Brian Wybourne and Schur functions

Ronald C King

School of Mathematics, University of Southampton

Southampton, SO17 1BJ, England

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Dedicated to Brian Wybourne - friend, colleague, collaborator

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Motivation for use of Schur functions

Theory of complex spectra

- Role of continuous groups - Racah, Judd, ...
- Classification of n-particle configurations
- Classification of N-particle operators
- Determination of selection rules

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Group theoretic framework

- Irreducible representations of U(n), O(n), SO(n), Sp(2n)
- Branching rules for restriction from a group to a subgroup
- Decomposition of tensor (Kronecker) products
- Symmetrised tensor products

Early papers involving Schur functions

P R Smith and B G Wybourne J Math Phys 1967, 1968

- *Selection rules and the decomposition of the Kronecker square of irreducible representations*.
- *Plethysm and the theory of complex spectra*.

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P H Butler and B G Wybourne J de Physique 1969

- *The configuration* $(d+s)^N$ *and the group* R_6.
- *Reduction of the Kronecker products for rotational groups*.
- *Applications of S-functional analysis to continuous groups*.

Schur functions

Definition Theory of symmetric functions

Partition $\lambda = (\lambda_1, \ldots, \lambda_n)$. Indeterminates $x = (x_1, \ldots, x_n)$.

$$\{\lambda\} = s_\lambda(x) = \frac{\left|x_i^{n+\lambda_j-j}\right|}{\left|x_i^{n-j}\right|}.$$

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Example Theory of angular momentum

$$\chi_\phi^j = e^{ij\phi} + e^{i(j-1)\phi} + \cdots + e^{-ij\phi} = \frac{e^{i(j+\frac{1}{2})\phi} - e^{-i(j+\frac{1}{2})\phi}}{e^{i\frac{1}{2}\phi} - e^{-i\frac{1}{2}\phi}}$$

$$= \frac{x_1^{2j+1} - x_2^{2j+1}}{x_1 - x_2} = \frac{\begin{vmatrix} x_1^{2j+1} & 1 \\ x_2^{2j+1} & 1 \end{vmatrix}}{\begin{vmatrix} x_1 & 1 \\ x_2 & 1 \end{vmatrix}} = s_{(2j,0)}(x_1, x_2) = \{2j\},$$

with $x_1 = e^{i\phi/2}$, $x_2 = e^{-i\phi/2}$, $x_1 x_2 = 1$.

Schur functions as characters

Weyl's character formula - Schur function

- Lie group $G = U(n)$. Irreducible representation V^λ. Highest weight λ.
- Weyl group $W = S_n$. Positive roots Δ^+. Weyl vector $\rho = \sum_{\alpha \in \Delta^+} \alpha = (n-1, \ldots, 1, 0)$.

$$\mathrm{ch}\, V^\lambda = \frac{\sum_{w \in W} (-1)^{\ell(w)}\ e^{w(\lambda+\rho)}}{\sum_{w \in W} (-1)^{\ell(w)}\ e^{w(\rho)}} = s_\lambda(x) = \frac{\left| x_i^{n+\lambda_j-j} \right|}{\left| x_i^{n-j} \right|}.$$

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Weyl's denominator formula - Vandermonde determinant

$$\sum_{w\in W} (-1)^{\ell(w)}\ e^{w(\rho)} = \prod_{\alpha\in\Delta^+} (e^{\alpha/2} - e^{-\alpha/2})$$

$$= \left| x_i^{n-j} \right| = \prod_{1\le i<j\le n} (x_i - x_j).$$

Schur function operations

- Outer product - Littlewood-Richardson coefficients
  $s_\lambda(x)\ s_\mu(x) = \sum_\nu c^\nu_{\lambda\mu}\ s_\nu(x)$.
- Quotient - Littlewood-Richardson coefficients
  $s_{\nu/\mu}(x) = \sum_\lambda c^\nu_{\lambda\mu}\ s_\lambda(x)$.
- Monomial expansion
  $s_\lambda(x) = \sum_k y_k$ with $y_k = x^\beta = x_1^{\beta_1}\cdots x_n^{\beta_n}$ for some β.
- Plethysm
  $s_{\lambda\otimes\mu}(x) = s_\mu(s_\lambda(x)) = s_\mu(y) = \sum_\nu p^\nu_{\lambda\mu}\ s_\nu(x)$.
- Symmetrised powers - plethysms
  $s_\lambda(x)^n = \sum_\mu f^\mu\ s_{\lambda\otimes\mu}(x)$ where $f^\lambda = \dim V^\lambda_{S_n}$.
- Resolution of Kronecker square
  $s_\lambda(x)^2 = s_{\lambda\otimes 2}(x) + s_{\lambda\otimes 1^2}(x) = s_2(y) + s_{1^2}(y)$.

