ON GENERATORS OF CRYSTALLOGRAPHIC GROUPS AND ACTIONS ON FLAT ORBIFOLDS

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ABSTRACT. We find new bounds on the minimal number of generators of crystallographic groups with p-group holonomy. We also show that similar bounds exist on the minimal number of generators of the abelianizations of arbitrary crystallographic groups. As a consequence, we show that this restricts the rank of elementary abelian p-groups that can act effectively on closed connected flat orbifolds.

1. INTRODUCTION

A closed connected flat n-orbifold M is a quotient of $\mathbb{R}^n$ by a cocompact action of a discrete subgroup of isometries of $\mathbb{R}^n$. A group Γ admitting such an action is known as a *crystallographic* group and it is the orbifold fundamental group of M. When Γ is torsion-free, it is called a *Bieberbach* group and it is the fundamental group of the corresponding flat manifold. By the first Bieberbach theorem (see [2]), every n-dimensional crystallographic group has a normal subgroup T of translations which is a lattice of $\mathbb{R}^n$ and the holonomy group Γ/T is finite.

According to a result of Gromov, $\sqrt{2n\pi}2^{n-2}$ is an upper bound on the minimal number of generators of the fundamental group of any nonnegatively curved n-manifold (see [10, p. 21]). In case of flat manifolds, the existing bounds are considerably lower. In [4], Dekimpe and Penninckx proved that every n-dimensional crystallographic group with elementary abelian p-group holonomy, for any odd prime p, is generated by n elements. They also showed that the same bound holds for every n-dimensional Bieberbach group with elementary abelian 2-group holonomy.

The aim of the present paper is to find attainable bounds on the minimal number of generators of crystallographic groups and their abelianizations and apply this in the study of finite subgroups of symmetries of closed connected flat orbifolds.

We always assume that all actions on orbifolds are by orbifold homeomorphisms. Also, throughout, if not specified, p denotes a prime number, A is an elementary abelian p-group and β_1 denotes the first Betti number of the crystallographic group under consideration. Let $a = 2$ when $p \leq 19$ and $a = 3$ otherwise.

Theorem A. Let Γ be an n-dimensional crystallographic group such that its holonomy group G is a p-group. Then Γ can be generated by $a(n - \beta_1)/(p-1) + \beta_1$ elements.

We note that this bound is $2n - \beta_1$ when $p = 2$ and it is at most n when p is odd. Hence, Theorem A is a generalization and an improvement of Theorem 4.1 of [4].

For any crystallographic group Γ, the center $Z(\Gamma)$ is a free abelian group of rank β_1 (see [2]). Moreover, the group $\Gamma/Z(\Gamma)$ is a crystallographic group with trivial center. Therefore, the problem of finding bounds on the minimal number of generators of crystallographic groups may be reduced to finding such bounds for crystallographic groups that have trivial center. For such n-dimensional crystallographic groups, the theorem asserts that they are $an/(p-1)$-generated. In fact, we can show that this bound is attained in arbitrarily large dimensions (see Example 2.4).

Given any, possibly trivial, finite abelian group F, we denote $rk(F) = \max\{\text{rk}(F \otimes \mathbb{Z}_q) \mid \text{prime } q\}$. Passing to the abelianizations of crystallographic groups, we obtain:

Theorem B. Let Γ be an n-dimensional crystallographic group with holonomy group G and F be the torsion subgroup of the abelianization Γ_{ab} of Γ. Let p be a prime such that $rk(F) = \text{rk}(F \otimes \mathbb{Z}_p)$ and Γ' to be the crystallographic subgroup of Γ which is the preimage of a Sylow p-subgroup of G under the standard projection of Γ onto G. Then the minimal number of generators of Γ_{ab} is at most $2(n - \beta_1(\Gamma'))/(p-1) + \beta_1(\Gamma')$.

Let us remark that the tables of abelianizations of crystallographic groups in [13] show that the bounds of Theorem B are sometimes sharp in dimensions 2 and 3.

Using Theorem B together with the fact that a finite group acts effectively on a closed connected flat n-orbifold M if and only if it acts effectively and

isometrically on an orbifold affinely equivalent to M (see Proposition 4.1), we derive:

Theorem C. Let A be an elementary abelian p-group. Suppose A acts effectively on a closed connected flat n-orbifold M. Then $\operatorname{rk}(A) \leq 2n$ when $p = 2$ and $\operatorname{rk}(A) \leq n$ when p is odd. Moreover, if the holonomy of M is a p-group and p is odd, then $\operatorname{rk}(A) \leq 2(n - \beta_1)/(p - 1) + \beta_1$ where β_1 is the first Betti number of M.

