Alternating sign matrices

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September 30, 2005

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1 Alternating Sign Matrices

Alternating sign matrices (ASMs) are square matrices such that:

- all the entries are 0, 1 or -1;
- the sum of the entries in each row and each column is 1;
- the nonzero entries alternate in sign across each row and down each column.

Example:

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}. \tag{1.1}$$

Let $\mathcal{A}(n)$ be the set of all $n \times n$ ASMs.

$$\mathcal{A}(1) = \left\{ \begin{bmatrix} 1 \end{bmatrix} \right\}$$

$$\mathcal{A}(2) = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right\}$$

$$\mathcal{A}(3) = \left\{ \begin{array}{l} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \\ \\ \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 0 \end{bmatrix} \end{array} \right\}$$

Theorem 1.1 *Let $A_n = \#\mathcal{A}(n)$ be the number of $n \times n$ ASMs. Then*

$$A_n = \prod_{k=0}^{n-1} \frac{(3k+1)!}{(n+k)!}. \tag{1.2}$$

- First conjectured by Mills, Robbins and Rumsey, 1983.
- Proved using different methods by Zeilberger, 1996 and Kuperberg, 1996.
- A complete history has been given by Bressoud, 1999.

There are two particular quite different combinatorial objects that are in one-to-one correspondence with ASMs.

- Square ice configurations C;
- Triangular semistandard shifted Young tableaux ST.

Square ice configurations (SICs): Each square ice configuration C is an assignment of arrows to each edge of a square array of vertices in such a way that:

- at each vertex two of the four arrows point towards the vertex;
- arrows at the top and bottom of the array point outwards, while those at the left and right point inwards.

Example:

$$C = \left[\begin{array}{ccccccccccc}
 & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & \\
\rightarrow & \cdot & \rightarrow & \cdot & \leftarrow & \cdot & \leftarrow & \cdot & \leftarrow & \cdot & \leftarrow \\
 & \uparrow & & \downarrow & & \uparrow & & \uparrow & & \uparrow & \\
\rightarrow & \cdot & \leftarrow & \cdot & \rightarrow & \cdot & \rightarrow & \cdot & \leftarrow & \cdot & \leftarrow \\
 & \downarrow & & \uparrow & & \uparrow & & \downarrow & & \uparrow & \\
\rightarrow & \cdot & \rightarrow & \cdot & \leftarrow & \cdot & \leftarrow & \cdot & \rightarrow & \cdot & \leftarrow \\
 & \downarrow & & \downarrow & & \uparrow & & \uparrow & & \downarrow & \\
\rightarrow & \cdot & \rightarrow & \cdot & \rightarrow & \cdot & \rightarrow & \cdot & \leftarrow & \cdot & \leftarrow \\
 & \downarrow & & \downarrow & & \uparrow & & \downarrow & & \downarrow & \\
\rightarrow & \cdot & \rightarrow & \cdot & \rightarrow & \cdot & \leftarrow & \cdot & \leftarrow & \cdot & \leftarrow \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow &
\end{array}\right] \tag{1.3}$$

- In this six vertex model the distinct vertices with two inward pointing arrows are:

$$\begin{array}{ccc} & \uparrow & \\ \rightarrow & \cdot & \leftarrow \\ & \downarrow & \end{array} \quad
\begin{array}{ccc} & \downarrow & \\ \leftarrow & \cdot & \rightarrow \\ & \uparrow & \end{array} \quad
\begin{array}{ccc} & \uparrow & \\ \rightarrow & \cdot & \rightarrow \\ & \uparrow & \end{array} \quad
\begin{array}{ccc} & \downarrow & \\ \leftarrow & \cdot & \leftarrow \\ & \downarrow & \end{array} \quad
\begin{array}{ccc} & \uparrow & \\ \leftarrow & \cdot & \leftarrow \\ & \uparrow & \end{array} \quad
\begin{array}{ccc} & \downarrow & \\ \rightarrow & \cdot & \rightarrow \\ & \downarrow & \end{array}$$

Bijection between square ice configurations and ASMs
This involves the following maps:

$$\begin{array}{ccc} & \uparrow & \\ \rightarrow & \cdot & \leftarrow \\ & \downarrow & \end{array} \Leftrightarrow WE \Leftrightarrow \begin{array}{ccc} & \cdot & \\ \cdot & 1 & \cdot \\ & \cdot & \end{array}$$

$$\begin{array}{ccc} & \downarrow & \\ \leftarrow & \cdot & \rightarrow \\ & \uparrow & \end{array} \Leftrightarrow NS \Leftrightarrow \begin{array}{ccc} & \cdot & \\ \cdot & -1 & \cdot \\ & \cdot & \end{array}$$

$$\begin{array}{ccc} & \uparrow & \\ \rightarrow & \cdot & \rightarrow \\ & \uparrow & \end{array} \Leftrightarrow NW \Leftrightarrow sw \Leftrightarrow \begin{array}{ccc} & \cdot & \\ 1 & 0 & \cdot \\ & 1 & \end{array}$$

$$\begin{array}{ccc} & \downarrow & \\ \leftarrow & \cdot & \leftarrow \\ & \downarrow & \end{array} \Leftrightarrow SW \Leftrightarrow nw \Leftrightarrow \begin{array}{ccc} & 1 & \\ 1 & 0 & \cdot \\ & \cdot & \end{array}$$

$$\begin{array}{ccc} & \uparrow & \\ \leftarrow & \cdot & \leftarrow \\ & \uparrow & \end{array} \Leftrightarrow NE \Leftrightarrow se \Leftrightarrow \begin{array}{ccc} & \cdot & \\ \cdot & 0 & 1 \\ & 1 & \end{array}$$

$$\begin{array}{ccc} & \downarrow & \\ \rightarrow & \cdot & \rightarrow \\ & \downarrow & \end{array} \Leftrightarrow SE \Leftrightarrow ne \Leftrightarrow \begin{array}{ccc} & 1 & \\ \cdot & 0 & 1 \\ & \cdot & \end{array}$$

The compass point directions N, E, W, S give the directions of the inward pointing arrows, while n, e, w, s give the directions in which each entry 0 has a nearest non-vanishing neighbour 1.

