]\!]_{-3} = -[\![1]\!]_3$
$[\![21^2]\!]_0 = 0$	$[\![21^3]\!]_{-9} = -[\![2]\!]_6 - [\![1^2]\!]_3$
$[\![2^21]\!]_0 = 0$	$[\![2^21^2]\!]_{-8} = -[\![21]\!]_8$
$[\![2^3]\!]_0 = 0$	$[\![2^31]\!]_0 = 0, \quad [\![2^4]\!]_3 = [\![1^2]\!]_3$

$\{\lambda\}$	$[\![\lambda/M_{1^3}]\!]$	$\{\lambda\}$	$[\![\lambda/M_{1^3}]\!]$	$\{\lambda\}$	$[\![\lambda/M_{1^3}]\!]$
$\{0\}_1$	$[\![0]\!]_1$	$\{2\}_6$	$[\![2]\!]_6$	$\{2^2\}_6$	$[\![2^2]\!]_6$
$\{1\}_3$	$[\![1]\!]_3$	$\{21\}_8$	$[\![21]\!]_8$	$\{2^21\}_3$	$[\![1^2]\!]_3$
$\{1^2\}_3$	$[\![1^2]\!]_3$	$\{21^2\}_3$	$[\![1]\!]_3$	$\{2^3\}_1$	$[\![0]\!]_1$
$\{1^3\}_1$	$[\![0]\!]_1$				

Modification rules and revised branchings

$[\![\lambda_1,\lambda_2,\lambda_3,\lambda_4]\!]$ with $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \lambda_4 \geq 0$
$[\![\lambda_1,\lambda_2,\lambda_3,0]\!] = 0$ if $\lambda_3 \geq 1$
$[\![\lambda_1,\lambda_1,1,1]\!] = -[\![\lambda_1,\lambda_1-1]\!]$
$[\![\lambda_1,\lambda_2,1,1]\!] = -[\![\lambda_1,\lambda_2-1]\!] - [\![\lambda_1-1,\lambda_2]\!] = 0$ if $\lambda_1 > \lambda_2$
$[\![\lambda_1,\lambda_2,2,1]\!] = 0$
$[\![\lambda_1,\lambda_2,2,2]\!] = [\![\lambda_1-1,\lambda_2-1]\!]$
$[\![\lambda_1,\lambda_2,\lambda_3,\lambda_4]\!] = 0$ if $\lambda_3 \geq 3$

$\{\lambda\}$	$[\![\lambda/M_{1^3}]\!]$	
$\{\lambda_1,\lambda_2\}$	$[\![\lambda_1,\lambda_2]\!]$	for $\lambda_1 \geq \lambda_2 \geq 0$
$\{\lambda_1,\lambda_2,\lambda_3\}$	$[\![\lambda_1-\lambda_3,\lambda_2-\lambda_3]\!]$	for $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq 0$

Tensor products for $H_{1^3}(3)$

General rule: $[\![\lambda]\!] \cdot [\![\mu]\!] = \sum_{\sigma,\tau} [\![(\lambda/(\sigma\cdot\tau(1^2))) \cdot (\mu/(\sigma(1^2)\cdot\tau))]\!]$

Example: Evaluation of $[\![2^2]\!] \cdot [\![21]\!]$ in $H_{1^3}(3)$

$$\begin{array}{ll} H_{1^3}(n) & [\![43]\!] + [\![421]\!] + [\![3^21]\!] + [\![32^2]\!] + [\![321^2]\!] + [\![2^31]\!] \\ & \quad + 2[\![31]\!] + 2[\![2^2]\!] + 3[\![21^2]\!] + [\![1^4]\!] + 2[\![1]\!] \\ H_{1^3}(3) & [\![43]\!] + [\![31]\!] + [\![2^2]\!] + [\![1]\!] \end{array}$$

Check: Direct evaluation of $\{2^2\} \cdot \{21\}$ in $SL(3)$

$$\begin{array}{ll} SL(n) & \{43\} + \{421\} + \{3^21\} + \{32^2\} + \{321^2\} + \{2^31\} \\ SL(3) & \{43\} + \{31\} + \{2^2\} + \{1\} \end{array}$$