2. Proof of Theorem A

Let G be a group and M be an integral G-lattice. We say that M is *irreducible* if $M \otimes \mathbb{Q}$ is an irreducible $\mathbb{Q}G$-module.

Lemma 2.1. Let G be a nontrivial p-group and assume there exists a faithful irreducible integral G-lattice M. Then there exists an integer $\alpha \geq 0$ such that $\operatorname{rk}(M) = p^\alpha(p - 1)$ and every subgroup of G can be generated by p^α elements.

Proof. The G-module $M \otimes \mathbb{Q}$ is faithful and irreducible. So, by Corollary 1.11 of [5], $\exists \alpha \geq 0$ such that $\operatorname{rk}(M) = p^\alpha(p - 1)$. Let $n = \operatorname{rk}(M)$. Viewing G as a subgroup of $\operatorname{GL}(n, \mathbb{Q})$, let P be a maximal p-subgroup of $\operatorname{GL}(n, \mathbb{Q})$ containing G. The representation of P into $\operatorname{GL}(n, \mathbb{Q})$ is irreducible. Now, it is a standard result (see [8, §4.5, p. 93]) that

$$P \cong \underbrace{C_p \wr \cdots \wr C_p}_{\alpha+1}.$$

But, by a result of Hulse (see [7, Theorem A]), any subgroup of such an iterated wreath product can be generated by p^α elements. $\square$

Proposition 2.2. Let G be a nontrivial p-group and M be a faithful integral G-lattice of rank n. Let $k = \operatorname{rk}(M^G)$. Then every subgroup of G can be generated by $(n - k)/(p - 1)$ elements.

Proof. Denote $M_{\mathbb{Q}} = M \otimes \mathbb{Q}$. As a $\mathbb{Q}G$-module, it decomposes into $(\oplus_{i=1}^l V_i) \oplus \mathbb{Q}^k$ where $n_1 + \cdots + n_l = n - k$ for $n_i = \dim(V_i)$, and each V_i is an irreducible $\mathbb{Q}G$-module. For each i, let $\psi_i : G \to \operatorname{GL}_{n_i}(\mathbb{Q})$ be the induced irreducible representation. Since the $\mathbb{Q}G$-module $M_{\mathbb{Q}}$ is faithful, these representations determine a monomorphism:

$$\psi : G \hookrightarrow \psi_1(G) \times \cdots \times \psi_l(G),$$
$$g \mapsto (\psi_1(g), \ldots, \psi_l(g)).$$

We claim that every subgroup of the group $D := \psi_1(G) \times \cdots \times \psi_l(G)$ can be generated by $(n-k)/(p-1)$ elements. We proceed by induction on l. Let S be a subgroup of D. By the previous lemma, each subgroup of $\psi_i(G)$ can be generated by $n_i/(p-1)$ elements. Hence, $l = 1$ case is clear. Now, let $l > 1$ and consider the standard projection $\pi : \psi_1(G) \times \cdots \times \psi_l(G) \to \psi_l(G)$. By induction, $\ker(\pi|_S)$ is generated by $(n_1 + \cdots + n_{l-1})/(p-1)$ elements and $\pi(S)$ is generated by $n_l/(p-1)$ elements. Therefore, S can be generated by $(n_1 + \cdots + n_{l-1})/(p-1) + n_l/(p-1) = (n-k)/(p-1)$ elements. $\square$

Lemma 2.3. Let C_p be a cyclic group of order p and let M be a $\mathbb{Z}C_p$-lattice of dimension n such that $M^{C_p} = 0$. Then M, as a C_p-module, can be generated by $(a-1)n/(p-1)$ elements.

Proof. By hypothesis, there exists a family of C_p-sublattices:

$$0 = V_0 \subset V_1 \subset \cdots \subset V_m = \mathbb{Z}^n$$

such that each quotient $Q_i = V_i/V_{i-1}$ is an irreducible, nontrivial C_p-lattice. It is well-known that every irreducible $\mathbb{Z}C_p$-module is of dimension $p - 1$ and it is isomorphic to an ideal I in $\mathbb{Z}[\zeta]$ where ζ is a primitive p-th root of unity (see [3]). For $p \leq 19$, the ideal class group of $\mathbb{Z}[\zeta]$ is trivial (see [14, 29.1.3]). Therefore, I is a principal ideal and $I \cong \mathbb{Z}[\zeta]$, which is 1-generated. For $p > 19$, since $\mathbb{Z}[\zeta]$ is a Dedekind domain, every ideal is 2-generated (see [14, 7.1-2]). This shows that each Q_i is $(a-1)$-generated. The result now follows by a simple induction on m. $\square$

