Mathematical
Sciences

UNIVERSITY OF
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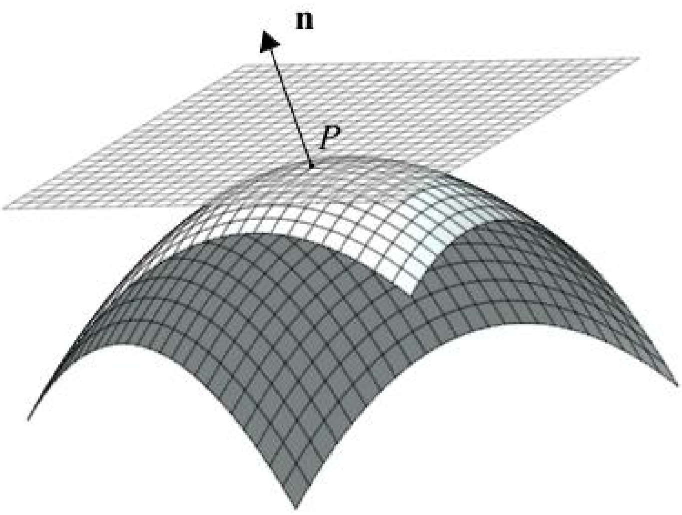
MATH1056 Calculus

Volume II



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MATH1056 Calculus

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Volume II

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Mathematical
Sciences Lecture Note Series

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Introduction to Volume II

Calculus is an old subject. Aspects of it go back to Archimedes, roughly 2500 years ago, and continue to be useful into the present day. The main development of calculus as the subject we now know goes back to the days of Newton and Leibniz, roughly 350 years ago. As such, it is unlikely that we will have anything mathematically original to say about calculus. Beyond that, there are hundreds, perhaps thousands, of books on different aspects of calculus that have been written over the centuries. It is impossible to encompass all of that thought about this single subject into a single book, nor is it wise to try, lest we hurt ourselves trying to lift it.

These are lectures notes for MATH1056 Calculus Part II. They consist largely of the material presented during the lectures, though we have taken the liberty of fleshing them out in some places and of being more cursory here than in the lectures in other places.

A standard approach taken by a mathematical textbook is to present a collection of standard facts, definitions and techniques in a standard order, moving from the simple to the less simple, and there is absolutely no problem with this. It is the route taken by thousands of authors of thousands of books. And in some sense, this is an inescapable approach, particularly for a subject like calculus which has been an object of our attention and study for several hundred years. We as mathematicians have been refining our approach to calculus and the teaching of calculus for this whole time. If that is the approach that you find most accessible, then there are a plethora of available sources.

In what follows, we would like to try something a little bit different, because we do not see the point of producing a set of notes that merely reproduces the same standard approach that already exists in multiple other texts. So, we take a different path. This is to start with a question and to explore what tools exist to address this question, what tools we can develop to address this question, what we already know that we can apply to the question.

The focus of these notes is multivariable calculus, by which we mean the application of the ideas from the calculus of functions of one variable that you have already seen to functions of several variables. Before we get started, though, we need to establish the questions that will be the focus of our journey, and to review the tools that we already have to hand.

The calculus of a function of one variable has two main pieces, the differential calculus and the integral calculus, differentiation and integration. There is a single basic idea that underlies both of these pieces, namely the notion of the limit, and these two pieces are linked through the Fundamental Theorem of Calculus. So, to what extent can we extend these ideas of differentiation and integration to functions of several variables, is there an analogue of the Fundamental Theorem of Calculus, and is there something new, that we do not see when considering functions of a single variable, that arises from the fact that we are working with several variables. We will only scratch the surface. There are many directions in which one can take these basic idea, through to, including and beyond the use that Einstein made of differential geometry in the formulation of the theory of relativity.

One of the first things we will do is to consider the extension of familiar ideas from differential calculus (that is, calculus involving derivatives) to functions of more than one variable. These familiar ideas include the definition of the derivative as well as the ways we use the derivative, such as maximising and minimising functions to solve problems. We will next extend some of these ideas to functions of a complex variable and discuss the differences between differentiation for real- and complex-valued functions.

We will then move to integration of functions of more than one variable. This will include developing different coordinate systems for Euclidean space and relating them to one another, and the calculation of areas and volumes of simple shapes.

These notes should be read in conjunction with the weekly tutorial sheets and solutions, as the problems in the weekly sheets provide additional examples of many of the things covered in the notes.

Notation and Terminology

The following is a summary of commonly used symbols and terminology.

Quantifiers

$\forall$ – for all

$\exists$ – there exists

Terminology

Theorem (or Proposition) – a proven mathematical statement

this is usually of the form *if such and such then so and so*

the Hypothesis is *such and such*

the Conclusion is *so and so*

Lemma – a little theorem.

Corollary – a mathematical statement which follows from a previous theorem.

Sets

$\mathbb{N}$ – natural numbers $\{1, 2, 3, \dots\}$

$\mathbb{Z}$ – integers $\{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$

$\mathbb{Q}$ – rational numbers e.g. $1, \frac{1}{2}, -\frac{1}{3}$ etc.

$\mathbb{R}$ – real numbers
(rational numbers and irrational numbers e.g.
$\pi = 3.14159\dots, \sqrt{2} = 1.41421\dots$)

$\mathbb{C}$ – complex numbers $a + ib$ where a, b are real and $i = \sqrt{-1}$

$\emptyset$ – the empty set $\{\}$

$\in$ – is a member of (is in e.g. $-1 \in \mathbb{Z}$, $\sqrt{2} \in \mathbb{R}$)

$\notin$ – is not a member of (is in e.g. $-1 \notin \mathbb{N}$, $\sqrt{2} \notin \mathbb{Q}$)

$\cup$ – union (things that are in either or both of the sets)

$\cap$ – intersection (things that are in both sets)

$\subseteq$ – is a subset of subset (is contained in, meaning one set is inside another)

$\subsetneq$ – is a subset of proper subset (is strictly contained in, meaning the sets are not equal)

Greek letters

In this module and throughout mathematics you will encounter numerous Greek letters. Here is a table so that you know what they all are and how they are called.

A	B	Γ	Δ	E	Z
α	β	γ	δ	ε	ζ
Alpha	Beta	Gamma	Delta	Epsilon	Zeta
H	Θ	I	K	Λ	M
η	θ	ι	κ	λ	μ
Eta	Theta	Iota	Kappa	Lambda	Mu
N	Ξ	O	Π	P	Σ
ν	ξ	o	π	ρ	σ
Nu	Xi	Omicron	Pi	Rho	Sigma
T	Y	Φ	X	Ψ	Ω
τ	υ	ϕ or φ	χ	ψ	ω
Tau	Upsilon	Phi	Chi	Psi	Omega

Chapter 1. Differential calculus for functions of several variables

The purpose of this chapter is to present the basics of the differential calculus for functions of several variables. To plant ourselves on firm ground, we start with a review of the basics from the differential calculus of functions of one variable, including the core fundamental idea that drives everything, namely the δ-ε definition of the limit. We will then consider different variations of the definition of the derivative of a function of more than one variable and explore their properties. As a focus of our activity in this chapter, we consider the question, given a function of several variables, how do we recognise, find, and classify its maxima and minima.

As model questions driving the material in this chapter, let us consider the following seemingly similar questions:

(1) Determine the maximum and minimum values of $f(x, y, z) = x^2 + y^2 + z^2 - 3$ on the plane $x + 2y + 3z = 1$.

(2) Determine the maximum and minimum values of $g(x, y, z) = x + 2y + 3z$ on the sphere $x^2 + y^2 + z^2 = 3$.

Even though these two questions are phrased similarly, in that both ask for the maxima and minima of a function subject to a constraint, they are in fact very different question, requiring different methods to attack. Part of what we do in this chapter is to highlight the ways in which these two questions are similar and the ways in which they are different.

1.1. Functions of several variables

In this section, we consider functions, which are the basic objects on which we will focus our attention for most of this book. Formally, a function $f : X \to Y$ between two sets X and Y is the assignment of one and only one member of Y, which we call $f(x)$, to each element $x \in X$. There is a standard collection of adjectives that we can associate to a function describing its basic attributes and its basic properties.

In the case that f associates different elements of X to different elements of Y, that is, $f(x_1) \neq f(x_2)$ for x_1, $x_2 \in X$ with $x_1 \neq x_2$, then f is said to be *injective* or *one-to-one*. In the case that f associates an element of X to every element of Y, so that for each $y \in Y$, there is $x \in X$ with $f(x) = y$, then f is said to be *surjective* or *onto*. A function that is both injective and surjective is called *bijective*.

The most general sort of function we will deal with is a function between the finite dimensional Euclidean spaces $\mathbb{R}^n$ and $\mathbb{R}^m$ for n, $m \geq 1$.

Definition 1.1.1. A *function* $F : D \to \mathbb{R}^m$ from a subset D of $\mathbb{R}^n$ to $\mathbb{R}^m$ is an assignment of a unique point $f(x_1, \ldots, x_n) \in \mathbb{R}^m$ for each point $(x_1, \ldots, x_n) \in D$. The set D is called the *domain* of F and the set of all points $F(x_1, \ldots, x_n)$ obtained from the domain is called the *range* of F.

Remark 1.1.2. Where there is no confusion, we will often write $F : \mathbb{R}^n \to \mathbb{R}^m$ instead of $F : D \to \mathbb{R}^m$ with the understanding that the function may not be defined on all of $\mathbb{R}^n$ but rather on its subset D, the domain.

A function $F : \mathbb{R}^n \to \mathbb{R}^m$ can be viewed as an ordered m-tuple of functions from $\mathbb{R}^n$ to $\mathbb{R}$, which we write as

$$F(x_1, \ldots, x_n) = (f_1(x_1, \ldots, x_n), \ldots, f_m(x_1, \ldots, x_n)),$$

where each $f_i : \mathbb{R}^n \to \mathbb{R}$ for $1 \le i \le m$ is itself a function from $\mathbb{R}^n$ to $\mathbb{R}$.

Example 1.1.3. Suppose $f(x, y) = \sqrt{4 - x^2 - y^2}$. The domain D is the disk of radius 2 centred at the origin:

$$D = \{(x, y) \mid x^2 + y^2 \le 4\}.$$

Example 1.1.4. Suppose $F(t) = (t\cos(t), t\sin(t))$. The domain is all real numbers and $F : \mathbb{R} \to \mathbb{R}^2$ is a spiral curve (see Figure 1).

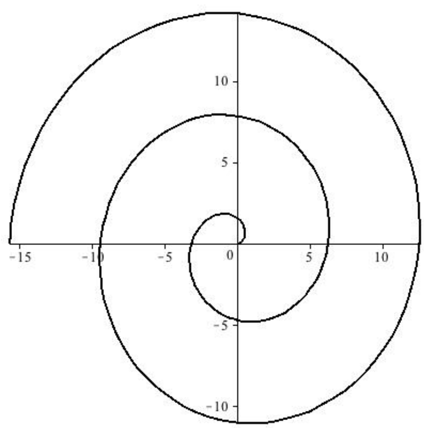


FIGURE 1. spiral curve for $t \ge 0$

Functions of a single variable have one of two basic flavours. A function is an *explicit function* if we have or are given an explicit description of $f(x)$ as a formula in terms of x, for instance a formula like

$$f(x) = x^2 \quad \text{or} \quad f(x) = \exp(\sin(x + (x^3 + 1)\cos(2x + 76))) + \ln(x^2 + 2),$$

or more generally if we have some other explicit description of how to calculate the value of $f(x)$ given a value of the input x.

An example of this later sort might be the greatest integer function, where we set $\lfloor x \rfloor$ to be the largest integer that is less than or equal to x. We have an explicit description for $\lfloor x \rfloor$ given x but no formula.

A function is an *implicit function* if we have a description of the relationship between two (or more) variables, such as x and y, but without the relationship being sufficiently simple to allow for us to solve for y explicitly in terms of x. A classical example comes from the equation of a circle in the plane with centre the origin and radius 1, namely $x^2 + y^2 = 1$. Here, we cannot solve for y as a function of x in a way that is valid for all appropriate values of x; specifically, if we solve for y, we get $y = \pm\sqrt{x^2 - 1}$, and so for each $-1 < x < 1$, there are two possible values of y and this violates the definition of function as discussed (albeit briefly) above.

The same loose split of functions between implicit functions and explicit functions holds for functions of more than one variable just as it does for functions of one variable.

An example of an explicit function of two variables is $f(x, y) = x^2 - y^2 + xy\cos(x + y)$.

1.1.1. **Graphs.** Recall that the graph of a function in one variable given by the equation $y = f(x)$ is a set of points $(x, f(x))$ in the xy-plane $\mathbb{R}^2$. More generally, the *graph* of function in n-variables given by the equation $w = f(x_1, \ldots, x_n)$ is the set of points $(x_1, \ldots, x_n, f(x_1, \ldots, x_n))$ in $\mathbb{R}^{n+1}$ where $(x_1, \ldots, x_n)$ is in the domain of f.

Example 1.1.5. Consider the function $f(x, y) = 1 - x - y$ where $0 \le x \le 1$, $0 \le y \le 1$. Its graph is is the set of points

$$\{(x, y, 1-x-y) \mid x \in \mathbb{R},\ y \in \mathbb{R}\} = \{(x, y, z) \mid x+y+z = 1,\ x \in \mathbb{R},\ y \in \mathbb{R},\ z \in \mathbb{R}\}.$$

This is plane in $\mathbb{R}^3$ intersecting the points $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$.

Example 1.1.6. Consider the function $f(x, y) = \sqrt{4 - x^2 - y^2}$. We saw that its domain is

$$D = \{(x, y) \mid x^2 + y^2 \le 4\}.$$

Now the the graph of this function is the set of points

$$\{(x, y, \sqrt{4 - x^2 - y^2} \mid x \in \mathbb{R},\ y \in \mathbb{R}\}$$

$$= \{(x, y, z) \mid x^2 + y^2 + z^2 = 4,\ x \in \mathbb{R},\ y \in \mathbb{R},\ z \ge 0\}.$$

This is the upper hemisphere of the sphere centred at the origin of radius 2 (see Figure 2).

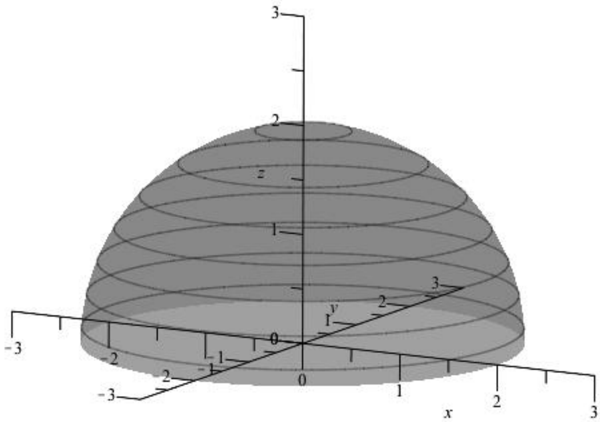


FIGURE 2. upper hemisphere

1.1.2. **Contours.** For a function $f(x)$ of one variable, there is a relatively straightforward process for sketching the graph of $y = f(x)$, gathering the information from the axis intercepts and any vertical or horizontal asymptotes, the first derivative test and the second derivative test. Doing something similar for functions of two or more variables is significantly more complicated.

One relatively efficient way of getting an idea of how the graph of a function $f(x, y)$ of two variables looks in $\mathbb{R}^3$ is to consider the *contours* or *level sets* of the graph of $z = f(x, y)$. By a contour or level set, we mean that for each $c \in \mathbb{R}$, we take the set

$$L_c = L_c(f) = \{(x, y) \in \mathbb{R}^2 \mid f(x, y) = c\}.$$

In words, the level set of height c is the set of points in the domain at which the function takes the value c. One immediate consequence of this is that level sets of different heights are necessarily disjoint: by the definition of function, there cannot exist a point at which the function takes on two different values.

Some caution is required in reconstructing the function from its level sets as it is seen in the next example.

Example 1.1.7. Determine the level sets of the functions $f(x, y) = \sqrt{x^2 + y^2}$ and $g(x, y) = x^2 + y^2$. The level set of the function $f(x, y) = \sqrt{x^2 + y^2}$ of height $c \in \mathbb{R}$ is

- for $c < 0$, the empty set $L_c =$ for $c < 0$, as there are no (real) solutions to $\sqrt{x^2 + y^2} = c$;
- for $c = 0$, the set $L_0 = \{(0, 0)\}$ containing only the origin;

- for $c > 0$, the circle $L_c = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = c^2\}$ of radius c centered at $(0, 0)$.

The level set of the function $g(x, y) = x^2 + y^2$ of height $c \in \mathbb{R}$ is

- for $c < 0$, the empty set $L_c =$ for $c < 0$, as there are no (real) solutions to $x^2 + y^2 = c$;
- for $c = 0$, the set $L_0 = \{(0, 0)\}$ containing only the origin;
- for $c > 0$, the circle $L_c = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = c^2\}$ of radius $\sqrt{c}$ centered at $(0, 0)$.

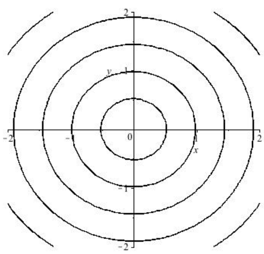


FIGURE 3. contours for $f(x, y) = \sqrt{x^2 + y^2}$

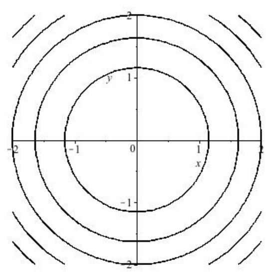


FIGURE 4. contours for $g(x, y) = x^2 + y^2$

Note that while the sets of level curves for these two functions $f(x, y) = \sqrt{x^2 + y^2}$ (see Figure 3) and $g(x, y) = x^2 + y^2$ (see Figure 4) are the same sets of circles, the curves are labeled differently. This labelling corresponds to the altitudes given to the contour lines on topographical maps. The graphs of the two functions with the corresponding level curves are shown below.

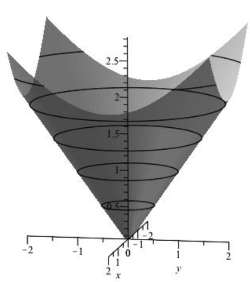


FIGURE 5. graph of $f(x, y) = \sqrt{x^2 + y^2}$

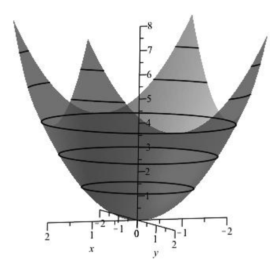


FIGURE 6. graph of $g(x, y) = x^2 + y^2$

Example 1.1.8. Consider the function $f(x, y) = x^2 - y^2 + xy\cos(x + y)$ for $x \geq 0$, $y \geq 0$.

The level set of height $c \in \mathbb{R}$ is the set

$$L_c = \{(x, y) \in \mathbb{R}^2 \mid x^2 - y^2 + xy\cos(x + y) = c\}.$$

The graph of $f(x, y)$ is too complicated to plot by hand. So, we use Maple Graphics (see Figure 7).

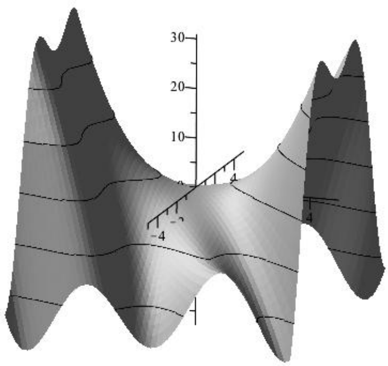


FIGURE 7. graph of $f(x, y) = x^2 - y^2 + xy\cos(x + y)$

Most of what we will cover in the rest of this chapter will help us better understand the behaviour of a multivariable function and to plot its graph.

1.2. LIMITS IN $\mathbb{R}^2$ AND CONTINUITY

The notion of a limit of a function of two (or more) variables is similar to that of a function of one variable. The limit describes the behaviour of the function as the input (x, y), which in this case is a point in $\mathbb{R}^2$, approaches a fixed point (a, b) in $\mathbb{R}^2$. But what does it mean for (x, y) to approach a fixed point in $\mathbb{R}^2$?

For functions of one variable we could make sense of what it meant for x to approach a point $a \in \mathbb{R}$, by understanding that the difference in absolute value $|x - a|$, which is the distance between x and a on the real line, got smaller and smaller. This led us to δ-ε-definition of the limit.

For functions of two variables, when the point (x, y) approaches (a, b), it means that the distance between these two points, which is

$$\sqrt{(x - a)^2 + (y - b)^2},$$

decreases. This leads us to the following formulation.

Definition 1.2.1. Given a function $f : \mathbb{R}^2 \to \mathbb{R}$. We say $\lim_{(x,y)\to(a,b)} f(x,y) = L$, whenever

(i) every neighbourhood of the point (a,b) contains a point of the domain of f different from (a,b), and

(ii) for every $\varepsilon > 0$, there exists $\delta > 0$ such that if (x,y) is in the domain and satisfies
$$0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta,$$
then $|f(x,y) - L| < \varepsilon$.

By a *neighbourhood* of a point (a,b) we mean an open disc centred at a point (a,b) of radius r, that is
$$D_r(a,b) = \{(x,y) \in \mathbb{R}^2 \mid \sqrt{(x-a)^2 + (y-b)^2} < r\}.$$

The condition (i) is included because we do not want to consider limits for isolated points of the domain as in that case there is no "limiting process". The condition (ii) implies that as the distance between (x,y) and (a,b) tends to zero, the distance between $f(x,y)$ and L also tends to zero.

Example 1.2.2. Let us find
$$\lim_{(x,y)\to(0,0)} (x^2 + y^2 + 7).$$

Note that as (x,y) tend to $(0,0)$, the function approaches the value 7. So, we shall try to prove that this is the limit.

The domain is the xy-plane. We need to show that for any $\varepsilon > 0$, there exists $\delta > 0$ such that if $0 < \sqrt{x^2 + y^2} < \delta$, then
$$|x^2 + y^2 + 7 - 7| < \varepsilon.$$

Setting $\delta = \sqrt{\varepsilon}$ gives the desired inequality.

Example 1.2.3. Show that
$$\lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^2 + y^2}} = 0.$$

Note that the domain is the whole xy-plane without the origin. So, condition (i) is trivially satisfied.

Now, we need to show that for $\varepsilon > 0$, there exists $\delta > 0$ such that if $0 < \sqrt{x^2 + y^2} < \delta$, then
$$\left| \frac{xy}{\sqrt{x^2 + y^2}} \right| < \varepsilon.$$

Since
$$\frac{\pm\, xy}{x^2 + y^2} \leq \frac{1}{2},$$

we obtain

$$\left|\frac{xy\sqrt{x^2+y^2}}{x^2+y^2}\right| \le \frac{1}{2}\sqrt{x^2+y^2} < \frac{\delta}{2}.$$

Therefore, if choose $\delta = 2\varepsilon$, then the condition (ii) would be satisfied.

Example 1.2.4. Show that the limit

$$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$$

does not exist.

Again the domain of the function is the whole xy-plane without the origin. So, we need to show that the condition (ii) of the definition does not hold.

Suppose to the contrary that the limit did exist. Let us say it equals to $L \in \mathbb{R}$. Since the function tends to L as (x, y) approaches $(0, 0)$ it should still tend to L no matter how the (x, y) approaches $(0, 0)$, i.e. as long as the distance between these points tends to zero. So, we can state that if (x, y) approaches $(0, 0)$ along the y-axes, then

$$L = \lim_{x=0,(x,y)\to(0,0)} \frac{xy}{x^2+y^2} = 0,$$

if it approaches along the x-axes, then

$$L = \lim_{y=0,(x,y)\to(0,0)} \frac{xy}{x^2+y^2} = 0.$$

But if it approaches along the line $y = \pm x$, then

$$L = \lim_{y=\pm x,(x,y)\to(0,0)} \frac{xy}{x^2+y^2} = \pm\frac{1}{2}.$$

This leads to a contradiction. Therefore, the limit does not exist.

Next, we give some properties of the limit.

Lemma 1.2.5. *Let $f : \mathbb{R}^2 \to \mathbb{R}$ and $g : \mathbb{R}^2 \to \mathbb{R}$. Suppose*

$$\lim_{(x,y)\to(a,b)} f(x,y) = L \quad \textit{and} \quad \lim_{(x,y)\to(a,b)} g(x,y) = M.$$

Then

(i) $\lim_{(x,y)\to(a,b)} (f(x,y) \pm g(x,y)) = L \pm M$*,*

(ii) $\lim_{(x,y)\to(a,b)} (f(x,y)g(x,y)) = LM$*,*

(iii) $\lim_{(x,y)\to(a,b)} (f(x,y)/g(x,y)) = L/M$*, as long as* $M \neq 0$.

(iv) Given a function $h : \mathbb{R} \to \mathbb{R}$ that is continuous at $L \in \mathbb{R}$, then $\lim_{(x,y)\to(a,b)} h(f(x,y)) = h(L)$.

Example 1.2.6. Find the limit of $f(x, y) = x^3 + x^2y^6 + 5$ as (x, y) approaches $(1, 0)$.

The domain contains all points of the xy-plane. Applying the assertions (i) and (ii) of Lemma 1.2.5, we have that the limit is $1^3 + 1^2 \cdot 0 + 0^6 + 5 = 6$.

Note that we could also only use part (iv) of Lemma 1.2.5 as the function $h(x) = x^3$ is continuos at $x = 1$ and the function $s(y) = y^6$ is continuous at $y = 0$.

Example 1.2.7. Find the limit of $f(x, y) = \dfrac{x^3 + x^2y^6 + 5}{x^4 + 3xy - y^2 + \sqrt{x - y}}$ as (x, y) approaches $(1, 0)$.

First observe that the point $(1, 0)$ is in the domain of the function. Also observe that it is not an isolated point of the domain as we can in every open disc centred at $(1, 0)$ always find points of the form $(x, 0) \neq (1, 0)$ that are in the domain. So, it makes sense to talk about the limit of the function at this point.

Since the function given by $h(t) = \sqrt{t}$ is continuous at $t = 1$, it follows from the assertions (i), (ii) and (iv) of Lemma 1.2.5, that

$$\lim_{(x,y)\to(1,0)} \sqrt{x - y} = \sqrt{1 - 0} = 1.$$

Then

$$\lim_{(x,y)\to(1,0)} x^4 + 3xy - y^2 + \sqrt{x - y} = 1 + 1 = 2$$

and

$$\lim_{(x,y)\to(1,0)} \frac{x^3 + x^2y^6 + 5}{x^4 + 3xy - y^2 + \sqrt{x - y}} = \frac{6}{2} = 3$$

by the assertion (iii).

Remember that continuity of a function of one variable was defined using limits. To define what it means for a function of two or more variables to be continuous at a point, we again rely on the notion of the limit.

Definition 1.2.8. The function $f : \mathbb{R}^2 \to \mathbb{R}$ is said to be *continuous at* $(a, b) \in \mathbb{R}^2$ if

$$\lim_{(x,y)\to(a,b)} f(x, y) = f(a, b).$$

If f is continuous at every point of a subset D of $\mathbb{R}^2$, then f is said to be *continuous on* D. If D is the whole domain, then f is simply said to be *continuous.*

It follows that sums and products of continuous functions are again continuous. That is, there are analogous statements to parts (i) and (ii) of Lemma 1.2.5 for continuity at a point. For quotients one needs to proceed with a bit of care as in part (iii) of Lemma 1.2.5.

Example 1.2.9. Determine in which of the previous examples the function in the limit is continuous at the limit point.

In Example 1.2.7, the function

$$f(x,y) = \frac{x^3 + x^2y^6 + 5}{x^4 + 3xy - y^2 + \sqrt{x-y}}$$

is continuous at $(1,0)$ since $\lim_{(x,y)\to(1,0)} f(x,y) = 3 = f(1,0)$.