Example

$$
\begin{bmatrix}
 & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & \\
\rightarrow & \cdot & \rightarrow & \cdot & \leftarrow & \cdot & \leftarrow & \cdot & \leftarrow & \cdot & \leftarrow \\
 & \uparrow & & \downarrow & & \uparrow & & \uparrow & & \uparrow & \\
\rightarrow & \cdot & \leftarrow & \cdot & \rightarrow & \cdot & \rightarrow & \cdot & \leftarrow & \cdot & \leftarrow \\
 & \downarrow & & \uparrow & & \uparrow & & \downarrow & & \uparrow & \\
\rightarrow & \cdot & \rightarrow & \cdot & \leftarrow & \cdot & \leftarrow & \cdot & \rightarrow & \cdot & \leftarrow \\
 & \downarrow & & \downarrow & & \uparrow & & \uparrow & & \downarrow & \\
\rightarrow & \cdot & \rightarrow & \cdot & \rightarrow & \cdot & \rightarrow & \cdot & \leftarrow & \cdot & \leftarrow \\
 & \downarrow & & \downarrow & & \uparrow & & \downarrow & & \downarrow & \\
\rightarrow & \cdot & \rightarrow & \cdot & \rightarrow & \cdot & \leftarrow & \cdot & \leftarrow & \cdot & \leftarrow \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow &
\end{bmatrix}
$$

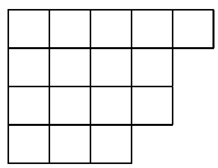
$$
\Leftrightarrow \begin{bmatrix}
NE & WE & NW & NW & NW \\
WE & NS & NE & WE & NW \\
SE & WE & NW & NS & WE \\
SE & SE & NE & WE & SW \\
SE & SE & WE & SW & SW
\end{bmatrix}
\Leftrightarrow \begin{bmatrix}
se & 1 & sw & sw & sw \\
1 & -1 & se & 1 & sw \\
ne & 1 & sw & -1 & 1 \\
ne & ne & se & 1 & nw \\
ne & ne & 1 & nw & nw
\end{bmatrix}
$$

$$
\Leftrightarrow \begin{bmatrix}
0 & 1 & 0 & 0 & 0 \\
1 & -1 & 0 & 1 & 0 \\
0 & 1 & 0 & -1 & 1 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0
\end{bmatrix}.
$$

Young diagram

- Each partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ defines a Young diagram consisting of n rows of boxes of lengths λ_i for $i = 1, 2, \ldots, n$ left adjusted to a vertical line.
- A semistandard Young tableaux T is a filling of the boxes of F^λ with entries weakly increasing across rows and strictly increasing down columns.

For example, for $n = 4$ and $\lambda = (5, 4, 4, 3)$ we have:

$F^\lambda = F^{5443} =$ 

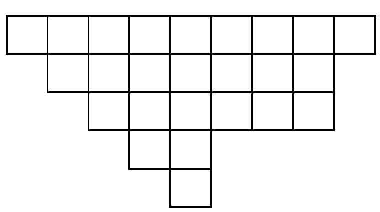
$T =$

1	1	1	2	4
2	2	3	4	
3	4	4	5	
5	5	5		

Shifted Young diagram

- Each partition $\mu = (\mu_1, \mu_2, \ldots, \mu_n)$ into distinct parts defines a shifted Young diagram SF^μ consisting of n rows of boxes of lengths μ_i for $i = 1, 2, \ldots, n$ left adjusted to a diagonal line.
- A semistandard shifted Young tableaux ST is a filling of the boxes of SF^μ with entries weakly increasing across rows and down columns, and strictly increasing along each diagonal.

For example, for $n = 5$ and $\mu = (9, 7, 6, 2, 1)$ we have

$SF^\mu =$ 

$ST =$

1	1	1	2	2	2	4	4	5
	2	3	3	3	4	4	5	
		3	4	4	4	5	5	
			4	5				
				5				

Triangular semistandard shifted Young tableaux and ASMs

Let $\mathcal{ST}(n)$ be the set of all semistandard shifted Young tableaux ST of triangular shape $\rho = (n, n-1, \ldots, 1)$

- with entries from $\{1, 2, \ldots, n\}$
- weakly increasing across rows and down columns
- strongly increasing down diagonals.

For example, for $n = 4$

$$ST = \begin{array}{ccccc} 1 & 1 & 2 & 2 & 3 \\ & 2 & 3 & 4 & 4 \\ & & 3 & 4 & 5 \\ & & & 4 & 5 \\ & & & & 5 \end{array}$$

Bijection between triangular STs and ASMs

- First map ST to a matrix whose (i, j)th entry is 1 or 0 according as the entry i appears or does not appear in the jth diagonal.
- Then map that matrix to A by replacing any 0 immediately to the left of any 1 by -1, and then replacing any 1 immediately to the left of any 1 by 0.

Example

$$\begin{array}{ccccc} 1 & 1 & 2 & 2 & 3 \\ & 2 & 3 & 4 & 4 \\ & & 3 & 4 & 5 \\ & & & 4 & 5 \\ & & & & 5 \end{array} \Leftrightarrow \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 \end{bmatrix}. \Leftrightarrow \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}.$$

Definition Let n be a fixed positive integer, a an arbitrary non-zero complex number and let $x = (x_1, x_2, \ldots, x_n)$ and $(y = (y_1, y_2, \ldots, y_n)$. Then the partition function $Z_n(x, y; a)$ for the square ice-model is given by

$$Z_n(x,y;a) = \sum_C \prod_{i,j=1}^{n} W(i,j;\frac{x_i}{y_j}),$$

where the sum is over all SICs C, and $C(i,j)$ is the vertex in the ith row and jth column of C, and

$$W(i,j;z) = \sigma(a^2) \quad \text{if} \quad C(i,j) = \begin{array}{ccc} & \uparrow & \\ \rightarrow & \cdot & \leftarrow \\ & \downarrow & \end{array} \quad \text{or} \quad \begin{array}{ccc} & \downarrow & \\ \leftarrow & \cdot & \rightarrow \\ & \uparrow & \end{array}$$

$$W(i,j;z) = \sigma(\frac{a}{z}) \quad \text{if} \quad C(i,j) = \begin{array}{ccc} & \uparrow & \\ \rightarrow & \cdot & \rightarrow \\ & \uparrow & \end{array} \quad \text{or} \quad \begin{array}{ccc} & \downarrow & \\ \leftarrow & \cdot & \leftarrow \\ & \downarrow & \end{array}$$

$$W(i,j;z) = \sigma(az) \quad \text{if} \quad C(i,j) = \begin{array}{ccc} & \uparrow & \\ \leftarrow & \cdot & \leftarrow \\ & \uparrow & \end{array} \quad \text{or} \quad \begin{array}{ccc} & \downarrow & \\ \rightarrow & \cdot & \rightarrow \\ & \downarrow & \end{array}$$

with

$$\sigma(t) = t - \frac{1}{t}.$$

Lemma Let neg(A) be the number of entries -1 in the alternating sign matrix A, and let

$$A_n(q) = \sum_{A \in \mathcal{A}_n} q^{\mathrm{neg}(A)}.$$

Then

$$A_n(q) = \frac{1}{\sigma(a)^{n^2-n}\,\sigma(a^2)^n}\; Z_n(1, 1, a)$$

where $1 = (1, 1, \ldots, 1)$ and $q = a^2 + 2 + a^{-2}$.

