Affine Weyl groups, coloured grids and characters of affine algebras

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Joint work with Trevor Welsh inspired by some preliminary results and conjectures made in collaboration with Amran Hussin

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Introduction

- Motivation to develop a combinatorial way of expressing both numerator and denominator of characters of irreps of affine Kac-Moody algebra in terms of characters of simple subalgebras
- Technique to study in detail the relationship between the corresponding finite and affine Weyl groups
- Discovery of a new combinatorial tool in the form of coloured grids and associated diagrams
- Byproduct an explicit realisation of the action of affine Weyl groups on arbitrary weights through their action on coloured diagrams
- Outcome a simple procedure to write down the required numerators and denominators

Kac-Moody algebras

- A matrix $A = (A_{ij})_{i,j\in I}$ with $A_{ij} \in \mathbb{Z}$ is said to be a generalised Cartan matrix if:
  - $A_{ii} = 2$;
  - $A_{ij} \leq 0$ for $i \neq j$;
  - $A_{ij} = 0 \Rightarrow A_{ji} = 0$;
- The associated Kac-Moody algebra **g**(A) is a complex Lie algebra with generators d, e_i, f_i, h_i for $i \in I$ satisfying

$$[h_i, h_j] = 0, \quad [e_i, f_j] = \delta_{ij} h_i,$$
$$[h_i, e_j] = A_{ij} e_j, \quad [h_i, f_j] = -A_{ij} f_j$$
$$(\mathrm{ad}\, e_i)^{1-A_{ij}} e_j = 0, \quad (\mathrm{ad}\, f_i)^{1-A_{ij}} f_j = 0, \quad \text{if } i \neq j,$$
$$[d, h_i] = 0, \quad [d, e_i] = \delta_{i0}\, e_i, \quad [d, f_i] = -\delta_{i0}\, f_i$$

where $(\mathrm{ad}\, x)y = [x, y]$ for all $x, y \in$ **g**(A).

Simple and affine algebras

- If A is indecomposable and $\det A > 0$ then $\mathbf{g}(A) \approx \bar{\mathbf{g}}$, a finite-dimensional complex simple Lie algebra:

classical	$A_\ell,\ B_\ell,\ C_\ell,\ D_\ell;$
exceptional	$E_6,\ E_7,\ E_8,\ F_4,\ G_2.$

- If A is indecomposable and there exists a vector δ with positive integer components such that $A\,\delta = 0$ then $\mathbf{g}(A) \approx \mathbf{g}$, an infinite-dimensional affine Kac-Moody algebra:

classical	$A_\ell^{(1)},\ B_\ell^{(1)},\ C_\ell^{(1)},\ D_\ell^{(1)},\ A_{2\ell}^{(2)},\ A_{2\ell-1}^{(2)},\ D_{\ell+1}^{(2)};$
exceptional	$E_6^{(1)},\ E_7^{(1)},\ E_8^{(1)},\ F_4^{(1)},\ G_2^{(1)},\ E_6^{(2)},\ D_4^{(3)}.$

Natural embeddings

- A simple Lie algebra $\overline{\mathbf{g}} = \mathbf{g}(\overline{A})$ is said to be naturally embedded in the affine Kac-Moody algebra $\mathbf{g} = \mathbf{g}(A)$ if there exists $I = \{0, 1, 2, \ldots, \ell\}$ and $\overline{I} = \{1, \ldots, \ell\}$ such that $A = (A_{ij})_{ij \in I}$ and $\overline{A} = (A_{ij})_{ij \in \overline{I}}$
- The natural embeddings $\overline{\mathbf{g}} \subset \mathbf{g}$ include

$$
\begin{array}{lcl}
A_\ell & \subset & A_\ell^{(1)} \\
B_\ell & \subset & B_\ell^{(1)}, A_{2\ell}^{(2)}, D_{\ell+1}^{(2)} \\
C_\ell & \subset & C_\ell^{(1)}, A_{2\ell-1}^{(2)} \\
D_\ell & \subset & D_\ell^{(1)}
\end{array}
\qquad
\begin{array}{lcl}
E_6 & \subset & E_6^{(1)} \\
E_7 & \subset & E_7^{(1)} \\
E_8 & \subset & E_8^{(1)} \\
F_4 & \subset & F_4^{(1)}, E_6^{(2)} \\
G_2 & \subset & G_2^{(1)}, D_4^{(3)}
\end{array}
$$

- Here we consider the 7 on the left indexed by their rank ℓ.

Key parameters for affine algebras

- Let the affine Kac-Moody algebra **g** $= \mathbf{g}(A)$ have
  - Cartan subalgebra **h**, with basis $\{d, h_i \,|\, i \in I\}$
  - Dual **h***, with basis $\{\Lambda_0, \alpha_i \,|\, i \in I\}$
  - Simple roots $\{\alpha_i \,|\, i \in I\}$
  - Weyl vector $\rho = \sum_{i \in I} \Lambda_i$
  - Marks $c_j \in \mathbb{N}$ such that $\sum_{j \in I} A_{ij}\, c_j = 0$
  - Comarks $c_i^\vee \in \mathbb{N}$ such that $\sum_{i \in I} c_i^\vee A_{ij} = 0$
  - Imaginary (or null) root $\delta = \sum_{i \in I} c_i\, \alpha_i$
  - Coxeter number h with dual $h^\vee = \sum_{i \in I} c_i^\vee$
  - Convention $c_0 = 1$, $c_0^\vee = 2$ if **g** $= A_{2\ell}^{(2)}$ and 1 otherwise
  - Modified Coxeter number $\tilde{h}^\vee = 2h^\vee$ if **g** $= C_\ell^{(1)}$ and $h^\vee$ otherwise

Bases for $\mathbf{h}^*$

- Let **g** have rank ℓ, and let $I = \{0, 1, , \ldots, \ell\}$
- Let $n = \ell + 1$ for **g** $= A_\ell^{(1)}$ and $n = \ell$ otherwise
- Let $N = \{1, 2, \ldots, n\}$
- Then there are three convenient bases for $\mathbf{h}^*$:
    - Simple root basis: $\{\Lambda_0,\ \alpha_i \,|\, i \in I\}$
    - Fundamental weight basis: $\{\delta,\ \Lambda_i \,|\, i \in I\}$
    - Euclidean or natural basis: $\{\Lambda_0,\ \delta,\ \epsilon_j \,|\, j \in N\}$
- In the case of $A_\ell^{(1)}$ we have $\epsilon_1 + \epsilon_2 + \cdots + \epsilon_n = 0$
- In all cases $\alpha_i = \sum_{j \in I} A_{ji}\, \Lambda_j + \delta_{i0}\, \delta$ and
- $\mathbf{h}^* = \overline{\mathbf{h}}^* \oplus \mathbb{C}\, \Lambda_0 \oplus \mathbb{C}\, \delta$ with $\overline{\mathbf{h}}^*$ spanned by $\{\alpha_1, \alpha_2, \ldots, \alpha_\ell\}$

Simple root systems of affine algebras

- In terms of δ and $\epsilon_1, \epsilon_2, \ldots, \epsilon_n$, the simple roots of **g** are

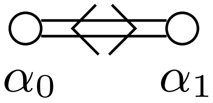
$$\alpha_i = \epsilon_i - \epsilon_{i+1} \quad \text{for } i = 1, 2, \ldots, \ell - 1$$

- together with

$$\begin{array}{lll}
A_\ell^{(1)}: & \alpha_0 = \delta - \epsilon_1 + \epsilon_{\ell+1} & \alpha_\ell = \epsilon_\ell - \epsilon_{\ell+1} \\
B_\ell^{(1)}: & \alpha_0 = \delta - \epsilon_1 - \epsilon_2 & \alpha_\ell = \epsilon_\ell \\
C_\ell^{(1)}: & \alpha_0 = \delta - 2\epsilon_1 & \alpha_\ell = 2\epsilon_\ell \\
D_\ell^{(1)}: & \alpha_0 = \delta - \epsilon_1 - \epsilon_2 & \alpha_\ell = \epsilon_{\ell-1} + \epsilon_\ell \\
A_{2\ell}^{(2)}: & \alpha_0 = \delta - 2\epsilon_1 & \alpha_\ell = \epsilon_\ell \\
A_{2\ell-1}^{(2)}: & \alpha_0 = \delta - \epsilon_1 - \epsilon_2 & \alpha_\ell = 2\epsilon_\ell \\
D_{\ell+1}^{(2)}: & \alpha_0 = \delta - \epsilon_1 & \alpha_\ell = \epsilon_\ell
\end{array}$$

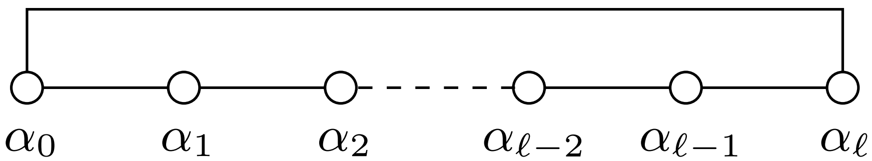
Dynkin diagrams of untwisted affine algebras