Similarly, in all of the other examples where the limit exists except Example 1.2.3 the functions in the limit are continuous at given points.

In Example 1.2.3 though, we have

$$f(x,y) = \frac{xy}{\sqrt{x^2+y^2}},$$

and $\lim_{(x,y)\to(0,0)} f(x,y) = 0$, yet $(0,0)$ is not in the domain. So, this function is not continuous at $(0,0)$.

Next, we give definitions of the limit and continuity more generally for functions $F : \mathbb{R}^n \to \mathbb{R}^m$ for $n \geq 1$ and $m \geq 1$. The definitions are completely analogous to the case where we have function of two variables $f : \mathbb{R}^2 \to \mathbb{R}$.

To shorten the notation, we will denote points in the n-dimensional Euclidean space $\mathbb{R}^n$ by bold letters, e.g. $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$. Given any points $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^n$, we will define the distance between these points to be the *norm* of the vector difference $\mathbf{x} - \mathbf{y} \in \mathbb{R}^n$, that is

$$||\mathbf{x} - \mathbf{y}|| = \sqrt{(x_1 - y_1)^2 + \cdots + (x_n - y_n)^2}.$$

By a *neighbourhood* of a point $\mathbf{a}$ in $\mathbb{R}^n$ we mean an open n-ball centred at a point $\mathbf{a}$ of radius r, that is

$$B_r(\mathbf{a}) = \{\mathbf{x} \in \mathbb{R}^n \mid ||\mathbf{x} - \mathbf{a}|| < r\}.$$

Definition 1.2.10. Given a function $F : \mathbb{R}^n \to \mathbb{R}^m$, a point $\mathbf{a} \in \mathbb{R}^n$ and $\mathbf{L} \in \mathbb{R}^m$. We say $\lim_{\mathbf{x}\to\mathbf{a}} F(\mathbf{x}) = \mathbf{L}$, whenever

(i) every neighbourhood of the point $\mathbf{a}$ contains a point of the domain of F different from (a,b), and

(ii) for every $\varepsilon > 0$, there exists $\delta > 0$ such that if $\mathbf{x}$ is in the domain and satisfies $0 < ||\mathbf{x} - \mathbf{a}|| < \delta$, then $||F(\mathbf{x}) - \mathbf{L}|| < \varepsilon$.

Using this we can make the definition of continuity.

Definition 1.2.11. The function $F : \mathbb{R}^n \to \mathbb{R}^m$ is said to be *continuous at* $\mathbf{a} \in \mathbb{R}^n$ if

$$\lim_{\mathbf{x}\to\mathbf{a}} F(\mathbf{x}) = F(\mathbf{a}).$$

If f is continuous at every point of the domain, then F is said to be *continuous.*

We end this section with a useful lemma which is a generalisation of the analogous result about compositions of functions of one variables.

Lemma 1.2.12. *Let $F : \mathbb{R}^n \to \mathbb{R}^m$ be continuous at $\mathbf{a} \in \mathbb{R}^n$ and let $G : \mathbb{R}^m \to \mathbb{R}^k$ be continuous at $F(\mathbf{a}) \in \mathbb{R}^m$. Then the composition $G \circ F$ is continuous at $\mathbf{a}$.*

Proof. By continuity of G at $F(\mathbf{a})$, we have that for any $\varepsilon > 0$, there exists $\sigma > 0$ such that if $||\mathbf{y} - F(\mathbf{a})|| < \sigma$, then $||G(y) - G(F(\mathbf{a}))|| < \varepsilon$.

Also, by continuity of F at $\mathbf{a}$, we have that for $\sigma > 0$, there exists $\delta > 0$ such that if $||\mathbf{x} - \mathbf{a}|| < \delta$, then $||y - F(\mathbf{a})|| < \sigma$.

Combining the two statements, we obtain that whenever $||\mathbf{x} - \mathbf{a}|| < \delta$, then $||G(y) - G(F(\mathbf{a}))|| < \varepsilon$. This finishes the proof.

□

1.3. Partial derivatives

We wish to generalise the basic notion of the derivative to a wider class of functions. We start with the real-valued functions of several variables $f : \mathbb{R}^2 \to \mathbb{R}$, or more generally $f : \mathbb{R}^n \to \mathbb{R}$ for some $n \geq 2$. One thing that we will see is that for functions of several variables, the derivative is a collection of functions. So, our initial goal is to explore how to define and calculate the derivative for a function of several variables, and how to organise these functions that make up the derivative and extract information from them.

Let $z = f(x, y)$ be a function of two variables. We proceed naively and attempt to directly generalise to functions of several variables the definition of the derivative of a function of a single variable. There are several ways in which we could attempt to mimic the definition of the derivative for a function of one variable. All of these ways involve the taking of limits.

One is a direct generalisation: namely, given a point (x_0, y_0), we could set

$$f'(x_0, y_0) = \lim_{(x,y)\to(x_0,y_0)} \frac{f(x, y) - f(x_0, y_0)}{||(x, y) - (x_0, y_0)||}.$$

(In the denominator, we need to take the norm of $(x, y) - (x_0, y_0)$, as we cannot divide by a vector.) However, this definition is difficult to work with as it stands. We will come back to it later.

Hence, we take for the time being a different tack and work with one variable at a time. The derivatives that we construct in this way we refer to as *partial derivatives.* The basic limit definition for the partial derivatives of $f(x, y)$

with respect to the independent variables x and y is a straightforward generalisation of the limit definition for a function of a single variable.

Definition 1.3.1. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function. Define *the partial derivative with respect to x* as

$$\frac{\partial f}{\partial x}(x, y) = \lim_{h \to 0} \frac{f(x+h, y) - f(x, y)}{h}$$

provided this limit exists. In words, the partial derivative $\frac{\partial f}{\partial x}(x, y)$ of $f(x, y)$ with respect to x is the function that we calculate by holding all other (independent) variables constant and differentiating the resulting function as normal with respect to x.

Similarly, define *the partial derivative with respect to y* as

$$\frac{\partial f}{\partial y}(x, y) = \lim_{h \to 0} \frac{f(x, y+h) - f(x, y)}{h}$$

provided this limit exists. In words, the partial derivative $\frac{\partial f}{\partial y}(x, y)$ of $f(x, y)$ with respect to y is the function that we calculate by holding all other (independent) variables constant and differentiating as normal with respect to y.

In particular, both $\frac{\partial f}{\partial x}(x, y)$ and $\frac{\partial f}{\partial y}(x, y)$ are in this case themselves functions from $\mathbb{R}^2$ to $\mathbb{R}$, just as $f(x, y)$.

Example 1.3.2. Find the partial derivatives of the function $f(x, y) = x^2 y$ at $(2, 3)$.

Following the definition, we have

$$\begin{aligned}\frac{\partial f}{\partial x}(x, y) &= \lim_{h \to 0} \frac{f(x+h, y) - f(x, y)}{h} \\ &= \lim_{h \to 0} \frac{(x+h)^2 y - x^2 y}{h} \\ &= \lim_{h \to 0} (2xy + hy) = 2xy\end{aligned}$$

Similarly,

$$\begin{aligned}\frac{\partial f}{\partial y}(x, y) &= \lim_{h \to 0} \frac{f(x, y+h) - f(x, y)}{h} \\ &= \lim_{h \to 0} \frac{x^2(y+h) - x^2 y}{h} \\ &= \lim_{h \to 0} x^2 = x^2.\end{aligned}$$

Substituting $(x, y) = (2, 3)$, we get $\frac{\partial f}{\partial x}(2, 3) = 12$ and $\frac{\partial f}{\partial y}(2, 3) = 4$.

We have the following useful lemma.

Lemma 1.3.3. *Let $u(x,y)$ be a function of the variables x and y, and assume that*

$$\frac{\partial u}{\partial x}(x,y) = \frac{\partial u}{\partial y}(x,y) = 0$$

for all $(x,y) \in \mathbb{R}^2$. Then, $u(x,y)$ is constant on $\mathbb{R}^2$.

Proof. We start from the assumption that $\frac{\partial u}{\partial x}(x,y) = 0$ for all $(x,y) \in \mathbb{R}^2$. Integrating with respect to x, we see that $u(x,y) = \psi(y)$, since the constant of integration with respect to x must depend only on y.

Using now the assumption that $\frac{\partial u}{\partial y}(x,y) = \psi'(y) = 0$, we see that $\psi(y)$ must be constant, and hence that $u(x,y)$ must be constant. □

We can define partial derivatives of more general functions in a similar way.

Definition 1.3.4. For a function $f : \mathbb{R}^n \to \mathbb{R}$ given by $f(x_1,\ldots,x_n)$, we have n first order partial derivatives, one with respect to each variable $x_1,\ldots,x_n$, is given by

$$\frac{\partial f}{\partial x_j}(x_1,\ldots,x_n) = \lim_{h\to 0} \frac{f(x_1,\ldots,x_{j-1},x_j+h,x_{j+1},\ldots,x_n) - f(x_1,\ldots,x_n)}{h}.$$

provided the limit exists.

As we will see, we do not normally use the definition to evaluate the partial derivatives of particular functions, any more than we use the definition for a function of one variable to calculate the derivative of that function. Rather, we use the definition to determine the derivatives of some basic functions and to prove properties of the derivative, and then we leverage these few calculations and these rules to differentiate a wide variety of functions.

The standard rules of differentiation, such as the product rule, the quotient rule, and the chain rule, continue to hold for functions of several variables. We first state them for functions of two variables.

Lemma 1.3.5. *Let $f(x,y)$ and $g(x,y)$ be functions of two variables.*

(i) Suppose $\frac{\partial f}{\partial x}(x,y)$ exists. Then for a constant $c \in \mathbb{R}$,

$$\frac{\partial}{\partial x}(cf(x,y)) = c\left(\frac{\partial}{\partial x}f(x,y)\right) = c\frac{\partial f}{\partial x}(x,y),$$

and similarly for the partial derivative with respect to y.

(ii) Suppose $\frac{\partial f}{\partial x}(x,y)$ and $\frac{\partial g}{\partial x}(x,y)$ exist. Then

$$\frac{\partial}{\partial x}(f(x,y)+g(x,y)) = \frac{\partial f}{\partial x}(x,y) + \frac{\partial g}{\partial x}(x,y),$$

and similarly for the partial derivative with respect to y.

(iii) *Suppose* $\frac{\partial f}{\partial x}(x,y)$ *and* $\frac{\partial g}{\partial x}(x,y)$ *exist. For two functions* $f(x,y)$ *and* $g(x,y)$*, we have the product rule:*

$$\frac{\partial}{\partial x}(f(x,y)g(x,y)) = \left(\frac{\partial f}{\partial x}(x,y)\right) g(x,y) + f(x,y)\left(\frac{\partial g}{\partial x}(x,y)\right),$$

and similarly for the partial derivative with respect to y*.*

(iv) *Suppose* $\frac{\partial f}{\partial x}(x,y)$ *and* $\frac{\partial g}{\partial x}(x,y)$ *exist. For two functions* $f(x,y)$ *and* $g(x,y)$*, we have the quotient rule:*

$$\frac{\partial}{\partial x}\left(\frac{f(x,y)}{g(x,y)}\right) = \frac{\left(\frac{\partial f}{\partial x}(x,y)\right) g(x,y) - f(x,y)\left(\frac{\partial g}{\partial x}(x,y)\right)}{(g(x,y))^2},$$

and similarly for the partial derivative with respect to y*.*

Proof. We only prove part (i) to show that the proofs are analogous to those when the function has only one variable.

$$\begin{aligned}\frac{\partial}{\partial x}(cf(x,y)) &= \lim_{h\to 0}\frac{cf(x+h,y) - cf(x,y)}{h}\\ &= c\left(\lim_{h\to 0}\frac{f(x+h,y)-f(x,y)}{h}\right)\\ &= c\left(\frac{\partial f}{\partial x}(x,y)\right).\end{aligned}$$

□

More generally, for two functions $f(x_1,\dots,x_n)$ and $g(x_1,\dots,x_n)$ of n variables, the basic rules of differentiation are as you would expect. Again, we assume that relevant partial derivatives of the two functions exist when stating a formula involving them.

Lemma 1.3.6. *Let* $f(x_1,\dots,x_n)$ *and* $g(x_1,\dots,x_n)$ *be functions of* n*-variables. Then*

(i) *For a constant* $c \in \mathbb{R}$ *and for each* $1 \le j \le n$*,*

$$\frac{\partial}{\partial x_j}(cf(x_1,\dots,x_n)) = c\frac{\partial f}{\partial x_j}(x_1,\dots,x_n);$$

(ii) *The derivative of a sum is the sum of the derivatives, namely*

$$\frac{\partial}{\partial x_j}(f(x_1,\dots,x_n)+g(x_1,\dots,x_n)) = \frac{\partial f}{\partial x_j}(x_1,\dots,x_n) + \frac{\partial g}{\partial x_j}(x_1,\dots,x_n)$$

for each $1 \le j \le n$*;*

(iii) *We have the product rule:*

$$\begin{aligned}&\frac{\partial}{\partial x_j}(f(x_1,\dots,x_n)g(x_1,\dots,x_n))\\ &= \left(\frac{\partial f}{\partial x_j}(x_1,\dots,x_n)\right) g(x_1,\dots,x_n) + f(x_1,\dots,x_n)\left(\frac{\partial g}{\partial x_j}(x_1,\dots,x_n)\right)\end{aligned}$$

for each $1 \leq j \leq n$*;*

(iv) We have the quotient rule:

$$\frac{\partial}{\partial x_j}\left(\frac{f(x_1,\ldots,x_n)}{g(x_1,\ldots,x_n)}\right)$$

$$= \frac{\left(\frac{\partial f}{\partial x_j}(x_1,\ldots,x_n)\right) g(x_1,\ldots,x_n) - f(x_1,\ldots,x_n)\left(\frac{\partial g}{\partial x_j}(x_1,\ldots,x_n)\right)}{(g(x_1,\ldots,x_n))^2}$$

for each $1 \leq j \leq n$*.*

Example 1.3.7. Find all the partial derivatives of the function

$$f(x,y,z) = 5x^3y^2z - y^2z^2 + 3x - 4$$

at $(-1,0,2)$.

Recall that when we differentiate with respect to x, we treat all other variables as constants and then differentiate normally. So, to calculate $\frac{\partial f}{\partial x}$, we treat y and z as constants and see that

$$\frac{\partial f}{\partial x}(x,y,z) = 5\frac{\partial}{\partial x}(x^3y^2z) - \frac{\partial}{\partial x}(y^2z^2) + 3\frac{\partial}{\partial x}(x) - \frac{\partial}{\partial x}(4)$$

$$= 15x^2y^2z + 3.$$

Similarly, to calculate $\frac{\partial f}{\partial y}$, we treat x and z as constants and see that

$$\frac{\partial f}{\partial y}(x,y,z) = 5\frac{\partial}{\partial y}(x^3y^2z) - \frac{\partial}{\partial y}(y^2z^2) + 3\frac{\partial}{\partial y}(x) - \frac{\partial}{\partial y}(4)$$

$$= 10x^3yz - 2yz^2.$$

Similarly, to calculate $\frac{\partial f}{\partial y}$, we treat x and y as constants and see that

$$\frac{\partial f}{\partial z}(x,y,z) = 5\frac{\partial}{\partial z}(x^3y^2z) - \frac{\partial}{\partial z}(y^2z^2) + 3\frac{\partial}{\partial z}(x) - \frac{\partial}{\partial z}(4)$$

$$= 5x^3y^2 - 2y^2z.$$

Then $\frac{\partial f}{\partial x}(-1,0,2) = 3$, $\frac{\partial f}{\partial y}(-1,0,2) = 0$, and $\frac{\partial f}{\partial z}(-1,0,2) = 0$.

Example 1.3.8. Find the partial derivatives of

$$f(x,y) = \cos(x)\sin(y) + y\exp(x).$$

$\frac{\partial f}{\partial x}(x,y) = \left(\frac{\mathrm{d}}{\mathrm{d}x}\cos(x)\right)\sin(y) + y\left(\frac{\mathrm{d}}{\mathrm{d}x}\exp(x)\right) = -\sin(x)\sin(y) + y\exp(x).$

$\frac{\partial f}{\partial y}(x,y) = \cos(x)\left(\frac{\mathrm{d}}{\mathrm{d}y}\sin(y)\right) + \left(\frac{\mathrm{d}}{\mathrm{d}y}y\right)\exp(x) = \cos(x)\cos(y) + \exp(x).$

1.4. The Chain Rule

The chain rule is in many ways the most interesting of our basic rules for differentiation, in that it has a number of different forms for functions of more than one variable. Recall that in words, the chain rule says that we first differentiate the outside function and evaluate this derivative at the inside function, and then multiply by the derivative of the inside function.

We are trying to capture how much the composition is changing, and this involves following the change of both functions in the composition. Our normal paradigm will involve differentiating with respect to each of the intermediate variables, and so the number of terms is determined by the number of intermediate variables.

We start with a simple version. Consider the composition

$$\mathbb{R}^2_{(x,y)} \xrightarrow{f} \mathbb{R}_{(t)} \xrightarrow{g} \mathbb{R}.$$

Here, we use the notation $\mathbb{R}^2_{(x,y)}$ to mean that we are using variables x and y on $\mathbb{R}^2$. In this composition, $g(t)$ is a function of a single variable, and so there is only one possible notion for the derivative of $g(t)$, namely its usual derivative $g'(t)$.

On the other hand, $f(x,y)$ is a function of two variables, and so it has two partial derivatives, namely $\frac{\partial f}{\partial x}(x,y)$ and $\frac{\partial f}{\partial y}(x,y)$. Here, x and y are the independent variables and t is the intermediate variable. Since there is only one intermediate variable, we expect that the chain rule in this case will involve only a single term, and in fact we see that

$$\frac{\partial}{\partial x} g(f(x,y)) = g'(f(x,y)) \frac{\partial f}{\partial x}(x,y).$$

To use a different notation, we can say that $w = w(t)$ is a function of the single variable t, and $t = t(x,y)$ is a function of the two variables x and y. Again, t is the intermediate variable and x and y are the independent variables, and so the composition is $w = w(t) = w(t(x,y))$. The chain rule in this case says that

$$\boxed{\frac{\partial w}{\partial x} = \frac{\mathrm{d}w}{\mathrm{d}t}\,\frac{\partial t}{\partial x} = \frac{\mathrm{d}w}{\mathrm{d}t}(t(x,y))\,\frac{\partial t}{\partial x}(x,y),} \tag{1}$$

where in the last term we are merely making explicit the variables on which each of the functions is evaluated. Similarly,

$$\boxed{\frac{\partial w}{\partial y} = \frac{\mathrm{d}w}{\mathrm{d}t}\,\frac{\partial t}{\partial y} = \frac{\mathrm{d}w}{\mathrm{d}t}(t(x,y))\,\frac{\partial t}{\partial y}(x,y).} \tag{2}$$

Example 1.4.1. Consider the composition of $w = w(t) = \exp(t^2 + 1)$ and $t = t(x,y) = x^2 y + \sin(xy)$. Use the chain rule to evaluate $\frac{\partial w}{\partial x}(x,y)$ and $\frac{\partial w}{\partial y}(x,y)$.

Expanding out, we see that the composition is

$$w = w(x, y) = \exp((x^2y + \sin(xy))^2 + 1).$$

Using the chain rule, the partial derivatives of the composition $w = w(t) = w(t(x, y))$ are

$$\begin{aligned}\frac{\partial w}{\partial x} &= \frac{\mathrm{d}w}{\mathrm{d}t}(t(x, y))\,\frac{\partial t}{\partial x}(x, y)\\ &= 2t\exp(t^2 + 1)(2xy + y\cos(xy))\\ &= 2(x^2y + \sin(xy))(2xy + y\cos(xy))\exp((x^2y + \sin(xy))^2 + 1)\end{aligned}$$

and

$$\begin{aligned}\frac{\partial w}{\partial y} &= \frac{\mathrm{d}w}{\mathrm{d}t}(t(x, y))\,\frac{\partial t}{\partial y}(x, y)\\ &= 2t\exp(t^2 + 1)(x^2 + x\cos(xy))\\ &= 2(x^2y + \sin(xy))(x^2 + x\cos(xy))\exp((x^2y + \sin(xy))^2 + 1).\end{aligned}$$

In this case, we can check that our answer by substituting the expression for $t = t(x, y)$ into the expression for $w = w(t)$ and differentiating directly. We will see below an example where we are not able to do this direct calculation.

Next, consider the composition $\mathbb{R}_{(t)} \to \mathbb{R}^2_{(x,y)} \to \mathbb{R}$. In this case, we can say that $w = w(x, y)$ is a function of two variables, while $x = x(t)$ and $y = y(t)$ are themselves both functions of t. Here, x and y are the intermediate variables and t is the independent variable. In this case, the composition $w = w(x(t), y(t))$ is a function of the single variable t.

$$\begin{aligned}\frac{\mathrm{d}w}{\mathrm{d}t} &= \lim_{h\to 0}\frac{w(t+h) - w(t)}{h}\\ &= \lim_{h\to 0}\frac{w(x(t+h), y(t+h)) - w(x(t), y(t))}{h}\\ &= \lim_{h\to 0}\frac{w(x(t+h), y(t+h)) - w(x(t), y(t+h))}{h}\\ &+ \lim_{h\to 0}\frac{w(x(t), y(t+h)) - w(x(t), y(t))}{h}\\ &= \frac{\partial w}{\partial x}(x(t), y(t))\,\frac{\mathrm{d}x}{\mathrm{d}t}(t) + \frac{\partial w}{\partial y}(x(t), y(t))\,\frac{\mathrm{d}y}{\mathrm{d}t}(t).\end{aligned}$$

We do not explain the last equality in detail here but merely point out that it is similar to the proof of the singe-variable Chain Rule.

So, the Chain Rule says that

$$\boxed{\frac{\mathrm{d}w}{\mathrm{d}t} = \frac{\partial w}{\partial x}\,\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial w}{\partial y}\,\frac{\mathrm{d}y}{\mathrm{d}t}.} \qquad (3)$$

In both this case and the case before, notice the difference in notation. For differentiation with respect to t for the functions of a single variable, we use the roman $\frac{\mathrm{d}}{\mathrm{d}t}$, while for differentiation with respect to x and y for the functions of more than one variable, we use the round $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$.

Example 1.4.2. As an example, consider the composition of the function $w = w(x, y) = x^2 y + \sin(xy)$ with $x = x(t) = \exp(t^2 + 1)$ and $y = y(t) = t^3 + t$. Use the chain rule to evaluate $\frac{\mathrm{d}w}{\mathrm{d}t}(t)$.

Using the chain rule, the derivative of the composition $w = w(t) = w(x(t), y(t))$ is

$$\begin{aligned}\frac{\mathrm{d}w}{\mathrm{d}t} &= \frac{\partial w}{\partial x}(x(t), y(t))\,\frac{\mathrm{d}x}{\mathrm{d}t}(t) + \frac{\partial w}{\partial y}(x(t), y(t))\,\frac{\mathrm{d}y}{\mathrm{d}t}(t)\\ &= (2xy + y\cos(xy))2t\exp(t^2+1) + (x^2 + x\cos(xy))(3t^2+1)\\ &= 2t\exp(t^2+1)[(t^3+t)(2\exp(t^2+1) + \cos((t^3+t)\exp(t^2+1))]\\ &\quad +(3t^2+1)\exp(t^2+1)[\exp(t^2+1) + \cos((t^3+t)\exp(t^2+1))].\end{aligned}$$

In this case, as above, we can check our answer by substituting the expressions for $x = x(t)$ and $y = y(t)$ into $w = w(x, y)$ to realise $w = w(t)$ directly as a function of t and differentiating without using the chain rule.

Consider now the composition $\mathbb{R}^2_{(s,t)} \to \mathbb{R}^2_{(x,y)} \to \mathbb{R}$. In this case, we have that $w = w(x, y)$ is a function of x and y, and both $x = x(s, t)$ and $y = y(s, t)$ are functions of s and t. Here, x and y are again the intermediate variables, while s and t are the independent variables. The composition $w = w(x(s, t), y(s, t))$ is then a function of s and t. The Chain Rule in this case says that

$$\text{(4)}\qquad \boxed{\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} \quad \text{and} \quad \frac{\partial w}{\partial t} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial t}.}$$

In its most general form, consider the situation of a function $w = w(x_1, \ldots, x_n)$, where each $x_j = x_j(t_1, \ldots, t_p)$ is itself a function of $(t_1, \ldots, t_p)$. We can also write this as a composition $\mathbb{R}^p \xrightarrow{f} \mathbb{R}^n \xrightarrow{w} \mathbb{R}$, where the variables on $\mathbb{R}^p$ are $(t_1, \ldots, t_p)$ and the variables on $\mathbb{R}^n$ are $(x_1, \ldots, x_n)$. In this set up, the $(t_1, \ldots, t_p)$ are the independent variables and the $(x_1, \ldots, x_n)$ are the intermediate variables.

We follow the same paradigm as above, by differentiating the composition

$$w = w(x_1, \ldots, x_n) = w(x_1(t_1, \ldots, t_p), \ldots, x_n(t_1, \ldots, t_p))$$

with respect to one of the independent variables $(t_1, \ldots, t_p)$, where the corresponding expression has one term for each of the intermediate variables $(x_1, \ldots, x_n)$. So, for each $1 \leq k \leq p$, we see that

$$\text{(5)}\qquad \boxed{\frac{\partial w}{\partial t_k} = \sum_{j=1}^{n} \frac{\partial w}{\partial x_j}\frac{\partial x_j}{\partial t_k},}$$

or if we were to add the arguments for each of the functions,

$$\frac{\partial w}{\partial t_k}(x_1(t_1,\ldots,t_p),\ldots,x_n(t_1,\ldots,t_p)) =$$

$$= \sum_{j=1}^{n} \frac{\partial w}{\partial x_j}(x_1(t_1,\ldots,t_p),\ldots,x_n(t_1,\ldots,t_p))\, \frac{\partial x_j}{\partial t_k}(t_1,\ldots,t_p).$$

Example 1.4.3. Suppose that the function $u = u(x,t)$ satisfies the differential equation

$$\frac{\partial u}{\partial t}(x,t) + u\frac{\partial u}{\partial x}(x,t) = 0$$

and that $x = x(t)$ as a function of t satisfies

$$\frac{\mathrm{d}x}{\mathrm{d}t}(t) = u(x,t).$$

Prove that $u(x(t),t)$ is constant as a function of t.

Since we wish to prove that $u(x(t),t)$ is constant as a function of t, let us start by calculating $\frac{\mathrm{d}}{\mathrm{d}t}u(x(t),t)$. Using the Chain Rule, we see that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}u(x(t),t) &= \frac{\partial u}{\partial x}(x(t),t)\frac{\mathrm{d}x}{\mathrm{d}t}(t) + \frac{\partial u}{\partial t}(x,t)\\ &= \frac{\partial u}{\partial x}(x(t),t)u(x,t) + \frac{\partial u}{\partial t}(x,t)\\ &= \frac{\partial u}{\partial x}(x(t),t)u(x,t) - u\frac{\partial u}{\partial x}(x,t) = 0.\end{aligned}$$

Therefore, $u(x(t),t)$ is constant as a function of t.

Example 1.4.4. Suppose that $z = z(u,v,r)$ is a function of the variables u, v and r; that $u = u(x,y,r)$ is a function of x, y and r; that $v = v(x,y,r)$ is a function of x, y and r, and that $r = r(x,y)$ is a function of x and y. Find $\frac{\partial z}{\partial x}$.

As with all examples of using the chain rule for functions of several variables, we first need to determine which are the independent variables (which are the variables that do not depend on any other variables) and which variables are the intermediate variables (which can depend either on the independent variables or on other intermediate variables).