Some characters of classical groups

Unitary groups $U(n), SU(n)$.

- $U(n)$ tensor irreps: $\mathrm{ch}\, V^\lambda = s_\lambda(x) = \{\lambda\}$.
- $SU(n)$ characters: $s_\lambda(x) = \{\lambda\}$ with $x_1 x_2 \cdots x_n = 1$.

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Orthogonal groups $O(n), SO(n)$.

- Tensor irreps: $[\lambda]$. Spinor irreps: $[\Delta; \lambda]$.
- Spin irreps: $SO(2n+1):\ \Delta$. $\quad SO(2n):\ \Delta_\pm$.
- Spin character: $\Delta = \Delta_+ + \Delta_- = \prod_{i=1}^n (x_i^{\frac{1}{2}} + x_i^{-\frac{1}{2}})$.
- Difference character: $\Delta'' = \Delta_+ - \Delta_- = \prod_{i=1}^n (x_i^{\frac{1}{2}} - x_i^{-\frac{1}{2}})$.

More characters of classical groups

Symplectic groups $Sp(2n)$.

- Tensor irreps: $\langle\lambda\rangle$. Harmonic irreps: $\langle\frac{k}{2}(\lambda)\rangle$.
- Metaplectic irreps: $\tilde{\Delta}_+ = \langle\frac{1}{2}(0)\rangle$, $\tilde{\Delta}_- = \langle\frac{1}{2}(1)\rangle$.
- $\tilde{\Delta} = \tilde{\Delta}_+ + \tilde{\Delta}_- = \prod_{i=1}^{n}(x_i^{-\frac{1}{2}} - x_i^{\frac{1}{2}})^{-1}$.
- $\tilde{\Delta}'' = \tilde{\Delta}_+ - \tilde{\Delta}_- = \prod_{i=1}^{n}(x_i^{-\frac{1}{2}} + x_i^{\frac{1}{2}})^{-1}$.

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Note Let $\varepsilon = (x_1 x_2 \ldots x_n)$, then for $SO(2n)$ and $Sp(2n)$:

- $\Delta = \varepsilon^{\frac{1}{2}}\,(\{0\} + \{1\} + \{1^2\} + \cdots + \{1^n\})$.
- $\Delta'' = \varepsilon^{\frac{1}{2}}\,(\{0\} - \{1\} + \{1^2\} + \cdots + (-1)^n\{1^n\})$.
- $\tilde{\Delta} = \varepsilon^{\frac{1}{2}}\,(\{0\} + \{1\} + \{2\} + \cdots)$.
- $\tilde{\Delta}'' = \varepsilon^{\frac{1}{2}}\,(\{0\} - \{1\} + \{2\} + \cdots)$

Classification of atomic states

f-shell Governed by the restriction $U(7) \to SO(3)$

- $f^1, S = \frac{1}{2}$: $\{1\} \to F$
- $f^2, S = 0$: $\{2\} \to SDGI$
- $f^2, S = 1$: $\{1^2\} \to PFH$
- $f^3, S = \frac{1}{2}$: $\{21\} \to PD^2F^2G^2H^2IKL$
- $f^3, S = \frac{3}{2}$: $\{1^3\} \to SDFGI$

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Corresponds to evaluating plethysms

- $F = [3] = x^3{+}x^2{+}x{+}1{+}\overline{x}{+}\overline{x}^2{+}\overline{x}^3$ with $x = e^{i\phi}, \overline{x} = e^{-i\phi}$.
- $\{1^3\} = s_{1^3}(y) = \sum_{1\leq i<j<k\leq 7} y_i y_j y_k$ with $y_k = x^{4-k}$
  $= x^6{+}x^5{+}2x^4{+}3x^3{+}4x^2{+}4x{+}5{+}4\overline{x}{+}4\overline{x}^2{+}3\overline{x}^3{+}2\overline{x}^4{+}\overline{x}^5{+}\overline{x}^6$
  $= [6] + [4] + [3] + [2] + [0] = I + G + F + D + S$.