$A_1^{(1)}$



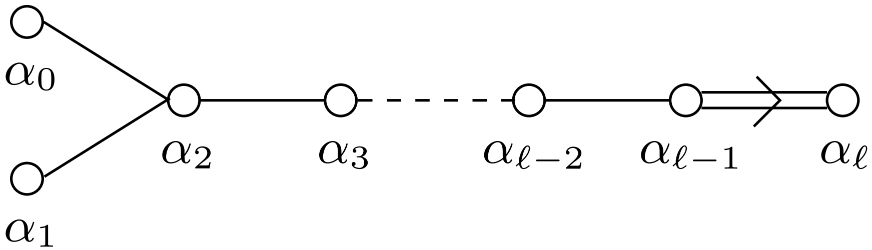
$\ell = 1$

$A_\ell^{(1)}$



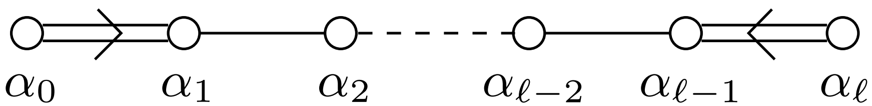
$\ell \geq 2$

$B_\ell^{(1)}$



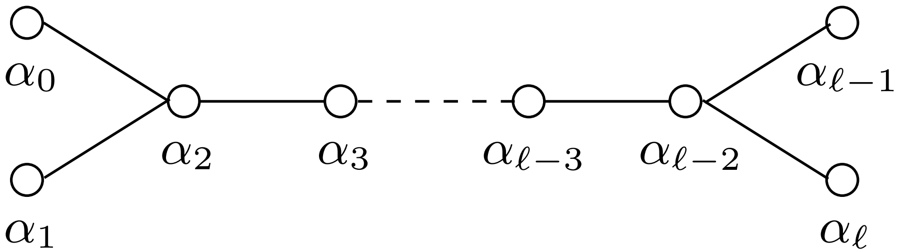
$\ell \geq 3$

$C_\ell^{(1)}$



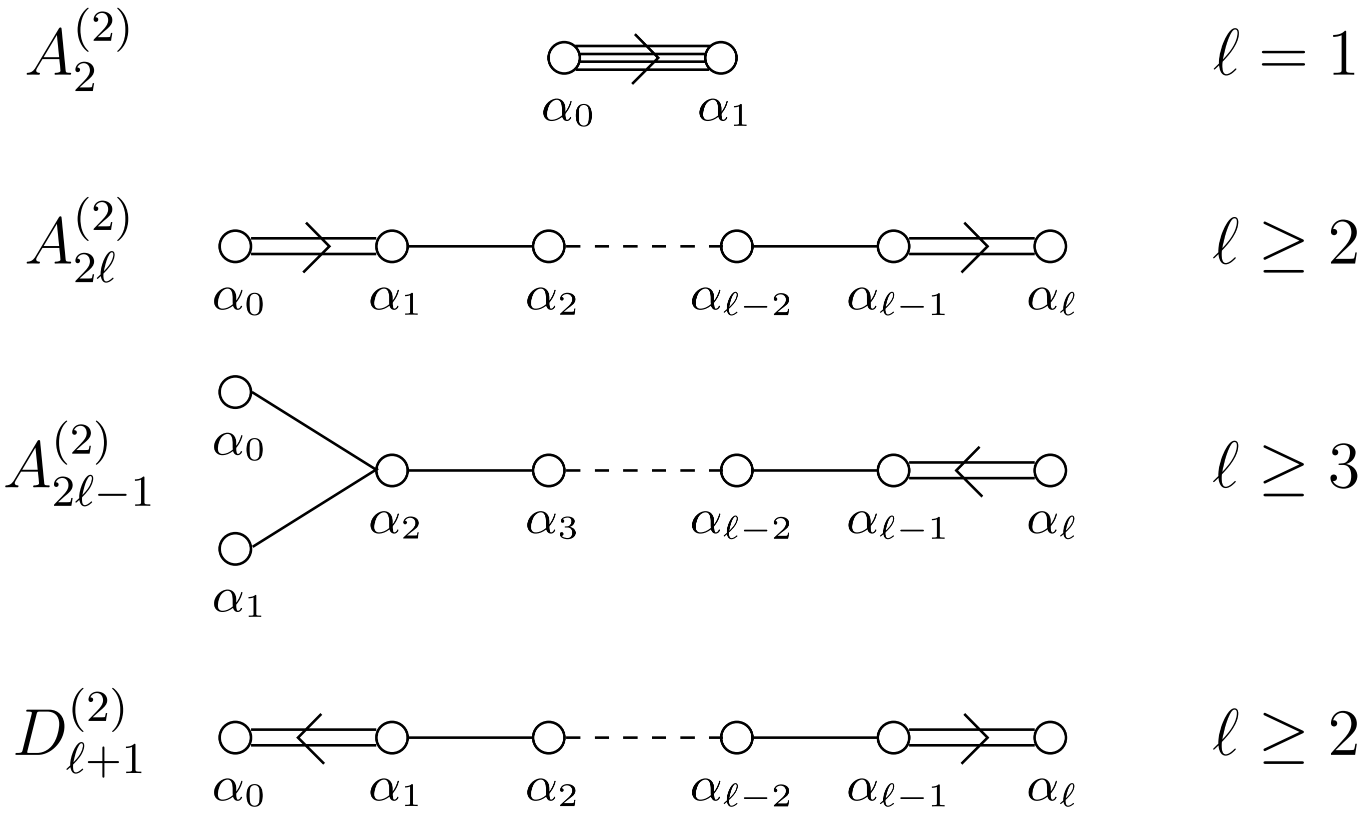
$\ell \geq 2$

$D_\ell^{(1)}$



$\ell \geq 4$

Dynkin diagrams of twisted affine algebras



Note: In each case the Dynkin diagram of the simple Lie algebra $\bar{\mathfrak{g}}$ is obtained by deleting the node α_0 and its edges

Root systems

- Let **g** and $\bar{\mathbf{g}}$ have positive roots Δ^+ and Δ_0^+
- Then $\Delta^+ = \Delta_{re}^+ + \Delta_{im}^+$ with
  - $\Delta_{im}^+ = \{m\,\delta \,|\, m \in \mathbb{Z}_{>0}\}$
  - $\Delta_{re}^+ = \{2m\,\delta \pm \alpha \,|\, m \in \mathbb{Z}_{\geq 0}, \alpha \in \Delta_0^+\}$
    $\cup \{(2m-1)\,\delta \pm \alpha \,|\, m \in \mathbb{Z}_{>0}, \alpha \in \Delta_1^+\}$
    where Δ_1^+ is either Δ_0^+ or a small modification and $\Delta_{re}^+ \supset \Delta_0^+$
- There exists a non-degenerate symmetric bilinear form $(\cdot|\cdot)$ on **h*** such that:
  - $(\alpha_j|\alpha_i^\vee) = A_{ij}$ for $i, j \in I$ where $\alpha_i^\vee = 2\alpha_i/(\alpha_i|\alpha_i)$
  - $(\alpha|\alpha) > 0$ for $\alpha \in \Delta_{re}$
  - $(\alpha|\alpha) = 0$ for $\alpha \in \Delta_{im}$

Weyl groups

- Let W and $\overline{W}$ denote the Weyl groups of **g** and $\overline{\mathbf{g}}$
- The finite Weyl group $\overline{W}$ is generated by $s_1, \ldots, s_\ell$
- The affine Weyl group W is generated by $s_0, s_1, \ldots, s_\ell$
- In each case, for any $\lambda \in \mathbf{h}^*$ we have

$$s_i(\lambda) = \lambda - (\lambda \,|\, \alpha_i^\vee)\, \alpha_i$$

- The length $\ell(w)$ of any Weyl group element w is the smallest $t \in N$ such that $w = s_{i_1} s_{i_2} \cdots s_{i_t}$
- The parity or sign of w is given by $\varepsilon(w) = (-1)^{\ell(w)}$
- If $w = s_{i_1} s_{i_2} \cdots s_{i_{\ell(w)}}$ then this is said to be a reduced expression for w

Right coset representatives

Lemma

- Let W and $\overline{W}$ be the Weyl groups of **g** and $\bar{\mathbf{g}}$,
- Let $W_s = \{\, w \in W \,|\, \ell(s_i w) > \ell(w)$ for all $i \in I \backslash \{0\} \,\}$
- Then $\overline{W} \subset W$ and W_s is a set of minimal length right coset representatives of $\overline{W}$ in W

Lemma

- Let $W_s = \cup_{t=0}^{\infty} W_s^{(t)}$ with $W_s^{(t)} = \{w \in W_s \,|\, \ell(w) = t\}$
- Then $W_s^{(0)} = \{1\}$ and $W_s^{(t+1)} = \{\, w s_k \,|\, w \in W_s^{(t)},\ k \in I,$ with

$$\ell(ws_k) = \ell(w) + 1, w(\alpha_k) \in \Delta_{re}^{+} \backslash \Delta_0^{+}\}.$$

Note: This defines W_s recursively.

Weight space

- For any weight $\lambda \in \mathbf{h}^*$ we have

$$\lambda \;=\; \sum_{i=0}^{\ell} m_i(\lambda)\,\Lambda_i - D(\lambda)\,\delta \;=\; \overline{\lambda} + \frac{1}{c_0^\vee}\,L(\lambda)\,\Lambda_0 - D(\lambda)\,\delta$$

where

  - $D(\lambda)$ is the depth of λ
  - $m_i(\lambda) = (\lambda\,|\,\alpha_i^\vee)$ for $i \in I$ are the Dynkin labels of λ
  - $L(\lambda) = \sum_{i=0}^{\ell} c_i^\vee\, m_i(\lambda)$ is the level of λ
  - $\overline{\lambda}$ is the restriction of λ from $\mathbf{h}^*$ to $\overline{\mathbf{h}}^*$
  - $\overline{\lambda} = \sum_{i=0}^{n} \lambda_i\,\epsilon_i$ where $\lambda_i = (\overline{\lambda}\,|\,\epsilon_i)$ for $i \in I\backslash\{0\}$

Irreducible representations and characters

Simple Lie algebra $\overline{\mathbf{g}}$

- $\overline{\lambda} \in \overline{\mathbf{h}}^*$ is integral if $(\overline{\lambda}\,|\,\alpha_i^\vee) \in \mathbb{Z}$ for all $i \in \overline{I}$
- $\overline{\lambda} \in \overline{h}^*$ is dominant integral if $(\overline{\lambda}\,|\,\alpha_i^\vee) \in \mathbb{Z}_{\geq 0}$ for all $i \in \overline{I}$
- Each dominant integral weight $\overline{\lambda}$ specifies a finite-dimensional irrep $V_0^{\overline{\lambda}}$ of $\overline{g}$ with character

$$\mathrm{ch}\, V_0^{\overline{\lambda}} = \sum_{w\in\overline{W}} (-1)^{\ell(w)}\, e^{w(\overline{\lambda}+\overline{\rho})-\overline{\rho}} \Big/ \prod_{\alpha\in\Delta_0^+} (1-e^{-\alpha})$$

where

$$\prod_{\alpha\in\Delta_0^+} (1-e^{-\alpha}) = \sum_{w\in\overline{W}} (-1)^{\ell(w)}\, e^{w(\overline{\rho})-\overline{\rho}}$$

Irreducible representations and characters

Affine Kac-Moody algebra **g**

- $\Lambda \in \mathbf{h}^*$ is integral if $(\Lambda \,|\, \alpha_i^\vee) \in \mathbb{Z}$ for all $i \in I$
- $\Lambda \in \mathbf{h}^*$ is dominant integral if $(\Lambda \,|\, \alpha_i^\vee) \in \mathbb{Z}_{\geq 0}$ for all $i \in I$
- Each dominant integral weight Λ specifies an infinite-dimensional irrep V^Λ of g with character

$$\mathrm{ch}\, V^\Lambda = \sum_{w \in W} (-1)^{\ell(w)} \, e^{w(\Lambda+\rho)-\rho} \Big/ \prod_{\alpha \in \Delta^+} (1 - e^{-\alpha})^{\mathrm{mult}\,(\alpha)}$$

where

$$\prod_{\alpha \in \Delta^+} (1 - e^{-\alpha})^{\mathrm{mult}\,(\alpha)} = \sum_{w \in W} (-1)^{\ell(w)} \, e^{w(\rho)-\rho}$$

Character formula

Proposition [Hussin & K, 97]