Example 2.4. Let C_p be a cyclic group of order p and let I be an irreducible $\mathbb{Z}C_p$-module. According to the above discussion, as a C_p-module, I is $(a-1)$-generated. Define Γ to be the semidirect product group $I \rtimes C_p$. Then Γ is $(p-1)$-dimensional crystallographic group. Since C_p is 1-generated, Γ can be generated by a elements. When $p \leq 19$, this means that Γ is 2-generated and it is clear that this is the minimal number of generators of Γ. We leave it as an easy exercise to the reader to check that when $p > 19$ and I is not a principal ideal, then Γ cannot be 2-generated. Hence, the bound 3 which we obtain for the minimal number of generators of Γ is again optimal.

Proposition 2.5. Let G be a p-group and $\mathbb{Z}^n \rightarrowtail \Gamma \twoheadrightarrow G$ be an extension of groups such that the induced representation $\phi : G \to \mathrm{GL}_n(\mathbb{Z})$ is faithful. Then $\mathbb{Z}^n$, as a G-module, can be generated by $(a-1)(n-\beta_1)/(p-1) + \beta_1$ elements.

Proof. Recall that $\mathrm{rk}((\mathbb{Z}^n)^G) = \beta_1$. There exists a family of G-sublattices:

$$(\mathbb{Z}^n)^G = W_0 \subset W_1 \subset \cdots \subset W_l = \mathbb{Z}^n$$

such that each quotient $M_i = W_i/W_{i-1}$ is an irreducible, nontrivial G-lattice. Let C_p be a central subgroup of the quotient of G by the kernel of the action of G on M_i. Since $M_i^{C_p}$ is a G-submodule of M_i, it must be trivial. So, by the previous lemma, we have that, as a C_p-module and hence also as a G-module, each M_i is $(a-1)\mathrm{rk}(M_i)/(p-1)$-generated. Again, an easy induction on l finishes the proof. $\square$

Proof of Theorem A. Let $\rho : G \to \mathrm{GL}(T)$ be the holonomy representation of Γ. Since Γ is crystallographic, ρ is faithful. By Proposition 2.2, G can be generated by $(n - \beta_1)/(p-1)$ elements. According to Proposition 2.5, T as a G-module, can be generated by $(a-1)(n-\beta_1)/(p-1) + \beta_1$. Hence, Γ can be generated by $a(n-\beta_1)/(p-1) + \beta_1$ elements. $\square$

3. Proof of Theorem B

Lemma 3.1. Let G be a p-group and let M be a $\mathbb{Z}G$-lattice of dimension n such that $M^G = 0$. Then, M_G is an abelian p-group of order at most $p^{n/(p-1)}$.

Proof. Let us first assume that $G = C_p$. By hypothesis, there exists a family of C_p-sublattices:

$$0 = V_0 \subset V_1 \subset \cdots \subset V_l = M$$

such that each quotient $Q_i = V_i/V_{i-1}$ is an irreducible, nontrivial C_p-lattice. Also, $(Q_i)_{C_p} \cong \mathbb{Z}_p$ (see [2, Ex. IV.4.5]). Proceeding by induction on l, we may assume that $(V_{l-1})_{C_p}$ is a p-group of order at most $p^{(n-\dim(Q_l))/(p-1)} = p^{(n-(p-1))/(p-1)}$. Now, by the exact sequence:

$$(V_{l-1})_{C_p} \to M_{C_p} \to (Q_l)_{C_p} \to 0,$$

we can deduce that M_{C_p} is a finite p-group of order at most $p \cdot p^{(n-(p-1))/(p-1)} = p^{n/(p-1)}$. In general, there exists a family of G-sublattices:

$$0 = M_0 \subset M_1 \subset \cdots \subset M_l = M$$

such that each quotient $N_i = M_i/M_{i-1}$ is an irreducible, nontrivial G-lattice. We take a quotient of G by the kernel of the action of G on N_i and let C_p be a central subgroup of this quotient. Since N_i is irreducible G-module, it implies $(N_i)^{C_p} = 0$. So, by the previous case, $(N_i)_{C_p}$ and hence $(N_i)_G$, is a p-group of order at most $p^{\mathrm{rk}(N_i)/(p-1)}$. The result now follows from a similar induction as before. $\square$

Lemma 3.2. Let G be a p-group and let M be an n-dimensional $\mathbb{Z}G$-lattice and denote $\dim(M^G) = k$. Then, $(M/M^G)_G$ is an abelian p-group of order at most $p^{(n-k)/(p-1)}$. Consequently, the rank of $M_G \otimes \mathbb{Z}_p$ is at most $(n-k)/(p-1) + k$.