In this example, the independent variables are x and y, as we do not express any of these variables as functions of other variables. Note that the actual names of the independent variables will vary from one example to another; therefore, we need to make this determination of independent versus intermediate variable separately for each exercise or example we consider. The variables u, v and r are the intermediate variables, as each are functions of other variables. Note that both u and v are in fact functions of the independent variables x and y and the intermediate variable r, which is in turn a function of x and y.

So we can write z as a function of the variables x and y only:

$$z = z\big(u(x, y, r(x, y)), v(x, y, r(x, y)), r(x, y)\big).$$

Letting $\overline{u} = u(x, y, r(x, y))$ and $\overline{v} = v(x, y, r(x, y))$, we can apply the Chain Rule to obtain an expression for $\frac{\partial z}{\partial x}$ as a sum of three terms, one coming from each of the intermediate variables $\overline{u}$, $\overline{v}$, and r as follows:

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial \overline{u}}\frac{\partial \overline{u}}{\partial x} + \frac{\partial z}{\partial \overline{v}}\frac{\partial \overline{v}}{\partial x} + \frac{\partial z}{\partial r}\frac{\partial r}{\partial x}.$$

Since x, y and r are intermediate variables that sit between $\overline{u}$ and x, and between $\overline{v}$ and x, we can break down the terms $\frac{\partial \overline{u}}{\partial x}$ and $\frac{\partial \overline{v}}{\partial x}$ further. Namely,

$$\frac{\partial \overline{u}}{\partial x} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial x} + \frac{\partial u}{\partial r}\frac{\partial r}{\partial x} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial r}\frac{\partial r}{\partial x}$$

and similarly

$$\frac{\partial \overline{v}}{\partial x} = \frac{\partial v}{\partial x}\frac{\partial x}{\partial x} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial x} + \frac{\partial v}{\partial r}\frac{\partial r}{\partial x} = \frac{\partial v}{\partial x} + \frac{\partial v}{\partial r}\frac{\partial r}{\partial x}.$$

Also, note that

$$\frac{\partial z}{\partial \overline{u}} = \frac{\partial z}{\partial u} \quad \text{and} \quad \frac{\partial z}{\partial \overline{v}} = \frac{\partial z}{\partial v}$$

evaluated at the point $\big(u(x, y, r(x, y)), v(x, y, r(x, y)), r(x, y)\big)$.

Bringing everything together, we see that

$$\begin{aligned}
\frac{\partial z}{\partial x} &= \frac{\partial z}{\partial u}\frac{\partial \overline{u}}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial \overline{v}}{\partial x} + \frac{\partial z}{\partial r}\frac{\partial r}{\partial x} \\
&= \frac{\partial z}{\partial u}\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial r}\frac{\partial r}{\partial x}\right) + \frac{\partial z}{\partial v}\left(\frac{\partial v}{\partial x} + \frac{\partial v}{\partial r}\frac{\partial r}{\partial x}\right) + \frac{\partial z}{\partial r}\frac{\partial r}{\partial x} \\
&= \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial u}\frac{\partial u}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial z}{\partial r}\frac{\partial r}{\partial x}.
\end{aligned}$$

1.5. THE GRADIENT AND THE JACOBIAN MATRIX

Now that we are equipped with the notion of the partial derivative of a function of several variables with respect to one of its variables, we can define a notion of a single derivative for a function $f : \mathbb{R}^n \to \mathbb{R}$ for any $n \geq 2$, and indeed for a function

$$F = (f_1, \ldots, f_m) : \mathbb{R}^n \to \mathbb{R}^m$$

for $n \geq 2$ and $m \geq 1$, where this single derivative is comprised of the partial derivatives of the component functions f_i, $1 \leq i \leq m$. We start with the former.

For this we need the notion of a gradient of a function which combines the terms of the partial derivatives into a single expression.

Definition 1.5.1. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function. Assume that both partial derivatives exist at a point $(x, y) \in \mathbb{R}^2$. We define the *gradient* of $f(x, y)$ to be the vector of partial derivatives

$$\nabla f(x,y) = \left(\frac{\partial f}{\partial x}(x,y), \frac{\partial f}{\partial y}(x,y) \right).$$

One thing to note is that, while $f : \mathbb{R}^2 \to \mathbb{R}$, its gradient $\nabla f(x, y)$ is a function $\nabla f : \mathbb{R}^2 \to \mathbb{R}^2$, since the value of the gradient $\nabla f(a, b)$ at the point $(a, b) \in \mathbb{R}^2$ is itself a vector, namely

$$\nabla f(a,b) = \left(\frac{\partial f}{\partial x}(a,b), \frac{\partial f}{\partial y}(a,b) \right),$$

which we can again view as a point in $\mathbb{R}^2$.

More generally, we can define the *gradient* of a function $f(x_1, \ldots, x_n)$ of n variables to again be the vector of partial derivatives

$$\nabla f(x_1,\ldots,x_n) = \left(\frac{\partial f}{\partial x_1}(x_1,\ldots,x_n), \ldots, \frac{\partial f}{\partial x_n}(x_1,\ldots,x_n) \right).$$

Example 1.5.2. Find the gradients of the functions $f(x, y) = x + y - 1$, $g(x, y) = e^x + xe^y$, and $h(x, y, z) = x^2 + y^2 + z^2 + 3$.

$\nabla f(x,y) = \left(\frac{\partial f}{\partial x}(x,y), \frac{\partial f}{\partial y}(x,y) \right) = (1,1),$

$\nabla g(x,y) = (e^x + e^y, xe^y),$

$\nabla h(x,y,z) = (2x, 2y, 2z).$

Up to this point, we have considered real-valued functions on $\mathbb{R}^n$, which are just functions of the form $f : \mathbb{R}^n \to \mathbb{R}$ for $n \geq 1$. Next, we expand our view to functions of the form $F : \mathbb{R}^n \to \mathbb{R}^p$, where $n \geq 1$ and $p \geq 1$.

The first thing to note is that a function $F : \mathbb{R}^n \to \mathbb{R}^p$ is composed of p functions $f_j : \mathbb{R}^n \to \mathbb{R}$ for $j = 1 \cdots , p$, namely

$$F(x_1,\ldots,x_n) = (f_1(x_1,\ldots,x_n),\ldots,f_p(x_1,\ldots,x_n)).$$

As when we introduced the gradient, we wish to have a single object that contains all of the partial derivatives of the component functions $f_1(x_1, \ldots, x_n), \ldots, f_p(x_1, \ldots, x_n)$ of $F(x_1, \ldots, x_n)$ with respect to $x_1, \ldots, x_n$ that can play the role of the derivative of $F(x_1, \ldots, x_n)$. To that end, we introduce the Jacobian matrix $J_F = J_F(x_1, \ldots, x_n)$ of $F(x_1, \ldots, x_n)$.

Definition 1.5.3. Let $F : \mathbb{R}^n \to \mathbb{R}^p$ be a function where $n \geq 1$ and $p \geq 1$. Assume $\frac{\partial f_i}{\partial x_j}$ exists for all $1 \leq i \leq p$ and $1 \leq j \leq n$ at a point $(x_1, \ldots, x_n) \in D$. The *Jacobian matrix* at $(x_1, \ldots, x_n)$ is the $p \times n$ matrix of first order partial derivatives of the component functions of $F(x_1, \ldots, x_n)$. Specifically, for

$$F(x_1,\ldots,x_n) = (f_1(x_1,\ldots,x_n),\ldots,f_p(x_1,\ldots,x_n)),$$

we set

$$J_F(x_1,\ldots,x_n) = \left(\frac{\partial f_i}{\partial x_j}\right) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(x_1,\ldots,x_n) & \cdots & \frac{\partial f_1}{\partial x_n}(x_1,\ldots,x_n) \\ \vdots & & \vdots \\ \frac{\partial f_p}{\partial x_1}(x_1,\ldots,x_n) & \cdots & \frac{\partial f_p}{\partial x_n}(x_1,\ldots,x_n) \end{pmatrix}.$$

Note that the rows of J_F are merely the gradients of the component functions $f_1,\ldots,f_n$ of F.

Remark 1.5.4. There is one important distinction between the gradient and the Jacobian matrix to note. For a function $f:\mathbb{R}^n \to \mathbb{R}$, the gradient $\nabla f(x_1,\ldots,x_n)$ of f and the Jacobian matrix $J_f(x_1,\ldots,x_n)$ of f are the same, as both are just the row vector of the partial derivatives of f. However, for a function $F:\mathbb{R}^n \to \mathbb{R}^m$ for $m \geq 2$, the gradient is undefined, and all we have is the Jacobian matrix.

Example 1.5.5. Find the Jacobian matrix of the function

$$F(x,y,z) = (x+y-1, e^x + xe^y, x^2+y^2+z^2+3).$$

$$J_F(x,y,z) =$$

$$= \begin{pmatrix} \frac{\partial}{\partial x}(x+y-1) & \frac{\partial}{\partial y}(x+y-1) & \frac{\partial}{\partial z}(x+y-1) \\ \frac{\partial}{\partial x}(e^x+xe^y) & \frac{\partial}{\partial y}(e^x+xe^y) & \frac{\partial}{\partial z}(e^x+xe^y) \\ \frac{\partial}{\partial x}(x^2+y^2+z^2+3) & \frac{\partial}{\partial y}(x^2+y^2+z^2+3) & \frac{\partial}{\partial z}(x^2+y^2+z^2+3) \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 1 & 0 \\ e^x+e^y & xe^y & 0 \\ 2x & 2y & 2z \end{pmatrix}.$$

1.6. The Chain Rule via the Jacobian matrix

One reason for introducing the gradient and the Jacobian is that they allow us to greatly simplify our presentation of the Chain Rule. Let us consider again the examples we went through in the previous section.

First consider the composition

$$\mathbb{R}^2_{(x,y)} \xrightarrow{f} \mathbb{R}_{(t)} \xrightarrow{g} \mathbb{R}.$$

In this composition, $g(t)$ is a function of a single variable, and so there is only one possible notion for the derivative of $g(t)$, namely its usual derivative $g'(t)$. On the other hand, $f(x,y)$ is a function of two variables, and so it has two partial derivatives, namely $\frac{\partial f}{\partial x}(x,y)$ and $\frac{\partial f}{\partial y}(x,y)$. The derivative of the composition is then

$$\nabla(g\circ f)(x,y) =$$

$$= \begin{pmatrix} g'(f(x,y))\,\frac{\partial f}{\partial x}(x,y) & g'(f(x,y))\,\frac{\partial f}{\partial y}(x,y) \end{pmatrix}$$
$$= g'(f(x,y)) \begin{pmatrix} \frac{\partial f}{\partial x}(x,y) & \frac{\partial f}{\partial y}(x,y) \end{pmatrix}.$$

Now, we can write the chain rule using the gradient and the Jacobian:

$$\boxed{\nabla(g\circ f)(x,y) = g'(f(x,y))\,\nabla f(x,y).} \tag{6}$$

Next, consider the composition $\mathbb{R}_{(t)} \xrightarrow{f} \mathbb{R}^2_{(x,y)} \xrightarrow{w} \mathbb{R}$. In this case, we can say that $w(x,y)$ is a function of two variables, while $f(t) = (x(t), y(t))$. In this case, the composition $w(f(t)) = w(x(t), y(t)$ is a function of the single variable t, so the Chain Rule in this case says that

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} w(f(t)) &= \frac{\partial w}{\partial x}(x(t),y(t))\,\frac{\mathrm{d}x}{\mathrm{d}t}(t) + \frac{\partial w}{\partial y}(x(t),y(t))\,\frac{\mathrm{d}y}{\mathrm{d}t}(t)\\
&= \begin{pmatrix} \frac{\partial w}{\partial x}(f(t)) & \frac{\partial w}{\partial y}(f(t)) \end{pmatrix} \begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix}\\
&= \nabla w(f(t)) \cdot J_f(t)
\end{aligned}$$

Again, we can write the chain rule using the Jacobian:

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}t}(w\circ f)(t) = \nabla w(f(t)) \cdot J_f(t).} \tag{7}$$

Consider now the composition $\mathbb{R}^2_{(s,t)} \xrightarrow{f} \mathbb{R}^2_{(x,y)} \xrightarrow{w} \mathbb{R}$. In this case, we have that $w(x,y)$ is a function of x and y, and $g(s,t) = (x(s,t), y(s,t))$ is a function of s and t. We have that

$$\begin{pmatrix} \frac{\partial(w\circ f)}{\partial s}(s,t) & \frac{\partial(w\circ f)}{\partial t}(s,t) \end{pmatrix} =$$

$$\begin{aligned}
&= \begin{pmatrix} \frac{\partial w}{\partial x}(f(s,t))\,\frac{\partial x}{\partial s}(s,t) + \frac{\partial w}{\partial y}(f(s,t))\,\frac{\partial y}{\partial s}(s,t) \\ \frac{\partial w}{\partial x}(f(s,t))\,\frac{\partial x}{\partial t}(s,t) + \frac{\partial w}{\partial y}(f(s,t))\,\frac{\partial y}{\partial t}(s,t) \end{pmatrix}\\
&= \begin{pmatrix} \frac{\partial w}{\partial x}(f(s,t)) & \frac{\partial w}{\partial y}(f(s,t)) \end{pmatrix} \begin{pmatrix} \frac{\partial x}{\partial s}(s,t) & \frac{\partial x}{\partial t}(s,t) \\ \frac{\partial y}{\partial s}(s,t) & \frac{\partial y}{\partial t}(s,t) \end{pmatrix}\\
&= \nabla w(f(s,t)) \cdot J_f(s,t).
\end{aligned}$$

Hence, we write the Chain Rule in this case:

$$\boxed{\nabla(w\circ f)(s,t) = \nabla w(f(s,t)) \cdot J_f(s,t).} \tag{8}$$

Next, we state the most general form of the Chain Rule.

Theorem 1.6.1 (The Chain Rule)**.** *Let $F : \mathbb{R}^m \to \mathbb{R}^n$ be differentiable at a point $\mathbf{a} \in \mathbb{R}^m$ and $G : \mathbb{R}^n \to \mathbb{R}^p$ be differentiable at the point $\mathbf{b} = F(\mathbf{a})$. Then $G \circ F : \mathbb{R}^m \to \mathbb{R}^p$ is differentiable at $\mathbf{a}$ and*

$$J_{G\circ F}(\mathbf{a}) = J_G(\mathbf{b}) \cdot J_F(\mathbf{a}).$$

We differ the definition of differentiability of a function $F : \mathbb{R}^m \to \mathbb{R}^n$ until Section 1.8. For now, let us see how this general form of the Chain Rule implies all the other special cases.

Example 1.6.2. For example, when $m = n = p = 1$, then we have that $F : \mathbb{R} \to \mathbb{R}$ is differentiable at $a \in R$ and $G : \mathbb{R} \to \mathbb{R}$ is differentiable at $b = F(a)$. By Theorem 1.6.1, we conclude that the composition $(G \circ F)(x)$ is differentiable at a and one has

$$\frac{\partial}{\partial x}(G \circ F)(a) = J_{G\circ F}(a) = J_G(F(a)) \cdot J_F(a) = G'(F(a))\frac{\partial F}{\partial x}(a).$$

which is the Chain Rule for composition of one variable function we are familiar with.

When $m = n = 2$ and $p = 1$, then $F : \mathbb{R}^2_{(s,t)} \to \mathbb{R}^2_{(x,y)}$ is differentiable at $(a_1, a_2) \in R^2$ and $G : \mathbb{R}^2_{(x,y)} \to \mathbb{R}$ is differentiable at $F(a_1, a_2)$. Using Theorem 1.6.1, it follows that the composition function $G \circ F : \mathbb{R}^2 \to \mathbb{R}$ is differentiable at (a_1, a_2) and at this point we have

$$\begin{aligned}
\begin{pmatrix} \frac{\partial}{\partial s}(G\circ F) & \frac{\partial}{\partial t}(G\circ F) \end{pmatrix} &= J_{G\circ F} \\
&\overset{1.6.1}{=} J_G \cdot J_F \\
&= \begin{pmatrix} \dfrac{\partial G}{\partial x} & \dfrac{\partial G}{\partial y} \end{pmatrix} \begin{pmatrix} \dfrac{\partial x}{\partial s} & \dfrac{\partial x}{\partial t} \\ \dfrac{\partial y}{\partial s} & \dfrac{\partial y}{\partial t} \end{pmatrix} \\
&= \begin{pmatrix} \dfrac{\partial G}{\partial x}\dfrac{\partial x}{\partial s} + \dfrac{\partial G}{\partial y}\dfrac{\partial y}{\partial s} & \dfrac{\partial G}{\partial x}\dfrac{\partial x}{\partial t} + \dfrac{\partial G}{\partial y}\dfrac{\partial y}{\partial t} \end{pmatrix}
\end{aligned}$$

Therefore,

$$\frac{\partial}{\partial s}(G\circ F) = \frac{\partial G}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial G}{\partial y}\frac{\partial y}{\partial s} \quad \text{and} \quad \frac{\partial}{\partial t}(G\circ F) = \frac{\partial G}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial G}{\partial y}\frac{\partial y}{\partial t}$$

as in the equation (4).

1.7. EQUATIONS OF NORMAL VECTORS AND TANGENT PLANES

Recall that in order to define a plane in $\mathbb{R}^3$, we need two pieces of information. The first is a point $P \in \mathbb{R}^3$ on the plane and the second is a normal vector $\mathbf{n}$ to the plane. The equation of the plane passing through P and normal to $\mathbf{n}$ is then given by

$$((x, y, z) - P) \cdot \mathbf{n} = 0.$$

So, let $f(x, y)$ be a function of two variables. Given a point (a, b) (in the domain of $f(x, y)$), the corresponding point on the graph of $f(x, y)$ is $(a, b, f(a, b))$. Figure 8 shows the normal vector and the tangent plane to the graph of the function $f(x, y)$ at the point $P = (a, b, f(a, b))$.

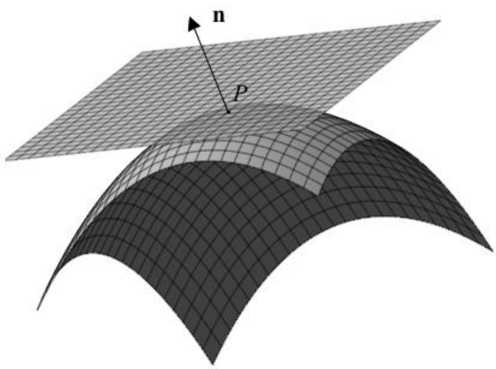


FIGURE 8. normal vector and tangent plane at P

Consider what happens when we fix one of the variables and let the other one vary. For instance, consider the behaviour of the curve on the graph of the function $w = f(x, y)$ defined by holding $y = b$ constant and letting x vary through values near a. We then get the curve $(x, b, f(x, b))$, and the tangent vector to this curve at the point $(a, b, f(a, b))$ is given by the vector $(1, 0, \frac{\partial f}{\partial x}(a, b))$.

Similarly, we can fix $x = a$ and let y vary near b to get the curve $(a, y, f(a, y))$, and the tangent vector to this curve at the point $(a, b, f(a, b))$ is given by the vector $(0, 1, \frac{\partial f}{\partial y}(a, b))$.

Definition 1.7.1. Let $\mathbf{n}(a, b)$ be the vector to the graph of $f(x, y)$ at a point $(a, b, f(a, b))$ that is the cross product of the above two vectors, namely

$$\mathbf{n}(a, b) = \left(1, 0, \frac{\partial f}{\partial x}(a, b)\right) \times \left(0, 1, \frac{\partial f}{\partial y}(a, b)\right) = \left(-\frac{\partial f}{\partial x}(a, b), -\frac{\partial f}{\partial y}(a, b), 1\right).$$

The *normal line* to the graph of $f(x, y)$ at $(a, b, f(a, b))$ is the unique line passing through this point that is parallel to $\mathbf{n}(a, b)$. We say that a nonzero vector $\mathbf{v}$ at $(a, b, f(a, b))$ is a *normal vector* to the graph of $f(x, y)$ at $(a, b, f(a, b))$ if it is parallel to the normal line. That is, there is a nonzero $\lambda \in \mathbb{R}$ such that $\mathbf{v} = \lambda \mathbf{n}(a, b)$. In particular, $\mathbf{n}(a, b)$ is a normal vector.

The equation of the tangent plane to the graph of $f(x,y)$ at the point $(a,b,f(a,b))$ is then given by

$$\begin{aligned} 0 &= ((x,y,z)-(a,b,f(a,b)))\cdot \mathbf{n}(a,b) \\ &= ((x,y,z)-(a,b,f(a,b)))\cdot \left(-\frac{\partial f}{\partial x}(a,b), -\frac{\partial f}{\partial y}(a,b), 1\right) \\ &= (x-a, y-b, z-f(a,b))\cdot \left(-\frac{\partial f}{\partial x}(a,b), -\frac{\partial f}{\partial y}(a,b), 1\right) \\ &= -\frac{\partial f}{\partial x}(a,b)(x-a) - \frac{\partial f}{\partial y}(a,b)(y-b) + z - f(a,b). \end{aligned}$$

Definition 1.7.2. The *tangent plane* to the graph of the function $w = f(x,y)$ at the point $(a,b) \in \mathbb{R}^2$ is given by the equation

$$z = \frac{\partial f}{\partial x}(a,b)(x-a) + \frac{\partial f}{\partial y}(a,b)(y-b) + f(a,b).$$

Example 1.7.3. Find a normal vector and the equation of the tangent plane to the graph of $f(x,y) = x^2 \exp(xy)$ at the point $(1,\pi,f(1,\pi))$.

We start by calculating the gradient of $f(x,y)$:

$$\nabla f(x,y) = \left(\frac{\partial f}{\partial x}(x,y), \frac{\partial f}{\partial y}(x,y)\right) = (2x\exp(xy) + x^2 y \exp(xy), x^3 \exp(xy)).$$

Hence, a normal vector at the point $(x,y,f(x,y))$ on the graph of $f(x,y)$ is

$$\mathbf{n}(x,y) = \left(-\frac{\partial f}{\partial x}(x,y), -\frac{\partial f}{\partial y}(x,y), 1\right) = (-(2x+x^2y)\exp(xy), -x^3\exp(xy), 1),$$

and so a normal vector to the graph of $f(x,y)$ at the point $(1,\pi,f(1,\pi)) = (1,\pi,\exp(\pi))$ is

$$\mathbf{n}(1,\pi) = (-(2+\pi)\exp(\pi), -\exp(\pi), 1).$$

Hence, the equation of the tangent plane to the graph of $f(x,y) = x^2\exp(xy)$ at the point $(1,\pi,f(1,\pi)) = (1,\pi,\exp(\pi))$ is

$$\begin{aligned} 0 &= ((x,y,z)-(1,\pi,\exp(\pi)))\cdot \mathbf{n}(1,\pi) \\ &= ((x,y,z)-(1,\pi,\exp(\pi)))\cdot(-(2+\pi)\exp(\pi), -\exp(\pi), 1) \\ &= -(x-1)(2+\pi)\exp(\pi) - (y-\pi)\exp(\pi) + z - \exp(\pi) \\ &= -(2+\pi)\exp(\pi)x - \exp(\pi)y + z - (1+2\pi)\exp(\pi). \end{aligned}$$

Example 1.7.4. Show that every tangent plane to the cone $z^2 = x^2 + y^2$ (see Figure 9) passes through the origin $\mathbf{0} = (0,0,0)$.

We start by expressing the cone $z^2 = x^2 + y^2$ as the union of the graphs of two functions. Namely, we can express the cone as the union of the graph of $f(x,y) = \sqrt{x^2+y^2}$ (part of the cone above the xy-plane) and the graph of $g(x,y) = -\sqrt{x^2+y^2}$ (part of the cone below the xy-plane), where both function $f(x,y)$ and $g(x,y)$ are defined on the whole plane $\mathbb{R}^2$ and where the

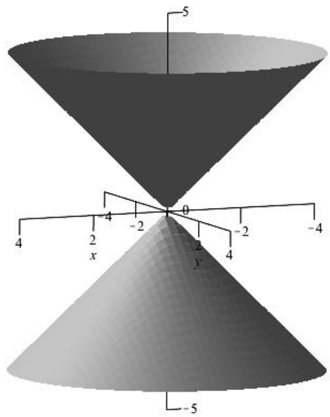


FIGURE 9. the cone given by the equation $z^2 = x^2 + y^2$

two graphs intersect at the single point which is the origin $(0,0,0)$ in $\mathbb{R}^3$.

We start with $f(x,y)$. At a point (a,b) in $\mathbb{R}^2$, the equation for the normal vector $\mathbf{n}_f(a,b)$ to the graph of $f(x,y)$ at the point $(a,b,f(a,b))$ is

$$\mathbf{n}_f(a,b) = \left(-\frac{\partial f}{\partial x}(a,b), -\frac{\partial f}{\partial y}(a,b), 1\right) = \left(-\frac{a}{\sqrt{a^2+b^2}}, -\frac{b}{\sqrt{a^2+b^2}}, 1\right).$$

We note that the normal vector is not defined at the origin $\mathbf{0} = (0,0)$. Hence, the equation of the tangent vector to the graph of $f(x,y)$ at $(a,b,f(a,b))$ is

$$((x,y,z) - (a,b,f(a,b))) \cdot \mathbf{n}_f(a,b) =$$
$$= ((x,y,z) - (a,b,f(a,b))) \cdot \left(-\frac{a}{\sqrt{a^2+b^2}}, -\frac{b}{\sqrt{a^2+b^2}}, 1\right) = 0.$$

This becomes the equation

$$0 = -\frac{a(x-a)}{\sqrt{a^2+b^2}} - \frac{b(y-b)}{\sqrt{a^2+b^2}} + z - \sqrt{a^2+b^2}.$$

In order to determine whether the origin $(0,0,0)$ in $\mathbb{R}^3$, we plug in the values $(x,y,z) = (0,0,0)$ and see whether the equation remains true.

Setting $(x,y,z) = (0,0,0)$, we get the equation

$$0 = -\frac{a(-a)}{\sqrt{a^2+b^2}} - \frac{b(-b)}{\sqrt{a^2+b^2}} + -\sqrt{a^2+b^2} = \frac{a^2+b^2}{\sqrt{a^2+b^2}} - \sqrt{a^2+b^2},$$

which is a true identity (meaning it is always true).

Though we do not go through the details, the discussion of $g(x,y) = -\sqrt{x^2+y^2}$ works in exactly the same way.

Example 1.7.5. Find every point on the surface of the ellipsoid given by the equation $x^2 + 4y^2 + 9z^2 = 16$ at which the normal line at the point passes through the centre $(0,0,0)$ of the ellipsoid.

We start by working on the upper half of the ellipsoid, which is the graph of the function

$$f(x, y) = \frac{1}{3}\sqrt{16 - x^2 - 4y^2}$$

(which is obtained by solving the equation for the ellipsoid for z and then imposing the constraint that $z > 0$).

Hence, at a point $(a, b, f(a, b))$ on the ellipsoid, the equation of a normal vector is

$$\mathbf{n}(a, b) =$$
$$= \left(-\frac{\partial f}{\partial x}(a, b), -\frac{\partial f}{\partial y}(a, b), 1\right) = \left(\frac{a}{3\sqrt{16 - a^2 - 4b^2}}, \frac{4b}{3\sqrt{16 - a^2 - 4b^2}}, 1\right).$$

The normal line through $(a, b, f(a, b))$ is given parametrically by the equation

$$(a, b, f(a, b)) + t\mathbf{n}(a, b) =$$
$$= \left(a, b, \frac{1}{3}\sqrt{16 - a^2 - 4b^2}\right) + t\left(\frac{a}{3\sqrt{16 - a^2 - 4b^2}}, \frac{4b}{3\sqrt{16 - a^2 - 4b^2}}, 1\right).$$

In order for this line to pass through the origin, we need to find a value of t so that $(a, b, f(a, b)) + t\mathbf{n}(a, b) = (0, 0, 0)$. If we consider the third coordinate, this tells us that

$$t = -\frac{1}{3}\sqrt{16 - a^2 - 4b^2}.$$

For this value of t, we then have that

$$-\frac{1}{3}\sqrt{16 - a^2 - 4b^2}\left(\frac{a}{3\sqrt{16 - a^2 - 4b^2}}, \frac{4b}{3\sqrt{16 - a^2 - 4b^2}}, 1\right) =$$
$$= \left(-\frac{a}{9}, -\frac{4b}{9}, -\frac{1}{3}\sqrt{16 - a^2 - 4b^2}\right),$$

and so for this value of t, we have that

$$(a, b, f(a, b)) + t\mathbf{n}(a, b) = \left(a - \frac{a}{9}, b - \frac{4b}{9}, 0\right) = \left(\frac{8a}{9}, \frac{5b}{9}, 0\right).$$

The only point values of a and b for which this can be $(0, 0, 0)$ are $a = 0$ and $b = 0$, which gives the point $(a, b, f(a, b)) = (0, 0, \frac{4}{3})$ on the ellipsoid, which is the topmost point.