Let $\Lambda \in \mathbf{h}^*$ be dominant integral, then $\mathrm{ch}\, V^\Lambda = M^\Lambda / \, M$ with

$$M^\Lambda = e^{\frac{1}{c_0^\vee} L(\Lambda)\, \Lambda_0} \sum_{w \in W_s} (-1)^{\ell(w)} \; e^{-D(w\cdot\Lambda)\,\delta} \, \mathrm{ch}\, V_0^{\overline{w\cdot\Lambda}}$$

and

$$M = M^0 = \sum_{w \in W_s} (-1)^{\ell(w)} \; e^{-D(w\cdot 0)\,\delta} \, \mathrm{ch}\, V_0^{\overline{w\cdot 0}}$$

where

$$w \cdot \Lambda = w(\Lambda + \rho) - \rho \quad \text{and} \quad w \cdot 0 = w(\rho) - \rho$$

with

$\overline{w \cdot \Lambda}$ and $\overline{w \cdot 0}$ dominant integral for all $w \in W_s$

Comments on the character formula

- Both the numerator, M^{Λ}, and the denominator, M^{0}, are expressed as signed sums of characters of the simple subalgebra, $\overline{\mathbf{g}}$, of the affine Kac-Moody algebra, $\mathbf{g}$.
- To exploit the character formula we need a good way of evaluating $\overline{w \cdot 0} = \overline{w(\rho) - \rho}$ and $\overline{w \cdot \Lambda} = \overline{w(\Lambda + \rho) - \rho}$
- This can be done by noting that if $w = w' s_k$ then

$$\begin{aligned} w(\rho) - \rho &= w'(\rho) - \rho - w'(\alpha_k) \\ w(\Lambda) - \Lambda &= w'(\Lambda) - \Lambda - (\Lambda \,|\, \alpha_k^{\vee})\, w'(\alpha_k) \end{aligned}$$

- One proceeds by mapping w and w' to certain coloured diagrams, $T(w)$ and $T(w')$, whose difference encodes $w'(\alpha_k)$

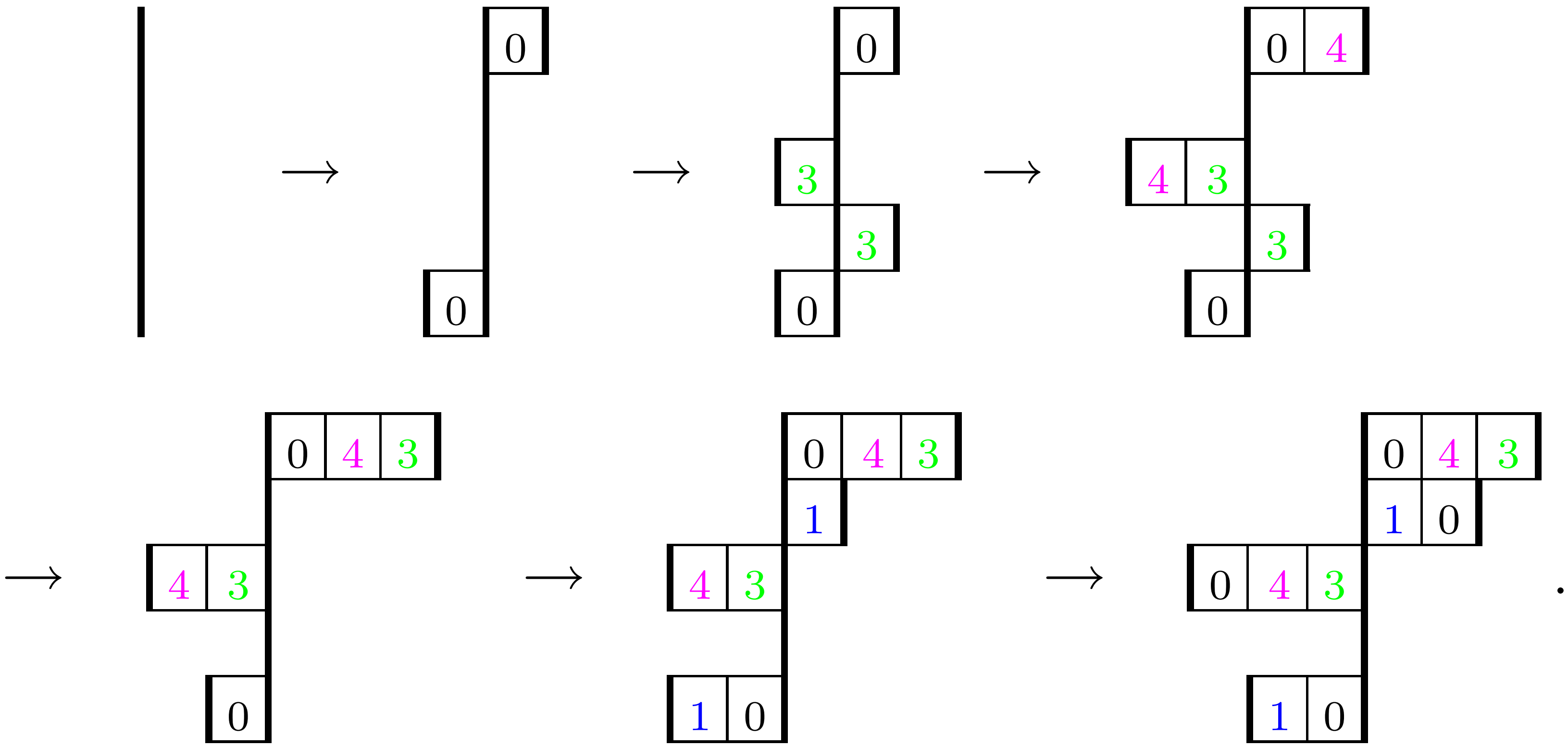
Generating the diagrams

Ex: Let **g** $= A_4^{(1)}$ and $w = s_0 s_3 s_4 s_3 s_1 s_0$. We find

w_s	$w_s \cdot 0 = w_s(\rho) - \rho$
1	0
s_0	$-\delta + \epsilon_1 - \epsilon_5$
$s_0 s_3$	$-\delta + \epsilon_1 - \epsilon_3 + \epsilon_4 - \epsilon_5$
$s_0 s_3 s_4$	$-2\delta + 2\epsilon_1 - 2\epsilon_3 + \epsilon_4 - \epsilon_5$
$s_0 s_3 s_4 s_3$	$-3\delta + 3\epsilon_1 - 2\epsilon_3 - \epsilon_5$
$s_0 s_3 s_4 s_3 s_1$	$-4\delta + 3\epsilon_1 + \epsilon_2 - 2\epsilon_3 - 2\epsilon_5$
$s_0 s_3 s_4 s_3 s_1 s_0$	$-5\delta + 3\epsilon_1 + 2\epsilon_2 - 3\epsilon_3 - 2\epsilon_5$

Unveiling the colours

Ex: Let $\mathbf{g} = A_4^{(1)}$ and $w = s_0 s_3 s_4 s_3 s_1 s_0$. Then $T(w)$ is obtained from $T(1)$ as follows:



where the number k of each entry corresponds to the index on the s_k which created the box

The map from w to $T(w)$

- Each $w \in W$ defines a weight $w(\rho) - \rho$
- that yields a restricted weight $\overline{w(\rho) - \rho} = \sum_{i=1}^{n} \lambda(w)_i \, \epsilon_i$
- that definines a generalised partition

$$\lambda(w) = (\lambda(w)_1, \lambda(w)_2, \ldots, \lambda(w)_n)$$

- that specifies a generalised Young diagram

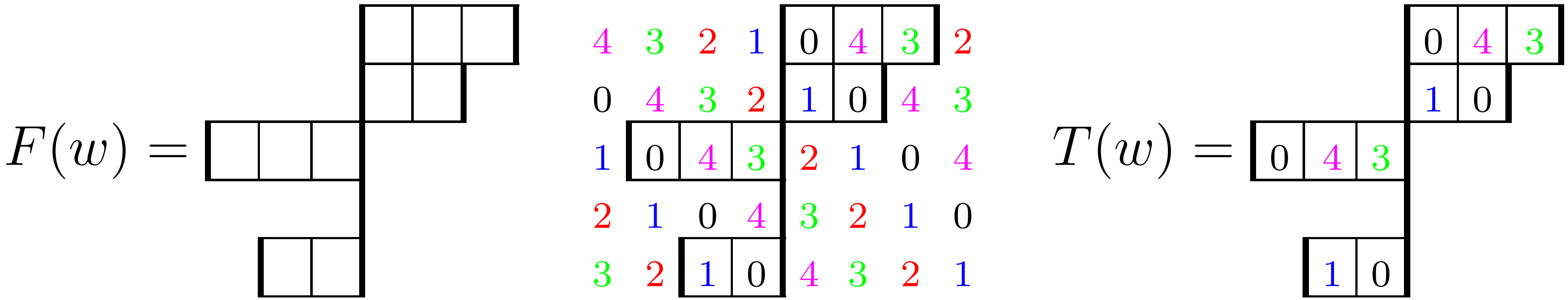
$$F(w) = F^{\lambda(w)}$$

with successive row lengths $\lambda(w)_i$ (positive, zero or negative) all adjusted to a vertical reference line

- In doing this it is observed that all coloured diagrams $T(w)$ may be created by superposing $F(w)$ on a predetermined coloured grid

Example of the use of a coloured grid

If **g** $= A_4^{(1)}$ and $w = s_0 s_3 s_4 s_3 s_1 s_0$ then $\lambda(w) = (3, 2, -3, 0, -2)$



- The numbers are identical on any given diagonal
- periodic along each row and distinct in each column
- and equally balanced to left and right of the vertical axis
- The vertical axis cuts each pair $1\,0$, $2\,1$, $3\,2$, $4\,3$, $0\,4$,
- The profile of $T(w)$, defined by vertical lines of double thickness, cuts the same pairs once each

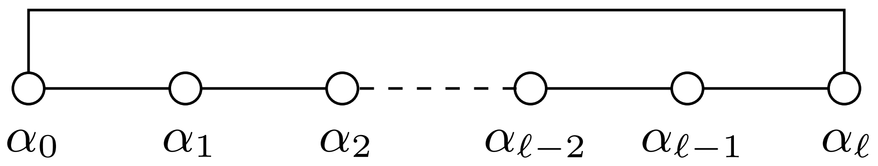
The full coloured grid

- The coloured grid

$$A_4^{(1)}: \qquad \tilde{h}^\vee = 5; \qquad \begin{array}{ccccc|ccccccccccccccc} 0 & 4 & 3 & 2 & 1 & 0 & 4 & 3 & 2 & 1 & 0 & 4 & 3 & 2 & 1 & 0 & 4 & 3 & 2 & 1 \\ 1 & 0 & 4 & 3 & 2 & 1 & 0 & 4 & 3 & 2 & 1 & 0 & 4 & 3 & 2 & 1 & 0 & 4 & 3 & 2 \\ 2 & 1 & 0 & 4 & 3 & 2 & 1 & 0 & 4 & 3 & 2 & 1 & 0 & 4 & 3 & 2 & 1 & 0 & 4 & 3 \\ 3 & 2 & 1 & 0 & 4 & 3 & 2 & 1 & 0 & 4 & 3 & 2 & 1 & 0 & 4 & 3 & 2 & 1 & 0 & 4 \\ 4 & 3 & 2 & 1 & 0 & 4 & 3 & 2 & 1 & 0 & 4 & 3 & 2 & 1 & 0 & 4 & 3 & 2 & 1 & 0 \end{array}$$