Branching rules

Restriction from G to H

- Characters of G: $[\lambda]$. Defining irrep character: $[1]$.
- Characters of H: (μ).
- Embedding defined by branching:
  $[1] \to (\alpha) + (\beta) + \cdots + (\omega)$
- Branching of arbitrary representation: $[\lambda] \to \sum_{\mu} b_{\lambda\mu} \, (\mu)$.

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Theorem [Wybourne and Butler]

$$[\lambda] \to ((\alpha) + (\beta) + \cdots + (\omega)) \otimes [\lambda].$$

The plethysm is to be evaluated by expressing both $(\alpha) + (\beta) + \cdots + (\omega)$ and $[\lambda]$ in terms of Schur functions.

Branching rules application

Group-subgroup chain: $SU(7) \supset SO(7) \supset G_2 \supset SO(3)$

- Characters of $SU(7)$: $\{\kappa\}$.
- Characters of $SO(7)$: $[\lambda]$.
- Characters of G_2: (μ).
- Characters of $SO(3)$: $[\nu]$.

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Specification of embeddings and relevant branchings

- $SU(7) \downarrow SO(7)$: $\{1\} \downarrow [1]$, $\{1^3\} \downarrow [1^3]$.
- $SO(7) \downarrow G_2$: $[1] \downarrow (1)$, $[1^3] \downarrow (2) + (1) + (0)$.
- $G_2 \downarrow SO(3)$: $(1) \downarrow [3]$, $(2) \downarrow [3] \otimes (2)$.

Branching rule for $G_2 \downarrow SO(3)$

Express plethysm in terms of Schur functions

- $G_2 \uparrow SU(7)$: $(2) \uparrow \{2\} - \{0\}$
- $SO(3) \uparrow SU(3)$: $[3] \uparrow \{3\} - \{1\}$.
- Required plethysm $[3] \otimes (2) = (\{3\} - \{1\}) \otimes (\{2\} - \{0\})$
  $= \{6\} + \{42\} - \{4\} - \{31\} + \{1^2\} - \{0\}$.
- $SU(3) \downarrow SO(3)$:
  $\{6\} \downarrow [6] + [4] + [2]$, $\{42\} \downarrow [4] + [3] + 2[2] + [0]$
  $\{4\} \downarrow [4] + [2] + [0]$, $\{31\} \downarrow [3] + [2] + [1]$,
  $\{1^2\} \downarrow [1]$, $\{0\} \downarrow [0]$.
- Hence $G_2 \downarrow SO(3)$: $(2) \downarrow [6] + [4] + [2] = I + G + D$.

Development of SCHUR

Purpose and scope

- Tensor products, branching rules, symmetrised powers
- Covers both classical and exceptional groups
- Deals with both finite and infinite-dimensional irreps
- Properties of symmetric group and symmetric functions

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Approach

- Express all characters in terms of Schur functions
- Use Schur function products, quotients, plethysms
- For classical groups work in arbitrary rank and modify
- Symmetric group use inner products, inner plethysms
- Exploit infinite series of Schur functions

Development of SCHUR

Implementation

- DUDLEY (Littlewood) by Butler, 1969 FORTRAN
- SCHUR by Black, 1982 PASCAL
- SCHUR 4 Black, Hirst, Wilson, Wybourne, 1986 PASCAL
- SCHUR 5 1990 TURBOPASCAL, SUNPASCAL, 1999 C
- Mathematical input: Butler, Black, King, Jarvis, Farmer, Cummins, Luan Dehuai, Yang Mei, Toumazet, ...

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Application

- Physics - atomic, nuclear, particle, GR,
- Mathematics - explore, conjecture, test

Conjectures on spin irreps of $SO(2n)$

Symmetrised cubes Wybourne

All can be found if $\Delta \otimes \{21\}$ and $\Delta'' \otimes \{21\}$ known.

Proof by induction: $\Delta \otimes \{21\} = \Delta \ \sum_x ([1^{n-1-3x}] + [1^{n-2-3x}])$.