The group $H_{13}(4)$

- Canonical form of third rank antisymmetric invariant in 4 dimensions: $\eta_{ijk} = \begin{cases} \epsilon_{ijk} & \text{if } i,j,k \in \{1,2,3\} \\ 0 & \text{otherwise} \end{cases}$
- For $A \in H_{13}(4)$ we have $A: \eta_{ijk} \mapsto A_i^p A_j^q A_k^r \epsilon_{pqr} = \eta_{ijk}$
- $\Rightarrow \quad A = \begin{bmatrix} B & D \\ 0 & C \end{bmatrix}$ where B is 3×3 and C is 1×1, with $\det B = 1$, D arbitrary and $\det C \neq 0$.
- Hence $H_{13}(4) = \begin{bmatrix} SL(3) & * \\ 0 & GL(1) \end{bmatrix}$

The group $H_{1^3}(4)$

- $H_{1^3}(4)$ is neither semisimple nor reductive
- In general its representations are not fully reducible

$\{\lambda\}_{\mathrm{dim}}$	$[\![\lambda/M_{1^3}]\!]_{\mathrm{dim}}$	$\{\lambda\}_{\mathrm{dim}}$	$[\![\lambda/M_{1^3}]\!]_{\mathrm{dim}}$
$\{0\}_1$	$[\![0]\!]_1$	$\{21^3\}_4$	$[\![21^3]\!]_{-12} + [\![2]\!]_{10} + [\![1^2]\!]_6$
$\{1\}_4$	$[\![1]\!]_4$	$\{2^2\}_{20}$	$[\![2^2]\!]_{20}$
$\{1^2\}_6$	$[\![1^2]\!]_6$	$\{2^21\}_{20}$	$[\![2^21]\!]_{14} + [\![1^2]\!]_6$
$\{1^3\}_4$	$[\![1^3]\!]_3 + [\![0]\!]_1$	$\{2^21^2\}_6$	$[\![2^21^2]\!]_{-17} + [\![21]\!]_{20} + [\![1^3]\!]_3$
$\{1^4\}_1$	$[\![1^4]\!]_{-3} + [\![1]\!]_4$	$\{2^3\}_{10}$	$[\![2^3]\!]_6 + [\![1^3]\!]_3 + [\![0]\!]_1$
$\{2\}_{10}$	$[\![2]\!]_{10}$	$\{2^31\}_4$	$[\![2^31]\!]_{-8} + [\![21^2]\!]_{1^2} + [\![1^4]\!]_{-3} + [\![1]\!]_4$
$\{21\}_{20}$	$[\![21]\!]_{20}$	$\{2^4\}_1$	$[\![2^4]\!]_3 + [\![21^3]\!]_{-12} + [\![2]\!]_{10}$
$\{21^2\}_{15}$	$[\![21^2]\!]_{1^2} + [\![1]\!]_4$		

Modification rules for $H_{1^3}(4)$

In $GL(4)$:

- $\{\lambda_1, \lambda_2, \lambda_3, \lambda_4\} = \epsilon^{\lambda_4}\{\lambda_1 - \lambda_4, \lambda_2 - \lambda_4, \lambda_3 - \lambda_4, 0\}$
- $\epsilon = \{1^4\}$ is the character of the determinant rep of $GL(4)$.
- Ex. $\{21^3\} = \epsilon\{1\}$

Under restriction from $GL(4)$ to $H_{1^3}(4)$:

- $\{1\}_4 \mapsto [\![1]\!]_4$
- $\{1^4\}_1 = \epsilon\{0\}_1 \mapsto \epsilon[\![0]\!]_1$
- $\{21^3\}_4 \mapsto [\![21^3]\!]_{-12} + [\![2]\!]_{10} + [\![1^2]\!]_6$

Hence in $H_{1^3}(4)$ we have the modification rule:

- $[\![21^3]\!]_{-12} = \epsilon[\![1]\!]_4 - [\![2]\!]_{10} - [\![1^2]\!]_6$

Tensor products for $H_{1^3}(4)$

General rule for $H_{1^3}(n)$:

$$[\![\lambda]\!] \cdot [\![\mu]\!] = \sum_{\sigma,\tau} [\![(\lambda/(\sigma \cdot \tau(1^2))) \cdot (\mu/(\sigma(1^2) \cdot \tau))]\!]$$

Example of product evaluated in $H_{1^3}(n)$:

$$[\![2]\!] \cdot [\![1^3]\!] = [\![31^2]\!] + [\![21^3]\!] + [\![2]\!] + [\![1^2]\!]$$

Example of product evaluated in $H_{1^3}(4)$:

$$\begin{aligned} [\![2]\!]_{10} \cdot [\![1^3]\!]_3 &= [\![31^2]\!]_{26} + [\![21^3]\!]_{-12} + [\![2]\!]_{10} + [\![1^2]\!]_6 \\ &= [\![31^2]\!]_{26} + (\epsilon\, [\![1]\!]_4 - [\![2^2]\!]_{10} - [\![1]\!]_6) + [\![2^2]\!]_{10} + [\![1]\!]_6 \\ &= [\![31^2]\!]_{26} + \epsilon\, [\![1]\!]_4 \end{aligned}$$

Note: In general $[\![\lambda]\!]$ is the character of an indecomposable but reducible representation.

Variety of groups $H_{1^3}(n)$

Canonical forms for η_{ijk} of symmetry $\pi = 1^3$

- Let $p, q, r, s, t, \dots$ be linearly independent vectors.
- Let $[pqr]$ be the tri-vector with components $p_{[i}q_j r_{k]}$ where $[\cdots]$ indicates antisymmetrisation of indices.
- Canonical forms for η (Gurevich)

n	η	n	η
3	$[pqr]$	6	$[pqr]$
4	$[pqr]$		$[pqr]+[pst]$
5	$[pqr]$		$[pqr]+[stu]$
	$[pqr]+[pst]$		$[pqr]+[pst]+[qsu]$

Variety of groups $H_{13}(n)$

Candidate groups: $\begin{bmatrix} A & * \\ 0 & B \end{bmatrix}$ with $*$ arbitrary

n	η	$A * B$
3	$[pqr]$	$SL(3)$
4	$[pqr]$	$SL(3) * GL(1)$
5	$[pqr]$	$SL(3) * GL(2)$
	$[pqr]+[pst]$	??
6	$[pqr]$	$SL(3) * GL(3)$
	$[pqr]+[pst]$	?? $* GL(1)$
	$[pqr]+[stu]$	$SL(3) * SL(3)$
	$[pqr]+[pst]+[qsu]$	??

The case $\pi = 3$

Implications of invariance of η_{ijk}

- Third rank symmetric invariant in 3 dimensions:

$$\eta_{ijk} = \delta_{ijk} = \begin{cases} 1 & \text{if } i = j = k \\ 0 & \text{otherwise} \end{cases}$$

- For $A \in H_3(n)$ we have $A : \delta_{ijk} \mapsto A_i^p\, A_j^q\, A_k^r\, \delta_{pqr} = \delta_{ijk}$
  - $\sum_{p=1}^{n} (A_i^p)^3 = 1$
  - $\sum_{p=1}^{n} (A_i^p)^2 A_j^p = 0$ for $i \neq j$
  - $\sum_{p=1}^{n} A_i^p\, A_j^p\, A_k^p = 0$ for $i \neq j \neq k \neq i$.
- For $n \leq 3$ we find $A_i^p \in \mathbb{Z}_3 = \{1, \omega, \omega^2\}$ with $\omega = e^{i2\pi/3}$ for all i, p.