- Comparison with the Dynkin diagram

$A_\ell^{(1)}$

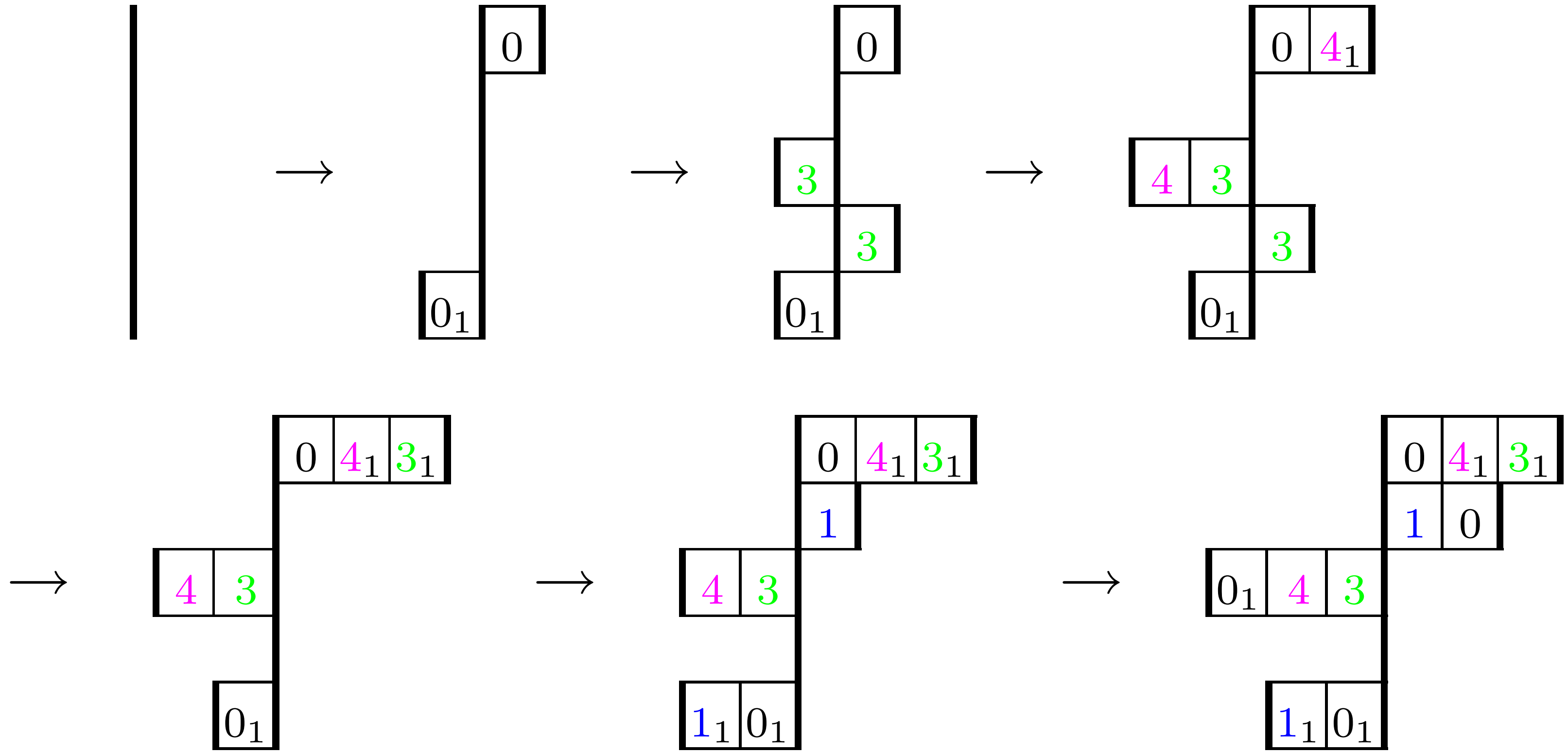


$\ell \geq 2$

- The grid entries across each row from left to right are obtained from the Dynkin diagram by reading subscripts on the simple roots taken in turn anti-clockwise

Laying the depth charges

Ex: Let $\mathbf{g} = A_4^{(1)}$ and $w = s_0 s_3 s_4 s_3 s_1 s_0$. We augment each box with a subscript d indicating a contribution to the depth:



Hence $w(\rho) - \rho = -5\delta + 3\epsilon_1 + 2\epsilon_2 - 3\epsilon_3 - 2\epsilon_5$

The full coloured grid with depth charges

- The contribution to the depth of each entry is given as a subscript d or $\bar{d}$

$$A_4^{(1)}: \qquad \tilde{h}^\vee = 5; \qquad \begin{array}{ccccc|ccccccccccccccc}
0_{\bar{1}} & 4_0 & 3_0 & 2_0 & 1_0 & 0_0 & 4_1 & 3_1 & 2_1 & 1_1 & 0_1 & 4_2 & 3_2 & 2_2 & 1_2 & 0_2 & 4_3 & 3_3 & 2_3 & 1_3 \\
1_{\bar{1}} & 0_{\bar{1}} & 4_0 & 3_0 & 2_0 & 1_0 & 0_0 & 4_1 & 3_1 & 2_1 & 1_1 & 0_1 & 4_2 & 3_2 & 2_2 & 1_2 & 0_2 & 4_3 & 3_3 & 2_3 \\
2_{\bar{1}} & 1_{\bar{1}} & 0_{\bar{1}} & 4_0 & 3_0 & 2_0 & 1_0 & 0_0 & 4_1 & 3_1 & 2_1 & 1_1 & 0_1 & 4_2 & 3_2 & 2_2 & 1_2 & 0_2 & 4_3 & 3_3 \\
3_{\bar{1}} & 2_{\bar{1}} & 1_{\bar{1}} & 0_{\bar{1}} & 4_0 & 3_0 & 2_0 & 1_0 & 0_0 & 4_1 & 3_1 & 2_1 & 1_1 & 0_1 & 4_2 & 3_2 & 2_2 & 1_2 & 0_2 & 4_3 \\
4_{\bar{1}} & 3_{\bar{1}} & 2_{\bar{1}} & 1_{\bar{1}} & 0_{\bar{1}} & 4_0 & 3_0 & 2_0 & 1_0 & 0_0 & 4_1 & 3_1 & 2_1 & 1_1 & 0_1 & 4_2 & 3_2 & 2_2 & 1_2 & 0_2
\end{array}$$

- Properties of the depth charge
  - Both d and $\bar{d}$ contribute d to the depth
  - d is constant along each diagonal
  - d is constant across rows in blocks of length $\tilde{h}^\vee$
  - d increases only between each pair $0_d\,\ell_{d+1}$

Weyl group action on an arbitrary weight

- For any $\Lambda = \sum_{k=0}^{\ell} (\Lambda \,|\, \alpha_k^\vee)\, \Lambda_k - D(\Lambda)\, \delta$, we would like to calculate

$$w(\Lambda) - \Lambda = \sum_{i=1}^{n} \lambda^{\Lambda}(w)_i \, \epsilon_i - d^{\Lambda}(w)\, \delta$$

- We know that for $\rho = \sum_{k=0}^{\ell} 1\, \Lambda_k$ we have

$$w(\rho) - \rho = \sum_{i=1}^{n} \lambda(w)_i \, \epsilon_i - d(w)\, \delta$$

  where $\lambda(w)$ and $d(w)$ are encoded in $T(w)$

- Note: $w(\Lambda) - \Lambda$ is independent of $D(\Lambda)$ since $w(\delta) = \delta$

Scaling factors

- However

$$w(\rho) - \rho \;=\; w'(\rho) - \rho - 1\, w'(\alpha_k)$$

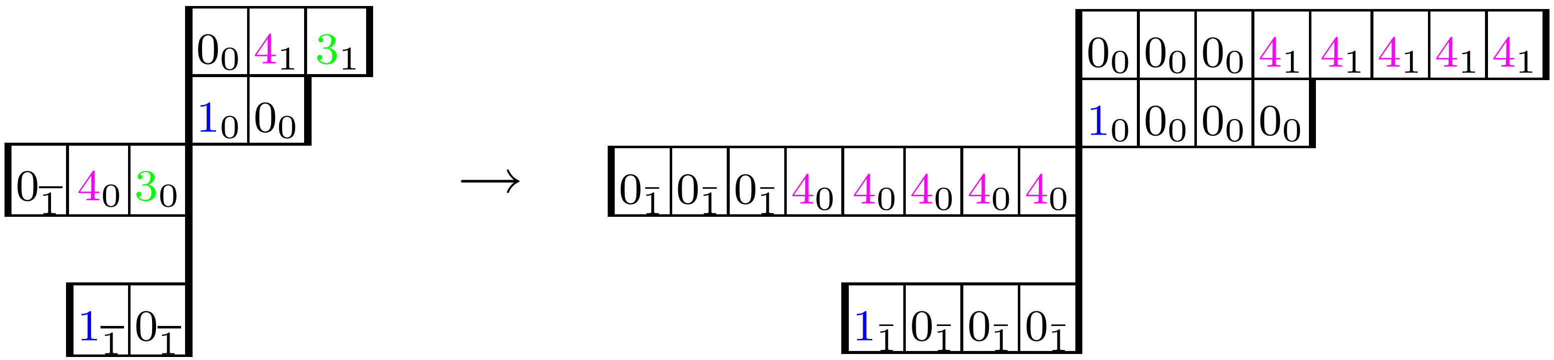
$$w(\Lambda) - \Lambda \;=\; w'(\Lambda) - \Lambda - (\Lambda\,|\,\alpha_k^\vee)\, w'(\alpha_k)$$

- For each $k = 0, 1, \ldots, \ell$ the contribution of the boxes with entries k in $T(w)$ of shape $\lambda(w)$ should be scaled by a factor $(\Lambda\,|\,\alpha_k^\vee)$ to create $T^\Lambda(w)$ of shape $\lambda^\Lambda(w)$
- The contributions to the depth should be scaled by the same factor
- Note also that $w(\Lambda+\rho) - \rho = \Lambda + (w(\Lambda) - \Lambda) + (w(\rho) - \rho)$

Stretched coloured diagrams

Ex: Let **g** $= A_4^{(1)}$ and $w = s_0 s_3 s_4 s_3 s_1 s_0$, (as before)

For $\Lambda = 3\Lambda_0 + \Lambda_1 + 4\Lambda_2 + 5\Lambda_4 = 13\Lambda_0 + 5\epsilon_1 + 4\epsilon_2 - 5\epsilon_5$, the passage from $T(w)$ to $T^\Lambda(w)$ takes the form:



- Thus $\lambda^\Lambda(w) = (8, 4, -8, 0, -4)$ and $d^\Lambda(w) = 12$
- so that $w(\Lambda) - \Lambda = 8\epsilon_1 + 4\epsilon_2 - 8\epsilon_3 - 4\epsilon_5 - 12\delta$
- Since $w(\rho) - \rho = 3\epsilon_1 + 2\epsilon_2 - 3\epsilon_3 - 2\epsilon_5 - 5\delta$ we find
- $w(\Lambda + \rho) - \rho = 13\Lambda_0 + 16\epsilon_1 + 10\epsilon_2 - 11\epsilon_3 - 11\epsilon_5 - 17\delta$