Data from use of SCHUR

$SO(4)$: $\Delta'' \otimes \{21\} = -\Delta'' ([1] - [0])$

$SO(6)$: $\Delta'' \otimes \{21\} = -\Delta'' ([1^2] - [1])$

$SO(8)$: $\Delta'' \otimes \{21\} = -\Delta'' ([1^3] - [1^2] - [0])$

$SO(10)$: $\Delta'' \otimes \{21\} = -\Delta'' ([1^4] - [1^3] - [1] + [0])$

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Conjecture for $SO(2n)$ Wybourne

$$\Delta'' \otimes \{21\} = -\Delta'' ([1^{n-1-6x}] - [1^{n-2-6x}] - [1^{n-4-6x}] + [1^{n-5-6x}]).$$

Related binomial coefficient conjecture Proved by Breach

$$3^{n-1} = \sum_x \left\{ \binom{2n}{n-1-6x} - \binom{2n}{n-2-6x} - \binom{2n}{n-4-6x} + \binom{2n}{n-5-6x} \right\}.$$

Symmetrised squares for $SO(2n)$ and $Sp(2n)$

Theorem Symmetrised squares for $SO(2n)$ - Littlewood

$$\Delta_+ \otimes \{2\} = [1^n]_+ + \sum_x [1^{n-4-4x}]$$

$$\Delta_+ \otimes \{1^2\} = \sum_x [1^{n-2-4x}]$$

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$\Delta_- \otimes \{1^2\} = \sum_x [1^{n-2-4x}]$

Conjecture Symmetrised squares for $Sp(2n)$ - Grudzinski and Wybourne

$\tilde{\Delta}_+ \otimes \{2\} = \langle 1; (0)\rangle + \sum_x \langle 1; (4+4x)\rangle$

$\tilde{\Delta}_+ \otimes \{1^2\} = \sum_x \langle 1; (2+4x)\rangle$

$\tilde{\Delta}_- \otimes \{2\} = \sum_x \langle 1; (2+4x)\rangle]$

$\tilde{\Delta}_- \otimes \{1^2\} = \langle 1; (1^2)\rangle + \sum_x \langle (1; (4+4x)\rangle$

Proved by Thibon, Toumazet and Wybourne

Further conjectures

Metaplectic plethysm identities

$\langle \frac{1}{2}(0)\rangle \otimes \{1^2\} = \langle \frac{1}{2}(1)\rangle \otimes \{2\}$ True (TTW)

$\langle \frac{1}{2}(0)\rangle \otimes \{21^2\} = \langle \frac{1}{2}(1)\rangle \otimes \{31\}$ True

$\langle \frac{1}{2}(0)\rangle \otimes \{\mu\} = \langle \frac{1}{2}(1)\rangle \otimes \{\mu'\}$ False.

$\langle \frac{1}{2}(0)\rangle \otimes \{\mu\} = \langle \frac{1}{2}(1)\rangle \otimes \{\mu'\})^*$ True (KW)

$\langle \frac{k}{2}(\lambda)\rangle^* \otimes \{\mu\} = (\langle \frac{k}{2}(\lambda)\rangle \otimes \{\mu^{\prime k}\})^*$ True (KW)

Note: $'$ and * signify the conjugate and associate.

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Note: $'$ and * signify the conjugate and associate.

$SO(2n)$and $Sp(2n)$ analogies - King and Wybourne

$\Delta_+ \sim \tilde{\Delta}_+ = \langle\frac{1}{2}(0)\rangle$, $\quad \Delta_- \sim \tilde{\Delta}_- = \langle\frac{1}{2}(1)\rangle = \langle\frac{1}{2}(0)\rangle^*$,

$\Delta_+ \otimes \{2\} \sim \tilde{\Delta}_+ \otimes \{2\}$, $\quad \Delta_- \otimes \{2\} \sim \tilde{\Delta}_- \otimes \{1^2\}$,

$\Delta_+ \otimes \{1^2\} \sim \tilde{\Delta}_+ \otimes \{1^2\}$, $\quad \Delta_- \otimes \{1^2\} \sim \tilde{\Delta}_- \otimes \{2\}$,

$[1^n]_+ \sim \langle(1(0)\rangle$, $\quad [1^n]_- \sim \langle(1(1^2)\rangle = \langle(1(0)\rangle^*$,

$[1^{n-t}] \sim \langle 1;(t)\rangle$ for $t < n$, $\quad [m^n/\lambda'] \sim \langle m(\lambda)\rangle$.

Square of Vandermonde determinant

- Vandermonde determinant
  $V_n(x) = \left|x_i^{n-j}\right| = \prod_{1\le i<j\le n}(x_i - x_j)$.
- Square of Vandermonde determinant
  - $V_n(x)$ is an antisymmetric function of the x_i.
  - $V_n^2(x)$ is a symmetric function of the x_i.
  - $V_n^2(x) = \sum_\lambda c_n^\lambda\, s_\lambda(x)$.