Proof

Let A be the ASM corresponding to the SIC C.

Let $\#NS(A)$, $\#WE(A)$, $\#NW(A)$, $\#SW(A)$,$\#NE(A)$ and $\#SE(A)$ be the numbers of entries NS, WE, NW, SW,NE and SE, respectively in the compass point matrix corresponding to both C and A. Then

$$\begin{aligned} Z_n(1, 1; a) \;\; = \;\; & \sum_{A \in \mathcal{A}_n} \sigma(a^2)^{\#NS(A)+\#WE(A)} \\ & \qquad \cdot \;\; \sigma(a)^{\#NW(A)+\#SW(A)+\#NE(A)+\#SE(A)}. \end{aligned}$$

From the properties of ASMs we have

$$\#NS(A) = \#(-1)s \in A = \mathrm{neg}(A)\,;$$

$$\#WE(A) = \#(\ 1)s \in A = n + \#(-1)s \in A = n + \mathrm{neg}(A)\,;$$

$$\#NW(A) + \#SW(A) + \#NE(A) + \#SE(A) = \#(\ 0)s \in A$$

$$= n^2 - \#(\ 1)s \in A - \#(-1)s \in A = n^2 - n - 2\,\mathrm{neg}(A)\,.$$

Hence

$$\begin{aligned} Z_n(1,1;a) &= \sum_{A\in\mathcal{A}_n} \sigma(a^2)^{n+2\,\mathrm{neg}(A)}\ \sigma(a)^{n^2-n-2\,\mathrm{neg}(A)} \\ &= \sum_{A\in\mathcal{A}_n} \left(\frac{\sigma(a^2)}{\sigma(a)}\right)^{2\,neg(A)} \sigma(a^2)^n\,\sigma(a)^{n^2-n}. \end{aligned}$$

But

$$\left(\frac{\sigma(a^2)}{\sigma(a)}\right)^2 = \left(\frac{a^2-a^{-2}}{a-a^{-1}}\right)^2 = (a+a^{-1})^2 = a^2+2+a^{-2} = q\,.$$

It follows, as required, that

$$\frac{1}{\sigma(a^2)^n\,\sigma(a)^{n^2-n}}\ Z_n(1,1;a) = \sum_{A\in\mathcal{A}_n} q^{\mathrm{neg}(A)}\,.$$

Corollary Let $\zeta = e^{i\,2\pi/6}$. Then the number of $n \times n$ ASMs is given by

$$A_n = \frac{1}{\sigma(\zeta)^{n^2-n}\,\sigma(\zeta^2)^n}\; Z_n(1,1;\zeta)$$

Proof Note that for $q = a^2 + 2 + a^{-2}$

$$a = \zeta = e^{i\,2\pi/6} \qquad \text{implies} \qquad q = 1\,;$$

$$a = \eta = e^{i\,2\pi/8} \qquad \text{implies} \qquad q = 2\,;$$

$$a = \theta = e^{i\,2\pi/12} \qquad \text{implies} \qquad q = 3\,.$$

Using the first of these, $a = \zeta$, in our previous result

$$\sum_{A \in \mathcal{A}_n} q^{\mathrm{neg}(A)} \;=\; \frac{1}{\sigma(a^2)^n\,\sigma(a)^{n^2-n}}\; Z_n(1,1;a)\,.$$

gives the required formula for A_n.

Conclusion All that we need do to count ASMs is to evaluate $Z_n(1,1;\zeta)$.

Theorem [Izergin and Korepin]

$$Z_n(x, y, ; a) = \sigma(a^2)^n \frac{\prod_{i,j=1}^{n} \sigma\left(a \frac{x_i}{y_j}\right) \sigma\left(a \frac{y_j}{x_i}\right)}{\prod_{1\le i<j\le n} \sigma\left(\frac{x_j}{x_i}\right) \sigma\left(\frac{y_i}{y_j}\right)} \times \det\left(\frac{1}{\sigma\left(a \frac{x_i}{y_j}\right) \sigma\left(a \frac{y_j}{x_i}\right)}\right)_{1\le i,j\le n}$$

This formula was used by Kuperberg to derive the following results:

Theorem [Kuperberg]

$$A_n = A_n(1) = \prod_{k=0}^{n-1} \frac{(3k+1)!}{(n+k)!} = (-3)^{n(n-1)/2} \prod_{i,j=1}^{n} \frac{1+3(j-i)}{n+(j-i)};$$

$$A_n(2) = 2^{n(n-1)/2};$$

$$A_n(3) = \left(\frac{3}{2}\right)^{n(n-1)} \prod_{i,j=1;(j-i)\equiv 1(\text{mod}2)}^{n} \frac{1+3(j-i)}{3(j-i)}.$$

Approach to proof We have the following identities

$$\begin{aligned}
\sigma\left(a\,\frac{x}{y}\right)\;\sigma\left(a\,\frac{y}{x}\right) &= \left(\frac{ax}{y}-\frac{y}{ax}\right)\left(\frac{ay}{x}-\frac{x}{ay}\right) \\
&= a^2-\frac{x^2}{y^2}-\frac{y^2}{x^2}+\frac{1}{a^2} = -\left(\frac{x^4+y^4}{x^2\,y^2}+2-q\right)\,. \\
\sigma\left(\frac{x_j}{x_i}\right) &= \left(\frac{x_j}{x_i}-\frac{x_i}{x_j}\right) = -\frac{1}{x_i\,x_j}\;(x_i^2-x_j^2)\,; \\
\sigma\left(\frac{y_i}{y_j}\right) &= \left(\frac{y_i}{y_j}-\frac{y_j}{y_i}\right) = \;\frac{1}{y_i\,y_j}\,(y_i^2-y_j^2)\,.
\end{aligned}$$

Hence

$$\begin{aligned}
&\frac{1}{\sigma(a^2)^n\,\sigma(a)^{n^2-n}}\;Z_n(x,y;a) \\
&= (-q)^{n(n-1)/2}\;\prod_{i=1}^{n}(x_i\,y_i)^{-n+1}\;\frac{\Pi_{i,j=1}^{n}\;\left(x_i^4+(2-q)x_i^2\,y_j^2+y_j^4\right)}{\Pi_{1\le i<j\le n}\;(x_i^2-x_j^2)\,(y_i^2-y_j^2)} \\
&\qquad\times\qquad \det\left(\frac{1}{x_i^4+(2-q)x_i^2\,y_j^2+y_j^4}\right)_{1\le i,j]\le n}
\end{aligned}$$

where

q	$x^4+(2-q)x^2\,y^2+y^4$
1	$(x^6-y^6)/(x^2-y^2)$
2	x^4+y^4
3	$(x^6+y^6)/(x^2+y^2)$

Theorem [Okada] Let

$x^2 = (x_1^2, x_2^2, \dots, x_n^2)$ and $y^2 = (y_1^2, y_2^2, \dots, y_n^2)$;

$s_\lambda(x^2, y^2)$ be the Schur function specified by a partition λ;

$\delta(n-1, n-1) = (n-1, n-1, n-2, n-2, \dots, 2, 2, 1, 1)$;

$\zeta = e^{i\,2\pi/6}$.