Outline of proof

- Recall that for $w = w' s_k$ we have

$$\begin{aligned} w(\Lambda) - \Lambda &= w'(\Lambda) - \Lambda - (\Lambda \,|\, \alpha_k^\vee)\, w'(\alpha_k) \\ w(\Lambda_j) - \Lambda_j &= w'(\Lambda_j) - \Lambda_j - \delta_{kj}\, w'(\alpha_k) \end{aligned}$$

- But $\alpha_k = \sum_{m=0}^{\ell} A_{mk}\, \Lambda_m + \delta_{k0}\, \delta$ and $w'(\delta) = \delta$, so that

$$w'(\alpha_k) = \alpha_k + \sum_{m=0}^{\ell} A_{mk}\, (w'(\Lambda_m) - \Lambda_m)$$

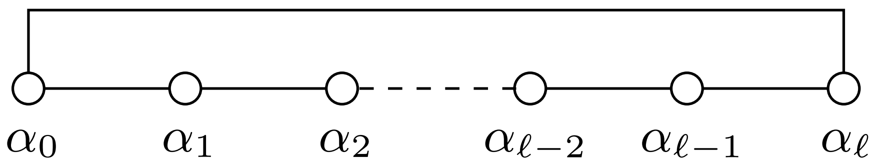
- Now let $w(\Lambda_j) - \Lambda_j = \sum_{i=1}^{n} N_{ij}(w)\, \epsilon_i - P_j(w)\, \delta$ where $N_{ij}(w)$ determines the number of boxes coloured j in the ith row, and $P_j(w)$ the corresponding depth charge
- We then have a recurrence relation for $N_{ij}(w)$ and $P_j(w)$ which can be related to the recursive way of constructing $T(w)$ from $T(w')$ using the A_{mk}

Coloured grid for $A_4^{(1)}$

$$A_4^{(1)}: \quad \tilde{h}^\vee = 5; \quad \begin{array}{ccccc|ccccccccccccccc}
0 & 4 & 3 & 2 & 1 & 0 & 4 & 3 & 2 & 1 & 0 & 4 & 3 & 2 & 1 & 0 & 4 & 3 & 2 & 1 \\
1 & 0 & 4 & 3 & 2 & 1 & 0 & 4 & 3 & 2 & 1 & 0 & 4 & 3 & 2 & 1 & 0 & 4 & 3 & 2 \\
2 & 1 & 0 & 4 & 3 & 2 & 1 & 0 & 4 & 3 & 2 & 1 & 0 & 4 & 3 & 2 & 1 & 0 & 4 & 3 \\
3 & 2 & 1 & 0 & 4 & 3 & 2 & 1 & 0 & 4 & 3 & 2 & 1 & 0 & 4 & 3 & 2 & 1 & 0 & 4 \\
4 & 3 & 2 & 1 & 0 & 4 & 3 & 2 & 1 & 0 & 4 & 3 & 2 & 1 & 0 & 4 & 3 & 2 & 1 & 0
\end{array}$$

- The corresponding Dynkin diagram:

$A_\ell^{(1)}$



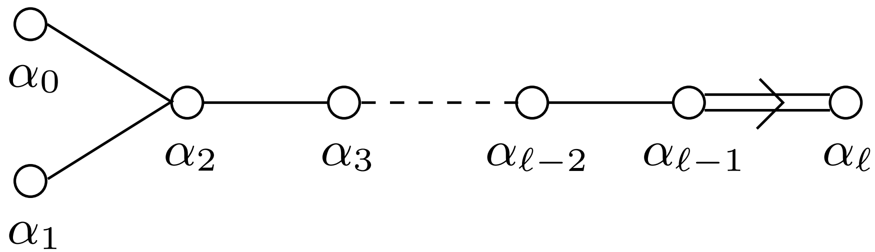
$\ell \geq 2$

Coloured grid for $B_4^{(1)}$

$$B_4^{(1)}: \quad \tilde{h}^\vee = 7; \quad \begin{array}{ccccc|cccccccccccccc}
3 & 4 & 3 & 2 & 1 & 0 & 2 & 3 & 4 & 3 & 2 & 1{\sim}0 & 2 & 3 & 4 & 3 & 2 & 1{\sim}0 \\
2 & 3 & 4 & 3 & 2 & 1{\sim}0 & 2 & 3 & 4 & 3 & 2 & 1{\sim}0 & 2 & 3 & 4 & 3 & 2 & 1{\sim} \\
{\sim}0 & 2 & 3 & 4 & 3 & 2 & 1{\sim}0 & 2 & 3 & 4 & 3 & 2 & 1{\sim}0 & 2 & 3 & 4 & 3 & 2 \\
1{\sim}0 & 2 & 3 & 4 & & 3 & 2 & 1{\sim}0 & 2 & 3 & 4 & 3 & 2 & 1{\sim}0 & 2 & 3 & 4 & 3
\end{array}$$

- The corresponding Dynkin diagram:

$B_\ell^{(1)}$



$\ell \geq 2$

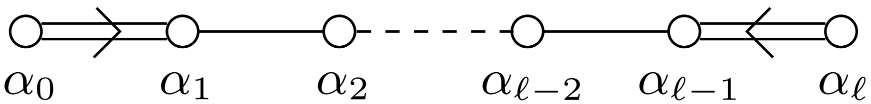
- Note the appearence of a tied pair $1{\sim}0$ which should be viewed as unordered and could be entered as $0{\sim}1$

Coloured grids

$$C_4^{(1)}: \quad \tilde{h}^\vee = 10; \qquad \begin{array}{ccccc|ccccccccccccccc}
4\!-\!4 & 3 & 2 & 1 & & 0\!-\!0 & 1 & 2 & 3 & 4\!-\!4 & 3 & 2 & 1 & 0\!-\!0 & 1 & 2 & 3 \\
3 & 4\!-\!4 & 3 & 2 & & 1 & 0\!-\!0 & 1 & 2 & 3 & 4\!-\!4 & 3 & 2 & 1 & 0\!-\!0 & 1 & 2 \\
2 & 3 & 4\!-\!4 & 3 & & 2 & 1 & 0\!-\!0 & 1 & 2 & 3 & 4\!-\!4 & 3 & 2 & 1 & 0\!-\!0 & 1 \\
1 & 2 & 3 & 4\!-\!4 & & 3 & 2 & 1 & 0\!-\!0 & 1 & 2 & 3 & 4\!-\!4 & 3 & 2 & 1 & 0\!-\!0
\end{array}$$

- The corresponding Dynkin diagram:

$C_\ell^{(1)}$



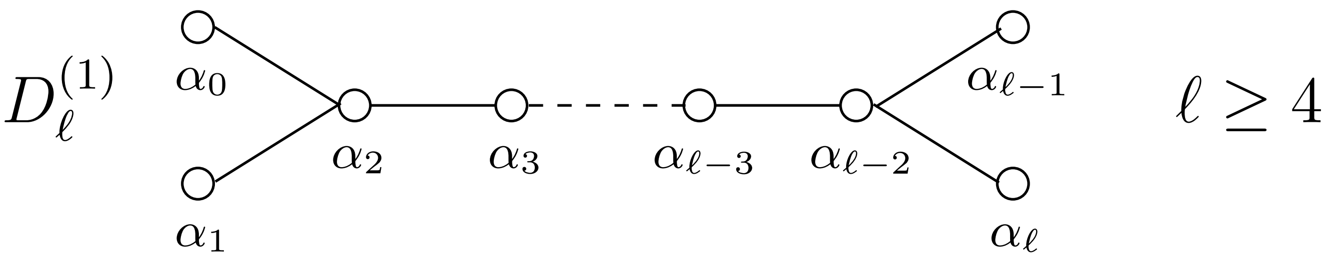
$\ell \geq 2$

- Note the appearence of joined pairs 0–0 and 4–4

Coloured grid for $D_4^{(1)}$

$$D_4^{(1)}: \quad \tilde{h}^\vee = 6; \quad \begin{array}{r|l} 2\ \ 4{\sim}3\ \ 2\ \ 1 & 0\ \ 2\ \ 4{\sim}3\ \ 2\ \ 1{\sim}0\ \ 2\ \ 4{\sim}3\ \ 2\ \ 1{\sim}0\ \ 2\ \ 4{\sim} \\ {\sim}0\ \ 2\ \ 4{\sim}3\ \ 2 & 1{\sim}0\ \ 2\ \ 4{\sim}3\ \ 2\ \ 1{\sim}0\ \ 2\ \ 4{\sim}3\ \ 2\ \ 1{\sim}0\ \ 2 \\ 1{\sim}0\ \ 2\ \ 4{\sim}3 & 2\ \ 1{\sim}0\ \ 2\ \ 4{\sim}3\ \ 2\ \ 1{\sim}0\ \ 2\ \ 4{\sim}3\ \ 2\ \ 1{\sim}0 \\ 2\ \ 1{\sim}0\ \ 2\ \ 4 & 3\ \ 2\ \ 1{\sim}0\ \ 2\ \ 4{\sim}3\ \ 2\ \ 1{\sim}0\ \ 2\ \ 4{\sim}3\ \ 2\ \ 1{\sim} \end{array}$$

- The corresponding Dynkin diagram:



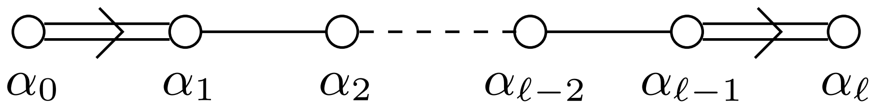
- Note the appearance of two tied pairs $1{\sim}0$ and $4{\sim}3$

Coloured grid for $A_8^{(2)}$

$$A_8^{(2)}: \quad \tilde{h}^\vee = 9; \quad \begin{array}{ccccc|cccccccccccccccc}
3 & 4 & 3 & 2 & 1 & 0\!-\!0 & 1 & 2 & 3 & 4 & 3 & 2 & 1 & 0\!-\!0 & 1 & 2 & 3 & 4 \\
2 & 3 & 4 & 3 & 2 & 1 & 0\!-\!0 & 1 & 2 & 3 & 4 & 3 & 2 & 1 & 0\!-\!0 & 1 & 2 & 3 \\
1 & 2 & 3 & 4 & 3 & 2 & 1 & 0\!-\!0 & 1 & 2 & 3 & 4 & 3 & 2 & 1 & 0\!-\!0 & 1 & 2 \\
-0 & 1 & 2 & 3 & 4 & 3 & 2 & 1 & 0\!-\!0 & 1 & 2 & 3 & 4 & 3 & 2 & 1 & 0\!-\!0 & 1
\end{array}$$

- The corresponding Dynkin diagram:

$A_{2\ell}^{(2)}$



$\ell \geq 2$

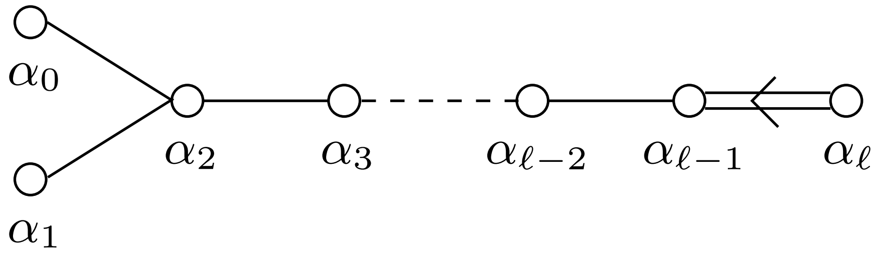
- Note the appearance of a joined pair 0–0

Coloured grids for $A_7^{(2)}$

$$A_7^{(2)}: \qquad \tilde{h}^\vee = 8; \qquad \begin{array}{ccccc|cccccccccccccccccc} 4\text{—}4 & 3 & 2 & 1 & & 0 & 2 & 3 & 4\text{—}4 & 3 & 2 & 1\sim 0 & 2 & 3 & 4\text{—}4 & 3 & 2 \\ 3 & 4\text{—}4 & 3 & 2 & & 1\sim 0 & 2 & 3 & 4\text{—}4 & 3 & 2 & 1\sim 0 & 2 & 3 & 4\text{—}4 & 3 & \\ 2 & 3 & 4\text{—}4 & 3 & & 2 & 1\sim 0 & 2 & 3 & 4\text{—}4 & 3 & 2 & 1\sim 0 & 2 & 3 & 4\text{—}4 & \\ \sim 0 & 2 & 3 & 4\text{—}4 & & 3 & 2 & 1\sim 0 & 2 & 3 & 4\text{—}4 & 3 & 2 & 1\sim 0 & 2 & 3 & 4\text{—} \end{array}$$

- The corresponding Dynkin diagram:

$A_{2\ell-1}^{(2)}$



$\ell \geq 3$

- Note the appearance of a joined pair $4\text{—}4$ and a tied pair $1{\sim}0$

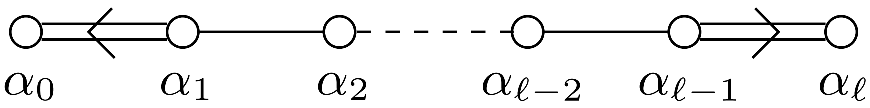
Coloured grid for $D_5^{(2)}$

$D_5^{(2)}: \quad \tilde{h}^\vee = 8;$

3	4	3	2	1	0	1	2	3	4	3	2	1	0	1	2	3	4	3	2
2	3	4	3	2	1	0	1	2	3	4	3	2	1	0	1	2	3	4	3
1	2	3	4	3	2	1	0	1	2	3	4	3	2	1	0	1	2	3	4
0	1	2	3	4	3	2	1	0	1	2	3	4	3	2	1	0	1	2	3

- The corresponding Dynkin diagram:

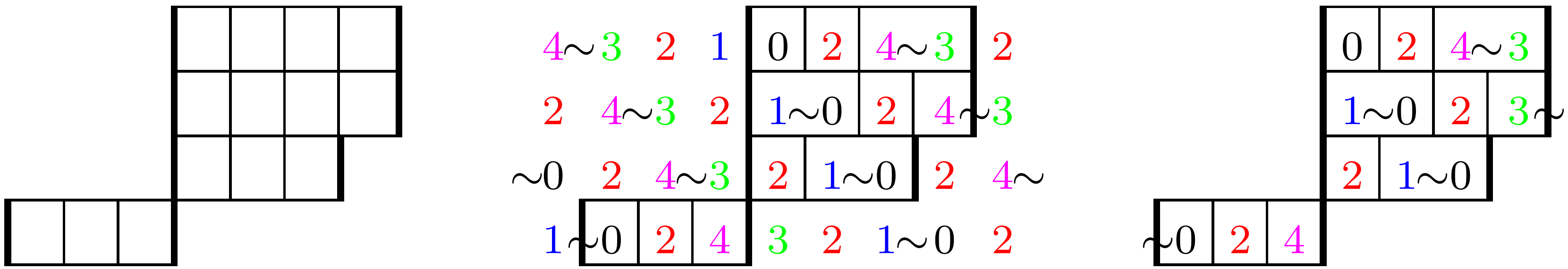
$D_{\ell+1}^{(2)}$



$\ell \geq 2$

Use of a coloured grid involving ties

Ex: For $\mathbf{g} = D_4^{(1)}$ and $w = s_0 s_2 s_1 s_4 s_2 s_3 s_0$, we find that $\lambda(w) = (4, 4, 3, -3)$ and the superposition of $F(w)$ on the coloured grid gives $T(w)$ as shown:



- Note that 3 and 0 have been included from the unordered tied pairs $4 \sim 3$ and $1 \sim 0$
- This selection has been made so that each number appears an even number of times in $T(w)$

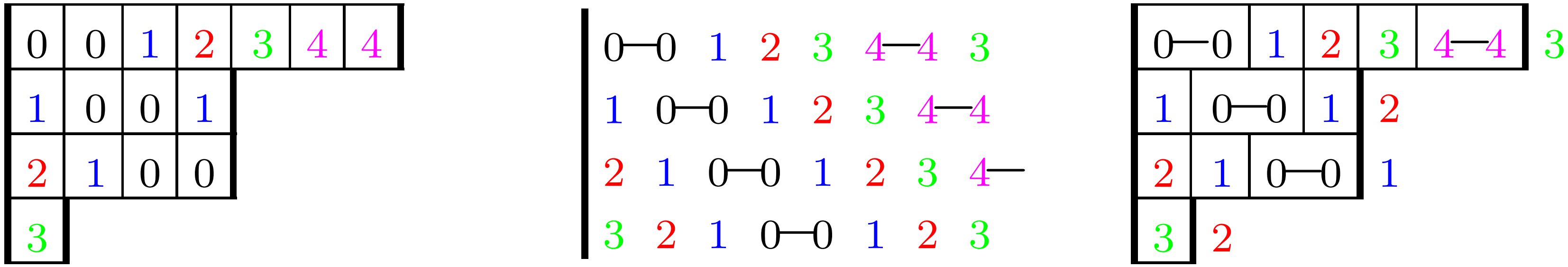
Action of the Weyl group on coloured diagrams

Theorem Let $w = w's_k$, then the action of s_k on $T(w')$ to give $T(w)$ is determined as follows:

- superpose $T(w')$ on the appropriate coloured grid
- identify all those k-nodes adjacent to the profile of $T(w')$
- where a k-node in the coloured grid is said to be adjacent to the profile of $T(w')$ if it is any one of the following:
  - next to one of the vertical edges that define the profile
  - one of a joined pair next to such an edge
  - one of a tied pair next to such an edge
- append to $T(w')$ all adjacent k-nodes outside $T(w')$
- delete from $T(w')$ all adjacent k-nodes inside $T(w')$

Action of the Weyl group on coloured diagrams

Ex: For $C_4^{(1)}$ with $w' = s_0 s_1 s_2 s_3 s_4 s_0 s_1 s_0$ the coloured diagram $T(w')$ on the left may be superposed on the full grid to identify the adjacent nodes as shown on the right



- For $w = w's_0$ delete the one adjacent joined pair 0–0
- For $w = w's_3$ append the one adjacent external 3-node and delete the one adjacent internal 3-node