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Examples

- $V_2^2(x) = \{2\} - 3\{1^2\}$
- $V_3^2(x) = \{42\} - 3\{41^2\} - 3\{3^2\} + 6\{321\} - 15\{2^3\}$
- $V_4^2(x) = \{642\} - 3\{641^2\} - 3\{63^2\} - 3\{5^2 2\} + \cdots + 105\{3^4\}$

Problems

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- Determine c_n^λ.
- Determine λ such that $c_n^\lambda > 0$.
- Determine $\sum_\lambda c_n^\lambda$ and $\sum_\lambda \left(c_n^\lambda\right)^2$.

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Observations

- Leading term $\{2\rho\} = \{2n-2, 2n-4, \ldots, 2, 0\}$.
- Final term $(-1)^{[n/2]}(2n-1)!! \, \{(n-1)^n\}$.
- λ is admissible if $\lambda - 2\rho$ is a sum of positive roots.
- $c_n^\lambda = 0$ unless λ is admissible.
- $c_n^\lambda \neq 0$ for all admissible λ for $n \leq 7$.

Further observations and a conjecture

Observations Wybourne

- For $n \geq 8$ there exist admissible λ for which $c_n^\lambda = 0$
- For $n = 8, 9, 10$ the numbers of such λ are $8, 66, 389$
- $\sum_{\lambda} \left(c_n^\lambda\right)^2 = \frac{(3n)!}{(3!)^n \, n!}$ for all n Di Francesco *et al*.

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Conjecture Wybourne

- $C_n = \sum_\lambda c_n^\lambda = (-1)^{n(n-1)/2} \frac{(3n-2)!!!}{2^{[n/2]}}$.
- Verified for all $n \leq 10$,
- $C_2 = -2$, $C_3 = -14$, $C_4 = 70$, $\cdots$, $C_{10} = -532532000$.

q-dependant generalisations

q-discriminant

- $D_n(q;x) = \prod_{1\leq i\neq j\leq n}(x_i - qx_j)$.
- $D_n(q;x) = \sum_\lambda d_n^\lambda(q)\; s_\lambda(x)$.
- $D_n(1;x) = (-1)^{n(n-1)/2}V_n^2(x)$.
- $d_n^\lambda(1) = (-1)^{n(n-1)/2}c_n^\lambda$.

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Observation

- $d_n^\lambda(q) \ne 0$ if and only if λ is admissible.
- $d_8^{13\,11\,985^2 41}(q) = q^{17}(q^2+1)^2(q^2+q+1)^3(q^4+q^2+1)^2(q-1)^4$.
- $\displaystyle\sum_\lambda d_n^\lambda(q)\ d_n^\lambda(q^2) = \frac{[3n]!_q}{([3]!_q)^n\,[n]!_{q^2}}$ for all n.

q-dependant sum

Definition and data

- $D_n(q) = \sum_\lambda d_n^\lambda(q)$
- $D_2(q) = (q^2 + 1)$
- $D_3(q) = (q^6 + q^5 + 4q^4 + 2q^3 + 4q^2 + q + 1)$.
- $D_4(q) = (q^{12} + 2q^{11} + 6q^{10} + \cdots + 1)$.

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Observation

- None of the above factorise over the integers
- $D_n(q) \neq \dfrac{[3n-2]!!!_q}{([2]_q)^{[n/2]}}$.

Missing conjecture

Conjectured conjecture King, Toumazet and Wybourne

- Let $D_{n,w}(q) = \sum_\lambda w_n^\lambda(q)\; d_n^\lambda(q) = \frac{[3n-2]!!!_q}{([2]_q)^{[n/2]}}$.
- Open problem - find $w_n^\lambda(q)$ or modify right hand side.

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Motivation and natural history of problem

- Physics - study of quantum Hall effect
- Mathematics - Laughlin's wavefunction ansatz
- Test admissibility-nonzero coeff hypothesis
- Sums of squares known - look at sum of coeffs
- Explore q-dependent case to explain unexpected zeros
- Generalise all known results to q-dependent case
- Still leaves open problems

Academic life with Brian

Personal involvement

- First aware of Brian's work about 1969.
- Met briefly Southampton 1978, then Christchurch 1979.
- First paper together 1981.
- Next 20 years 17 joint publications - more to appear.

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Brian’s input

- Immensely stimulating.
- Full of challenges.
- Plenty of surprises.
- Great fun - he is greatly missed.