Then

$$\frac{Z_n(x, y; \zeta)}{\sigma(\zeta^2)^n\, \sigma(\zeta)^{n^2-n}} = \frac{1}{3^{n(n-1)/2}} \prod_{i=1}^{n} (x_i\, y_i)^{-n+1}\; s_{\delta(n-1,n-1)}(x^2, y^2)\,.$$

Corollary [Okada]
Let $(1, 1, \dots, 1)$ be a vector of length $2n$. Then

$$A_n = A_n(1) = \frac{1}{3^{n(n-1)/2}}\; s_{\delta(n-1,n-1)}(1, 1, \dots, 1)\,.$$

Challenge
Find a bijection from the semistandard Young tableaux determining $s_{\delta(n-1,n-1)}(x^2, y^2)$ and a pair consisting of an ASM A and triangular array M, all of whose $n(n-1)/2$ entries may take on precisely three values, for example 1, 0 and -1, without any restriction.

The application of Okada's formula

For any partition λ it is known that

$$s_\lambda(1,1,\ldots,1) = \prod_{(i,j)\in F^\lambda} \frac{2n-i-j}{h_{ij}(\lambda)},$$

where $h_{ij}(\lambda) = \lambda_i + \lambda'_i - i - j + 1$ is the hook length of the box in the ith row and jth column of F^λ. Hence

$$A_n = A_n(1) = \frac{1}{3^{n(n-1)/2}} \prod_{(i,j)\in F^{\delta(n-1,n-1)}} \frac{2n-i+j}{h_{ij}(\delta(n-1,n-1))}.$$

Example For $n=5$ we have $\delta(n-1,n-1) = (4,4,3,3,2,2,1,1)$ and we find

$$A_5 = \frac{1}{3^{5\cdot 4/2}} \begin{matrix} 10 & 11 & 12 & 13 \\ 9 & 10 & 11 & 12 \\ 8 & 9 & 10 & \\ 7 & 8 & 9 & \\ 6 & 7 & & \\ 5 & 6 & & \\ 4 & & & \\ 3 & & & \end{matrix} \Bigg/ \begin{matrix} 11 & 8 & 5 & 2 \\ 10 & 7 & 4 & 1 \\ 8 & 5 & 2 & \\ 7 & 4 & 1 & \\ 5 & 2 & & \\ 4 & 1 & & \\ 2 & & & \\ 1 & & & \end{matrix} = 429.$$

Recurrence relation The hook length formula implies

$$A_n = A_{n-1} \frac{\binom{3n-2}{n-1}}{\binom{2n-2}{n-1}}.$$

Semistandard primed shifted Young tableaux

Let $\mathcal{PST}(n)$ be the set of all semistandard primed shifted Young tableaux T of triangular shape $\rho = (n, n-1, \ldots, 1)$

- with entries from the ordered set $\{1', 1, 2', 2, \ldots, n', n\}$
- weakly increasing across rows and down columns
- no two identical unprimed entries in the same column
- no two primed entries in the same row
- there are no primed entries on the main diagonal.

Primed shifted diagrams

Let $\mathcal{PSD}(n)$ be the set of all primed shifted diagrams D of triangular shape $\rho = (n, n-1, \ldots, 1)$

- with entries from the ordered set $\{1', 1, 2', 2, \ldots, n', n\}$
- each unprimed entry k appears in the kth row
- each primed entry k' appears in the kth column
- there are no primed entries on the main diagonal.

Example For $n = 4$

$$T = \begin{array}{ccccc} 1 & 1 & 2' & 2 & 3' \\ & 2 & 3' & 4' & 4 \\ & & 3 & 4' & 5' \\ & & & 4 & 5' \\ & & & & 5 \end{array} \qquad D = \begin{array}{ccccc} 1 & 2' & 3' & 1 & 5' \\ & 2 & 3' & 4' & 2 \\ & & 3 & 4' & 5' \\ & & & 4 & 4 \\ & & & & 5 \end{array}$$

Theorem [Chapman]

Let $x = (x_1, x_2, \ldots, x_n)$ and $(y = (y_1, y_2, \ldots, y_n)$.
Let $NE_k(A)$, $SE_k(A)$ and $NS_k(A)$ denote the number of entries NE, SE and NS in the kth row of the compass point matrix corresponding to the ASM A.
Then

$$\prod_{1\le i<j\le n} (x_i + y_j) = \sum_{A\in\mathcal{A}(n)} \prod_{k=1}^{n} \; x_k^{NE_k(A)} \; y_k^{SE_k(A)} (x_k + y_k)^{NS_k(A)} \,.$$

By setting $x_k = y_k = 1$ for all k, and noting that $\sum_{k=1}^n NS_k(A) = \mathrm{neg}(A)$, the total number of entries -1 in A, we immediately have:

Corollary

$$A_n(2) = \sum_{A\in\mathcal{A}(n)} \; 2^{\mathrm{neg}(A)} \; = 2^{n(n-1)/2} \,.$$

Proof [Hamel and King]

First a bijection is established between the elements $T \in \mathcal{PST}(n)$ and the elements $D \in \mathcal{PSD}(n)$. Then it is shown that associating x_k and y_k with unprimed entries k and primed entries k' leads to the required result.

The bijective steps - jeu de taquin

- To establish the required bijection from each $T \in \mathcal{PST}(n)$ to each $D \in \mathcal{PSD}(n)$ we must move all primed entries k' to the kth column.