$T(w's_0) =$

0	0	1	2	3	4	4
1	0	0	1			
2	1					
3						

and $T(w's_3) =$

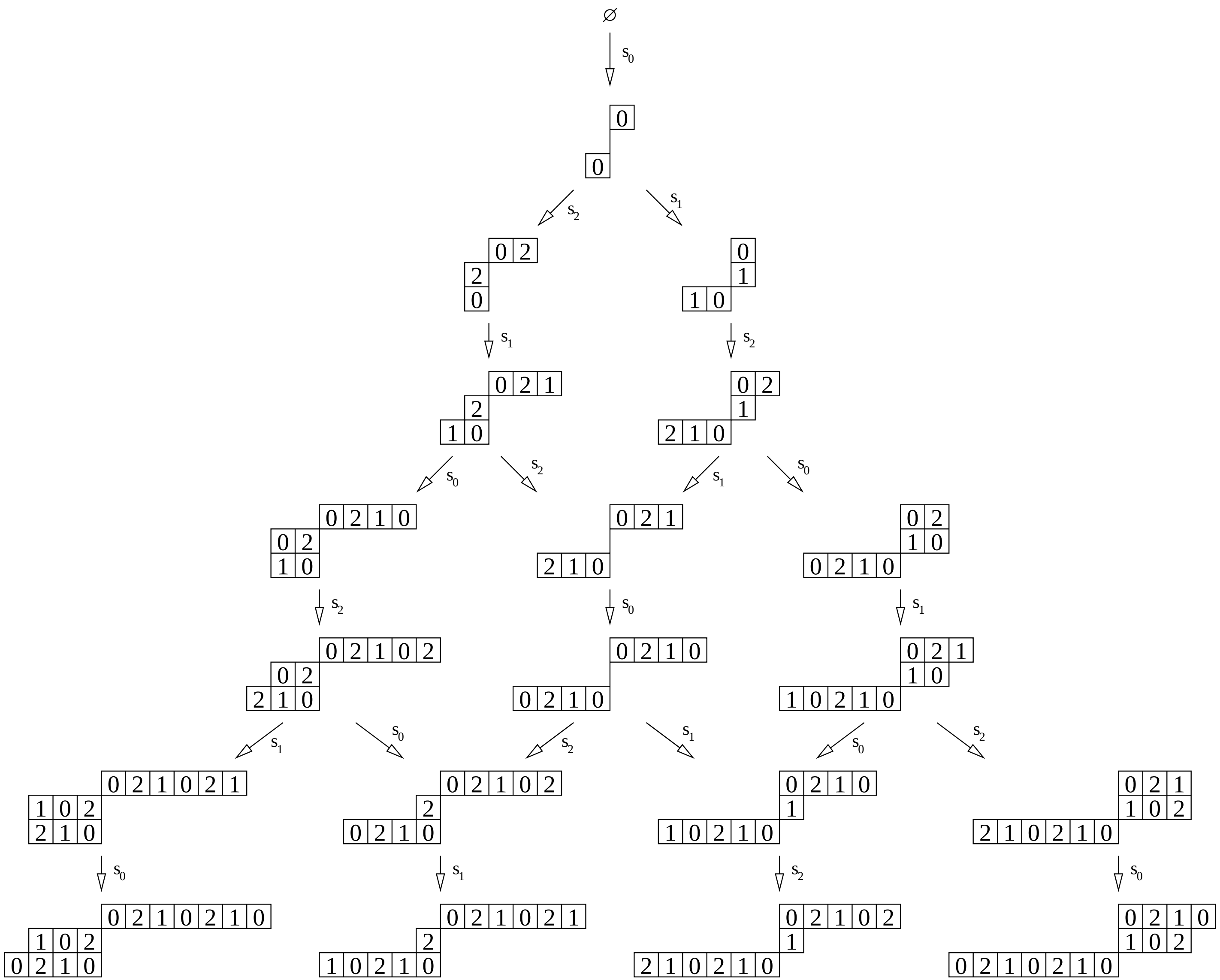
0	0	1	2	3	4	4	3
1	0	0	1				
2	1	0	0				

Action of coset representatives

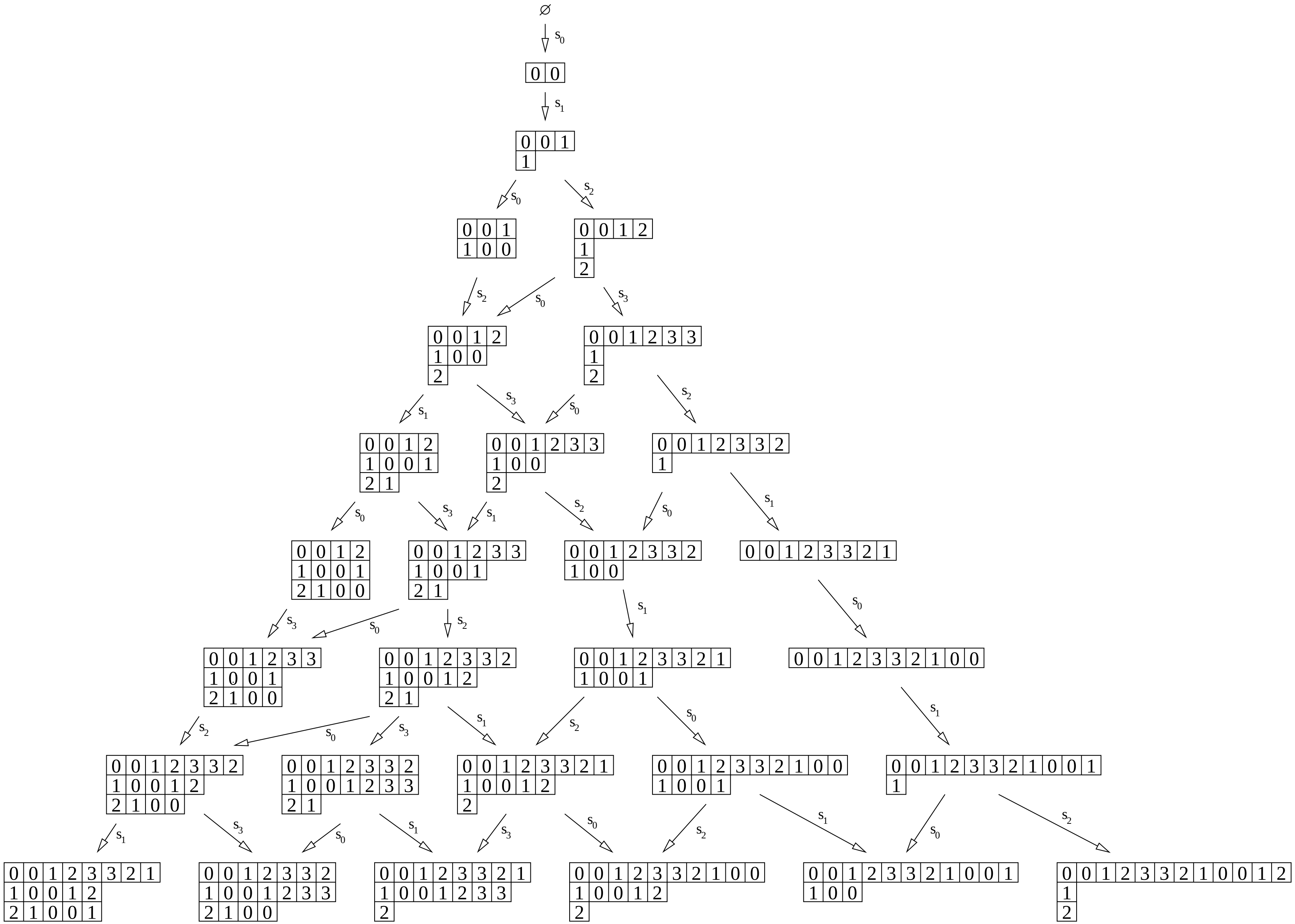
- In our character formula we only require the action of $w \in W_s$
- Our coloured diagrams $T(w)$ provide a way of evaluating $w(\rho) - \rho$
- The generalised partitions $\lambda(w)$ specify $\overline{w(\rho) - \rho}$ in the natural basis
- It is therefore important to determine $\lambda(w)$ for all $w \in W_s$
- This may be done recursively by using the action of s_k on $T(w')$ to give $T(w)$ for $w = w's_k$ with $\ell(w) = \ell(w') + 1$

The results can be exhibited by means of a Bruhat graph

Bruhat graph for $A_2^{(1)}$

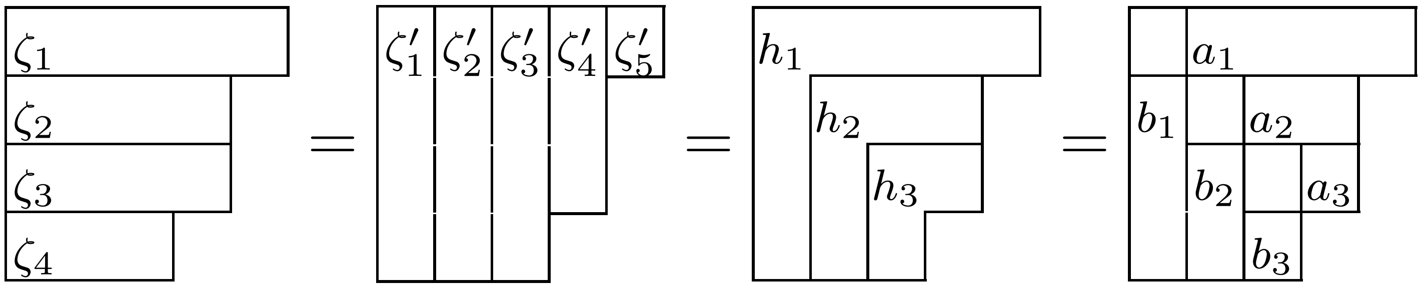


Bruhat graph for $C_3^{(1)}$



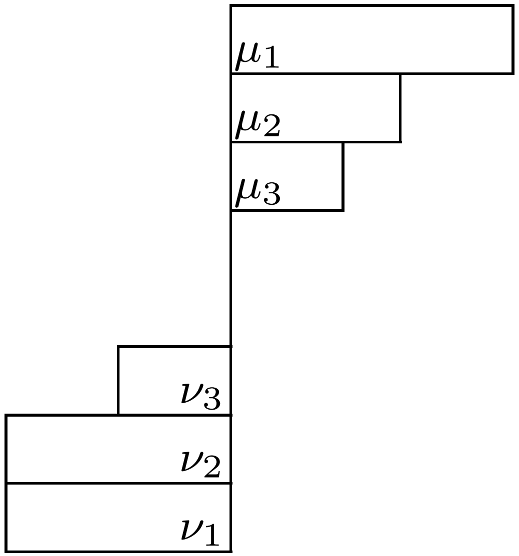
Young diagrams

- Let $\mathcal{P}$ be the set of all partitions ζ, including (0).
- Each partition ζ specifies a Young diagram F^ζ
- The row lengths of F^ζ are the parts ζ_i
- The column lengths ζ'_j define the conjugate partition ζ'.
- In Frobenius notation $\zeta = \begin{pmatrix} a_1 & a_2 & \cdots & a_r \\ b_1 & b_2 & \cdots & b_r \end{pmatrix}$ where F^ζ consists of r hooks of lengths $h_k = a_k + b_k + 1$

$F^\zeta =$ 

Generalised Young diagrams

- The generalised partition $\lambda = (\mu; \overline{\nu})$, with μ and ν partitions, specifies a composite Young diagram $F^{\mu;\overline{\nu}}$ with positive, zero and negative row lengths



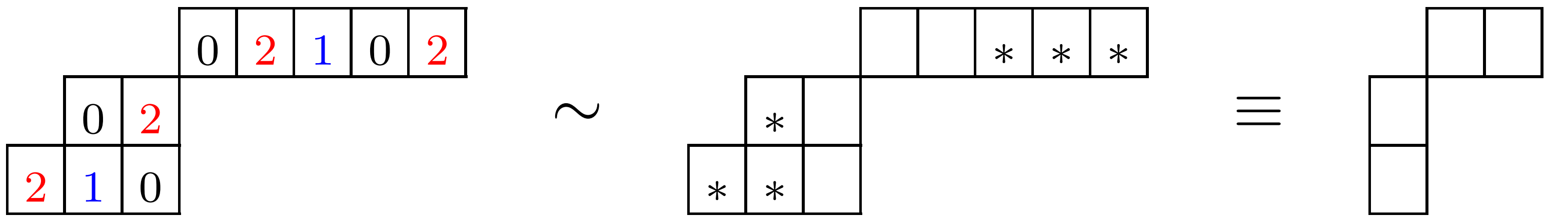
- For $\lambda = (\mu; \overline{\nu})$ to be a generalised partition with n parts, we must have $\ell(\mu) + \ell(\nu) \leq n$, where $\ell(\mu)$ and $\ell(\nu)$ are the number of non-zero parts of μ and ν

Some partitions and their cores

- For any partitions λ and ζ we say that ζ is the h-core of λ and write $\lambda \equiv \zeta \bmod (h)$ if the removal of all possible continuous boundary strips of length h from F^λ yields F^ζ
- For any generalised partitions $(\mu;\overline{\nu})$ and $(\sigma;\overline{\tau})$ we say that $(\sigma;\overline{\tau})$ is the balanced h-core of $(\mu;\overline{\nu})$ and write $(\mu;\overline{\nu}) \equiv (\sigma;\overline{\tau}) \bmod (h)$ if the removal of all possible pairs of continuous boundary strips of length h, one from F^μ and one from F^ν, yields F^σ and F^τ, respectively.
- Let $\mathcal{P}_t$ denote the set of all partitions which in Frobenius notation have $a_k - b_k = t$ for all $k = 1, 2, \ldots, r$, together with the partition (0).