The groups $H_3(n)$ and their characters

n	$H_3(n)$
2	$\mathbb{Z}_3 \wr S_2$
3	$\mathbb{Z}_3 \wr S_3$
4	$?? \supseteq \mathbb{Z}_3 \wr S_4$
n	$?? \supseteq \mathbb{Z}_3 \wr S_n$

Branching rule for $GL(n) \supset H_3(n)$

- $\{\lambda\} \to [\![\lambda/M_3]\!] = [\![\lambda/(0+3+6+42+\cdots)]\!]$

Tensor product rule for $H_3(n)$

- $[\![\lambda]\!] \cdot [\![\mu]\!] = \sum_{\sigma,\tau} [\![(\lambda/(\sigma \cdot \tau(2))) \cdot (\mu/(\sigma(2) \cdot \tau))]\!]$
  $= [\![(\lambda \cdot \mu]\!] + [\![(\lambda/1) \cdot (\mu/2)]\!] + [\![(\lambda/2) \cdot (\mu/1]\!] + \cdots$

Examples in the case $H_3(3)$

Typical restrictions from $GL(3)$ to $H_3(3)$

$\{\lambda\}_{\text{dim}}$	$[\![\lambda/M_3]\!]_{\text{dim}}$
$\{5\}_{21}$	$[\![5]\!]_{15} + [\![2]\!]_6$
$\{41\}_{24}$	$[\![41]\!]_{15} + [\![2]\!]_6 + [\![1^2]\!]_3$
$\{32\}_{15}$	$[\![32]\!]_9 + [\![2]\!]_6$

Typical tensor product decomposition for $H_3(3)$

$$[\![3]\!]_9 \cdot [\![2]\!]_6 = [\![5]\!]_{15} + [\![41]\!]_{15} + [\![32]\!]_9 + 2[\![2]\!]_6 + [\![1^2]\!]_3$$

Problems:

- $H_3(3) = \mathbb{Z}_3 \wr S_3$ is finite
- All its representations are fully reducible
- The dimensions of the irreducible reps are all 1 or 2

More than one invariant

Subgroup $H_{\pi,\rho}(n) \supset GL(n)$ leaving invariant both $\eta_{ij\cdots}$ and $\zeta_{ij\cdots}$ of symmetry π and ρ.

- Example: $\pi = 2$ and $\rho = 1^n$
- $\eta = g_{ij}$ symmetric, rank 2 and
  $\zeta = \epsilon_{ij\cdots k}$ antisymmetric, rank n
- $H_{2,1^n}(n) = SO(n)$

Characters

- Branching rule $\{\lambda\} \to [\lambda/M_2]$
- Tensor product $[\lambda] \cdot [\mu] = \sum_\sigma [(\lambda/\sigma) \cdot (\mu/\sigma)]$
- Additional modification rule $[\lambda] = [\lambda]_+ + [\lambda]_-$
  if $n = 2k$ and $\ell(\lambda) = k$

Subgroup $H_{2,3}(n)$ of $GL(n)$

Invariants: $\eta_{ij} = \delta_{ij}$, $\zeta_{ijk} = \delta_{ijk}$ of symmetry $\pi = 2$, $\rho = 3$

- Further invariants
  $\delta_k = \sum_{i,j} \delta_{ij}\, \delta_{ijk}$ and $\delta_{ijkl} = \sum_m \delta_{ijm}\, \delta_{mkl}$, etc.
- $H_{2,3}(n) = S_n$ (Littlewood)

Characters of S_n in reduced notation

- Branching rule for $GL(n) \supset S_n$
  $\{\lambda\} \to \sum_{\sigma,\tau,\dots} \langle (\lambda/(M \cdot M_2 \cdot \sigma(2) \cdot M_3 \cdot \tau(3) \cdots) \cdot (\sigma \cdot \tau \cdots) \rangle$
- Tensor product rule for S_n
  $\langle\lambda\rangle \cdot \langle\mu\rangle = \sum_{\nu,\sigma,\tau,} \langle (\lambda/(\nu \cdot \sigma)) \cdot (\mu/(\nu \cdot \tau)) \cdot (\sigma \circ \tau) \rangle$
- Modification rule for S_n:
  $\langle\lambda\rangle = (-1)^c \langle (\lambda' - h)' \rangle$ with $h = |\lambda| + \lambda_1 - n - 1$

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