- The allowed moves are transpositions of k' with their north or west unprimed neighbours, i or j, respectively.

$$\begin{array}{cc} & \boxed{j} \\ \boxed{i} & \boxed{k'} \end{array} \quad \longrightarrow \quad \begin{array}{cc} & \boxed{k'} \\ \boxed{i} & \boxed{j} \end{array} \qquad \text{if } i \leq j\,;$$

$$\begin{array}{cc} & \boxed{j} \\ \boxed{i} & \boxed{k'} \end{array} \quad \longrightarrow \quad \begin{array}{cc} & \boxed{j} \\ \boxed{k'} & \boxed{i} \end{array} \qquad \text{if } i > j\,.$$

- The chosen move is the unique one that ensures that the unprimed entries remain weakly increasing across rows and strongly increasing down columns

- k' must move west if it is already in the 1st row but not yet in the kth column.

$$\boxed{i}\boxed{k'} \quad \longrightarrow \quad \boxed{k'}\boxed{i}$$

- k' must also move west if it is immediately below some j that is in the jth row unless k' is already in the kth column.

$$\begin{array}{cc} & \boxed{j} \\ \boxed{i} & \boxed{k'} \end{array} \quad \longrightarrow \quad \begin{array}{cc} & \boxed{j} \\ \boxed{k'} & \boxed{i} \end{array} \qquad \text{necessarily } i > j\,.$$

- If k' is already in the kth column then it moves north unless j is in the jth row, in which case the moves of k' have been completed.

$$\begin{array}{c} \boxed{j} \\ \boxed{k'} \end{array} \quad \longrightarrow \quad \begin{array}{c} \boxed{k'} \\ \boxed{j} \end{array} \quad .$$

The bijective algorithm

- Apply the above moves to each tableau $T \in \mathcal{PST}(n)$ so that all primed entries move to their own column and unprimed to their own row to give a diagram $D \in \mathcal{PSD}(n)$.
- Do this in turn for $k' = 1', 2', \ldots, n'$, and for given k' deal in turn with each entry k' from top to bottom.

Example $n = 5$:

$$T \in \mathcal{SPT}(5) \; = \; \begin{array}{ccccc} 1 & 1 & 2' & 2 & 3' \\ & 2 & 3' & 4' & 4 \\ & & 3 & 4' & 5' \\ & & & 4 & 5' \\ & & & & 5 \end{array} \quad \overset{2'}{\longrightarrow} \quad \begin{array}{ccccc} 1 & 2' & 1 & 2 & 3' \\ & 2 & 3' & 4' & 4 \\ & & 3 & 4' & 5' \\ & & & 4 & 5' \\ & & & & 5 \end{array}$$

$$\overset{3'}{\longrightarrow} \quad \begin{array}{ccccc} 1 & 2' & 1 & 3' & 2 \\ & 2 & 3' & 4' & 4 \\ & & 3 & 4' & 5' \\ & & & 4 & 5' \\ & & & & 5 \end{array} \quad \overset{3'}{\longrightarrow} \quad \begin{array}{ccccc} 1 & 2' & 3' & 1 & 2 \\ & 2 & 3' & 4' & 4 \\ & & 3 & 4' & 5' \\ & & & 4 & 5' \\ & & & & 5 \end{array}$$

$$\overset{5'}{\longrightarrow} \quad \begin{array}{ccccc} 1 & 2' & 3' & 1 & 2 \\ & 2 & 3' & 4' & 5' \\ & & 3 & 4' & 4 \\ & & & 4 & 5' \\ & & & & 5 \end{array} \quad \overset{5'}{\longrightarrow} \quad \begin{array}{ccccc} 1 & 2' & 3' & 1 & 5' \\ & 2 & 3' & 4' & 2 \\ & & 3 & 4' & 4 \\ & & & 4 & 5' \\ & & & & 5 \end{array}$$

$$\overset{5'}{\longrightarrow} \quad \begin{array}{ccccc} 1 & 2' & 3' & 1 & 5' \\ & 2 & 3' & 4' & 2 \\ & & 3 & 4' & 5' \\ & & & 4 & 4 \\ & & & & 5 \end{array} \quad = \; D \in \mathcal{PSD}(5)\,.$$

To establish that the above map is bijective

- Need to show that each k' always reaches the kth column
- Only possible obstruction is another k' to the immediate west of a moving k'. However, at all stages there are no two identical primed entries in the same row.
- Need to show that each unprimed entry i reaches the ith row.
- At all stages each unprimed entry i can rise no further than the ith row. At the final stage each i in the kth column can be moved to the ith row by moving k's up this column. There are always enough k's to do this.
- Need to specify an inverse map taking each $D \in \mathcal{PSD}(n)$ to some $T \in \mathcal{PST}(n)$
- This inverse map is defined by inverting each individual move and taking them in reverse order.
- Need to show that the correct standardness conditions are satisfied by the final T.
- At all stages they are satisfied for all primed and all unprimed entries treated separately, and the final moves always ensure that the $k' < k$ order conditions are also satisfied.

Maps governing the x_k and y_k weighting

- Unprimed entries: $k \longrightarrow x_k$
- Obligatory primed entries: $k' \longrightarrow y_k$
- Optional primed entries: $\mathbf{k}'' \longrightarrow (x_k + y_k)$

$$
\begin{array}{ccccc}
1 & 1 & 2 & 2 & 3 \\
  & 2 & 3 & 4 & 4 \\
  &   & 3 & 4 & 5 \\
  &   &   & 4 & 5 \\
  &   &   &   & 5
\end{array}
\quad \Leftrightarrow \quad
\begin{array}{ccccc}
1 & 1 & \mathbf{2}'' & 2 & \mathbf{3}'' \\
  & 2 & 3' & 4' & 4 \\
  &   & 3 & 4' & 5' \\
  &   &   & 4 & 5' \\
  &   &   &   & 5
\end{array}
$$

$$
\Updownarrow \qquad\qquad\qquad\qquad \Updownarrow
$$

$$
\begin{bmatrix}
1 & 1 & 0 & 0 & 0 \\
1 & 0 & 1 & 1 & 0 \\
1 & 1 & 0 & 0 & 1 \\
1 & 1 & 1 & 1 & 0 \\
1 & 1 & 1 & 0 & 0
\end{bmatrix}.
\quad \Leftrightarrow \quad
\begin{bmatrix}
1 & 1 & 0 & 0 & 0 \\
2 & \mathbf{0} & \mathbf{2}'' & 2 & 0 \\
3 & 3' & 0 & \mathbf{0} & \mathbf{3}'' \\
4 & 4' & 4' & 4 & 0 \\
5 & 5' & 5' & 0 & 0
\end{bmatrix}
$$

$$
\Updownarrow \qquad\qquad\qquad\qquad \Updownarrow
$$

$$
\begin{bmatrix}
0 & 1 & 0 & 0 & 0 \\
1 & -1 & 0 & 1 & 0 \\
0 & 1 & 0 & -1 & 1 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0
\end{bmatrix}
\quad \Leftrightarrow \quad
\begin{bmatrix}
NE & 1 & NW & NW & NW \\
1 & -\mathbf{1} & NE & 1 & NW \\
SE & 1 & NW & -\mathbf{1} & 1 \\
SE & SE & NE & 1 & SW \\
SE & SE & 1 & SW & SW
\end{bmatrix}
$$

Weightings for $A \in \mathcal{A}(n)$

- Map from A to a doubly primed shifted tableau, and then to $2^{\text{neg}(A)}$ shifted primed tableaux.