We can run the same argument for the lower half of the ellipsoid given by

$$g(x, y) = -\frac{1}{3}\sqrt{16 - x^2 - 4y^2}$$

to get that the only point on the lower half of the ellipsoid through which the normal line passes through the origin is $(0, 0, -\frac{4}{3})$, the bottommost point.

Similarly, we can run the same argument on the front and back halves of the ellipsoid, so viewing x as a function of y and z, to get front-most and back-most points $(4, 0, 0)$ and $(-4, 0, 0)$, and on the left and right halves of the ellipsoid, so viewing y as a function of x and z, to get the leftmost and rightmost points $(0, 2, 0)$ and $(0, -2, 0)$. So there are in all 6 points on the ellipsoid through which the normal line passes through the origin, and these are the points at which the ellipsoid intersects the three coordinate axes.

The reason that we solve the equation of the ellipsoid for x, y, and z respectively and then consider the resulting 2 equations in each case is the following.

Each of theses 6 equations describes the ellipsoid in a particular region. Hence, in each case, when we find the points on the ellipsoid such that the normal line passes through $(0, 0, 0)$, they are only the points on one of the 6 regions of the ellipsoid. For example, when we consider

$$f(x, y) = \frac{1}{3}\sqrt{16 - x^2 - 4y^2}$$

and its partial derivatives, the region on which they are all defined is when $z > 0$. This is the portion of the ellipsoid which is above and not intersecting the xy-plane. So, in order to find all the points, we need to consider regions that cover the whole ellipsoid. Solving for x, y, and z respectively and then considering the resulting 2 equations in each case does exactly this as the 6 regions cover the whole ellipsoid.

Example 1.7.6. Find the distance from the origin $O = (0, 0, 0)$ to the ellipsoid given by the equation $x^2 + 4y^2 + 9z^2 = 16$.

Note that the distance between the origin and the ellipsoid is the length of the shortest line segment connecting a point on the ellipsoid to the origin. Let P be such a point on the ellipsoid. Then the vector $\overrightarrow{OP}$ must be normal to the surface and hence on the normal line passing through the origin. By Example 1.7.5, we conclude that $P \in \{(\pm 4, 0, 0), (0, \pm 2, 0), (0, 0, \pm\frac{4}{3})\}$. Calculating the distance between each point and the origin, we see that the minimum is $\frac{4}{3}$. Thus, the minimum distance from the origin $O = (0, 0, 0)$ to the ellipsoid is $\frac{4}{3}$ which is attained at points $(0, 0, \pm\frac{4}{3})\}$ of the surface.

1.8. Differentiability

We go back to the definition of the derivative for a function $f(x)$ of one variable at a point $x = a$, to whit

$$f'(a) = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}.$$

Since $f'(a)$ is a constant, we can carry it over to the right hand side and use basic properties of limits to see that we can rewrite this equation as

$$\lim_{h \to 0} \frac{f(a + h) - f(a) - f'(a)h}{h} = 0. \tag{9}$$

We can then say that the function $f : \mathbb{R} \to \mathbb{R}$ is *differentiable* at a point $x = a$ if there exists $f'(a) \in \mathbb{R}$ so that

$$\lim_{h \to 0} \frac{f(a+h) - f(a) - f'(a)h}{h} = 0.$$

We also know that the derivative $f'(a)$ has a geometric interpretation as the slope of the line tangent to the graph of the function at the point $(a, f(a)) \in \mathbb{R}^2$. So, the function is differentiable when such a tangent line exists and hence its slope is well-defined.

If we now consider a function of two variables $f : \mathbb{R}^2 \to \mathbb{R}$ instead, then the tangent line is replaced by the tangent plane at a given point $(a, b, f(a, b))$. So, the function would be differentiable at (a, b) when such a tangent plane exists. The tangent plane does not have a single slope but rather many slopes coming from different directions. We then compute the slopes in the x and y directions, that is, when we fix $y = b$ and $x = a$ respectively and consider the slopes of the lines $(x, b, f(x, b))$ and $(a, y, f(a, y))$ in the tangent plane. These slopes are precisely the partial derivatives of $f(x, y)$ at (a, b). So, for a function $f(x, y)$ to be differentiable at (a, b), the gradient $\nabla f(a, b)$ needs to exist.

One should expect the converse not to be true as we only choose two specific lines on the tangent plane in order to describe the gradient vector. But as we shall see, this is almost true.

1.8.1. **Differentiability in the case of two variable functions.** Now, let us make the precise definition of differentiability.

Definition 1.8.1. A function $f : \mathbb{R}^2 \to \mathbb{R}$ is *differentiable* at a point $(a, b) \in \mathbb{R}^2$ if there is a vector $\mathbf{v} = (v_1, v_2) \in \mathbb{R}^2$ so that

$$\lim_{(h,k) \to (0,0)} \frac{f(a+h, b+k) - f(a, b) - \mathbf{v} \cdot (h, k)}{\sqrt{h^2 + k^2}} = 0. \tag{10}$$

Theorem 1.8.2. *If the function $f : \mathbb{R}^2 \to \mathbb{R}$ is differentiable at a point $(a, b) \in \mathbb{R}^2$, then it is continuous at (a, b). Moreover, the vector $\mathbf{v}$ defined by (10) is unique and is equal to $\nabla f(a, b)$.*

Proof. To see the first assertion note that $\lim_{(h,k) \to (0,0)} \sqrt{h^2 + k^2} = 0$. This together with (10) gives us

$$\lim_{(h,k) \to (0,0)} \sqrt{h^2 + k^2} \cdot \lim_{(h,k) \to (0,0)} \frac{f(a+h, b+k) - f(a, b) - \mathbf{v} \cdot (h, k)}{\sqrt{h^2 + k^2}} = 0,$$

$$\lim_{(h,k) \to (0,0)} f(a+h, b+k) - f(a, b) - \mathbf{v} \cdot (h, k) = 0,$$

$$\lim_{(h,k) \to (0,0)} f(a+h, b+k) - f(a, b) = 0,$$

$$\lim_{(h,k)\to(0,0)} f(a+h,b+k) = f(a,b).$$

To see the second assertion, note that we get the same limit if we let $k = 0$ in (10). So,

$$\lim_{(h,0)\to(0,0)} \frac{f(a+h,b) - f(a,b) - (v_1,v_2)\cdot(h,0)}{|h|} = 0.$$

But this implies that

$$\lim_{h\to 0} \left| \frac{f(a+h,b) - f(a,b) - (v_1,v_2)\cdot(h,0)}{h} \right| = 0,$$

which in turn implies

$$\lim_{h\to 0} \frac{f(a+h,b) - f(a,b) - (v_1,v_2)\cdot(h,0)}{h} = 0,$$

$$\lim_{h\to 0} \frac{f(a+h,b) - f(a,b)}{h} - v_1 = 0,$$

$$v_1 = \lim_{h\to 0} \frac{f(a+h,b) - f(a,b)}{h} = \frac{\partial f}{\partial x}(a,b).$$

Similarly, by letting $h = 0$, we get

$$v_2 = \lim_{k\to 0} \frac{f(a,b+k) - f(a,b)}{h} = \frac{\partial f}{\partial y}(a,b).$$

Note that the vector $\mathbf{v} = (v_1, v_2)$ is thus the gradient of $f(x,y)$ at (a,b). □

We have just shown that if a function $f : \mathbb{R}^2 \to \mathbb{R}$ is differentiable at a point $(a,b) \in \mathbb{R}^2$, then it is continuous at (a,b) and its partial derivatives $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$ exist at that point. There is a useful criteria that almost states the converse, guaranteeing that a function is differentiable at a given point. We will not prove this theorem here, but we will rather use it in the subsequent examples.

Theorem 1.8.3. *If both partial derivatives* $\dfrac{\partial f}{\partial x}$ *and* $\dfrac{\partial f}{\partial y}$ *of a function* $f : \mathbb{R}^2 \to \mathbb{R}$ *exist and are continuous on a neighbourhood of a point* (a,b) *then* f *is differentiable at* (a,b).

Example 1.8.4. Show that the function

$$f(x,y) = x^2 + y^2 + xy\exp\left(x^2+y^2\right)$$

is differentiable.

First, we compute the partial derivatives.

$$\frac{\partial f}{\partial x}(x,y) = 2x + y\exp(x^2+y^2) + 2x^2y\exp(x^2+y^2),$$

$$\frac{\partial f}{\partial y}(x,y) = 2y + x\exp(x^2+y^2) + 2xy^2\exp(x^2+y^2).$$

As these functions are obtained from continuous functions via products, sums and compositions they are also continuous. So, by Theorem 1.8.3, $f(x,y)$ is differentiable.

1.8.2. **Derivative as a linear transformation (section is not assessed).** At this point, we shift our point of view. For a function $f : \mathbb{R} \to \mathbb{R}$ of one variable differentiable at $x = a$, we have that the map $h \mapsto f'(a)h$ is a linear map from $\mathbb{R}$ to $\mathbb{R}$. Think of it as the line passing through the origin and parallel to the tangent line of $f(x)$ at $x = a$. This together with the equation (9) leads us to the definition of the derivative of $F : \mathbb{R}^n \to \mathbb{R}^p$.

Definition 1.8.5. Given a function $F : \mathbb{R}^n \to \mathbb{R}^p$, the *derivative* of $F(x_1, \dots, x_n)$ at $\mathbf{a} = (a_1, \dots, x_n)$ is a linear map (or linear transformation) $A_F : \mathbb{R}^n \to \mathbb{R}^p$ satisfying

$$\lim_{\mathbf{h}\to\mathbf{0}} \frac{F(\mathbf{a}+\mathbf{h}) - F(\mathbf{a}) - A_F(\mathbf{h})}{||\mathbf{h}||} = \mathbf{0}.$$

We say that F is *differentiable at* $\mathbf{a}$ if its derivative exists at this point. If F is differentiable at every point of the domain we simply say that it is *differentiable.*

The reason we need to use $||\mathbf{h}||$ in the denominator is that we cannot divide by vectors as we can by numbers.

As in the case when $n = 2$ and $p = 1$, it follows from the definition that if a function $F : \mathbb{R}^n \to \mathbb{R}^p$ is differentiable at a point $\mathbf{a} \in \mathbb{R}^n$, then F is continuous at $\mathbf{a}$ and all partial derivatives $\dfrac{\partial f_i}{\partial x_j}$ exist at $\mathbf{a}$. Again there is a useful criteria that almost states the converse, guaranteeing that a function is differentiable at a given point.

Theorem 1.8.6. *If all partial derivatives* $\dfrac{\partial f_i}{\partial x_j}$ *of a function* $F = (f_1, \dots, f_n) : \mathbb{R}^n \to \mathbb{R}^p$ *exist and are continuous on a neighbourhood of a point* $\mathbf{a} \in \mathbb{R}^n$ *then* f *is differentiable at* $\mathbf{a}$*.*

Next, we show the matrix corresponding to the "mysterious" linear transformation A_F appearing in the definition of the differentiability is nothing other than the Jacobian matrix.

Theorem 1.8.7. *If a function* $F = (f_1, \dots, f_p) : \mathbb{R}^n \to \mathbb{R}^p$ *is differentiable at* $\mathbf{a} \in \mathbb{R}^n$*, then* $A_F = J_F(\mathbf{a})$*.*

Proof. By definition, we have

$$\lim_{\mathbf{h}\to\mathbf{0}} \frac{F(\mathbf{a}+\mathbf{h}) - F(\mathbf{a}) - A_F(\mathbf{h})}{||\mathbf{h}||} = \mathbf{0}.$$

We can let $\mathbf{h} = (h_1, \ldots, h_n)$ and $F = (f_1, \ldots, f_p)$. Let $(A_F)_i$ be the i-th row and let $(A_F)_{ij}$ be the ij-the entry of the matrix corresponding to the linear transformation A_F for $1 \le i \le p$. The above limit implies that for i-th component function f_i of F one has

$$\lim_{\mathbf{h}\to\mathbf{0}} \frac{f_i(\mathbf{a}+\mathbf{h}) - f_i(\mathbf{a}) - (A_F)_i \cdot \mathbf{h}}{||\mathbf{h}||} = 0.$$

Let $\mathbf{h} = (0, \ldots, 0, h_j, 0, \ldots, 0)$ where all but the j-th entry are zero for $1 \le j \le n$. This implies that

$$\lim_{h_j\to 0} \frac{f_i(\mathbf{a}+\mathbf{h}) - f_i(\mathbf{a}) - (A_F)_{ij} \cdot h_j}{h_j} = 0.$$

Then we get

$$(A_F)_{ij} = \lim_{h_j\to 0} \frac{f_i(\mathbf{a}+\mathbf{h}) - f_i(\mathbf{a})}{h_j} = \frac{\partial f_i}{\partial x_j}(\mathbf{a})$$

for $1 \le i \le p$ and $1 \le j \le n$. Therefore, $A_F = J_F(\mathbf{a})$. □

As an immediate application we obtain the following.

Corollary 1.8.8. *Given a function $F : \mathbb{R}^n \to \mathbb{R}^p$, if the derivative of $F(x_1, \ldots, x_n)$ exists at $\mathbf{a} = (a_1, \ldots, x_n)$, then it is unique.*

Example 1.8.9. Let $A : \mathbb{R}^n \to \mathbb{R}^p$. With this definition of derivative, show that A is its own derivative; that is, show that $A = A_A$ for all $\mathbf{a} \in \mathbb{R}^n$.

We only need to show that A_A satisfies the definition given above, namely that

$$\lim_{\mathbf{h}\to\mathbf{0}} \frac{A(\mathbf{a}+\mathbf{h}) - A(\mathbf{a}) - A(\mathbf{h})}{||\mathbf{h}||} = \mathbf{0}.$$

Since A is a linear map, we have that $A(\mathbf{a}+\mathbf{h}) = A(\mathbf{a}) + A(\mathbf{h})$, and so the numerator in the limit is always $\mathbf{0}$, and we are done.

1.9. Directional derivative

For a function of two or more variables, we have already seen that its partial derivatives describe the rate of change of the function in either x or y directions depending which partial derivative we consider. In this section we discuss how one can measure the rate of change of such a function in any given direction.

Definition 1.9.1. Let $f : \mathbb{R}^2 \to \mathbb{R}$ and $\mathbf{u}$ be a unit vector in $\mathbb{R}^2$. The *directional derivative* (or *rate of change*) of f in the direction $\mathbf{u}$ at the point $(a, b) \in \mathbb{R}^2$ is defined to be

$$D_{\mathbf{u}}(f)(a, b) = \lim_{h\to 0} \frac{f((a, b) + h\mathbf{u}) - f(a, b)}{h}.$$

provided this limit exists.

By working through the definition of the derivative, we can see that there is an alternative formula for the directional derivative.

Theorem 1.9.2. *Suppose $f : \mathbb{R}^2 \to \mathbb{R}$ is differentiable at $(a,b) \in \mathbb{R}^2$ and $\mathbf{u}$ is a unit vector. Then the directional derivative of f in the direction $\mathbf{u}$ at (a,b) is given by*

$$D_{\mathbf{u}}(f)(a,b) = \nabla f(a,b) \cdot \mathbf{u}.$$

Proof. We can write $\mathbf{u} = (\alpha, \beta) \in \mathbb{R}^2$ where $\alpha^2 + \beta^2 = 1$, because $\mathbf{u}$ is a unit vector. Since $f : \mathbb{R}^2 \to \mathbb{R}$ is differentiable at (a,b), by (10) and Theorem 1.8.2, we have

$$\lim_{h\to 0} \frac{f(a+h\alpha, b+h\beta) - f(a,b) - \nabla f(a,b) \cdot (h\alpha, h\beta)}{\sqrt{(h\alpha)^2 + (h\beta)^2}} = 0$$

Since $\alpha^2 + \beta^2 = 1$, it follows that

$$\lim_{h\to 0} \frac{f(a+h\alpha, b+h\beta) - f(a,b) - \nabla f(a,b) \cdot (h\alpha, h\beta)}{h} = 0.$$

This gives us

$$\lim_{h\to 0} \frac{f(a+h\alpha, b+h\beta) - f(a,b)}{h} = \nabla f(a,b) \cdot (\alpha, \beta).$$

But the left-hand side of the above identity is exactly the directional derivative and thus

$$D_{\mathbf{u}}(f)(a,b) = \nabla f(a,b) \cdot \mathbf{u}$$

as desired. □

Remark 1.9.3. To see that it is important that $\mathbf{u}$ be a unit vector, let $\mathbf{w}$ be any non-zero vector in $\mathbb{R}^2$ and let $\mathbf{u}$ be the unit vector in the direction of $\mathbf{w}$, namely $\mathbf{u} = \frac{1}{|\mathbf{w}|}\mathbf{w}$. If we compare $\nabla f(a,b) \cdot \mathbf{u}$ with $\nabla f(a,b) \cdot \mathbf{w}$, we see that

$$\nabla f(a,b) \cdot \mathbf{u} = \nabla f(a,b) \cdot \left(\frac{1}{|\mathbf{w}|}\mathbf{w}\right) = \frac{1}{|\mathbf{w}|}\nabla f(a,b) \cdot \mathbf{w},$$

which differs from $\nabla f(a,b) \cdot \mathbf{w}$ by a multiplicative constant. In particular, if we do not impose a restriction on the vector $\mathbf{u}$, such as requiring that $\mathbf{u}$ be a unit vector, then the directional derivative is not well-defined.

The geometric meaning of the gradient comes from the definition of the directional derivative. We use the property of dot products that

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}|\,|\mathbf{b}| \cos(\theta)$$

where θ is the angle between $\mathbf{a}$ and $\mathbf{b}$. Letting θ be the angle between $\nabla f(a,b)$ and $\mathbf{u}$, we see that

$$D_{\mathbf{u}}(f)(a,b) = \nabla f(a,b) \cdot \mathbf{u} = |\nabla f(a,b)|\,|\mathbf{u}| \cos(\theta) = |\nabla f(a,b)| \cos(\theta),$$

since $\mathbf{u}$ is a unit vector.

We see that since $\nabla f(a,b)$ is constant, to maximise $D_{\mathbf{u}}(f)(a,b)$, we only need to maximise $\cos(\theta)$. Therefore, we obtain the following geometric properties of the gradient.

(i) The function f increase the most rapidly at (a,b) when $\theta = 0$, which is when $\mathbf{u}$ is pointing in the same direction as $\nabla f(a,b)$.

(ii) The function f decrease the most rapidly at (a,b) when $\theta = \pi$, which is when $\mathbf{u}$ is pointing in the opposite direction as $\nabla f(a,b)$.

(iii) The rate of change of the function f is zero (a,b) when $\theta = \frac{\pi}{2}$, which is when $\mathbf{u}$ is perpendicular to the direction of $\nabla f(a,b)$.

For a given function $f(x,y)$, the sorts of calculations we do with the directional derivative involve solving the equation $D_{\mathbf{u}}(f)(a,b) = c$ when two of the parameters $\mathbf{u}$, (a,b) and c are specified.

Example 1.9.4. For the function $f(x,y) = x^2 + y^2(1-x)^3$ at the point $(0,1)$ and the direction $\mathbf{u} = (\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}})$, determine $D_{\mathbf{u}}(f)$.

Calculating, we see that

$$\nabla f(x,y) = (2x - 3y^2(1-x)^2, 2y(1-x)^3),$$

so that

$$\nabla f(0,1) = (-3,2),$$

and hence

$$D_{(\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}})}(f)(0,1) = \nabla f(0,1) \cdot \left(\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right) = (-3,2) \cdot \left(\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right) = \frac{1}{\sqrt{5}}.$$

Example 1.9.5. For the function $f(x,y) = x^2 + y^2(1-x)^3$ at the point $(0,1)$, find all directions $\mathbf{u}$ in which $D_{\mathbf{u}}(f)(0,1) = 1$.

Since we must restrict to consider only unit vectors $\mathbf{u}$, we set $\mathbf{u} = (\alpha, \beta)$ where $\alpha^2 + \beta^2 = 1$. Using the calculations from Example 1.9.4, we have that $\nabla f(0,1) = (-3,2)$, and so we have the equation

$$1 = D_{\mathbf{u}} f(0,1) = \nabla f(0,1) \cdot \mathbf{u} = (-3,2) \cdot (\alpha, \beta).$$

Therefore, we have two equations in α and β, namely $\alpha^2 + \beta^2 = 1$ and $-3\alpha + 2\beta = 1$. We have two equations in two unknowns, though not both linear equations, and so we expect generically for there to be finitely many solutions. Solving $-3\alpha + 2\beta = 1$ for β and substituting into the other equation, we see that

$$1 = \alpha^2 + \left(\frac{1}{2} + \frac{3}{2}\alpha\right)^2,$$

which simplifies to $13\alpha^2 + 6\alpha - 3 = 0$. By the quadratic formula, we have two possible α, namely

$$\alpha = \frac{-6 \pm \sqrt{36 - 4(13)(-3)}}{26},$$

and for each α, we have a corresponding β. Hence, there are two directions.

Example 1.9.6. For the function $f(x,y) = x^2 + y^2(1-x)^3$, find all points (a,b) at which there exists a direction $\mathbf{u}$ in which $D_{\mathbf{u}}(f)(a,b) = 1$.

We start by setting up the equation $D_{\mathbf{u}}(f)(a,b) = 1$. For this, we need to know $\nabla f(a,b)$. For $f(x,y) = x^2 + y^2(1-x)^3$, we have that

$$\nabla f(x,y) = (2x - 3y^2(1-x)^2, 2y(1-x)^3),$$

and so for an arbitrary point (a,b), we have that

$$\nabla f(a,b) = (2a - 3b^2(1-a)^3, 2b(1-a)^3).$$

Let $\mathbf{u} = (\alpha, \beta)$ be a unit vector, so that $\alpha^2 + \beta^2 = 1$.

The equation $D_{\mathbf{u}}(f)(a,b) = 1$ becomes the equation $\nabla f(a,b) \cdot \mathbf{u} = 1$, and so we are left with the equation

$$(2a - 3b^2(1-a)^3, 2b(1-a)^3) \cdot (\alpha, \beta) = 1,$$

together with the condition $\alpha^2 + \beta^2 = 1$. Note that there are no conditions on a and b. This gives us a system of two (non-linear) equations in 4 variables, and so we generically expect that we will have infinitely many solutions.

The main geometric significance of the directional derivative comes in fact from its relationship to the level curves of a function of two variables.

Theorem 1.9.7. *Let $f : \mathbb{R}^2 \to \mathbb{R}$ be differentiable at $(a,b) \in \mathbb{R}^2$. Set $c = f(a,b)$ and note that by construction, the level curve L_c of $f(x,y)$ passes through (a,b). At (a,b), we have that L_c is perpendicular to $\nabla f(a,b)$.*

Proof. To show this, we start by parametrising L_c by a pair of functions $(x(t), y(t)) = \ell(t)$, where $\ell(t)$ is defined on $(-\varepsilon, \varepsilon)$ for some $\varepsilon > 0$ and where $\ell(0) = (x(0), y(0)) = (a,b)$. The fact that we can do this parametrisation is a fact that we have not proven in class, but is a rabbit that we are pulling out of a hat, for the moment. Now we calculate.

We know that since $\ell(t)$ is parametrising the level curve L_c of $f(x,y)$, the composition

$$f(\ell(t)) = f(x(t), y(t)) = c$$

is constant; this is just the definition of level curve. Using the chain rule,

$$0 = \frac{\mathrm{d}}{\mathrm{d}t} f(\ell(t)) = \frac{\partial f}{\partial x}(x(t), y(t))x'(t) + \frac{\partial f}{\partial y}(x(t), y(t))y'(t) = \nabla f(\ell(t)) \cdot \ell'(t),$$

and so at $t = 0$ we see that

$$0 = \nabla f(\ell(0)) \cdot \ell'(0) = \nabla f(a,b) \cdot (x'(0), y'(0)).$$

Since $(x'(0), y'(0))$ is just the tangent vector to the level curve L_c at (a,b), we are done. □

1.10. Higher order derivatives

When calculating partial derivatives of two or higher order, we iteratively use the rules for partial differentiation of a single order. Given a function $w = f(x, y)$, we have four possible order two partial derivatives.

$$\frac{\partial^2 w}{\partial x^2} = \frac{\partial}{\partial x}\frac{\partial w}{\partial x},$$
$$\frac{\partial^2 w}{\partial y^2} = \frac{\partial}{\partial y}\frac{\partial w}{\partial y},$$
$$\frac{\partial^2 w}{\partial x \partial y} = \frac{\partial}{\partial x}\frac{\partial w}{\partial y},$$
$$\frac{\partial^2 w}{\partial y \partial x} = \frac{\partial}{\partial y}\frac{\partial w}{\partial x}.$$

The last two are called *mixed* partials.

Example 1.10.1. Find all the second order partial derivatives of the function $f(x, y) = -x^3y^5$.

First, we compute the first order partials:

$$\frac{\partial f}{\partial x} = -3x^2y^5 \quad \text{and} \quad \frac{\partial f}{\partial x} = -5x^3y^4.$$

For the second order partials we get

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(-3x^2y^5) = -6xy^5.$$
$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(-5x^3y^4) = -20x^3y^3.$$
$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\frac{\partial f}{\partial y} = \frac{\partial}{\partial x}(-5x^3y^4) = -15x^2y^4.$$
$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\frac{\partial f}{\partial x} = \frac{\partial}{\partial y}(-3x^2y^5) = -15x^2y^4.$$

Example 1.10.2. Given $w = f(x, y) = x\cos(z) + \sin(xy)$, calculate $\frac{\partial^3 w}{\partial z \partial x^2}$, $\frac{\partial^3 w}{\partial x \partial z \partial x}$, and $\frac{\partial^3 w}{\partial x^2 \partial z}$.

$$\frac{\partial^3 w}{\partial z \partial x^2} = \frac{\partial^2 w}{\partial z \partial x}(\cos(z) + y\cos(xy)) = \frac{\partial w}{\partial z}(-y^2\sin(x)) = 0,$$

$$\frac{\partial^3 w}{\partial x \partial z \partial x} = \frac{\partial^2 w}{\partial x \partial z}(\cos(z) + y\cos(xy)) = \frac{\partial w}{\partial x}(-\sin(z)) = 0,$$

$$\frac{\partial^3 w}{\partial x^2 \partial z} = \frac{\partial^2 w}{\partial x^2}(-x\sin(z)) = \frac{\partial w}{\partial x}(-\sin(z)) = 0.$$

Observe that in the above example, the mixed partial derivatives involved the same variables but in different order. We have the following useful theorem on the order of partials differentiation.

Theorem 1.10.3. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a function. Suppose for some positive integer k, there exists a neighbourhood U of a point $P \in \mathbb{R}^n$ such that f and all its partial derivatives of order less than k are continuous on U. Then any two mixed partial derivatives of order k involving the same variables but in different sequential orders are equal at P, provided those partial derivatives are continuous at P.*

The chain rule for second and higher order partial derivatives is complicated mainly because of the number of terms that we need to track. We will conduct this discussion in the special case of a function $w = w(x, y)$, where both $x = x(s, t)$ and $y = y(s, t)$ are functions of s and t. Again here, the independent variables are s and t, and the intermediate variables are x and y.