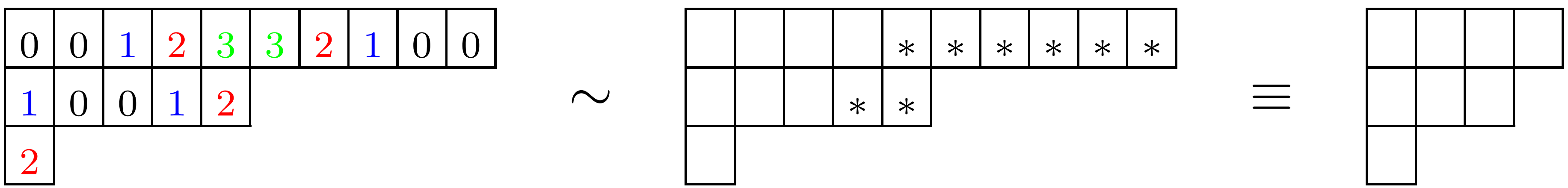
Examples

Ex: For $A_2^{(1)}$ we have $\tilde{h}^\vee = 3$. In the case $\lambda(w) = (5, -2, -3)$, its 3-core is of the form $(\zeta; \overline{\zeta'})$ with $\zeta = (2)$ and $\zeta' = (1, 1)$



Ex: For $C_3^{(1)}$ we have $\tilde{h}^\vee = 8$. In the case $\lambda(w) = (10, 5, 1)$,

its 8-core is $\zeta = (4, 3, 1) = \begin{pmatrix} 3 & 1 \\ 2 & 0 \end{pmatrix} \in \mathcal{P}_1$



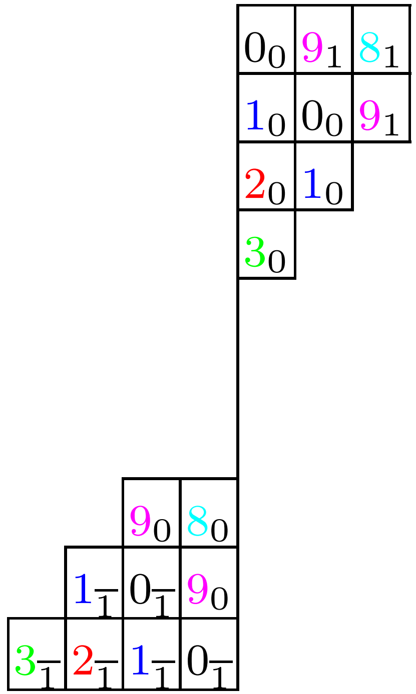
Theorem

Let **g** be an affine algebra and $\overline{\mathbf{g}}$ a naturally embedded simple subalgebra with minimal right coset representatives W_s. Let $\mathcal{P}(\mathbf{g}) = \{\, \lambda(w) \,|\, w \in W_s\}$, then all $\lambda(w)$ are listed as follows:

g	$\lambda(w)$	$\lambda(w) \bmod (h^\vee)$	$\tilde{h}^\vee$	constraint
$A_\ell^{(1)}$:	$\mu;\overline{\nu}$	$\zeta;\overline{\zeta'}\ \ \zeta \in \mathcal{P}$	$\ell+1$	$\ell(\mu)+\ell(\nu) \le \ell+1$
$B_\ell^{(1)}$:	λ	$\zeta \in \mathcal{P}_{-1}$	$2\ell-1$	$\ell(\lambda) \le \ell$
$C_\ell^{(1)}$:	λ	$\zeta \in \mathcal{P}_1$	$2\ell+2$	$\ell(\lambda) \le \ell$
$D_\ell^{(1)}$:	λ	$\zeta \in \mathcal{P}_{-1}$	$2\ell-2$	$\ell(\lambda) \le \ell,\ \ell(\zeta) < \ell$
$A_{2\ell}^{(2)}$:	λ	$\zeta \in \mathcal{P}_1$	$2\ell+1$	$\ell(\lambda) \le \ell$
$A_{2\ell-1}^{(2)}$:	λ	$\zeta \in \mathcal{P}_{-1}$	2ℓ	$\ell(\lambda) \le \ell$
$D_{\ell+1}^{(2)}$:	λ	$\zeta \in \mathcal{P}_0$	2ℓ	$\ell(\lambda) \le \ell$

Evaluation of $w(\rho) - \rho$

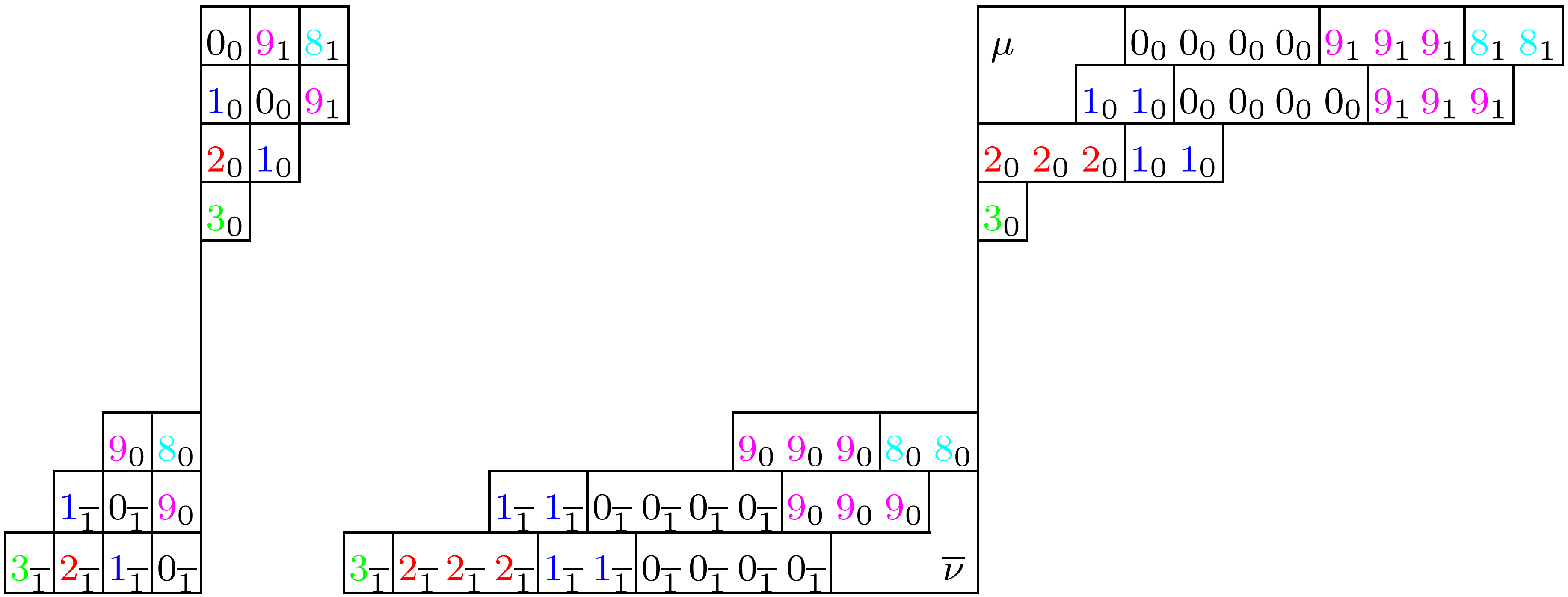
Ex: For $A_9^{(1)}$ and $\lambda(w) = (3,3,2,1,0,0,0,\overline{2},\overline{3},\overline{4}) = (\zeta;\overline{\zeta'})$ with $\zeta = (3,3,2,1)$ and $\zeta' = (4,3,2)$ we can write down $T(w)$ as shown:



$$w(\rho) - \rho = 3\epsilon_1 + 3\epsilon_2 + 2\epsilon_3 + \epsilon_4 - 2\epsilon_7 - 3\epsilon_8 - 5\epsilon_9 - 9\delta$$

Evaluation of $w(\Lambda+\rho)-\rho$

Let $\Lambda = 3\Lambda_0 + \Lambda_1 + 2\Lambda_2 + \Lambda_8 + 2\Lambda_9 = 9\Lambda_0 + 3\epsilon_1 + 2\epsilon_2 - \epsilon_9 - 3\epsilon_{10}$ $= 9\Lambda_0 + (\mu;\overline{\nu}) = 9\Lambda_0 + (3,2;\overline{1},\overline{3})$, with $L(\Lambda)=9$, $D(\Lambda)=0$.
Stretching the diagram horizontally we find



$$w(\Lambda+\rho)-\rho = 9\Lambda_0 + 12\epsilon_1 + 11\epsilon_2 + 5\epsilon_3 + \epsilon_4 - 5\epsilon_7 - 10\epsilon_8 - 13\epsilon_9 - 24\delta$$

Evaluation of $w(\rho) - \rho$

Ex: For $C_4^{(1)}$ andand $\lambda(w) = (5, 4, 2, 1)$ we can identify $T(w)$ as shown:

0_0	0_1	1_1	2_1	3_1
1_0	0_0	0_1	1_1	
2_0	1_0			
3_0				

Hence $w(\rho) - \rho = 5\epsilon_1 + 4\epsilon_2 + 2\epsilon_3 + \epsilon_4 - 6\delta$

Evaluation of $w(\Lambda + \rho) - \rho$

Let $\Lambda = 2\Lambda_0 + 3\Lambda_1 + 2\Lambda_3$

$$= 7\Lambda_0 + 5\epsilon_1 + 2\epsilon_2 + 2\epsilon_3$$

$$= 9\Lambda_0 + \mu = 9\Lambda_0 + (5, 2, 2) \text{ with } L(\Lambda) = 7, \ D(\Lambda) = 0.$$

Stretching the diagram horizontally we find

0_0	0_1	1_1	2_1	3_1
1_0	0_0	0_1	1_1	
2_0	1_0			
3_0				

μ		$0_0\ 0_0\ 0_0$	$0_1\ 0_1\ 0_1$	$1_1\ 1_1\ 1_1\ 1_1$	2_1	$3_1\ 3_1\ 3_1$
	$1_0\ 1_0\ 1_0\ 1_0$	$0_0\ 0_0\ 0_0$	$0_1\ 0_1\ 0_1$	$1_1\ 1_1\ 1_1\ 1_1$		
	2_0	$1_0\ 1_0\ 1_0\ 1_0$				
$3_0\ 3_0\ 3_0$						

$$w(\Lambda + \rho) - \rho = 7\Lambda_0 + 19\epsilon_1 + 16\epsilon_2 + 7\epsilon_3 + 3\epsilon_4 - 18\delta$$

Conclusions

- Coloured grids enable us to define a bijection between elements w of the Weyl group W of a classical Kac-Moody algebra and coloured diagrams $T(w)$
- The action of the generators s_k of W on $T(w)$ involves just appending and deleting k-nodes adjacent to the profile of $T(w)$
- The shapes of $T(w)$ as determined by the generalised partitions $\lambda(w)$ have a particularly simple form for all $w \in W_s$, the set of minimal right coset representatives associated with the natural embedding of $\overline{\mathbf{g}}$ in $\mathbf{g}$

Conclusions

- Using these $T(w)$ one can write down both $w(\rho) - \rho$ and $w(\Lambda + \rho) - \rho$ and thereby express both the numerator, M^Λ, and the denominator, M, of characters of **g** in terms of those of $\overline{\mathbf{g}}$
- If M is treated as an infinite series in powers of $q = e^{-\delta}$ this series may be inverted to give M^{-1} to any required depth
- Then $\mathrm{ch}\, V^\Lambda = M^\Lambda\, M^{-1}$ may be evaluated to give characters of the affine algebras **g** using nothing more than well known products of characters of simple Lie algebras $\overline{\mathbf{g}}$

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