- **Ex**:

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix} \Leftrightarrow \begin{array}{|c|c|c|c|c|} \hline 1 & 1 & \mathbf{2''} & 2 & \mathbf{3''} \\ \hline & 2 & 3' & 4' & 4 \\ \hline & & 3 & 4' & 5' \\ \hline & & & 4 & 5' \\ \hline & & & & 5 \\ \hline \end{array}$$

$$\Rightarrow \begin{array}{|c|c|c|c|c|} \hline 1 & 1 & 2 & 2 & 3 \\ \hline & 2 & 3' & 4' & 4 \\ \hline & & 3 & 4' & 5' \\ \hline & & & 4 & 5' \\ \hline & & & & 5 \\ \hline \end{array} + \begin{array}{|c|c|c|c|c|} \hline 1 & 1 & 2' & 2 & 3 \\ \hline & 2 & 3' & 4' & 4 \\ \hline & & 3 & 4' & 5' \\ \hline & & & 4 & 5' \\ \hline & & & & 5 \\ \hline \end{array}$$

$$+ \begin{array}{|c|c|c|c|c|} \hline 1 & 1 & 2 & 2 & 3' \\ \hline & 2 & 3' & 4' & 4 \\ \hline & & 3 & 4' & 5' \\ \hline & & & 4 & 5' \\ \hline & & & & 5 \\ \hline \end{array} + \begin{array}{|c|c|c|c|c|} \hline 1 & 1 & 2' & 2 & 3' \\ \hline & 2 & 3' & 4' & 4 \\ \hline & & 3 & 4' & 5' \\ \hline & & & 4 & 5' \\ \hline & & & & 5 \\ \hline \end{array}$$

- Now weight with $k \longrightarrow x_k$ and $k' \longrightarrow y_k$

- **Ex**:

$$\begin{array}{|c|c|c|c|c|} \hline x_1 & x_1 & x_2+y_2 & x_2 & x_3+y_3 \\ \hline & x_2 & y_3 & y_4 & x_4 \\ \hline & & x_3 & y_4 & y_5 \\ \hline & & & x_4 & y_5 \\ \hline & & & & x_5 \\ \hline \end{array} \longrightarrow \begin{array}{l} x_1^2 \\ x_2^2\, y_2\, (x_2 + y_2) \\ x_3\, y_3\, (x_3 + y_3) \\ x_4^2\, y_4 \\ x_5\, y_5^2 \end{array}$$

- In general, the contribution from kth row of A is

$$x_k^{NE_k(A)}\; y_k^{SE_k(A)} (x_k + y_k)^{NS_k(A)}\,.$$

Weightings for $D \in \mathcal{PSD}(n)$

- Unprimed entries: $\quad k \longrightarrow x_k$
- Primed entries: $\quad k' \longrightarrow y_k$
- **Example**:

$$D \;=\; \begin{array}{|c|c|c|c|c|} \hline 1 & 2' & 3' & 1 & 5' \\ \hline \multicolumn{1}{c|}{} & 2 & 3' & 4' & 2 \\ \cline{2-5} \multicolumn{2}{c|}{} & 3 & 4' & 5' \\ \cline{3-5} \multicolumn{3}{c|}{} & 4 & 4 \\ \cline{4-5} \multicolumn{4}{c|}{} & 5 \\ \cline{5-5} \end{array} \quad \longrightarrow \quad \begin{array}{|c|c|c|c|c|} \hline x_1 & y_2 & y_3 & x_1 & y_5 \\ \hline \multicolumn{1}{c|}{} & x_2 & y_3 & y_4 & x_2 \\ \cline{2-5} \multicolumn{2}{c|}{} & x_3 & y_4 & y_5 \\ \cline{3-5} \multicolumn{3}{c|}{} & x_4 & x_4 \\ \cline{4-5} \multicolumn{4}{c|}{} & x_5 \\ \cline{5-5} \end{array}$$

- Summing over all $D \in \mathcal{PSD}(n)$ gives $\displaystyle\prod_{1 \le i < j \le n} (x_i + y_j)$.

Combining the use of our bijection from $\mathcal{PST}(n)$ to $\mathcal{PSD}(n)$ this gives our required result:

Theorem

$$\prod_{1 \le i < j \le n} (x_i + y_j) = \sum_{A \in \mathcal{A}(n)} \prod_{k=1}^{n} \; x_k^{NE_k(A)} \; y_k^{SE_k(A)} (x_k + y_k)^{NS_k(A)} \, .$$

As pointed out earlier, by setting $x_k = y_k = 1$ for all k, and noting that $\sum_{k=1}^{n} NS_k(A) = \mathrm{neg}(A)$, the total number of entries -1 in A, we immediately have:

Corollary

$$A_n(2) = \sum_{A \in \mathcal{A}(n)} 2^{\mathrm{neg}(A)} \; = 2^{n(n-1)/2} \, .$$

Generalisation to arbitrary shifted tableaux

Let μ be a partition of length $\ell(\mu) = n$ whose parts are all distinct.

Shifted tableaux $ST \in \mathcal{ST}^{\mu}(n)$

$$
\begin{array}{ccccccccc}
1 & 1 & 1 & 2 & 3 & 3 & 4 & 4 & 4 \\
 & 2 & 2 & 2 & 3 & 4 & 5 & 5 & 5 \\
 & & 3 & 4 & 4 & 4 & 5 & 6 & \\
 & & & 4 & 5 & 5 & 6 & & \\
 & & & & 5 & 6 & 6 & & \\
 & & & & & 6 & & &
\end{array}
\quad \Leftrightarrow \quad
\begin{bmatrix}
1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
2 & 2 & 2 & 2 & 0 & 0 & 0 & 0 & 0 \\
3 & 0 & 0 & 3 & 3 & 3 & 0 & 0 & 0 \\
4 & 4 & 4 & 4 & 4 & 0 & 4 & 4 & 4 \\
5 & 5 & 5 & 0 & 5 & 5 & 5 & 5 & 0 \\
6 & 6 & 6 & 6 & 0 & 6 & 0 & 0 & 0
\end{bmatrix}
$$

μ-alternating sign matrices $A \in \mathcal{A}^{\mu}(n)$

- Row sums are 1 for all rows.
- Column sums are 1 or 0 according as the column number is or is not, respectively, a part of μ.

