Sequential Construction of a Confidence of Given Width

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1 Confidence Intervals of Given Width; σ^2 Known

We consider the sequential construction of a confidence interval of *given* width w, with *given* level of confidence.

Consider a sequence of observations

$$X_1, X_2, ..., X_n$$

where

$$X_i = \mu + \varepsilon_i$$

with all the ε_i having the same distribution with mean 0 and variance σ^2. (Typically but not necessarily normal.) Let the sample mean after n observations be

$$D_n = n^{-1} \sum_{i=1}^{n} X_i = \mu + n^{-1} \sum_{i=1}^{n} \varepsilon_i. \tag{1}$$

Consider the construction of a confidence interval for the unknown μ of form $(D_n - h_1,\, D_n + h_2)$ where $h_1, h_2 > 0$ so that the width of the interval is

$$w = h_1 + h_2.$$

We can obtain such a confidence interval from the probability statement

$$\Pr(z_\beta \leq \sqrt{n}(D_n - \mu)/\sigma \leq z_{1-\alpha}) = 1 - \alpha - \beta \tag{2}$$

where $\sqrt{n}(D_n - \mu)/\sigma$ is the standardised form of D_n so that it is distributed as a $N(0,1)$ variable, and where z_β and $z_{1-\alpha}$ are the β and $1-\alpha$ quantiles of the $N(0,1)$ distribution.

We shall assume that

$$\alpha,\ \beta < 1/2 \tag{3}$$

so that $z_\beta < 0$ and $z_{1-\alpha} > 0$. The probability statement (2) can be written as

$$\Pr(D_n - h_1 \leq \mu \leq D_n + h_2) = 1 - \alpha - \beta$$

provided we set

$$h_1 = -\frac{z_{1-\alpha}}{z_\beta} h_2 \tag{4}$$

and

$$h_2 = \sigma z_\beta / \sqrt{n}. \tag{5}$$

Both are positive under assumption (3) and the width of the interval is then

$$w = h_1 + h_2.$$

which is dependent only on α, β , σ^2 and n. The values of α and β are user selected. Thus if σ^2 is known, a confidence interval of given width w can be obtained simply by selecting n to satisfy (5). (Rounding up may be needed to ensure that n is integer.): We call this the *Known Sigma Stopping Rule*:

$$\text{Stop at } n = \sigma^2 z_\beta^2 / h_2^2 \tag{6}$$

The relationship between σ and n can be made explicit by combining (4) and (5) to eliminate h_2.This yields

$$n = \frac{(z_{1-\alpha} - z_\beta)^2 \sigma^2}{w^2}. \tag{7}$$

2 Case σ^2 Unknown

Consider now the situation where σ^2 is not known.

The sample variance,

$$s_n^2 = \frac{1}{n-1} \sum_{i=1}^{n} (d_i - D_n)^2$$

can now be used as a first order approximation for σ^2. This changes with n. A simple way to determine n is by the stopping rule:

$$\text{Take } N \text{ as the first } n \text{ for which } s_n^2 \leq \frac{nw^2}{(z_{1-\alpha} - z_\beta)^2}. \tag{8}$$

Here N is a random variable. Chow and Robbins (1965) give the theoretical justification for selecting N in this way, at least for the case $\alpha = \beta$. Further properties of the distribution of N are discussed by Starr (1966a and 1966b) and Woodroofe (1977) (again for the case $\alpha = \beta$). In particular Starr (1966a) showed that use of N gives reasonably consistent coverage and that it is more efficient than the well-known two-stage procedure proposed by Stein (1945). However this choice of N does seem liable to produce somewhat narrow confidence intervals so that the coverage is less than the nominal value of $(1 - 2\alpha)$.Use of N based on (8) does not seem to be much used in practice.

Anscombe (1953) provides a second order accurate version. The method is not often mentioned but performs significantly better and moreover is very easily implemented. It is this version that we recommend and which we now describe.

We transform the $X_1,\ X_2, ...X_n$ of (1) to independent (actually $\sigma\chi_1^2$) variates

$$U_i = \frac{1}{i(i+1)}\{iX_{i+1} - \sum_{j=1}^{i} X_j\}^2 \quad i = 1, 2, ..., n-1. \tag{9}$$

so that if additional observations $X_{n+1},\ X_{n+2}, ...$ are observed, additional $U_n, U_{n+1}, ...$can be added without changing the previous U_i. Anscombe (1953) discusses different second order versions of the rule (8). We use Anscombe's (2.6) rule, only for general $\alpha,\ \beta$:

Take N as the first n ($\geq$ 3) for which (10)

$$\sum_{i=1}^{n-1} U_i \ \leq \ \frac{w^2}{(z_{1-\alpha} - z_\beta)^2}(n-3)(n + 0.324 - \frac{1}{8}(z_{1-\alpha} - z_\beta)^2)$$

3 References

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