We begin with the chain rule for first order partial derivatives in this case, namely

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s}$$

and

$$\frac{\partial w}{\partial t} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial t}.$$

which is as always a sum in which we have one term for each intermediate variable.

A point worth stressing here, because we will make much use of it, is that as we have set things up, for any function $\Theta = \Theta(x, y)$ we have that

$$(11) \qquad \frac{\partial \Theta}{\partial s} = \frac{\partial \Theta}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial \Theta}{\partial y}\frac{\partial y}{\partial s} \quad \text{and} \quad \frac{\partial \Theta}{\partial t} = \frac{\partial \Theta}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial \Theta}{\partial y}\frac{\partial y}{\partial t}.$$

Note that this makes sense, as $x = x(s, t)$ and $y = y(s, t)$ are functions of s and t, and so $\Theta(x, y)$ is secretly also a function of s and t.

In order to calculate second order derivatives, we just calculate, using the product rule and the above. So:

$$\begin{aligned}
\frac{\partial^2 w}{\partial s^2} &= \frac{\partial}{\partial s}\left(\frac{\partial w}{\partial x}\,\frac{\partial x}{\partial s}+\frac{\partial w}{\partial y}\,\frac{\partial y}{\partial s}\right)\\
&= \frac{\partial}{\partial s}\left(\frac{\partial w}{\partial x}\,\frac{\partial x}{\partial s}\right)+\frac{\partial}{\partial s}\left(\frac{\partial w}{\partial y}\,\frac{\partial y}{\partial s}\right)\\
&= \left(\frac{\partial}{\partial s}\frac{\partial w}{\partial x}\right)\,\frac{\partial x}{\partial s}+\frac{\partial w}{\partial x}\,\frac{\partial^2 x}{\partial s^2}+\left(\frac{\partial}{\partial s}\frac{\partial w}{\partial y}\right)\,\frac{\partial y}{\partial s}+\frac{\partial w}{\partial y}\,\frac{\partial^2 y}{\partial s^2}\\
&= \left(\frac{\partial^2 w}{\partial x^2}\,\frac{\partial x}{\partial s}+\frac{\partial^2 w}{\partial y\,\partial x}\,\frac{\partial y}{\partial s}\right)\,\frac{\partial x}{\partial s}\\
&\quad +\frac{\partial w}{\partial x}\,\frac{\partial^2 x}{\partial s^2}+\left(\frac{\partial^2 w}{\partial x\,\partial y}\,\frac{\partial x}{\partial s}+\frac{\partial^2 w}{\partial y^2}\,\frac{\partial y}{\partial s}\right)\,\frac{\partial y}{\partial s}+\frac{\partial w}{\partial y}\,\frac{\partial^2 y}{\partial s^2}\\
&= \frac{\partial^2 w}{\partial x^2}\,\left(\frac{\partial x}{\partial s}\right)^2+\frac{\partial^2 w}{\partial y\,\partial x}\,\frac{\partial x}{\partial s}\,\frac{\partial y}{\partial s}\\
&\quad +\frac{\partial w}{\partial x}\,\frac{\partial^2 x}{\partial s^2}+\frac{\partial^2 w}{\partial x\,\partial y}\,\frac{\partial x}{\partial s}\,\frac{\partial y}{\partial s}+\frac{\partial^2 w}{\partial y^2}\,\left(\frac{\partial y}{\partial s}\right)^2+\frac{\partial w}{\partial y}\,\frac{\partial^2 y}{\partial s^2}.
\end{aligned}$$

Here, we use that the derivative of a sum is the sum of the derivatives to move from the first line to the second and use the product rule to show that

$$\frac{\partial}{\partial s}\left(\frac{\partial w}{\partial x}\,\frac{\partial x}{\partial s}\right)=\left(\frac{\partial}{\partial s}\frac{\partial w}{\partial x}\right)\,\frac{\partial x}{\partial s}+\frac{\partial w}{\partial x}\,\frac{\partial^2 x}{\partial s^2}$$

and

$$\frac{\partial}{\partial s}\left(\frac{\partial w}{\partial y}\,\frac{\partial y}{\partial s}\right)=\left(\frac{\partial}{\partial s}\frac{\partial w}{\partial y}\right)\,\frac{\partial y}{\partial s}+\frac{\partial w}{\partial y}\,\frac{\partial^2 y}{\partial s^2}$$

to move from the second line to the third line. We use the chain rule (Equation (11) with $\Theta=\frac{\partial w}{\partial x}$ and $\Theta=\frac{\partial w}{\partial y}$, respectively) to show that

$$\frac{\partial}{\partial s}\,\frac{\partial w}{\partial x}=\left(\frac{\partial}{\partial x}\,\frac{\partial w}{\partial x}\right)\,\frac{\partial x}{\partial s}+\left(\frac{\partial}{\partial y}\,\frac{\partial w}{\partial x}\right)\,\frac{\partial y}{\partial s}=\frac{\partial^2 w}{\partial x^2}\,\frac{\partial x}{\partial s}+\frac{\partial^2 w}{\partial y\,\partial x}\,\frac{\partial y}{\partial s}$$

and

$$\frac{\partial}{\partial s}\,\frac{\partial w}{\partial y}=\left(\frac{\partial}{\partial x}\,\frac{\partial w}{\partial y}\right)\,\frac{\partial x}{\partial s}+\left(\frac{\partial}{\partial y}\,\frac{\partial w}{\partial y}\right)\,\frac{\partial y}{\partial s}=\frac{\partial^2 w}{\partial x\,\partial y}\,\frac{\partial x}{\partial s}+\frac{\partial^2 w}{\partial y^2}\,\frac{\partial y}{\partial s}$$

to move from the third line to the fourth line. We can similarly show that

$$\begin{aligned}
\frac{\partial^2 w}{\partial s\,\partial t} &= \frac{\partial^2 w}{\partial x^2}\frac{\partial x}{\partial t}\frac{\partial x}{\partial s}+\frac{\partial^2 w}{\partial y\partial x}\frac{\partial y}{\partial t}\frac{\partial x}{\partial s}+\frac{\partial^2 w}{\partial x\partial y}\frac{\partial x}{\partial t}\frac{\partial y}{\partial s}\\
&\quad +\frac{\partial^2 w}{\partial y^2}\frac{\partial y}{\partial t}\frac{\partial y}{\partial s}+\frac{\partial w}{\partial x}\frac{\partial^2 x}{\partial t\partial s}+\frac{\partial w}{\partial y}\frac{\partial^2 y}{\partial t\partial s}=\frac{\partial^2 w}{\partial t\,\partial s},
\end{aligned}$$

and

$$\frac{\partial^2 w}{\partial t^2}=\frac{\partial^2 w}{\partial x^2}\,\left(\frac{\partial x}{\partial t}\right)^2+\frac{\partial^2 w}{\partial y\,\partial x}\,\frac{\partial x}{\partial t}\,\frac{\partial y}{\partial t}+\frac{\partial w}{\partial x}\,\frac{\partial^2 x}{\partial t^2}$$

$$+\frac{\partial^2 w}{\partial x\,\partial y}\frac{\partial x}{\partial t}\frac{\partial y}{\partial t}+\frac{\partial^2 w}{\partial y^2}\left(\frac{\partial y}{\partial t}\right)^2+\frac{\partial w}{\partial y}\frac{\partial^2 y}{\partial t^2}.$$

It is important here to remark that these are not formulae to memorise. Rather, they are guides to the methods. Let us use them to do two examples.

Example 1.10.4. Consider the function $w = w(x, y) = \sin(x^2 y)$, where $x = x(s, t) = st^2$ and $y = y(s, t) = s^2 + \frac{1}{t}$. Without substituting in the expressions for $x(s, t)$ and $y(s, t)$, use the Chain Rule to calculate $\frac{\partial^2 w}{\partial s^2}$. We start with the first order partial derivatives, namely

$$\begin{aligned}\frac{\partial w}{\partial s} &= \frac{\partial}{\partial s}(\sin(x^2 y))\\ &= \cos(x^2 y)\frac{\partial}{\partial s}(x^2 y)\\ &= \cos(x^2 y)\left(2xy\frac{\partial x}{\partial s}+x^2\frac{\partial y}{\partial s}\right)\end{aligned}$$

and

$$\begin{aligned}\frac{\partial w}{\partial t} &= \frac{\partial}{\partial t}(\sin(x^2 y))\\ &= \cos(x^2 y)\frac{\partial}{\partial t}(x^2 y)\\ &= \cos(x^2 y)\left(2xy\frac{\partial x}{\partial t}+x^2\frac{\partial y}{\partial t}\right).\end{aligned}$$

The requested second order partial derivative is then

$$\begin{aligned}\frac{\partial^2 w}{\partial s^2} &= \frac{\partial}{\partial s}\left(\frac{\partial w}{\partial s}\right)\\ &= \frac{\partial}{\partial s}\left(\cos(x^2 y)\left(2xy\frac{\partial x}{\partial s}+x^2\frac{\partial y}{\partial s}\right)\right)\\ &= \frac{\partial}{\partial s}\left(2xy\cos(x^2 y)\frac{\partial x}{\partial s}\right)+\frac{\partial}{\partial s}\left(x^2\cos(x^2 y)\frac{\partial y}{\partial s}\right)\\ &= \left(2\left(\frac{\partial}{\partial s}x\right)y\cos(x^2 y)+2x\left(\frac{\partial}{\partial s}y\right)\cos(x^2 y)+2xy\left(\frac{\partial}{\partial s}\cos(x^2 y)\right)\right)\frac{\partial x}{\partial s}\\ &\quad +2xy\cos(x^2 y)\frac{\partial^2 x}{\partial s^2}+\left(\left(\frac{\partial}{\partial s}x^2\right)\cos(x^2 y)+x^2\left(\frac{\partial}{\partial s}\cos(x^2 y)\right)\right)\frac{\partial y}{\partial s}\\ &\quad +x^2\cos(x^2 y)\frac{\partial^2 y}{\partial s^2}\end{aligned}$$

$$\begin{aligned}
&= \left(2\frac{\partial x}{\partial s} y\cos(x^2y) + 2x\frac{\partial y}{\partial s}\cos(x^2y) - 2xy\sin(x^2y)\left(2x\frac{\partial x}{\partial s}y + x^2\frac{\partial y}{\partial s}\right)\right)\frac{\partial x}{\partial s} \\
&\quad +2xy\cos(x^2y)\frac{\partial^2 x}{\partial s^2} + \left(2x\frac{\partial x}{\partial s}\cos(x^2y) - x^2\sin(x^2y)\left(2x\frac{\partial x}{\partial s}y + x^2\frac{\partial y}{\partial s}\right)\right)\frac{\partial y}{\partial s} \\
&\quad +x^2\cos(x^2y)\frac{\partial^2 y}{\partial s^2} \\
&= 2\left(\frac{\partial x}{\partial s}\right)^2 y\cos(x^2y) + 2x\frac{\partial x}{\partial s}\frac{\partial y}{\partial s}\cos(x^2y) \\
&\quad -2xy\sin(x^2y)\left(2x\left(\frac{\partial x}{\partial s}\right)^2 y + x^2\frac{\partial x}{\partial s}\frac{\partial y}{\partial s}\right) + 2xy\cos(x^2y)\frac{\partial^2 x}{\partial s^2} \\
&\quad +2x\frac{\partial x}{\partial s}\frac{\partial y}{\partial s}\cos(x^2y) - x^2\sin(x^2y)\left(2x\frac{\partial x}{\partial s}\frac{\partial y}{\partial s}y + x^2\left(\frac{\partial y}{\partial s}\right)^2\right) + x^2\cos(x^2y)\frac{\partial^2 y}{\partial s^2} \\
&= 2t^4\left(s^2+\frac{1}{t}\right)\cos\left(s^2t^4\left(s^2+\frac{1}{t}\right)\right) + 4s^2t^4\cos\left(s^2t^4\left(s^2+\frac{1}{t}\right)\right) \\
&\quad +2st^2\left(s^2-\frac{1}{t}\right)\sin\left(s^2t^4\left(s^2+\frac{1}{t}\right)\right)\left(2st^6\left(s^2+\frac{1}{t}\right)+2s^3t^6\right) \\
&\quad +4s^2t^4\cos\left(s^2t^4\left(s^2+\frac{1}{t}\right)\right) \\
&\quad -s^2t^4\sin\left(s^2t^4\left(s^2+\frac{1}{t}\right)\right)\left(4s^2t^4\left(s^2+\frac{1}{t}\right)+4s^4t^4\right) \\
&\quad +2s^2t^4\cos\left(s^2t^4\left(s^2+\frac{1}{t}\right)\right)
\end{aligned}$$

Example 1.10.5. Consider an arbitrary function $w = w(x, y)$, where $x = x(r,\theta) = r\cos(\theta)$ and $y = y(r,\theta) = r\sin(\theta)$. Show that

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = \frac{\partial^2 w}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2} + \frac{1}{r}\frac{\partial w}{\partial r}.$$

As always, we start by differentiating with respect to the independent variables, which in this case are r and θ. Calculating, we see that

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r} = \cos(\theta)\frac{\partial w}{\partial x} + \sin(\theta)\frac{\partial w}{\partial y}$$

and

$$\frac{\partial w}{\partial \theta} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial \theta} = (-r\sin(\theta))\frac{\partial w}{\partial x} + (r\cos(\theta))\frac{\partial w}{\partial y}.$$

We now need to find the two second order partial derivatives $\frac{\partial^2 w}{\partial r^2}$ and $\frac{\partial^2 w}{\partial \theta^2}$; we do not need to find the mixed partial derivatives $\frac{\partial^2 w}{\partial r\,\partial \theta}$ and $\frac{\partial^2 w}{\partial \theta\,\partial r}$, as they are not part of what we are asked to show.

So, we calculate:

$$\begin{aligned}
\frac{\partial^2 w}{\partial r^2} &= \frac{\partial}{\partial r}\frac{\partial w}{\partial r} \\
&= \frac{\partial}{\partial r}\left(\cos(\theta)\frac{\partial w}{\partial x} + \sin(\theta)\frac{\partial w}{\partial y}\right) \\
&= \cos(\theta)\left(\frac{\partial}{\partial r}\frac{\partial w}{\partial x}\right) + \sin(\theta)\left(\frac{\partial}{\partial r}\frac{\partial w}{\partial y}\right) \\
&= \cos(\theta)\left(\left(\frac{\partial}{\partial x}\frac{\partial w}{\partial x}\right)\frac{\partial x}{\partial r} + \left(\frac{\partial}{\partial y}\frac{\partial w}{\partial x}\right)\frac{\partial y}{\partial r}\right) \\
&\quad + \sin(\theta)\left(\left(\frac{\partial}{\partial x}\frac{\partial w}{\partial y}\right)\frac{\partial x}{\partial r} + \left(\frac{\partial}{\partial y}\frac{\partial w}{\partial y}\right)\frac{\partial y}{\partial r}\right) \\
&= \cos(\theta)\left(\frac{\partial^2 w}{\partial x^2}\cos(\theta) + \frac{\partial^2 w}{\partial y\,\partial x}\sin(\theta)\right) \\
&\quad + \sin(\theta)\left(\frac{\partial^2 w}{\partial x\,\partial y}\cos(\theta) + \frac{\partial^2 w}{\partial y^2}\sin(\theta)\right) \\
&= \cos^2(\theta)\frac{\partial^2 w}{\partial x^2} + \sin(\theta)\cos(\theta)\left(\frac{\partial^2 w}{\partial y\,\partial x} + \frac{\partial^2 w}{\partial x\,\partial y}\right) + \sin^2(\theta)\frac{\partial^2 w}{\partial y^2}
\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial^2 w}{\partial \theta^2} &= \frac{\partial}{\partial \theta}\frac{\partial w}{\partial \theta} \\
&= \frac{\partial}{\partial \theta}\left((-r\sin(\theta))\frac{\partial w}{\partial x} + (r\cos(\theta))\frac{\partial w}{\partial y}\right) \\
&= (-r\cos(\theta))\frac{\partial w}{\partial x} - r\sin(\theta)\left(\frac{\partial}{\partial \theta}\frac{\partial w}{\partial x}\right) - r\sin(\theta)\frac{\partial w}{\partial y} + r\cos(\theta)\left(\frac{\partial}{\partial \theta}\frac{\partial w}{\partial y}\right) \\
&= (-r\cos(\theta))\frac{\partial w}{\partial x} - r\sin(\theta)\left(\left(\frac{\partial}{\partial x}\frac{\partial w}{\partial x}\right)\frac{\partial x}{\partial \theta} + \left(\frac{\partial}{\partial y}\frac{\partial w}{\partial x}\right)\frac{\partial y}{\partial \theta}\right) \\
&\quad - r\sin(\theta)\frac{\partial w}{\partial y} + r\cos(\theta)\left(\left(\frac{\partial}{\partial x}\frac{\partial w}{\partial y}\right)\frac{\partial x}{\partial \theta} + \left(\frac{\partial}{\partial y}\frac{\partial w}{\partial y}\right)\frac{\partial y}{\partial \theta}\right) \\
&= (-r\cos(\theta))\frac{\partial w}{\partial x} - r\sin(\theta)\left(\frac{\partial^2 w}{\partial x^2}(-r\sin(\theta)) + \frac{\partial^2 w}{\partial y\,\partial x}(r\cos(\theta))\right) \\
&\quad - r\sin(\theta)\frac{\partial w}{\partial y} + r\cos(\theta)\left(\frac{\partial^2 w}{\partial x\,\partial y}(-r\sin(\theta)) + \frac{\partial^2 w}{\partial y^2}(r\cos(\theta))\right) \\
&= (-r\cos(\theta))\frac{\partial w}{\partial x} - r\sin(\theta)\frac{\partial w}{\partial y} + r^2\sin^2(\theta)\frac{\partial^2 w}{\partial x^2} \\
&\quad - r^2\cos(\theta)\sin(\theta)\left(\frac{\partial^2 w}{\partial y\,\partial x}\frac{\partial^2 w}{\partial x\,\partial y}\right) + r^2\cos^2(\theta)\frac{\partial^2 w}{\partial y^2}.
\end{aligned}$$

Combining these, we see that

$$\begin{aligned}
&\frac{\partial^2 w}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2} + \frac{1}{r}\frac{\partial w}{\partial r} \\
&= \cos^2(\theta)\frac{\partial^2 w}{\partial x^2} + \sin(\theta)\cos(\theta)\left(\frac{\partial^2 w}{\partial y\,\partial x} + \frac{\partial^2 w}{\partial x\,\partial y}\right) + \sin^2(\theta)\frac{\partial^2 w}{\partial y^2} \\
&+ \frac{1}{r}\left(\cos(\theta)\frac{\partial w}{\partial x} + \sin(\theta)\frac{\partial w}{\partial y}\right) \\
&+ \frac{1}{r^2}\left((-r\cos(\theta))\frac{\partial w}{\partial x} - r\sin(\theta)\frac{\partial w}{\partial y} + r^2\sin^2(\theta)\frac{\partial^2 w}{\partial x^2}\right. \\
&\quad \left. - r^2\cos(\theta)\sin(\theta)\left(\frac{\partial^2 w}{\partial y\,\partial x} + \frac{\partial^2 w}{\partial x\,\partial y}\right)\right) + \frac{1}{r^2}\left(r^2\cos^2(\theta)\frac{\partial^2 w}{\partial y^2}\right) \\
&= (\cos^2(\theta) + \sin^2(\theta))\frac{\partial^2 w}{\partial x^2} + (\cos^2(\theta) + \sin^2(\theta))\frac{\partial^2 w}{\partial y^2} \\
&= \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2},
\end{aligned}$$

as desired.

1.11. Review of maximisation and minimisation of a function of one variable

There are two basic cases where we attempt to find the maxima and minima of a function of a single variable. They have the same starting point and the same first steps, but they finish differently.

Let f be a function defined on a subset $S \subseteq \mathbb{R}$ of the real line. We do not care about the specific structure of the set S; it might be an open interval, a closed interval, half-open interval, a ray, the whole real line, or a very complicated subset of the real line. Regardless of the structure of S, we can define what it means for f to achieve its maximum on S.

Definition 1.11.1. Let $S \subseteq \mathbb{R}$ be a set of real numbers and let $f : S \to \mathbb{R}$ be a function. Say that f achieves its *maximum* at $x_0 \in S$ if $f(x) \le f(x_0)$ for all $x \in S$.

We have the analogous definition of minimum.

Definition 1.11.2. Let $S \subseteq \mathbb{R}$ be a set of real numbers and let $f : S \to \mathbb{R}$ be a function. Say that f achieves its *minimum* at $x_0 \in S$ if $f(x) \ge f(x_0)$ for all $x \in S$.

For a general function f on a general subset $S \subseteq \mathbb{R}$, we can normally say nothing about the existence of points of S at which f achieves its maximum or minimum, much less provide any method for finding such points, even should they exist. In order to be able to give a sensible answer to this question, we need to impose a bit more structure on f and S.

The easiest case, though not a particularly interesting one, is the case that S contains a single point $S = \{s\}$, in which case there is not much to do. That is, f achieves both its maximum and its minimum at s, since the only value that f takes on any point of S is $f(s)$.

For a function $f(x)$ on the real line $\mathbb{R}$, the basic process is that we first find the critical points, which are the places at which $f'(x) = 0$, and then examine the behaviour of $f(x)$ as $x \to \pm\infty$ to then see whether the critical points are global maxima or minima. We could also use the second derivative test before determining the behaviour of $f(x)$ at infinity, to determine whether the critical points are local maxima and minima, but whether we do this may depend on the complexity of the problem.

For a function $f(x)$ on a closed interval $[a, b]$, the process is slightly different. We first find the critical points in the open interval (a, b), find the value of $f(x)$ at each of these critical points, and then compare these values to the values of $f(x)$ at the endpoints a and b of $[a, b]$. In this case, since we will be comparing the values of $f(x)$ at a finite set of points, we do not need to invoke the second derivative test, though as before we could if we so desired.

There are some general statement we can make about maximisation and minimisation regardless of the dimension in which we are working.

Theorem 1.11.3. *Let $f : [a, b] \to \mathbb{R}$ be a function that is continuous on the closed interval $[a, b]$. There then exists a number $c_{\max}$ in $[a, b]$ at which f achieves its maximum value, so that $f(x) \le f(c_{\max})$ for all $a \le x \le b$.*

It follows that the function $f : D \subseteq \mathbb{R} \to \mathbb{R}$ has an extreme (i.e. absolute or local minimum or maximum) value at $a \in D$ only if either

(i) a is a *critical point* of f, that is, when $f'(a) = 0$, or

(ii) a is a *singular point* of f, that is, when $f'(a)$ does not exist, or

(iii) a is a *boundary point* of D.

For example, when $D = [a, b]$ then the boundary consists of the points a and b.

1.12. Extreme Values

The analysis of the optimisation of functions of one variable leads us to the following characterisation of the types of extreme values for a given function.

Theorem 1.12.1. *A function $f : D \subseteq \mathbb{R}^n \to \mathbb{R}$ has an absolute or local minimum or maximum value at $\mathbf{a} \in D$ only if either*

(i) $\mathbf{a}$ is a critical point of f, that is, when $\nabla f(\mathbf{a}) = \mathbf{0}$, or

(ii) $\mathbf{a}$ is a singular point of f, that is, when $\nabla f(\mathbf{a})$ does not exist, or

(iii) **a** *is a* <u>*boundary point*</u> *of the domain* D.

Given a subset $D \subseteq \mathbb{R}^n$, a point $\mathbf{a} \in D$ is called an *interior point* of D, if there exists $r > 0$ such that the open n-ball of radius r centred at $\mathbf{a}$:

$$B_r(\mathbf{a}) = \{\mathbf{x} \in \mathbb{R}^n \mid ||\mathbf{x} - \mathbf{a}|| < r\}.$$

is contained in D. A point of D that is not an interior point is a *boundary point* of D.

Proof of Theorem 1.12.1. Suppose $\mathbf{a}$ is an interior point of the set D. If it is not a singular point, then $\nabla f(\mathbf{a})$ exists. If $\mathbf{a}$ is also not a critical point, then $\nabla f(\mathbf{a}) \neq 0$, therefore the function f has positive direction derivative in the direction of $\nabla f(\mathbf{a})$ so it increases in this direction and f has a negative directional derivative in the direction of $-\nabla f(\mathbf{a})$ so it decreases in that direction (see Section 1.9). We, conclude that f has neither a maximum nor a minimum value at $\mathbf{a}$ in this case. This shows that any point at which an extreme value occurs must be either a boundary point of D, a singular point of f or a critical point of f. □

Let us continue by considering a particularly straightforward class of max-min problems. We do not give a formal definition, but we will do a couple of examples and then extrapolate some of their common features. We begin with an observation. In the same way that the tangent line to the graph of a function $y = f(x)$ of one variable is horizontal at (local) maxima and minima of $f(x)$, we have that the tangent plane to the graph of a function $z = f(x, y)$ of two variables is horizontal at the (local) maxima and minima of $f(x, y)$. This is not to say that all points at which the tangent plane is horizontal are (local) maxima or minima, but in order to find (local) maxima and minima, the approach we start with is to find all points where the tangent plane is horizontal, which are the critical points of $f(x, y)$, and then do a bit of argument and apply some thought to see whether these points are indeed local or global maxima or minima.

Consider the following example.

Example 1.12.2. Find the points on the surface $xyz = 1$ that lie closest to the origin $(0, 0, 0)$ in $\mathbb{R}^3$.

We first note that we are not working with the graph of a function. Since the only information we have been given is that we are considering the points $(x, y, z) \in \mathbb{R}^3$ satisfying $xyz = 1$, we need to consider all such points. This surface has 4 pieces, one corresponding to each quadrant of the xy-plane. Since $xyz = 1$, we cannot have that any of x, y, and z are 0. Moreover, for any pair (x, y) of non-zero values, there is a unique value of z satisfying $xyz = 1$, obtained by setting $z = \frac{1}{xy}$. So, one way to view this surface is as a collection of 4 graphs, one over each component (piece) of the xy-plane with the x- and y-axes removed.

For any point (x, y, z) in $\mathbb{R}^3$, the distance to the origin is given by $D(x, y, z) = \sqrt{x^2 + y^2 + z^2}$. So, one way of phrasing the question is that we are asked to find the points in $\mathbb{R}^3$ that minimize $D(x, y, z)$, subject to the condition that they satisfy the constraint $xyz = 1$.

In this case, the constraint is sufficiently simple that we can solve for one of the variables in terms of the others. It doesn't matter which we solve for, but in this case let's solve for $z = \frac{1}{xy}$. We can plug this into the function being minimised, so that we are trying to minimize now the function

$$D(x, y) = \sqrt{x^2 + y^2 + \frac{1}{x^2 y^2}}.$$

We need to know the possible values of x and y that we need to consider. Since $xyz = 1$, we know that all of x, y, and z are nonzero, and that this is the only constraint (since given nonzero values of x and y, we can always find a value of z satisfying $xyz = 1$).

So, charging boldly forth, we differentiate and find the points at which the gradient is 0; these are the only points at which the function can have a maximum or minimum, since it is only at points where the gradient is the 0 vector that we can have a horizontal tangent plane to the graph of the function.