Ex: Typical $A \in \mathcal{A}^{\mu}(n)$ for $n = 6$ and $\mu = (9, 8, 6, 4, 3, 1)$.

$$
A = \begin{bmatrix}
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 1 \\
0 & 0 & 1 & -1 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & -1 & 1 & 0 & 0 & 0
\end{bmatrix}
\begin{matrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{matrix}
$$

$$
\begin{matrix}
1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 1 \\
\\
1 & - & 3 & 4 & - & 6 & - & 8 & 9
\end{matrix}
$$

Theorem [Hamel and King] Let $\mu = \lambda + \rho$ be a strict partition of length $\ell(\mu) = n$ with λ a partition of length $\ell(\lambda) \leq n$ and $\rho = (n, n-1, \ldots, 1)$. Then

$$\prod_{1 \leq i < j \leq n} (x_i + y_j)\, s_\lambda(x) = \sum_{A \in \mathcal{A}^\mu(n)} \prod_{k=1}^{n} x_k^{NE_k(A)}\, y_k^{SE_k(A)} (x_k + y_k)^{NS_k(A)}\,.$$

Proof

- First establish the bijection between $A \in \mathcal{A}^\mu(n)$ and shifted tableaux $ST \in \mathcal{ST}^\mu(n)$ by means of the matrix giving specifying the lists of entries on each diagonal.
- Then add primes as usual, with $2^{\mathrm{neg}(A)}$ of these optional.
- Then use the jeu de taquin moves to provide a bijection from primed shifted tableaux $PST \in \mathcal{PST}^\mu(n)$ to a pair (D, T) consisting of a primed shifted diagram $D \in \mathcal{PSD}$ and a semistandard Young tableau $T \in \mathcal{T}^\lambda(n)$.
- Finally, use the same weightings as before.

Ex: $PST \quad \Leftrightarrow \quad D \;,\; T$

PST:

1	1	1	$2'$	2	2	3	3	5
	2	2	$3'$	3	$4'$	$5'$	5	$6'$
		3	3	$4'$	4	$5'$	6	
			4	$5'$	5	5		
				5	$6'$	6		
					6			

$\Leftrightarrow$

D:

1	$2'$	1	$4'$	$5'$	$6'$
	2	$3'$	2	$5'$	2
		3	$4'$	3	3
			4	$5'$	$6'$
				5	5
					6

,

T:

1	2	3
3	5	5
4	6	
5		
6		

Note Since all the primed entries in PST are moved into D, the remaining portion T contains only unprimed entries. These entries automatically satisfy the usual semistandardness conditions so that the sum over all T gives $s_\lambda(x)$.

Generalisation to the symplectic case

Let $\mu = \lambda + \rho$ be a partition of length $\ell(\mu) = n$ whose parts are all distinct.

Symplectic shifted tableaux

A $sp(2n)$-standard shifted tableaux $ST \in \mathcal{ST}^{\mu}(n, \overline{n})$ is a numbering of the boxes of SF^{μ} with entries taken from the set $\{\overline{1}, 1, \overline{2}, \ldots, \overline{n}, n\}$ subject to the ordering $\overline{1} < 1 < \overline{2} < \ldots < \overline{n} < n$ and such that the entries are

- weakly increasing across each row from left to right;
- weakly increasing down each column from top to bottom;
- strictly increasing down each diagonal from top left to bottom right;
- with k and $\overline{k}$ appearing no lower than the kth row.

Ex: Typically, for $n = 5$ and $\mu = (9, 7, 6, 2, 1)$ we have:

$$ST = \begin{array}{|c|c|c|c|c|c|c|c|c|} \hline \overline{1} & 1 & \overline{2} & 2 & \overline{3} & \overline{3} & \overline{4} & 4 & 5 \\ \hline \multicolumn{1}{c|}{} & \overline{2} & \overline{2} & 2 & 3 & \overline{4} & \overline{4} & 4 & \multicolumn{1}{c}{} \\ \cline{2-8} \multicolumn{2}{c|}{} & 3 & \overline{4} & 4 & 4 & 4 & 4 & \multicolumn{1}{c}{} \\ \cline{3-8} \multicolumn{3}{c|}{} & 4 & 4 & \multicolumn{4}{c}{} \\ \cline{4-5} \multicolumn{4}{c|}{} & \overline{5} & \multicolumn{4}{c}{} \\ \cline{5-5} \end{array} \quad \in \ \mathcal{ST}^{97621}(5, \overline{5})$$

Symplectic primed shifted tableaux

A $sp(2n)$-standard primed shifted tableaux $QST \in \mathcal{QST}^\mu(n,\overline{n})$ is a numbering of the boxes of SF^μ with entries taken from the set $\{\overline{1}',\overline{1},1',1,\overline{2}',\overline{2},2',2,\ldots,\overline{n},\overline{n},n',n\}$, subject to the total ordering

$$\overline{1}' < \overline{1} < 1' < 1 < \overline{2}' < \overline{2} < 2' < 2 < \cdots < \overline{n}' < \overline{n} < n' < n.$$

and such that the entries are:

- weakly increasing across each row from left to right;
- weakly increasing down each column from top to bottom;
- with no two identical unprimed entries in any column;
- with no two identical primed entries in any row;
- with the kth entry on the main diagonal in the set $\{\overline{k}',k',\overline{k},k\}$.
- Notice that primes are now allowed on the main diagonal.

Ex: Typically, for $n=5$ and $\mu=(9,7,6,2,1)$ we have

$$QST = \begin{array}{ccccccccc}
\overline{1} & 1 & \overline{2}' & 2' & \overline{3}' & \overline{3} & \overline{4}' & 4' & 5 \\
 & \overline{2}' & \overline{2} & 2 & 3 & \overline{4}' & \overline{4} & 4' & \\
 & & 3' & \overline{4}' & 4' & 4 & 4 & 4 & \\
 & & & 4' & 4 & & & & \\
 & & & & \overline{5}' & & & &
\end{array} \quad \in \mathcal{QST}^{97621}(5,\overline{5})$$

Bijective map to new U–turn ASMs

- First map to a $2n$-rowed matrix of 0s and 1s whose columns record the entries on the corresponding diagonal of QST.
- Replace each 0 immediately to the left of a 1 by -1, and then replace each 1 to the left of another 1 by 0.

μ-U-turn alternating sign matrices $UA \in \mathcal{UA}^\mu(n)$

- Non-zero matrix elements alternate in sign from right to left across each odd row and back from left to right along the next row.
- Non-zero matrix elements alternate in sign down each column.
- Row sums are 0 or 1 for all rows, with a sum of 1 for each U-turn pair.
- Column sums are 1 or 0 according as the column number is or is not, respectively, a part of μ.