Proof. One can easily observe that (M/M^G) is torsion-free and $(M/M^G)^G = 0$. Hence, by Lemma 3.1, $(M/M^G)_G$ is an abelian p-group of order at most $p^{(n-k)/(p-1)}$. Now, let us consider the exact sequence:

$$(M^G)_G \to M_G \to (M/M^G)_G \to 0.$$

Since $(M^G)_G \cong M^G$, the rank of $(M^G)_G \otimes \mathbb{Z}_p$ equals k. Also, $(M/M^G)_G \otimes \mathbb{Z}_p$ has rank at most $(n-k)/(p-1)$. This implies that the rank of $M_G \otimes \mathbb{Z}_p$ is at most $(n-k)/(p-1) + k$. $\qquad\square$

Proof of Theorem B. By a standard decomposition of abelian groups:

$$\Gamma_{ab} \cong \mathbb{Z}^k \times \mathbb{Z}_{d_1} \times \cdots \times \mathbb{Z}_{d_s}$$

where $k \in \mathbb{Z}_{\geq 0}$, $d_i \in \mathbb{Z}_{\geq 2}$ and d_i divides d_{i+1} for each i. Note that Γ_{ab} is $(s+k)$-generated and the prime p divides each d_i. By a homological transfer and restriction argument, it follows that there is a monomorphism from $H_1(\Gamma, \mathbb{Z}_p)$ into $H_1(\Gamma', \mathbb{Z}_p)$. Hence, $s+k = \mathrm{rk}(\Gamma_{ab} \otimes \mathbb{Z}_p) \leq \mathrm{rk}(\Gamma'_{ab} \otimes \mathbb{Z}_p)$.

Next, we let P be the holonomy group of Γ' and note that T is also the subgroup of translations of Γ'. By the 5-term homological exact sequence associated to the extension $T \rightarrowtail \Gamma' \twoheadrightarrow P$, we obtain:

$$\cdots \to H_1(T, \mathbb{Z}_p)_P \to H_1(\Gamma', \mathbb{Z}_p) \to H_1(P, \mathbb{Z}_p) \to 0$$

which implies:

$$\mathrm{rk}(\Gamma'_{ab} \otimes \mathbb{Z}_p) \leq \mathrm{rk}(P_{ab} \otimes \mathbb{Z}_p) + \mathrm{rk}(T \otimes \mathbb{Z}_p)_P.$$

By Proposition 2.2, it follows that $P_{ab} \otimes \mathbb{Z}_p$ can be generated by $(n - \beta_1(\Gamma'))/(p-1)$ elements. We claim that $\mathrm{rk}(T \otimes \mathbb{Z}_p)_P \leq (n - \beta_1(\Gamma'))/(p-1) + \beta_1(\Gamma')$.

To this end, consider the short exact sequence:

$$0 \to T \xrightarrow{p} T \to T \otimes \mathbb{Z}_p \to 0.$$

Applying the coinvariants functor $(-)_P$, we obtain the exact sequence:

$$T_P \xrightarrow{p} T_P \to (T \otimes \mathbb{Z}_p)_P \to 0.$$

This shows that $(T \otimes \mathbb{Z}_p)_P \cong T_P \otimes \mathbb{Z}_p$. But, by the previous lemma, $T_P \otimes \mathbb{Z}_p$ has rank at most $(n - \beta_1(\Gamma'))/(p-1) + \beta_1(\Gamma')$, which finishes the claim.

Combining with a previous inequality, we obtain $s + k \leq \mathrm{rk}(\Gamma'_{ab} \otimes \mathbb{Z}_p) \leq 2(n - \beta_1(\Gamma'))/(p-1) + \beta_1(\Gamma')$. $\qquad\square$

4. Proof of Theorem C

First, we need a similar result as Theorem 6 of [9] which applies to orbifolds.

Proposition 4.1. Let F be a finite group. Suppose it acts effectively on a closed connected flat n-orbifold M. Then F can also act effectively and isometrically on an orbifold affinely equivalent to M.

Proof. The orbifold M arises as a quotient of $\mathbb{R}^n$ by a discrete group Γ acting effectively, isometrically, properly discontinuously and cocompactly on $\mathbb{R}^n$ (see [12, 13.3.3-4]). Then Γ is an n-dimensional crystallographic group and the natural quotient map $p : \mathbb{R}^n \to \mathbb{R}^n/\Gamma$ is an orbifold covering map where $\mathbb{R}^n$ is the orbifold universal cover.