For this function, the gradient is

$$\nabla D(x, y) = \left(\frac{x - \frac{1}{x^3 y^2}}{\sqrt{x^2 + y^2 + \frac{1}{x^2 y^2}}}, \frac{y - \frac{1}{x^2 y^3}}{\sqrt{x^2 + y^2 + \frac{1}{x^2 y^2}}} \right),$$

and so the points at which $\nabla D(x, y) = (0, 0)$ satisfy the two equations

$$x - \frac{1}{x^3 y^2} = 0 \text{ and } y - \frac{1}{x^2 y^3} = 0.$$

Doing some algebraic massage, the first equation becomes $x^4 y^2 = 1$ and the second $x^2 y^4 = 1$. Substituting $x^2 = \frac{1}{y^4}$ into the first equation, we see that $y^6 = 1$, and so $y = \pm 1$. Hence, $x = \pm 1$ as well, and so $D(x, y)$ has four critical points, at $(1, 1)$, $(1, -1)$, $(-1, 1)$, and $(-1, -1)$.

Although we have found the critical points, we do not yet know whether they are maxima or minima of the function $D(x, y)$. For this, we need a bit of argument. First, we calculate to see that

$$D(1, 1) = D(1, -1) = D(-1, 1) = D(-1, -1) = \sqrt{3}.$$

This is still not enough information, though, to determine whether these points are maxima, minima, or neither.

We need to consider the behaviour of $D(x, y)$ as the point (x, y) approaches the edges of the domain of definition. Since in this case the domain of definition is the whole of $\mathbb{R}^2$, minus the coordinate axes, the edges of the domain of definition are the coordinate axes and the far edge at infinity that we experience by moving away from the origin. In a bit more mathematical language, we need to consider the behaviour of $D(x, y)$ as either $x \to \pm\infty$, $y \to \pm\infty$, $x \to 0$ or $y \to 0$ (or some combination of these possibilities; for instance, we could have that $x \to \infty$ and $y \to 0$ if we were to approach the x-axis asymptotically).

Recall that

$$D(x, y) = \sqrt{x^2 + y^2 + \frac{1}{x^2 y^2}},$$

and note that each of x^2, y^2 and $\frac{1}{x^2y^2}$ is non-negative. In other words, we do not need to worry about any cancellation of a positive term by a negative term, which is relevant because x or y can be negative. So, we can consider the possibilities one at a time:

As $x \to \pm\infty$, we see that $x^2 \to \infty$. We do not know anything about the behaviour of either y^2 or $\frac{1}{x^2y^2}$, because we have no information about the behaviour of y, but this does not matter. As $x^2 \to \infty$, we see that

$$x^2 + y^2 + \frac{1}{x^2 y^2} \to \infty,$$

and so $D(x, y) \to \infty$, regardless of the behaviour of y. Similarly, as $y \to \pm\infty$, we see that $y^2 \to \infty$. Again, we do not know anything about the behaviour of either x^2 or $\frac{1}{x^2y^2}$, because we have no information about the behaviour of x, but this does not matter. As $y^2 \to \infty$, we see that

$$x^2 + y^2 + \frac{1}{x^2 y^2} \to \infty,$$

and so $D(x, y) \to \infty$, regardless of the behaviour of x.

As $x \to 0$, we approach the y-axis. Here, one of two things happens. As $x \to 0$, either y stays bounded (that is, there is some $B > 0$ so that $|y| \leq B$) or not. If y stays bounded, then $\frac{1}{x^2y^2} \to \infty$ as $x \to 0$, as the whole of the denominator is going to 0. If y does not stay bounded, then we can find values of y that are arbitrarily large, and so y^2 is unbounded. In either case, we have that the whole of $D(x, y)$ is unbounded. A similar argument shows that as $y \to 0$, we get that $D(x, y)$ is unbounded, regardless of the behaviour of x.

Since $D(x, y)$ is unbounded as (x, y) approaches the boundary of its domain of definition, we can see that the critical points must be global minima, and so the distance of points satisfying $xyz = 1$ from the origin is minimised at these four points.

Example 1.12.3. Determine the dimensions of an open rectangular box of maximal and of minimal surface area, with a fixed volume of 100.

By an open box, we mean a box with no top. By an rectangular box, we mean a box whose sides meet pairwise at right angles, like a shoe box. Label the lengths of the edges of the box as x, y and z, with the edges sitting on the coordinate axes in $\mathbb{R}^3$. The volume constraint then gives us the relationship $xyz = 100$.

The area of the box is the sum of the areas of the five sides, and so the area is given by the equation

$$A(x, y, z) = xy + 2xz + 2yz.$$

We start by considering a box of maximal surface area. Our intuition tells us that we can find a box with large area by choosing a base (given by x and y) of very large area, and then choosing the appropriate height (given by z) so that the volume constraint $xyz = 100$ is satisfied. In this case, we can simplify our analysis by considering boxes with square bases, so that $x = y$. The formula for the area then simplifies to

$$A(x) = x^2 + 2x \cdot \frac{100}{x^2} + 2x \cdot \frac{100}{x^2} = x^2 + \frac{400}{x}.$$

Letting $x \to \infty$ (or letting $x \to 0^+$) gives us boxes with bases of large area, and hence large areas, and with volume fixed at 100.

We now consider the question of minimising the area. We know that for each $x > 0$ and $y > 0$, we get a box, by using the volume constraint $z = \frac{100}{xy}$. The formula for the area of the box then becomes

$$A(x, y) = xy + 2x \cdot \frac{100}{xy} + 2y \cdot \frac{100}{xy} = xy + \frac{200}{x} + \frac{200}{y}.$$

We can find the possible minima by setting the derivative, in this case the gradient, equal to $(0, 0)$ and solving.

The gradient is

$$\nabla A(x, y) = \left(\frac{\partial A}{\partial x}(x, y), \frac{\partial A}{\partial y}(x, y) \right) = \left(y - \frac{200}{x^2}, x - \frac{200}{y^2} \right) = (0, 0).$$

Simultaneously solving the two equations $y = \frac{200}{x^2}$ and $x = \frac{200}{y^2}$ yields that $x = y = \sqrt[3]{200}$. At this point, the function takes the value $A(\sqrt[3]{200}, \sqrt[3]{200}) = 3\sqrt[3]{400}$.

This is indeed a minimum value for the surface area since the function

$$A(x) = x^2 + \frac{400}{x}.$$

tends to infinity as $x \to \infty$.

In general with maximisation or minimisation problems, we have the function we are attempting to maximise or minimise (in the example above, this is the distance to the origin) and we have some constraints on the set of points where we are attempting the maximisation or minimisation (in the example above, the points satisfy the constraint $xyz = 1$). The straightforward approach we took above tends to work when we can solve the constraint for one of the variables and substitute this into the function being optimised. So, consider the two questions we had as model questions in Section 1:

(1) Determine the maximum and minimum values of $f(x, y, z) = x^2 + y^2 + z^2 - 3$ on the plane $x + 2y + 3z = 1$.

(2) Determine the maximum and minimum values of $g(x, y, z) = x + 2y + 3z$ on the sphere $x^2 + y^2 + z^2 = 3$.

For the first of these, we can solve the constraint for one of the variables, say $x = 1 - 2y - 3z$, substitute it into the function being optimised to get

$$f(y, z) =$$

$$= f(1-2y-3z, y, z) = (1-2y-3z)^2+y^2+z^2-3 = -2+5y^2+10z^2-4y-6z+12yz.$$

This function is defined for all values of y and z, and so we can attempt to maximise and minimize it over all of $\mathbb{R}^2$. Proceeding as above, we start by finding the points where $\nabla f(y, z) = (0, 0)$. Since

$$\nabla f(y, z) = (10y - 4 + 12z, 20z - 6 + 12y),$$

this becomes a question of solving the linear system

$$(10y - 4 + 12z, 20z - 6 + 12y) = (0, 0)$$

of two equations in two unknowns, which has the unique solution $(y, z) = (\frac{2}{14}, \frac{3}{14})$. Since $f(y, z)$ clearly can be as large as we want, being the sum of three squares minus a small constant, we see that the unique critical point $(\frac{2}{14}, \frac{3}{14})$ is in fact a global minimum.

However, for the other question, we cannot solve the constraint directly for one of the variables in terms of the others. Before developing the means for solving this sort of question, we will spend a bit of time developing some more tools.

1.13. Compactness

We go back to the basic result that underlies the whole process we have laid out for maximising and minimising functions of a single variable on a closed interval.

Theorem 1.13.1. *Let $f : [a, b] \to \mathbb{R}$ be a continuous function on the closed interval $[a, b]$. There exists a point $c \in [a, b]$ so that $f(x)$ achieves its maximum at c, so that $f(c) \geq f(x)$ for all $a \leq x \leq b$.*

We can see that $f(x)$ achieves its minimum on $[a, b]$ by applying Theorem 1.13.1 to the negative $-f(x)$ of $f(x)$. We do not give a proof of Theorem 1.13.1 here, but we do ask the question:

What is the property of a region in higher dimensional Euclidean space $\mathbb{R}^n$ for some $n \geq 2$, that allows us to make a true statement similar to Theorem 1.13.1?

This property is *compactness.* In order to understand this property, we first need a few definitions.

A set $S \subset \mathbb{R}^n$ is *bounded* if there exists a constant $r > 0$ and a point $\mathbf{a} \in \mathbb{R}^n$ so that S is contained in the open n-ball of radius r centred at $\mathbf{a}$, that is, $S \subseteq B_r(\mathbf{a})$ where we recall that

$$B_r(\mathbf{a}) = \{\mathbf{x} \in \mathbb{R}^n \mid ||\mathbf{x} - \mathbf{a}|| < r\}.$$

A set $U \subseteq \mathbb{R}^n$ is *open* if for every $\mathbf{u} \in U$, there is $\varepsilon > 0$ (depending on $\mathbf{u}$) so that $B_\varepsilon(\mathbf{u}) \subseteq U$.

For a set $X \subset \mathbb{R}^n$, its *complement* X^c is the set $X^c = \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{x} \notin X\}$ consisting of all of the points in $\mathbb{R}^n$ that do not lie in X.

In order to define the complement of a set, we must know the set in which we are taking the complement, as the act of taking the complement of a set makes no sense in a vacuum. Unless we say so explicitly, we will as a matter of course assume that we are taking complements in the appropriate $\mathbb{R}^n$.

The reason we define the notion of openness and the act of taking the complement is so that we can define what it means for a set to be *closed.* A subset $X \subseteq \mathbb{R}^n$ for some $n \geq 1$ is *closed* if its complement is open.

We note that in the definition of open, the constant ε may and almost always does depend on the point $\mathbf{u}$. As an example, consider the open unit ball $U = B_1(\mathbf{0})$ of radius 1 and centre the origin $\mathbf{0}$. For any $\mathbf{u} \in U$, we see that if we set $\varepsilon = 1 - |\mathbf{u}|$, we have that $U_\varepsilon(\mathbf{u}) \subset U$. However, there is no constant ε independent of $\mathbf{u}$ that satisfies this definition, as the value of ε must get smaller as $\mathbf{u}$ gets closer to the boundary of the ball U.

This definition is extremely flexible. In particular, this definition gives us both that all of $\mathbb{R}^n$ is closed, as its complement the empty set $\emptyset$ is (vacuously) open, and that the empty set $\emptyset$ is closed, as its complement $\mathbb{R}^n$ is open.

One standard example of a closed set is the inverse image $f^{-1}(c) = \{\mathbf{x} \in \mathbb{R}^n \mid f(\mathbf{x}) = c\}$ for a continuous function $f : \mathbb{R}^n \to \mathbb{R}$. We leave this argument as an exercise.

We note that the two notions of closed and bounded are distinct, and neither implies the other. There are closed subsets of $\mathbb{R}^n$ that are not bounded and there are bounded subsets of $\mathbb{R}^n$ that are not closed. We combine these two notions to get the desired notion of compactness.

Definition 1.13.2. A subset $X \subset \mathbb{R}^n$ for some $n \geq 1$ is *compact* if it is both closed and bounded.

A standard example of a compact set is the sphere $S_r(\mathbf{a})$ in $\mathbb{R}^n$ centred at $\mathbf{a}_0 \in \mathbb{R}^n$ with radius $r > 0$, which is the set

$$S_r(\mathbf{a}) = \{\mathbf{x} \in \mathbb{R}^n \mid ||\mathbf{x} - \mathbf{a}|| = r\}.$$

The fact that $S_r(\mathbf{a})$ is bounded is explicit in its definition, since $S_r(\mathbf{a}) \subset B_{2r}(\mathbf{a})$. The fact that $S_r(\mathbf{a})$ is closed comes from the fact that $S_r(\mathbf{a}) = f^{-1}(r)$ for the function $f : \mathbb{R}^n \to \mathbb{R}$ given by $f(\mathbf{x}) = ||\mathbf{x} - \mathbf{a}||$.

We are now able to give a more general version of Theorem 1.13.1.

Theorem 1.13.3. *Let X be a compact subset of $\mathbb{R}^n$ for $n \geq 1$ and let $f : X \to \mathbb{R}$ be a continuous function. Then, $f(x)$ achieves its maximum value on X; that is, there exists a point $\mathbf{x}_0 \in X$ so that $f(\mathbf{x}_0) \geq f(\mathbf{x})$ for all $\mathbf{x} \in X$.*

Remark 1.13.4. As with Theorem 1.13.1, we can obtain the minimum value theorem on general compact sets by applying Theorem 1.13.3 to the function $-f(\mathbf{x})$. A consequence of this is that for a compact set $X \subset \mathbb{R}^n$ and a continuous function $f : X \to \mathbb{R}$, we must have that $f(\mathbf{x})$ has a maximum at some point $\mathbf{a} \in \mathbb{R}^n$ and a minimum at some point $\mathbf{b} \in \mathbb{R}^n$.

Example 1.13.5. Find the extreme values of the function $f(x, y) = xy \exp(x - y)$ on the triangular region

$$T = \{(x, y, z) \mid -3 \leq x \leq 0, 0 \leq y \leq 3, y \leq x + 3\}.$$

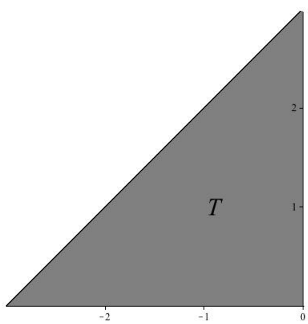


FIGURE 10. region T

Let us first find the critical points:

$$(0, 0) = \nabla f(x, y) = ((y + xy) \exp(x - y), (x - xy) \exp(x - y))$$

which shows that $y + xy = 0$ and $x - xy = 0$. This implies that either $x = -1, y = 1$ or $x = 0, y = 0$. At the point $(0, 0)$ the function takes the value $f(0, 0) = 0$ and at $(-1, 1)$ it is $f(-1, 1) = -e^{-2}$

Next, we look on the boundary of T which consists of three line segments. Note that when either $x = 0$ or $y = 0$, then $f(x, y) = 0$ which is not the minimum value since we already know that function attains $-e^{-2}$, but it could still be the maximum value of the function on this region. To check this we need to look at the third boundary line segment which is given by $y = x + 3$ so that $-3 < x < 0$. On this segment the we can express the function in terms of the variable x only:

$$g(x) = f(x, x + 3) = x(x + 3)e^{-3}.$$

The critical points of this function are given by:

$$0 = g'(x) = (2x+3)e^{-3}.$$

So, $x = -\frac{3}{2}$ and $g(-\frac{3}{2}) = f(-\frac{3}{2}, \frac{3}{2}) = -\frac{9}{4}e^{-3}$.

Since $0 > -\frac{9}{4}e^{-3} > -e^{-2}$, the function $f(x,y)$ has neither a minimum nor a maximum value at $(-\frac{3}{2}, \frac{3}{2})$.

Therefore, the function $f(x,y)$ attaints it maximum value of 0 on the boundary lines $x = 0$ or $y = 0$ of T and its minimum value of $-e^{-2}$ at the interior point $(-1, 1)$ of T.

1.14. Lagrange multipliers

Lagrange multipliers are a piece of mathematical machinery that provide us with a means of handling maximisation and minimisation questions that are more complicated than those that can be handled using the methods of Section 1.12. That is, we need a way of handling constraints that are sufficiently complicated that we cannot simply solve for one of the variables explicitly in terms of the others or parametrically by parametrising the given variables.

We begin with a simple case, that of maximising and minimising a function $f(x,y)$ subject to a single constraint $g(x,y) = 0$. That is, we wish to find the maximum and minimum values of the function $f(x,y)$ over all the points (x,y) satisfying the constraint $g(x,y) = 0$.

The basic idea behind Lagrange multipliers in this case is contained in the following result.

Theorem 1.14.1. *Suppose that $f(x,y)$ and $g(x,y)$ have continuous first order partial derivatives on a neighbourhood of a point (a,b). If the maximum or minimum of $f(x,y)$ subject to the constraint that $g(x,y) = 0$ occurs at (a,b) where $\nabla g(a,b) \neq (0,0)$, then there is some constant $\lambda \in \mathbb{R}$ so that $\nabla f(a,b) = \lambda \nabla g(a,b)$.*

Proof. The assumptions on partial derivatives guarantee that the curve C in $\mathbb{R}^2$ defined by the equation $g(x,y) = 0$ has a tangent line at the point (a,b). Suppose by a way of contradiction that $\nabla f(a,b)$ is not parallel to $\nabla g(a,b)$. Hence, it has a nonzero vector projection $\mathbf{v}$ on the tangent line to the curve C (see Figure 11). This means that f has a positive directional derivative in the direction of the vector $\mathbf{v}$: $D_{\mathbf{v}}(f) > 0$ and f has a negative directional derivative in the direction of the vector $-\mathbf{v}$: $D_{-\mathbf{v}}(f) < 0$. So, f increases as we move away from the point (a,b) along the curve C in the direction of $\mathbf{v}$ and it decreases as we move in the opposite direction along C. Thus the function f cannot have a maximum or a minimum value at (a,b). This is a contradiction. Therefore, we conclude that $\nabla f(a,b)$ is parallel to $\nabla g(a,b)$. □

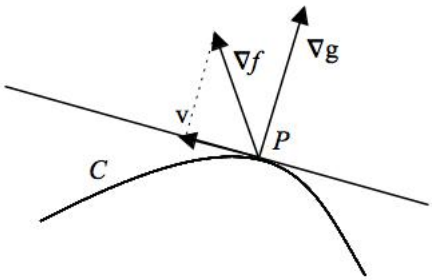


FIGURE 11. shows that the point $P = (a, b)$ is neither a maximum nor a minimum point on the curve C since $\nabla f(a, b)$ is not parallel to $\nabla g(a, b)$.

Now, we can give a general procedure for finding a maximum or a minimum for a function $f(x, y)$ subject to the simple constraint $g(x, y) = 0$ using Lagrange multipliers, which follows a basic pattern:

Step 1. We find the points (a, b), if any, at which $g(a, b) = 0$ and $\nabla g(a, b) = (0, 0)$. These points, which we may as well call *exceptional points*, are the points at which Theorem 1.14.1 does not apply;

Step 2. We introduce a new variable λ and form the function $L(x, y, \lambda) = f(x, y) - \lambda g(x, y)$;

Step 3. We find the critical points of $L(x, y, \lambda)$. Note that the equation $\nabla L(x, y, \lambda) = (0, 0, 0)$ contains both the equation $\nabla f(x, y) = \lambda \nabla g(x, y)$ in its first two variables and the constraint equation $g(x, y) = 0$ in the third variable;

Step 4. We evaluate $f(x, y)$ at the first two coordinates of each of the critical points of $L(x, y, \lambda)$ found in Step 3. and at each of the exceptional points found in Step 1., and we then eyeball to see where the maximum and minimum values occur.

Note that we follow here the same basic plan as before, when we were finding the maxima and minima of a continuous function on a closed interval. We find the critical points of $f(x, y)$ in the interior and we then consider the behaviour of $f(x, y)$ on the boundary. Here, though, we have that the behaviour of $f(x, y)$ on the boundary is where all of the interesting behaviour occurs. As in the case of finding unconstrained maxima and minima, we will also sometimes find points where $f(x, y)$ subject to the constraint $g(x, y) = 0$ does not have a maximum or minimum.

To illustrate this, consider the following example.

Example 1.14.2. Find the maximum and minimum values of $f(x, y) = 4xy$ on the circle $x^2 + y^2 = 1$.

We need to first rephrase the question in terms of a function being optimised and a function giving the constraint. In this example, the function being optimised is $f(x, y) = 4xy$, and the constraint is $g(x, y) = x^2 + y^2 - 1 = 0$. It is important that we always phrase the constraint as $g(x, y) = 0$ for some function $g(x, y)$.

We now look for any exceptional points. The critical points of the gradient $\nabla g(x, y)$ are the solutions to $\nabla g(x, y) = (2x, 2y) = (0, 0)$, of which there is only one, namely $(a, b) = (0, 0)$. Since $g(0, 0) = -1 \neq 0$, there are no exceptional points.

So, we form

$$L(x, y, \lambda) = f(x, y) - \lambda g(x, y) = 4xy - \lambda(x^2 + y^2 - 1)$$

and we find the critical points of $L(x, y, \lambda)$. Calculating, we see that

$$\nabla L(x, y, \lambda) = (4y - 2\lambda x, 4x - 2\lambda y, 1 - x^2 - y^2),$$

and so setting $\nabla L(x, y, \lambda) = (0, 0, 0)$ yields 3 equations in the unknowns x, y and λ, namely

(1) $4y = 2\lambda x$;

(2) $4x = 2\lambda y$;

(3) $x^2 + y^2 = 1$ (which is just the condition on the constraint that $g(x, y) = 0$).

Note that equations (1) and (2) combined yield that $x = 0$ if and only if $y = 0$, and equation (3) yields that we cannot have $x = y = 0$ because $0 \neq 1$. Hence, we see that both $x \neq 0$ and $y \neq 0$. We can therefore solve both of equations (1) and (2) for λ to get that

$$\frac{2y}{x} = \lambda = \frac{2x}{y}.$$

(We need to check that $x \neq 0$ before we can divide by x, as otherwise we have to consider the case that $x = 0$ as a separate case. The same comment holds for y. Do remember that we can only divide by non-zero quantities, and we can never divide by 0.)

Cross-multiplying and dividing by 2, this yields that $x^2 = y^2$. Plugging this into equation 3. yields that $2x^2 = 1$ and so $x = \pm\frac{1}{\sqrt{2}}$. Hence, we also have that $y = \pm\frac{1}{\sqrt{2}}$. Hence, we have 4 points at which to evaluate the value of $f(x, y) = 4xy$, namely:

- At $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$, where $f\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = 2$;

- At $\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$, where $f\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = -2$;

- At $\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$, where $f\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right) = -2$;
- At $\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$, where $f\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right) = 2$.

Hence, we see that $f(x, y) = 4xy$ subject to the constraint that $x^2 + y^2 = 1$ takes its maximum value of 2 at two points, namely $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ and $\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$, and that $f(x, y) = 4xy$ subject to the constraint that $x^2 + y^2 = 1$ takes its minimum value of -2 at two points, namely $\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ and $\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$.

Note that while we could easily have found the values of λ corresponding to these 4 points, and hence found the critical points of $L(x, y, \lambda)$, we normally do not need to determine the critical points of $L(x, y, \lambda)$. We are really concerned with the extreme points of $f(x, y)$ subject to the constraint that $g(x, y) = 0$, and we introduce the additional variable λ and the auxiliary function $L(x, y, \lambda)$ in order to make the process of finding these extreme points a bit smoother.

Example 1.14.3. Let us go back to one of the questions given at the beginning of this chapter and in Section 1.12, namely determining the maximum and minimum values of $f(x, y, z) = x + 2y + 3z$ on the sphere $x^2 + y^2 + z^2 = 3$.

We proceed in exactly the same way as above, regardless of the number of variables we are dealing with. We have the one constraint

$$g(x, y, z) = x^2 + y^2 + z^2 - 3 = 0.$$

We first look for any exceptional points. The only point at which

$$\nabla g(x, y, z) = (2x, 2y, 2z) = (0, 0, 0)$$

is $(a, b, c) = (0, 0, 0)$, and $g(0, 0, 0) = -3 \neq 0$.

Namely, we introduce one new variable λ, as we have the one constraint $g(x, y, z) = 0$. We then form the function

$$L(x, y, z, \lambda) = f(x, y, z) - \lambda g(x, y, z) = x + 2y + 3z - \lambda(x^2 + y^2 + z^2 - 3),$$

and we find the critical points of $L(x, y, z, \lambda)$. The gradient of $L(x, y, z, \lambda)$ is

$$\nabla L(x, y, z, \lambda) = (1 - 2\lambda x, 2 - 2\lambda y, 3 - 2\lambda z, 3 - x^2 - y^2 - z^2).$$

Setting $\nabla L(x, y, z, \lambda) = (0, 0, 0, 0)$ yields 4 equations in the 4 unknowns x, y, z, and λ, namely

(1) $1 - 2\lambda x = 0$;

(2) $2 - 2\lambda y = 0$;

(3) $3 - 2\lambda z = 0$;

(4) $x^2 + y^2 + z^2 = 3$.

First, note that equations (1), (2), and (3) imply that $x \neq 0$, $y \neq 0$, and $z \neq 0$, respectively. Solving equations (1), (2) and (3) for λ yields that

$$\lambda = \frac{1}{2x} = \frac{1}{y} = \frac{3}{2z},$$

and so $2x = y = \frac{2}{3}z$. Since $y = 2x$ and $z = 3x$, equation (4) yields that $14x^2 = 3$, and so $x = \pm\sqrt{\frac{3}{14}}$. Hence, we get two points, namely $\left(\sqrt{\frac{3}{14}}, 2\sqrt{\frac{3}{14}}, 3\sqrt{\frac{3}{14}}\right)$ and $\left(-\sqrt{\frac{3}{14}}, -2\sqrt{\frac{3}{14}}, -3\sqrt{\frac{3}{14}}\right)$.

Since

$f\left(\sqrt{\frac{3}{14}}, 2\sqrt{\frac{3}{14}}, 3\sqrt{\frac{3}{14}}\right) = \sqrt{42}$, $f\left(-\sqrt{\frac{3}{14}}, -2\sqrt{\frac{3}{14}}, -3\sqrt{\frac{3}{14}}\right) = -\sqrt{42}$,

we see that, subject to the constraint that

$$g(x, y, z) = x^2 + y^2 + z^2 - 3,$$

the function $f(x, y, z) = x + 2y + 3z$ achieves its maximum value of $\sqrt{42}$ at the point $\left(\sqrt{\frac{3}{14}}, 2\sqrt{\frac{3}{14}}, 3\sqrt{\frac{3}{14}}\right)$ and achieves its minimum value of $-\sqrt{42}$ at the point $\left(-\sqrt{\frac{3}{14}}, -2\sqrt{\frac{3}{14}}, -3\sqrt{\frac{3}{14}}\right)$.

We can reframe the machinery of Lagrange multipliers in great generality, for functions of n variables for any $n \geq 2$ and for any (finite) number of constraints. The formal formulation is very similar to that given for functions of two variables with a single constraint.

Theorem 1.14.4. *Suppose that* $f(x_1, \ldots, x_n)$, $g_1(x_1, \ldots, x_n), \ldots, g_m(x_1, \ldots, x_n)$ *have continuous first order partial derivatives on a neighbourhood of a point* $\mathbf{a} = (a_1, \ldots, a_n)$. *If the maximum or minimum of* $f(x_1, \ldots, x_n)$ *subject to the constraints that* $g_1(x_1, \ldots, x_n) = \cdots g_m(x_1, \ldots, x_n) = 0$ *occurs at* $\mathbf{a}$, *where* $\nabla g_1(\mathbf{a}), \ldots, \nabla g_m(\mathbf{a})$ *are linearly independent, then there are constants* $\lambda_1, \ldots, \lambda_m \in \mathbb{R}$ *so that*

$$\nabla f(\mathbf{a}) = \sum_{j=1}^{m} \lambda_j \nabla g_j(\mathbf{a}).$$

Example 1.14.5. Find the maximum and minimum values of $f(x, y, z) = z$ subject to the constraints $x^2 + y^2 = 1$ and $2x + 2y + z = 5$.