Ex: Typically, for $n = 5$ and $\mu = (9, 7, 6, 2, 1)$ we have:

$$UA \; = \; \left[\begin{array}{rrrrrrrrr} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 \end{array}\right] \begin{array}{c} 1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 1 \\ 1 \\ 0 \end{array}$$

$$\begin{array}{ccccccccc} 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ \\ 1 & 2 & - & - & - & 6 & 7 & - & 9 \end{array}$$

Primed and barred shifted diagrams

Let $\mathcal{QSD}(n,\overline{n})$ be the set of all primed and barred shifted diagrams QD of triangular shape $\rho = (n, n-1, \ldots, 1)$ with barred and unbarred, primed and unprimed entries such that

- each unprimed entry k or $\overline{k}$ appears in the kth row;
- each primed entry k' or $\overline{k}'$ appears in the kth column.
- with the kth entry on the main diagonal in the set $\{\overline{k}', k', \overline{k}, k\}$.
- Notice that primes are again allowed on the main diagonal.

Ex:

$$QD \;=\; \begin{array}{|c|c|c|c|c|} \hline \overline{1} & 1 & \overline{3}' & 4' & \overline{1} \\ \hline \multicolumn{1}{c|}{} & 2 & 2 & 2 & \overline{2} \\ \cline{2-5} \multicolumn{2}{c|}{} & \overline{3}' & 3 & \overline{3} \\ \cline{3-5} \multicolumn{3}{c|}{} & 4' & 4 \\ \cline{4-5} \multicolumn{4}{c|}{} & \overline{5}' \\ \cline{5-5} \end{array} \;\in \mathcal{QSD}(5,\overline{5})$$

Symplectic characters

The symplectic analogue of Schur functions, $s_\lambda(x)$ which can be identified with characters of $sl(n)$, are the characters $sp_\lambda(x)$ of $sp(2n)$.

- Let $\mathcal{T}^\lambda(n,\overline{n})$ be the set of $sp(2n)$-standard tableaux, T, of shape λ in which the entries, barred and unbarred, are weakly increasing across rows, and strictly increasing down columns, with no entry k or $\overline{k}$ lying below the kth row.
- Then $sp_\lambda(x) = \sum_{T \in \mathcal{T}^\lambda(n,\overline{n})} x^{\mathrm{wgt}(T)}$. Weighting each such tableaux by an additional factor of $t^{2\mathrm{bar}(T)}$, where $\mathrm{bar}(T)$ is the number of barred entries in T, then defines $sp_\lambda(x;t^2)$.

The new bijective steps - jeu de taquin

- To establish the required bijection from each $QST \in \mathcal{QST}(n,\overline{n})$ to a pair (QD, T) with $QD \in \mathcal{QSD}(n,\overline{n})$ and $T \in \mathcal{T}^{\lambda}(n,\overline{n})$ we must move all primed entries k' and $\overline{k}'$ to the kth column.

- The allowed moves are those of the jeu de taquin as before.

- There are no obstacles to moving first all the $\overline{k}'$s to the kth column.

- However, in moving the k's one may encounter a final move in which the destination site in the ith row and kth column is already occupied by an $\overline{k}'$. In this case we have the following transposition.

$$\begin{array}{|c|c|}\hline \overline{k}' & k' \\ \hline \end{array} \quad \longrightarrow \quad \begin{array}{|c|c|}\hline i & \overline{i} \\ \hline \end{array}$$

- Similarly, to avoid a pair i and $\overline{i}$ in the kth column, which would be non-standard in that they cannot both be in the ith row, one replaces such a pair by a $\overline{k}'$ and k' pair

$$\begin{array}{|c|}\hline \overline{i} \\ \hline i \\ \hline \end{array} \quad \longrightarrow \quad \begin{array}{|c|}\hline k' \\ \hline \overline{k}' \\ \hline \end{array}$$

- It is this last transformation that forces one to allow primed entries on the main diagonal.

- If these moves are to be weight preserving, then it is necessary that $\mathrm{wgt}(k')\,\mathrm{wgt}(\overline{k}') = \mathrm{wgt}(i)\,\mathrm{wgt}(\overline{i})$ for all i and k.

Combining the bijections from μ-U–turn ASMs to $QST \in \mathcal{QST}^\mu(n)$ and then to a pair (QD, T) with $QD \in \mathcal{QSD}^\mu(n)$ and $T \in \mathcal{T}^\lambda(n, \overline{n})$, and the weightings

- Unprimed unbarred entries: $k \longrightarrow x_k$
- Primed unbarred entries: $k' \longrightarrow y_k$
- Unprimed barred entries: $\overline{k} \longrightarrow t^2\,\overline{x}_k = t^2\,x_k^{-1}$
- Primed barred entries: $\overline{k}' \longrightarrow t^2\,\overline{y}_k = t^2\,y_k^{-1}$

we obtain the following

Theorem [Hamel and King]

Let $x = (x_1, x_2, \ldots, x_n)$, $y = (y_1, y_2, \ldots, y_n)$, $\overline{x} = (\overline{x}_1, \overline{x}_2, \ldots, \overline{x}_n)$ and $\overline{y} = (\overline{y}_1, \overline{y}_2, \ldots, \overline{y}_n)$, with $\overline{x}_i = x_i^{-1}$ and $\overline{y}_i = y_i^{-1}$ for all $i = 1, 2, \ldots, n$. Then

$$\prod_{1\leq i\leq j\leq n} (x_i + t^2\,\overline{x}_i + y_j + t^2\,\overline{y}_j)\ sp_\lambda(x; t^2)$$
$$= \sum_{UA\in\mathcal{UA}^\mu(n)} \prod_{k=1}^{n} x_k^{NE_k(UA)}\ (t^2\,\overline{x}_k)^{NE_{\overline{k}}(UA)} y_k^{SE_k(A)} (t^2\,\overline{y}_k)^{NE_{\overline{k}}(UA)}$$
$$\times \quad (x_k + y_k)^{NS_k(A)}\ (\overline{x}_k + \overline{y}_k)^{NS_{\overline{k}}(A)}\ .$$

In the case $\lambda = 0$, by setting $t = 1$ and $x_k = y_k = 1$ for all k, and noting that $\sum_{k=1}^{n}(NS_k(UA) + NS_{\overline{k}}(UA)) = \mathrm{neg}(UA)$, the total number of entries -1 in UA, we immediately have:

Corollary

$$\sum_{UA\in\mathcal{UA}(n)} 2^{\mathrm{neg}(UA)} = 2^{n(n+1)}\,.$$