On the other hand, the quotient M/F is a closed connected orbifold and the quotient map $q : M \to M/F$ is an orbifold covering map (see [11, p. 40, Ex. 2.18], [15, 13.2]). So, the projection $q \circ p : \mathbb{R}^n \to M/F$ is again an orbifold covering map (see [15, 13.2.4]). This shows that there exists an extension $\Gamma \rightarrowtail \overline{\Gamma} \twoheadrightarrow F$ where $\overline{\Gamma}$ is the orbifold fundamental group of M/F.

We claim that $\overline{\Gamma}$ is an n-dimensional crystallographic group. According to theorems of Bieberbach ([2]) and Zassenhaus ([16]), a finitely generated virtually abelian group is crystallographic if and only if it does not contain a finite nontrivial normal subgroup. Let us suppose that S is a finite normal subgroup of $\overline{\Gamma}$. Let T be the subgroup of translations of Γ and $\Gamma' := TS \leq \overline{\Gamma}$. Since $T \cap S = 1$, there is an extension $T \rightarrowtail \Gamma' \twoheadrightarrow S$. This extension induces an orbifold covering map $\mathbb{R}^n/T \to \mathbb{R}^n/\Gamma'$ which arises from an action of S on the n-torus $\mathbb{R}^n/T$. Let K be the kernel of this action. By passing to a subgroup of S, we can without loss of generality assume that $S = K$. Then $\mathbb{R}^n/\Gamma'$ is equivalent as an orbifold to $\mathbb{R}^n/T$ and hence they have isomorphic orbifold fundamental groups. This shows that $\Gamma' \cong T$ and so $K = 1$. We deduce that S acts effectively on $\mathbb{R}^n/T$. Then, by a result of Lee and Raymond (see [9, Cor., p. 256]) the extension $T \rightarrowtail \Gamma' \twoheadrightarrow S$ is admissible and therefore Γ' is a crystallographic group (see [9, Prop. 2]). Since S is a finite normal subgroup of Γ', S must be trivial. We can therefore conclude that $\overline{\Gamma}$ is an n-dimensional crystallographic group.

Let $\overline{\Gamma}$ act cocompactly, effectively, isometrically and properly discontinuously on $\mathbb{R}^n$ and denote by $\mathbb{R}^n /\!\!/ \Gamma$ the quotient of the action of the subgroup

Γ of $\overline{\Gamma}$ on $\mathbb{R}^n$. We observe that the extension $\Gamma \rightarrowtail \overline{\Gamma} \twoheadrightarrow F$ gives rise to an effective and isometric action of F on the orbifold $\mathbb{R}^n /\!\!/ \Gamma$. By the second Bieberbach theorem (see [2]), $\mathbb{R}^n/\Gamma$ and $\mathbb{R}^n /\!\!/ \Gamma$ are affinely equivalent. This finishes the proof. $\qquad\square$

Proof of Theorem C. By the proposition, we can assume that A acts isometrically on M. Let $\overline{M} = M/A$. It is again a compact flat orbifold. Let $\overline{\Gamma} = \pi_1^{orb}(\overline{M})$. Then $\overline{M} \cong \mathbb{R}^n/\overline{\Gamma}$ and $\overline{\Gamma}$ is again a crystallographic group. We also have an extension $\Gamma \rightarrowtail \overline{\Gamma} \twoheadrightarrow A$. Observe that the epimorphism of $\overline{\Gamma}$ onto A factors through the group $H := \overline{\Gamma}/T$ where T is the subgroup of translations of Γ. Let $\phi : H \to A$ be the induced epimorphism and note that the restriction of ϕ to a Sylow p-subgroup P of H is again surjective. Let Γ' be the preimage of P under the projection of $\overline{\Gamma}$ onto H. Then Γ' is an n-dimensional crystallographic group with a p-group holonomy. So, by applying Theorem B to Γ' it follows that Γ'_{ab} is n-generated when p is odd and $2n$-generated when $p = 2$. But since Γ'_{ab} surjects onto A, this gives us the desired bounds on $\mathrm{rk}(A)$.

If Γ has only p-group holonomy and p is odd, then $P = H$ and $\Gamma' = \overline{\Gamma}$. In this case, Theorem B entails $\mathrm{rk}(A) \le 2(n-\overline{\beta}_1)/(p-1)+\overline{\beta}_1 \le 2(n-\beta_1)/(p-1) + \beta_1$ where the second inequality follows from the fact that $\beta_1 \ge \overline{\beta}_1$. $\quad\square$

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