A Lagrange multipliers question with two constraints $g(x, y, z) = 0$ and $h(x, y, z) = 0$ is set up and solved in a manner very similar to the case of one constraint. Namely, we introduce two new variables, one for each constraint. In this case, the constraints are

$$g(x, y, z) = x^2 + y^2 - 1 = 0 \quad \text{and} \quad h(x, y, z) = 2x + 2y + z - 5 = 0.$$

The first step is to look for any exceptional points. With two constraints, this is a bit more involved than for one variable. We need to see whether there are points (x, y, z) at which $\nabla g(x, y, z)$ and $\nabla h(x, y, z)$ are parallel.

For $g(x, y, z)$, we have that $\nabla g(x, y, z) = (2x, 2y, 0)$ and for $h(x, y, z)$, we have that $\nabla h(x, y, z) = (2, 2, 1)$. The only way these two gradient vectors would be parallel if $x = y = 0$. But this set of points does not satisfy the first constraint equation. Hence, we conclude that there are no exceptional points.

We then form the function

$$L(x, y, z, \lambda, \mu) =$$

$$= f(x, y, z) - \lambda g(x, y, z) - \mu h(x, y, z) = z - \lambda(x^2 + y^2 - 1) - \mu(2x + 2y + z - 5),$$

and our task is to find the critical points of $L(x, y, z, \lambda, \mu)$. The gradient of $L(x, y, z, \lambda, \mu)$ is

$$\nabla L(x, y, z, \lambda, \mu) = (-2\lambda x - 2\mu, -2\lambda y - 2\mu, 1 - \mu, 1 - x^2 - y^2, 5 - 2x - 2y - z).$$

Setting $\nabla L(x, y, z, \lambda, \mu) = (0, 0, 0, 0, 0)$ yields 5 equations in the 5 unknowns x, y, z, λ, and μ, namely

(1) $-2\lambda x - 2\mu = 0$;

(2) $-2\lambda y - 2\mu = 0$;

(3) $1 - \mu = 0$;

(4) $x^2 + y^2 = 1$;

(5) $2x + 2y + z = 5$.

Equation (3) yields that $\mu = 1$. Equations (1) and (2) immediately imply that $\lambda \neq 0$, $x \neq 0$ and $y \neq 0$, since $-2\lambda x = 2 = -2\lambda y$. Solving this equation for λ, we see that $\lambda = -\frac{1}{x} = -\frac{1}{y}$, and so $x = y$. Equation (4) then immediately implies that $2x^2 = 1$, and so $x = \pm\frac{1}{\sqrt{2}}$.

For the two choices of sign for x, equation (5) then yields the corresponding value of $z = 5 - 2x - 2y$, and so the two points we need to consider are $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 5 - \frac{4}{\sqrt{2}}\right)$ and $\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 5 + \frac{4}{\sqrt{2}}\right)$. Since

$$f\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 5 - \frac{4}{\sqrt{2}}\right) = 5 - \frac{4}{\sqrt{2}} \quad \text{and} \quad f\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 5 + \frac{4}{\sqrt{2}}\right) = 5 + \frac{4}{\sqrt{2}},$$

we see that, subject to the constraints that $x^2 + y^2 = 1$ and $2x + 2y + z = 5$, the function $f(x, y, z) = z$ achieves its maximum value of $5 + \frac{4}{\sqrt{2}}$ at $\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 5 + \frac{4}{\sqrt{2}}\right)$ and achieves its minimum value of $5 - \frac{4}{\sqrt{2}}$ at $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 5 - \frac{4}{\sqrt{2}}\right)$.

Note that geometrically, this answer makes sense. The function $f(x, y, z) = z$ on $\mathbb{R}^3$ is measuring the vertical height of the point (x, y, z). The constraint $x^2 + y^2 = 1$ restricts our attention to points on the cylinder, while the constraint $2x + 2y + z = 5$ restricts our attention to points on this plane. The plane intersects the cylinder at an angle, and so we expect one maximum and one minimum.

The main complication of this most general version is that the algebraic massage involved in finding the solutions to the system of equations $\nabla L(x_1, \ldots, x_n, \lambda_1, \ldots, \lambda_m) = (0, \ldots, 0)$ might prove to be prohibitively difficult or impossible to carry out by hand, and we might need to resort to technological means (such as Maple) to find a numerical solution.

One interesting application of the method of Lagrange multipliers is to verify inequalities.

Example 1.14.6. Show that the arithmetic mean of n positive real numbers is always greater than or equal to the geometric mean of the numbers, with equality if and only if all the numbers are equal.

In coordinates $(x_1, \ldots, x_n)$ on $\mathbb{R}^n$, the geometric mean is

$$GM(x_1, \ldots, x_n) = \sqrt[n]{x_1 \cdot \cdots \cdot x_n},$$

while the arithmetic mean is the average

$$AM(x_1, \ldots, x_n) = \frac{1}{n}(x_1 + \cdots + x_n)$$

of the n numbers. We wish to show that $GM(x_1, \ldots, x_n) \leq AM(x_1, \ldots, x_n)$, with equality if and only if $x_1 = \cdots = x_n = a$. We need first to decide which function is the function we wish to maximise or minimise and which function gives the constraint.

Let us start by maximising the geometric mean $GM(x_1, \ldots, x_n)$ and letting the arithmetic mean provide the constraint. That is, we fix some positive number a and let the constraint be

$$g(x_1, \ldots, x_n) = AM(x_1, \ldots, x_n) - a = \frac{1}{n}(x_1 + \cdots + x_n) - a = 0.$$

As always, we start by forming the auxiliary function

$$L(x_1, \ldots, x_n, \lambda) = GM(x_1, \ldots, x_n) - \lambda g(x_1, \ldots, x_n) =$$

$$= \sqrt[n]{x_1 \cdot \cdots \cdot x_n} - \lambda \left(\frac{1}{n}(x_1 + \cdots + x_n) - a \right).$$

The gradient of $L(x_1, \ldots, x_n, \lambda)$ is

$$\nabla L(x_1, \ldots, x_n, \lambda) =$$

$$= \left(\frac{x_2 \cdot \cdots \cdot x_n}{n \sqrt[n]{(x_1 \cdot \cdots \cdot x_n)^{n-1}}} - \frac{\lambda}{n}, \ldots, \frac{x_1 \cdot \cdots \cdot x_{n-1}}{n \sqrt[n]{(x_1 \cdot \cdots \cdot x_n)^{n-1}}} - \frac{\lambda}{n}, a - \frac{1}{n}(x_1 + \cdots + x_n) \right).$$

Setting $\nabla L(x_1, \ldots, x_n, \lambda) = (0, \ldots, 0)$ yields the system of equations involving the equation

$$\frac{x_2 \cdot \cdots \cdot x_n}{n\sqrt[n]{(x_1 \cdot \cdots \cdot x_n)^{n-1}}} - \frac{\lambda}{n} = 0;$$

together with the $n-2$ equations

$$\frac{x_1 \cdot \cdots \cdot x_{j-1} \cdot x_{j+1} \cdot \cdots \cdot x_n}{n\sqrt[n]{(x_1 \cdot \cdots \cdot x_n)^{n-1}}} - \frac{\lambda}{n} = 0$$

for $2 \leq j \leq n-1$; together with the equation

$$\frac{x_1 \cdot \cdots \cdot x_{n-1}}{n\sqrt[n]{(x_1 \cdot \cdots \cdot x_n)^{n-1}}} - \frac{\lambda}{n} = 0;$$

and the equation

$$a - \frac{1}{n}(x_1 + \cdots + x_n) = 0.$$

A bit of algebraic manipulation applied to the first n equations, including moving the $\frac{\lambda}{n}$ term to the other side and taking the n^{th} power of both sides, yields that

$$x_1 \cdot \cdots \cdot x_n = \lambda^n x_1^n = \cdots = \lambda^n x_n^n.$$

Since we have assumed that each $x_j > 0$, this then yields that $x_1 = \cdots = x_n$. Plugging this into the last equation yields that $x_1 = \cdots = x_n = a$. Hence, the unique extreme point for the geometric mean $GM(x_1, \ldots, x_n)$ occurs at $(x_1, \ldots, x_n) = (a, \ldots, a)$, where the geometric mean takes the value $GM(a, \ldots, a) = a = AM(a, \ldots, a)$.

To see that this is indeed a maximum point for the geometric mean (rather than a minimum point), we can take a different point with the same arithmetic mean a, such as $(a+\varepsilon, a-\varepsilon, a, \cdots, a)$ for some $0 < \varepsilon < a$, and evaluate the geometric mean at this point. Calculating, we see that

$$GM(a+\varepsilon, a-\varepsilon, a, \cdots, a) =$$
$$= \sqrt[n]{(a+\varepsilon)(a-\varepsilon)a^{n-2}} = \sqrt[n]{a^n - \varepsilon^2 a^{n-2}} < \sqrt[n]{a^n} = GM(a, \ldots, a).$$

1.15. Shapes of maxima and minima, the Hessian, and the second derivative test

As with a function of one variable, we can determine local maxima and minima of a function of more than one variable.

To review, let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function of one variable. We are assuming here that the domain of f is all of $\mathbb{R}$, but this is for notational convenience and is not essential to the argument we are about to give. To find the local maxima and minima of $f(x)$, we first find all of the *critical points* of $f(x)$, which are the points at which its derivative $f'(x)$ satisfies $f'(x) = 0$. At each critical point, we then apply the *second derivative test*.

Theorem 1.15.1. *Let $f(x)$ be a function of one variable and let a be a critical point. Assume that $f(x)$ has continuous second order derivative on the open interval $(a-\varepsilon, a+\varepsilon)$ for some $\varepsilon > 0$. Then,*

(i) If $f''(a) > 0$, then $f(x)$ has a local minimum at a;

(ii) If $f''(a) < 0$, then $f(x)$ has a local maximum at a;

(iii) If $f''(a) = 0$, then we have no information.

For functions of more than one variable, we have a similar test. Before we can state the second derivative test for a function of two variables, we need to have an appropriate notion of what to use in place of the second derivative. To that end, we define the *Hessian matrix* (or *Hessian*) of a function $f(x,y)$. The Hessian is the matrix of second order partial derivatives. Specifically, for a function $f(x,y)$ of two variables, the Hessian $H(f)(x,y)$ is the 2×2 square matrix

$$H(f)(x,y) = \begin{pmatrix} \frac{\partial^2 f}{\partial x^2}(x,y) & \frac{\partial^2 f}{\partial x \partial y}(x,y) \\ \\ \frac{\partial^2 f}{\partial y \partial x}(x,y) & \frac{\partial^2 f}{\partial y^2}(x,y) \end{pmatrix}.$$

The second derivative test is stated in terms of the Hessian as follows.

Theorem 1.15.2. *Let $f(x,y)$ be a function of two variables and let (a,b) be a critical point. Assume that $f(x,y)$ has continuous second order partial derivatives on a neighbourhood of (a,b). Set $\Delta = \det H(f)(a,b)$.*

(i) *If $\Delta > 0$ and $\frac{\partial^2 f}{\partial x^2}(a,b) > 0$, then $f(x,y)$ has a local minimum at (a,b).*

(ii) *If $\Delta > 0$ and $\frac{\partial^2 f}{\partial x^2}(a,b) < 0$, then $f(x,y)$ has a local maximum at (a,b).*

(iii) *If $\Delta < 0$, then $f(x,y)$ has neither a local maximum nor a local minimum at (a,b).*

(iv) *If $\Delta = 0$, then we have no information.*

Remark 1.15.3. It seems that there is a missing case, in that we have not considered what happens if $\Delta > 0$ and $\frac{\partial^2 f}{\partial x^2}(a,b) = 0$. The reason is, this case never occurs. The hypothesis about the continuity of the second order partial derivatives is sufficient to imply that the mixed second order partial derivatives are equal at (a,b); that is, $\frac{\partial^2 f}{\partial x \partial y}(a,b) = \frac{\partial^2 f}{\partial y \partial x}(a,b)$. Since

$$\Delta = \frac{\partial^2 f}{\partial x^2}(a,b)\frac{\partial^2 f}{\partial y^2}(a,b) - \left(\frac{\partial^2 f}{\partial x \partial y}(a,b)\right)^2,$$

we see that if $\frac{\partial^2 f}{\partial x^2}(a,b) = 0$, then $\Delta \leq 0$.

The proof of the above theorem is beyond the scope of these course. We give some examples instead.

Example 1.15.4. Consider $f(x,y) = x^2 + y^2$. In this case, the Hessian is the constant matrix

$$H(f)(x,y) = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}.$$

At the single critical point $(0,0)$ of $f(x,y)$, we see that $\Delta = \det H(f)(0,0) = 4 > 0$ and $\frac{\partial^2 f}{\partial x^2}(0,0) = 2 > 0$, and so the second derivative test gives that $f(x,y) = x^2 + y^2$ has a local minimum at $(0,0)$.

Example 1.15.5. Consider now $f(x,y) = -x^2 - y^2$. In this case, the Hessian is the constant matrix

$$H(f)(x,y) = \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix}.$$

At the single critical point $(0,0)$ of $f(x,y)$, we see that $\Delta = \det H(f)(0,0) = 4 > 0$ and $\frac{\partial^2 f}{\partial x^2}(0,0) = -2 < 0$, and so the second derivative test gives that $f(x,y) = -x^2 - y^2$ has a local maximum at $(0,0)$.

Example 1.15.6. Consider now $f(x,y) = x^2 - y^2$. In this case, the Hessian is the constant matrix

$$H(f)(x,y) = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}.$$

At the single critical point $(0,0)$ of $f(x,y)$, we see that $\Delta = \det H(f)(0,0) = -4 < 0$, and so the second derivative test gives that $f(x,y) = x^2 - y^2$ has neither a local maximum nor a local minimum at $(0,0)$. We call a critical point satisfying this conclusion of the second derivative test a *saddle point*. The reason for the name is easily seen by considering the graph of $f(x,y) = x^2 - y^2$ near $(0,0)$.

To see why $\Delta = 0$ yields no information, consider the following functions:

- $h(x,y) = x^4 + y^4$. Like the paraboloid $f(x,y) = x^2 + y^2$, $h(x,y)$ has a local minimum at its single critical point $(0,0)$, but its Hessian matrix at $(0,0)$ is the 0 matrix;
- $h(x,y) = -x^4 - y^4$. Like the inverted paraboloid $f(x,y) = -x^2 - y^2$, $h(x,y)$ has a local maximum at its single critical point $(0,0)$, but its Hessian matrix at $(0,0)$ is the 0 matrix;
- $h(x,y) = x^4 - y^4$. Like the saddle $f(x,y) = x^2 - y^2$, $h(x,y)$ has neither local maximum nor a local minimum at its single critical point $(0,0)$, but its Hessian matrix at $(0,0)$ is the 0 matrix.

We see that three different phenomena all occur at critical points where the determinant of the Hessian matrix is 0.

Example 1.15.7. Find and classify the critical points of the function $h(x,y) = x^2 - y^2 - 2\exp(-x^2 - y^2)$.

As always for this sort of question, we begin by calculating the gradient of $h(x,y)$ and solving $\nabla h(x,y) = (0,0)$:

$$\nabla h(x,y) = (2x + 4x\exp(-x^2 - y^2), -2y + 4y\exp(-x^2 - y^2)) = (0,0),$$

which has three solutions. One solution is easy to see, namely $(x,y) = (0,0)$. In fact, since the first coordinate of $\nabla h(x,y)$ is

$$2x + 4x\exp(-x^2 - y^2) = 2x(1 + 2\exp(-x^2 - y^2))$$

and since $\exp(-x^2 - y^2) > 0$ for all points $(x,y) \in \mathbb{R}^2$, we see that the only solution to

$$2x(1 + 2\exp(-x^2 - y^2)) = 0$$

is $x = 0$, and so the x-coordinate of every critical point is 0.

Setting $x = 0$, we see that the second coordinate of $\nabla h(x,y)$ is

$$-2y + 4y\exp(-y^2)) = -2y(1 - 2\exp(-y^2)).$$

Setting $-2y(1 - 2\exp(-y^2)) = 0$, we see there are 3 solutions, namely $y = 0$ (which we had already found), and the two solutions to $1 - 2\exp(-y^2) = 0$, namely $y = \pm\sqrt{\ln(2)}$.

So, we have 3 critical points in all: $(0,0)$, $(0, \sqrt{\ln(2)})$, and $(0, -\sqrt{\ln(2)})$. Let us see what the second derivative has to see about their respective classifications.

We start by calculating the Hessian matrix $H(h)(x,y)$:

$$H(h)(x,y) =$$

$$\begin{pmatrix} 2 + 4\exp(-x^2 - y^2) - 8x^2\exp(-x^2 - y^2) & -8xy\exp(-x^2 - y^2) \\ -8xy\exp(-x^2 - y^2) & -2 + 4\exp(-x^2 - y^2) - 8y^2\exp(-x^2 - y^2) \end{pmatrix}$$

Evaluating the determinant of $H(h)(x,y)$ are each of the critical points, we see that:

- At $(0,0)$, we see that

$$\det(H(h)(0,0)) = \det\begin{pmatrix} 6 & 0 \\ 0 & 2 \end{pmatrix} = 12 > 0$$

  and that $\frac{\partial^2 h}{\partial x^2}(0,0) = 6 > 0$, and so $h(x,y)$ has a local minimum at $(0,0)$;

- At $(0, \sqrt{\ln(2)})$, we see that

$$\det(H(h)(0, \sqrt{\ln(2)})) = \det\begin{pmatrix} 4 & 0 \\ 0 & -4(\ln(2))^2 \end{pmatrix} = -16(\ln(2))^2 < 0$$

  and so $h(x,y)$ has a saddle point at $(0, \ln(2))$;

- At $(0, -\sqrt{\ln(2)})$, we see that

$$\det(H(h)(0, -\sqrt{\ln(2)})) = \det\begin{pmatrix} 4 & 0 \\ 0 & -4(\ln(2))^2 \end{pmatrix} = -16(\ln(2))^2 < 0$$

  and so $h(x,y)$ has a saddle point at $(0, -\ln(2))$.

This is an interesting example as the function has a single local minimum at $(0,0)$, no global or local maxima, and no global minimum (as can be seen by setting $x=0$ and letting $y \to \infty$).

1.16. When the second derivative test fails ...

In this section we combine different techniques we have discussed thus far to determine the extreme values of a given function with possible constraints.

Example 1.16.1. Consider the function $f(x,y) = 6xy^2 - 2x^3 - 3y^4$. Find and classify the critical points of $f(x,y)$.

The gradient is

$$\nabla f(x,y) = (6y^2 - 6x^2, 12xy - 12y^3)$$

and so $f(x,y)$ has three critical points, at $(0,0)$, $(1,1)$ and $(1,-1)$. The Hessian matrix of $f(x,y)$ is

$$H(f)(x,y) = \begin{pmatrix} \frac{\partial^2 f}{\partial x^2}(x,y) & \frac{\partial^2 f}{\partial x \partial y}(x,y) \\ \\ \frac{\partial^2 f}{\partial y \partial x}(x,y) & \frac{\partial^2 f}{\partial y^2}(x,y) \end{pmatrix} = \begin{pmatrix} -12x & 12y \\ \\ 12y & 12x - 36y^2 \end{pmatrix}.$$

At $(1,1)$, we see that $\Delta = \det(H(f)(1,1)) = 144 > 0$ and $\frac{\partial^2 f}{\partial x^2}(1,1) = -12 < 0$ and so by the second derivative test, $(1,1)$ is a local maximum.

Similarly, at $(1,-1)$, we see that $\Delta = \det(H(f)(1,-1)) = 144 > 0$ and $\frac{\partial^2 f}{\partial x^2}(1,-1) = -12 < 0$ and so by the second derivative test, $(1,-1)$ is a local maximum as well.

However, at $(0,0)$, we see that $\Delta = 0$, and so the second derivative test gives us no information. So, what other means do we have to determine whether $f(x,y)$ has a local maximum or local minimum at $(0,0)$?

One way to proceed is to consider what happens on the graph of $f(x,y)$ above lines in the (x,y)-plane.

We note here that this method does not give enough information to determine whether a critical point (or even a non-critical point) is a local maximum or local minimum, but it may provide enough information to show that a point is neither a local maximum nor a local minimum.

So, consider the line $y = x$, and let us see what the graph of

$$f(x,y) = 6xy^2 - 2x^3 - 3y^4$$

looks like above this line. Along the line, the function becomes

$$g(x) = f(x,x) = 6x^3 - 2x^3 - 3x^4 = 4x^3 - 3x^4.$$

Considering the behaviour of $f(x, y)$ along any line allows us to reduce questions about its behaviour to a one-dimensional question. Note that the derivative of $g(x)$ is

$$g'(x) = 12x^2 - 12x^3 = 12x^2(1 - x).$$

For any x satisfying $|x| < 1$, for instance, we see that $g'(x) \geq 0$, and so $g(x)$ is increasing on $(-1, 1)$. In particular, there are values of x arbitrarily close to 0 for which $g(x) < 0$ and values of x arbitrarily close to 0 so that $g(x) > 0$, and so $x = 0$ is neither a local maximum nor a local minimum for $g(x)$. Going back to $f(x, y)$, we see that along the line $y = x$, we can find points (a, b) arbitrarily close to $(0, 0)$ at which $f(a, b) > 0$ and points (a, b) arbitrarily close to $(0, 0)$ at which $f(a, b) < 0$. Hence, even though we get no information from the second derivative test, we can see that $(0, 0)$ is neither a local maximum nor a local minimum for

$$f(x, y) = 6xy^2 - 2x^3 - 3y^4.$$

We note that this method of considering all lines through a critical point is very much an ad hoc method, and it is a method that cannot be used to determine whether a point is a local maximum or local minimum. Rather, this method is sometimes useful for determining that a critical point at which the second derivative test gives no information is neither a local maximum nor a local minimum. To see that understanding the behaviour of a function at all lines through a critical point is not enough information, consider the following example.

Example 1.16.2. Consider the function $f(x, y) = (y - x^2)(y - 3x^2)$.

Show that the origin is a critical point of $f(x, y)$ and that the restriction of $f(x, y)$ to every line through the origin has a local minimum at the origin. (That is, show that $g(x) = f(x, ax)$ has a local minimum at the origin for all $a \in \mathbb{R}$ and that $h(y) = f(0, y)$ also has a local minimum at the origin.)

Show that the second derivative test gives no information about the classification of this critical point. By considering what happens to $f(x, y)$ on the parabola $y = 2x^2$, show that $f(x, y)$ does not have a local minimum at the origin.

To see that the origin is a critical point of $f(x, y)$, we calculate its gradient. Since

$$f(x, y) = (y - x^2)(y - 3x^2) = y^2 - 4x^2y + 3x^4,$$

we see that

$$\nabla f(x, y) = (-8xy + 12x^3, 2y - 4x^2).$$

To see that the origin is a critical point of $f(x, y)$, we notice that $\nabla f(0, 0) = (0, 0)$, as desired.

Consider now the restriction of $f(x,y)$ to the y-axis. This yields the function $h(y) = f(0,y) = y^2$ of the single variable y, which we know has a local minimum (in fact a global minimum) at $y = 0$.

We now fix $a \in \mathbb{R}$ and consider the function $g_a(x) = f(x, ax)$ of the single variable x. Calculating, we see that

$$g_a(x) = a^2x^2 - 4ax^3 + 3x^4.$$

To see that $g_a(x)$ has a local minimum at $x = 0$, we consider 2 cases. If $a = 0$, then $g_0(x) = 3x^4$, which has a local minimum at the origin as it is positive away from the origin. In fact, for $a = 0$, we see that $g_0(x)$ in fact has a global minimum at $x = 0$.

If $a \neq 0$, we notice that

$$g_a'(x) = 2a^2x - 12ax^2 + 12x^3$$

and so $g_a'(0) = 0$, and that

$$g_a''(x) = 2a^2 - 24ax + 36x^2$$

and so $g_a''(0) = 2a^2 > 0$. The second derivative test for functions of a single variable then implies that $g_a(x)$ has a local minimum at $x = 0$. Hence, if we restrict our attention to any line through $(0,0)$, we can see that $f(x,y)$ is positive on this line near $(0,0)$ except at $(0,0)$, where it takes the value 0.

We now apply the second derivative test to $f(x,y)$. The Hessian of $f(x,y)$ is

$$Hf(x,y) = \begin{pmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial y \partial x} \\ \\ \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial y^2} \end{pmatrix} = \begin{pmatrix} -8y + 36x^2 & -8x \\ \\ -8x & 2 \end{pmatrix},$$

and so $\Delta = \det(H(f)(0,0)) = 0$. Hence, the second derivative test gives us no information about the behaviour of $f(x,y)$ at $(0,0)$.

If we consider the behaviour of $f(x,y)$ along the parabola $y = 2x^2$, we see that $f(x,y) = (2x^2 - x^2)(2x^2 - 3x^2) = -x^4$. In particular, we see that if we restrict our attention to this parabola, we see that $f(x,y)$ is negative near the origin. Hence, there are points arbitrarily close to $(0,0)$ at which $f(x,y)$ is positive, and points arbitrarily close to $(0,0)$ at which $f(x,y)$ is negative, and so $(0,0)$ is neither a local maximum nor a local minimum of $f(x,y)$.

Example 1.16.3. Find the extrema of $f(x,y) = 6xy^2 - 2x^3 - 3y^4$ on the region

$$B = \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 5\}.$$

In this example we cannot apply Lagrange multipliers directly, as the constraint is not of the right form. When applying Lagrange multipliers, the constraint has to be of the form $g(x,y) = 0$, and in particular the constraint does not involve an inequality. Instead, we break the question into two pieces which we solve separately and then bring them together at the end.

This is analogous to how we solve extremisation questions for a function of one variable on a closed interval.

We start by finding the critical points of $f(x, y)$ inside the region B, so in the region $(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 < 5\}$. The gradient of $f(x, y)$ is

$$\nabla f(x, y) = (6y^2 - 6x^2, 12xy - 12y^3).$$

Setting $\nabla f(x, y) = (0, 0)$ yields the two equations

$$6y^2 - 6x^2 = 0 \text{ and } 12xy - 12y^3 = 0.$$

The first equation yields that $x^2 = y^2$, so that $y = \pm x$. The second equation factors to

$$12xy - 12y^3 = 12y(x - y^2) = 0,$$

so that either $y = 0$ or $x = y^2$. If $y = 0$, then $x = 0$. If $x = y^2$, then we see that $x = (\pm x)^2 = x^2$, so that $x = 0$ or $x = 1$. For $x = 1$, we see that there are two possible values of y, namely $y = \pm 1$. Combining all of these, we see that there are three critical points, namely

- $(0, 0)$, at which $f(0, 0) = 0$;
- $(1, 1)$, at which $f(1, 1) = 1$; and
- $(1, -1)$, at which $f(1, -1) = 1$.

We now apply Lagrange multipliers to determine the behaviour of $f(x, y)$ on the boundary of B, which is given by the constraint $g(x, y) = x^2 + y^2 - 5 = 0$. We now dump this question into our Lagrange multipliers machine. Set

$$L(x, y, \lambda) = f(x, y) - \lambda g(x, y) = 6xy^2 - 2x^3 - 3y^4 - \lambda(x^2 + y^2 - 5),$$

so that

$$\nabla L(x, y, \lambda) = (6y^2 - 6x^2 - 2\lambda x, 12xy - 12y^3 - 2\lambda y, 5 - x^2 - y^2) = (0, 0, 0).$$

This yields three equations:

(1) $6y^2 - 6x^2 - 2\lambda x = 0$;

(2) $12xy - 12y^3 - 2\lambda y = 0$:

(3) $x^2 + y^2 = 5$.

We can solve equations (1) and (2) for λ, so that either

$$\lambda = \frac{6y^2 - 6x^2}{2x} = \frac{3y^2}{x} - 3x \text{ or } x = 0,$$

and either

$$\lambda = \frac{12xy - 12y^3}{2y} = 6x - 6y^2 \text{ or } y = 0.$$

If $x = 0$, then equation (3) yields that $y = \pm\sqrt{5}$ but equation 1. yields that $y = 0$, and so there are no solutions with $x = 0$. If $y = 0$, we get no information from equation (1) (because of the presence of λ) or equation (2), and equation (3) yields that $x = \pm\sqrt{5}$. This discussion yields two possible solutions, namely

- $(\sqrt{5}, 0)$, at which $f(\sqrt{5}, 0) = -10\sqrt{5}$; and
- $(-\sqrt{5}, 0)$, at which $f(\sqrt{5}, 0) = 10\sqrt{5}$.

Assuming that both $x \neq 0$ and $y \neq 0$, we can set the two expressions for λ equal to one another and solve. So, we have that

$$\frac{3y^2}{x} - 3x = \lambda = 6x - 6y^2.$$

Multiplying through by x, using equation 3., we see that x satisfies the cubic equation

$$2x^3 + 4x^2 - 10x - 5 = 0.$$

This cubic has three real roots:

$$x_1 = -3.289085012,\ x_2 = -.4396737325,\ x_3 = 1.728758744.$$

Note that x_1 does not satisfy equation (3), and so we can discount it. For each of x_2 and x_3, we get two possible solutions from equation (3), namely

$$y^2 = 5 - x_2^2 \text{ yields } y_2^+ = 2.192415793 \text{ and } y_2^- = -2.192415793,$$

while

$$y^2 = 5 - x_3^2 \text{ yields } y_3^+ = 1.4182359483 \text{ and } y_3^- = -1.418235948.$$

This gives us four points, namely

- $(x_2, y_2^+) = (-.4396737325, 2.192415793)$, at which $f(-.4396737325, 2.192415793) = -81.82297485$;
- $(x_2, y_2^-) = (-.4396737325, -2.192415793)$, at which $f(-.4396737325, 2.192415793) = -81.82297485$;
- $(x_3, y_3^+) = (1.728758744, 1.4182359483)$, at which $f(1.728758744, 1.4182359483) = -1.60698660$;
- $(x_3, y_3^-) = (1.728758744, -1.4182359483)$, at which $f(1.728758744, 1.4182359483) = -1.60698660$.

(We should check that at each of these four points, the two values of λ, namely $\lambda = \frac{3y^2}{x} - 3x$ and $\lambda = 6x - 6y^2$, are in fact equal, as we are solving for the critical points of $L(x, y, \lambda)$. Checking, we see that this is in fact the case, and so none of these four points need to be disqualified.)

Bringing all of this together, we see that for the function $f(x, y) = 6xy^2 - 2x^3 - 3y^4$ on the region

$$B = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 5\},$$

the maximum occurs at the boundary point $(-\sqrt{5}, 0)$, at which $f(\sqrt{5}, 0) = 10\sqrt{5}$, while the minimum occurs at the boundary point $(x_2, \pm y_2^+) = (-.4396737325, \pm 2.192415793)$, at which $f(-.4396737325, \pm 2.192415793) = -81.82297485$.

Chapter 2. Differentiable calculus of functions of complex variable

2.1. Introduction to complex numbers

Complex numbers historically arise from the attempt to resolve the conundrum that some quadratic equations, such as $x^2 - 1 = 0$, have two real solutions, while other equations, such as $x^2 + 1 = 0$, have no real solutions. More generally, this is the conundrum that for a polynomial $p(x) = \sum_{k=0}^{d} a_k x^k$ of degree d, the number of real solutions (counting multiplicities in the case of repeated roots, such as the root 1 having multiplicity 2 for the polynomial $x^2 - 2x + 1 = (x-1)^2$) lies somewhere between 0 and d, and for any $d \geq 1$ it is possible to construct examples illustrating each of these possibilities.

To this end, we introduce the new quantity $i = \sqrt{-1}$ (so that $i^2 = -1$). Using i, the equation $x^2 + 1 = 0$ has 2 solutions, namely $x = \pm i$, since we can write

$$x^2 + 1 = (x - i)(x + i).$$

The standard quadratic formula for the solutions of a quadratic equation still holds, so that the equation

$$x^2 - 2x + 2 = 0$$

has solutions, namely

$$x = \frac{2 \pm \sqrt{4-8}}{2} = \frac{2 \pm \sqrt{4(-1)}}{2} = \frac{2 \pm \sqrt{4}\sqrt{-1}}{2} = \frac{2 \pm 2i}{2} = 1 \pm i$$

In general, the equation $\alpha x^2 + \beta x + c = 0$, with $\alpha \neq 0$, has solutions

$$x = \frac{-\beta \pm \sqrt{\beta^2 - 4\alpha c}}{2\alpha},$$

where we allow the discriminant $\beta^2 - 4\alpha c$ to be negative. (In the case that the discriminant $\beta^2 - 4\alpha c = 0$, we have that $-\frac{\beta}{2\alpha}$ is a root of multiplicity 2.)

Definition 2.1.1. Numbers of the form $a + bi$, where a and b are themselves real numbers, are called *complex numbers*, and the set of all complex numbers is denoted by $\mathbb{C}$.

To any complex number $z = x + iy$, we can naturally associate two real quantities, its *real part* $\mathrm{Re}(z) = x$ and its *imaginary part* $\mathrm{Im}(z) = y$.

For instance, the complex number $z = 2 + 3i$ has real part $\mathrm{Re}(z) = 2$ and imaginary part $\mathrm{Im}(z) = 3$. Please note that the imaginary part of $z = 2 + 3i$ is **not** $3i$,. Rather, the real and imaginary parts of a complex number are themselves real numbers.

A complex number z with $\mathrm{Im}(z) = 0$ is *real* (and so the real number $\mathbb{R}$ lie naturally as a subset inside the complex numbers $\mathbb{C}$), while a complex number z with $\mathrm{Re}(z) = 0$, such as $z = 2i$, is *purely imaginary.* In fact, the complex numbers are an augmentation of the real numbers by the inclusion of i, together with the condition that the arithmetic of i behaves well with respect to the standard arithmetic of real numbers.

We add or subtract complex numbers by adding or subtracting (respectively) real and imaginary parts respectively, for instance

Example 2.1.2.

$$(2 - 3i) + (4 + 5i) = (2 + 4) + (-3 + 5)i = 6 + 2i$$

and

$$(-3 + 4i) - (2 + 5i) = (-3 - 2) + (4 - 5)i = -5 - i.$$

Multiplication is slightly more work, but again is straightforward. As noted above, the same commutative, associative, and distributive laws hold for the complex numbers as for real numbers, and using $i^2 = -1$, we proceed naively forth, so for instance

Example 2.1.3.

$$(2 + 3i)(4 - 5i) = (2)(4) + (3i)(4) + 2(-5i) + 3i(-5i) =$$

$$= 8 + 12i - 10i - 15i^2 = 8 + 2i - 10(-1) = 18 + 2i.$$

In general,

$$(x + iy) + (x' + y'i) = (x + x') + (y + y')i$$

and

$$(x + iy)(x' + y'i) = (xx' - yy') + (xy' + x'y)i$$

Implicit in this discussion is the observation that, should we wish to do so, we can view the complex numbers $\mathbb{C}$ as a vector space over the real numbers $\mathbb{R}$ with basis $\{1, i\}$, and addition in $\mathbb{C}$ as defined above is then just the standard vector addition in $\mathbb{C}$ as a vector space over $\mathbb{R}$. This is not an observation we will make use of to any significant extent. Recasting this discussion, we have the following lemma, which expresses what we already have seen about the addition and multiplication of complex numbers, but phrased in terms of their real and imaginary parts.

Lemma 2.1.4. *Let z and w be complex numbers. Then*

(i) $\mathrm{Re}(z + w) = \mathrm{Re}(z) + \mathrm{Re}(w)$ *and* $\mathrm{Im}(z + w) = \mathrm{Im}(z) + \mathrm{Im}(w)$*;*

(ii) $\mathrm{Re}(zw) = \mathrm{Re}(z)\mathrm{Re}(w) - \mathrm{Im}(z)\mathrm{Im}(w)$ *and* $\mathrm{Im}(zw) = \mathrm{Re}(z)\mathrm{Im}(w) + \mathrm{Re}(w)\mathrm{Im}(z)$*.*

Every complex number $z = x + iy$ has a *complex conjugate* $\overline{z}$, defined by $\overline{z} = x - iy$. Note that complex conjugation is an involution, in that doing it twice returns us to where we began, so that $\overline{\overline{z}} = z$. Two complex numbers are *complex conjugates* if each is the complex conjugate of the other; hence, $2 + 3i$ and $2 - 3i$ are complex conjugates of one another.

Related to the complex conjugate is the *norm* or *modulus* $|z|$ of the complex number z, where $|z|$ is defined as the nonnegative real number

$$|z| = \sqrt{z\overline{z}} = \sqrt{\mathrm{Re}(z)^2 + \mathrm{Im}(z)^2}$$

The complex conjugate behaves well with respect to the basic operations of addition and multiplication of complex numbers. Some of the more useful properties are given below, with an indication of how they can be shown to hold.

Lemma 2.1.5. *Let z and w be complex numbers. We then have that*

(i) $\overline{z + w} = \overline{z} + \overline{w}$*;*

(ii) $\overline{zw} = \overline{z} \cdot \overline{w}$*;*

(iii) $\overline{z^n} = (\overline{z})^n$ *for* $n \in \mathbb{Z}$*;*

(iv) $z\overline{z} = (\mathrm{Re}(z))^2 + (\mathrm{Im}(z))^2 = |z|^2$*;*

(v) $\frac{z+\overline{z}}{2} = \mathrm{Re}(z)$*, and* $\frac{z-\overline{z}}{2} = i\mathrm{Im}(z)$*;*

(vi) $\overline{z} = z$ *if and only if* $z \in \mathbb{R}$*.*

Proof. The proofs of these facts are by direct calculation.

(i). We calculate, so that

$$\begin{aligned}
\overline{z + w} &= \overline{(\mathrm{Re}(z) + \mathrm{Im}(z)i) + (\mathrm{Re}(w) + \mathrm{Im}(w)i)} \\
&= \overline{\mathrm{Re}(z) + \mathrm{Re}(w) + (\mathrm{Im}(z) + \mathrm{Im}(w))i} \\
&= \mathrm{Re}(z) + \mathrm{Re}(w) - (\mathrm{Im}(z) + \mathrm{Im}(w))i \\
&= \mathrm{Re}(z) - \mathrm{Im}(z)i + \mathrm{Re}(w) - \mathrm{Im}(w)i \\
&= \overline{z} + \overline{w}
\end{aligned}$$

(ii). This is a direct calculation which we leave to the reader.

(iii). This is a direct calculation using induction.

(iv). Here we have
$z\overline{z} = (\mathrm{Re}(z) + i\mathrm{Im}(z))(\mathrm{Re}(z) - i\mathrm{Im}(z)) = (\mathrm{Re}(z))^2 + (\mathrm{Im}(z))^2$.

(v). This is a again a direct calculation whose prove will leave to the reader.

(vi). If $z \in \mathbb{R}$, then $\mathrm{Im}(z) = 0$ and hence $\overline{z} = z$. In the other direction, if $\overline{z} = z$, then

$$\mathrm{Re}(z) - i\mathrm{Im}(z) = \mathrm{Re}(z) + i\mathrm{Im}(z),$$

and subtracting the right hand side from the left hand side, we see that $\mathrm{Im}(z) = 0$.

□

Example 2.1.6. We use the properties of the complex conjugate in the arithmetic of dividing complex numbers. To whit, if z and w are complex numbers with $w \neq 0$, then

$$\frac{z}{w} = \frac{z}{w}\frac{\overline{w}}{\overline{w}} = \frac{z\overline{w}}{|w|^2} = \frac{z\overline{w}}{(\mathrm{Re}(w))^2 + (\mathrm{Im}(w))^2}.$$

To illustrate this with a specific examples, let us evaluate

$$\frac{3+4i}{2-i}.$$

Calculating, we see that

$$\frac{3+4i}{2-i} = \frac{(3+4i)(2+i)}{(2-i)(2+i)} = \frac{(3+4i)(2+i)}{5} = \frac{2}{5} + \frac{11}{5}i.$$

For another example, we calculate

$$\frac{2-3i}{1+i} = \frac{(2-3i)}{(1+i)} \cdot \frac{(1-i)}{(1-i)} = \frac{2-3i+2i-3i^2}{1-i^2} = \frac{5-i}{2} = \frac{5}{2} - \frac{1}{2}i.$$

The fact that complex conjugation behaves well with respect to both addition and multiplication has the following interesting consequence.

Lemma 2.1.7. *Let*

$$p(z) = a_n z^n + \cdots + a_1 z + a_0$$

be a non-constant polynomial in the complex variable z with real coefficients. If z_0 is a root of $p(z)$ satisfying $z_0 \in \mathbb{C}$ and $z_0 \notin \mathbb{R}$, then $\overline{z_0}$ is also a root of $p(z)$.

As a consequence, the roots of a quadratic polynomial $p(x) = ax^2 + bx + c$ with real coefficients and no real roots are always complex conjugates. More generally, the complex (non-real) roots of a non-constant polynomial in z with real coefficients always come in complex conjugate pairs, and so every non-constant polynomial in z with real coefficients can be factored as a product of linear and quadratic factors.

Proof of Lemma 2.1.7. Since z_0 is a root of $p(z)$, we have that $p(z_0) = a_n z_0^n + \cdots + a_1 z_0 + a_0 = 0$. Taking the complex conjugate of the equation $p(z_0) = 0$, using the properties above of complex conjugation, and remembering that the coefficients $a_n, \ldots, a_0$ are real numbers, we see that

$$0 = \overline{a_n z_0^n + \cdots + a_1 z_0 + a_0} = a_n \overline{z_0}^n + \cdots + a_1 \overline{z_0} + a_0 = p(\overline{z_0}),$$

and so $\overline{z_0}$ is also a root of $p(z)$.

Calculating, we see that

$$(z - z_0)(z - \overline{z_0}) = z^2 - z_0 z - \overline{z_0} z + z_0 \overline{z_0} = z^2 - 2 \cdot \mathrm{Re}(z_0) z + |z_0|^2,$$

which is a quadratic polynomial with real coefficients. □

To see that any non-constant polynomial with real coefficients can be factored as the product of linear and quadratic factors, we need the Fundamental Theorem of Algebra:

Theorem 2.1.8 (The Fundamental Theorem of Algebra)**.** *Let $p(z)$ be a non-constant polynomial with complex coefficients. Then $p(z)$ has a complex root.*

We do not prove the Fundamental Theorem of Algebra which is outside of our scope here. Rather, let us apply it to understand how to factor polynomials.

So, let $p(z)$ be a non-constant polynomial of degree $d \geq 1$ with real coefficients. By the Fundamental Theorem of Algebra, $p(z)$ has a root z_0.

If z_0 is real, then we can factor out the linear factor $z - z_0$ from $p(z)$ to express $p(z) = (z - z_0)q(z)$, where $q(z)$ is a polynomial with complex coefficients and degree $d - 1$.

If z_0 is not real, then $\overline{z_0}$ is also a root of $p(z)$, and so we can factor out the quadratic factor

$$(z - z_0)(z - \overline{z_0}) = z^2 - 2 \cdot \mathrm{Re}(z_0)z + |z_0|^2$$

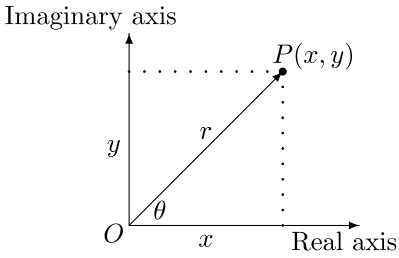
from $p(z)$ to express

$$p(z) = (z - z_0)(z - \overline{z_0})q(z),$$

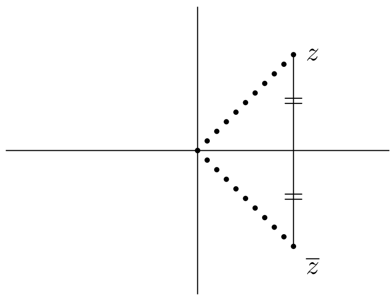
where $q(z)$ is a polynomial with complex coefficients and degree $d - 2$.

We now apply induction to see that any non-constant polynomial in z with real coefficients can be factored as a product of linear and quadratic factors.

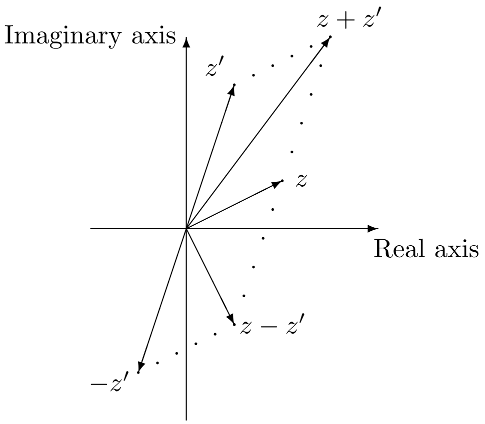
2.1.1. **Geometric representation of complex numbers.** We can represent complex numbers geometrically in the complex plane $\mathbb{C}$ using the *Argand diagram.* We associate the point (x, y) in the xy-plane with the complex number $x + iy$. Thus the x-axis is the *real* axis and the y-axis the *imaginary* axis in the complex plane. This goes back to the observation made earlier, that we can view $\mathbb{C}$ as a vector space over $\mathbb{R}$ with basis $\{1, i\}$, in which we again view the real as the horizontal and the imaginary as the vertical.



We can use the Argand diagram to give a geometric interpretation of the complex conjugate $\overline{z}$ of a complex number z as the reflection of z across the real axis $\mathbb{R}$ in $\mathbb{C}$; that is, $\overline{z}$ is the complex number on the other side of the real axis $\mathbb{R}$ from z and the same distance from the real axis as z.



Thinking of $\mathbb{C}$ as a vector space with basis 1 and i, we can identify $\mathbb{C}$ with $\mathbb{R}^2$. In this way, we can give OP a direction from O to P and identify z with the vector $\overrightarrow{OP}$. Addition in the complex plane can then be interpreted geometrically through the parallelogram law. We take the origin as a vertex of a parallelogram with adjacent sides z and z': the sum $z + z'$ is the diagonal of the parallelogram:



2.1.2. **Polar form of a complex number.** There are two natural coordinate systems on the plane, the standard cartesian coordinates x and y that we have been working with, and polar coordinates, in which we locate a point using the quantities a distance from the origin r and an angle θ measured counterclockwise from the positive real axis.

More explicitly, if we write $z = x + iy$ in terms of polar coordinates, we then have

$$x = r\cos(\theta), \qquad y = r\sin(\theta),$$

where $r = |z|$ and θ is a counterclockwise angle formed by the positive real axis and the segment OP from the origin 0 to the complex number z.

Thus, we have another way of describing complex numbers:

Definition 2.1.9. For a complex number $z = x + iy$, the *polar form* of z is given by

$$z = r(\cos(\theta) + i\sin(\theta))$$

where $x = r\cos(\theta)$, $y = r\sin(\theta)$, $r = |z|$ and θ is an angle, called *an argument of* z, measured counterclockwise from the positive real axis to the vector defined by z.

The discussion of the argument introduces some subtleties. The first of these subtleties is that the argument of a complex number is not unique. This is because both $\cos(\theta)$ and $\sin(\theta)$ are periodic with period 2π, so that $\cos(\theta + 2\pi k) = \cos(\theta)$ and $\sin(\theta + 2\pi k) = \sin(\theta)$ for any $k \in \mathbb{Z}$. Thus, the argument $\arg(z)$ of a complex number z is not a function of the sort we have dealt with before. Rather, it is a *set valued function*, as the argument $\arg(z)$ of z is

$$\arg(z) = \{\theta_0 + 2\pi k \mid k \in \mathbb{Z}\},$$

where $z = |z|(\cos(\theta_0) + i\sin(\theta_0))$ is one polar representation of z.

Write the non-zero complex number z as $z = |z|(\cos(\theta) + i\sin(\theta))$, and note that the angle θ then satisfies

$$\tan(\theta) = \frac{\mathrm{Im}(z)}{\mathrm{Re}(z)}$$

provided $\mathrm{Re}(z) \neq 0$.

However, because θ is determined only up to multiples of 2π but $\tan(x)$ is determined up to multiplies of π, we see that

$$\theta \neq \tan^{-1}\left(\frac{\mathrm{Im}(z)}{\mathrm{Re}(z)}\right),$$

as $\tan^{-1}\left(\frac{\mathrm{Im}(z)}{\mathrm{Re}(z)}\right)$ does not distinguish which quadrant of the plane contains the complex number z. That is, if we are given z in cartesian form as $z = \mathrm{Re}(z) + i\mathrm{Im}(z)$, we need not only the formula for θ in terms of $\tan^{-1}\left(\frac{\mathrm{Im}(z)}{\mathrm{Re}(z)}\right)$ but we also need the additional piece of information about the location of z in the complex plane.

Example 2.1.10. As an example of this ambiguity and how to resolve it, consider $z = 1 + i$ and $w = -z = -1 - i$. Calculating, we see that $|z| = |w| = \sqrt{2}$. With regards to arguments, though, we have that

$$\frac{\pi}{4} = \tan^{-1}\left(\frac{1}{1}\right) = \tan^{-1}\left(\frac{-1}{-1}\right),$$

even though z lies in the first quadrant and w lies in the third quadrant. Therefore, we have that

$$\arg(z) = \left\{\frac{\pi}{4} + 2\pi k \mid k \in \mathbb{Z}\right\}$$

and

$$\arg(w) = \left\{\frac{5\pi}{4} + 2\pi k \mid k \in \mathbb{Z}\right\}.$$

As this example illustrates, it is convenient to be able to pick out a single best value from $\arg(z)$. The standard approach is to take the value of $\theta \in \arg(z)$ satisfying $-\pi < \theta \leq \pi$; this value in $\arg(z)$ is called the *principal argument* of z and is denoted $\operatorname{Arg}(z)$.

- If $\operatorname{Re}(z) = 0$, then z is purely imaginary and $\operatorname{Arg}(z) = \pm\frac{\pi}{2}$ depending on the quadrant z lies in.

- If $\operatorname{Re}(z) \neq 0$, we calculate the principal argument of z as above, by taking first

$$\tan^{-1}\left(\frac{\operatorname{Im}(z)}{\operatorname{Re}(z)}\right)$$

  and then applying the geometric information of which quadrant z lies in, that is, adding or subtracting π from this angle, if necessary, to bring it into the same quadrant as z while keeping it in the range $(-\pi, \pi]$.

Note that we have

$$\arg(z) = \{\operatorname{Arg}(z) + 2\pi k \mid k \in \mathbb{Z}\}.$$

Example 2.1.11. Some examples of principal arguments are

$$\operatorname{Arg}(1 + i) = \frac{\pi}{4},$$

$$\operatorname{Arg}(i) = \frac{\pi}{2},$$

$$\operatorname{Arg}(-1) = \pi,$$

and

$$\operatorname{Arg}(-1 - i) = -\frac{3\pi}{4}$$

It is left as an exercise to check that these are correct by plotting the points in an Argand diagram.

One of the most basic formulae for working with complex numbers, particularly for working with them in polar form, is Euler's formula.

Definition 2.1.12 (Euler's formula)**.** For any real number θ, we define $e^{i\theta}$ also denoted $\exp(i\theta)$ by

$$e^{i\theta} = \exp(i\theta) = \cos(\theta) + i\sin(\theta).$$

It should be remarked that $e^{i\theta}$ is generally defined using power series. Therefore, the above expression is indeed a formula rather than a definition though its proof is outside the scope of this course.

Using Euler's formula and what we have already shown, we can express complex numbers in *exponential form*, by calculating their norm and their argument.

Example 2.1.13. For example, let $z = 1 + i\sqrt{3}$. Calculating, we see that $|1 + \sqrt{3}i| = \sqrt{1+3} = 2$ and that $\mathrm{Arg}(1 + \sqrt{3}i) = \frac{\pi}{3}$. Hence we have that

$$1 + i\sqrt{3} = 2\left(\cos\left(\frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{3}\right)\right) = 2e^{i\pi/3}.$$

The next lemma shows that $e^{i\theta}$ has all the expected properties of an exponential function.

Lemma 2.1.14. *Let $z_1 = r_1e^{i\theta_1}$ and $z_2 = r_2e^{i\theta_2}$ be two complex numbers, expressed in complex exponential form and let n be an integer. We then have*

(i) $z_1z_2 = r_1e^{i\theta_1}r_2e^{i\theta_2} = r_1r_2e^{i(\theta_1+\theta_2)}$;

(ii) $|z_1z_2| = |z_1|\,|z_2|$;

(iii) $\arg(z_1z_2) = \arg(z_1) + \arg(z_2)$;

(iv) $\frac{z_1}{z_2} = \frac{r_1e^{i\theta_1}}{r_2e^{i\theta_2}} = \frac{r_1}{r_2}e^{i(\theta_1-\theta_2)}$ *provided* $z_2 \neq 0$;

(v) $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$ *provided* $z_2 \neq 0$;

(vi) $\arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2)$ *provided* $z_2 \neq 0$;

(vii) $z_1^n = (r_1e^{i\theta_1})^n = r_1^ne^{in\theta_1}$;

(viii) $|z_1^n| = |z_1|^n$;

(ix) $\arg(z_1^n) = n \cdot \arg(z_1)$.

These statements can be verified in many ways, such as by multiplying or dividing in polar form and using addition theorems for sine and cosine.

Note that in each case, the resultant angle, such as $\arg(z_1 z_2)$ or $\arg(z^n)$, does not necessarily refer to the principal argument $\operatorname{Arg}(z_1 z_2)$ or $\operatorname{Arg}(z^n)$, respectively, but may instead equal one of many equivalent angles $\operatorname{Arg}(\cdots) + 2n\pi$. This distinction will become clearer in the next section.

Example 2.1.15. Given $z = 2 + 3i$ and $w = 3 - 4i$, calculate $\left|\frac{(2+3i)^6}{(3-4i)^4}\right|$.

Begin with $|z| = \sqrt{13}$ and $|w| = 5$ so that

$$\left|\frac{z^6}{w^4}\right| = \frac{|z|^6}{|w|^4} = \frac{\sqrt{13}^6}{5^4} = \frac{2197}{625}$$

2.2. Roots of complex numbers

The following important result will help us to find roots of complex numbers.

Theorem 2.2.1 (De Moivre's Theorem)**.** *For any real number θ and any integer n, we have*

$$(\cos(\theta) + i\sin(\theta))^n = \cos(n\theta) + i\sin(n\theta). \tag{12}$$

Proof. Let $z = \cos(\theta) + i\sin(\theta)$. Note that $|z| = 1$ and $\theta \in \arg(z)$.

Now, by Lemma 2.1.14, we have that $|z^n| = |z|^n = 1$ and $n\theta \in n\arg(z) = \arg(z^n)$. This shows that $z^n = \cos(n\theta) + i\sin(n\theta)$. □

Example 2.2.2. Find the n^{th} roots of unity.

We need to find the solutions to the equation $z^n = 1$. Start with

$$z = r(\cos(\theta) + i\sin(\theta)), \text{ so that } z^n = r^n(\cos(n\theta) + i\sin(n\theta)).$$

Taking the modulus of both sides of $z^n = 1$ yields $r^n = 1$, which in turn gives $r = 1$. Taking real parts of $z^n = 1$ then yields that $\cos(n\theta) = 1$. This in its turn yields that $\theta = 0, \frac{2\pi}{n}, \frac{4\pi}{n}, ..., \frac{(2n-2)\pi}{n}$.

(Note that by taking imaginary parts of both sides of $z^n = 1$, we see that $\sin(n\theta) = 0$, which yields a larger set of possible angles, namely $\theta = \frac{k\pi}{n}$ for $k = 0, \ldots, n-1$. However, not all of these angles are possible, as only some of them satisfy the equation $\cos(n\theta) = 1$ and the rest satisfy $\cos(n\theta) = -1$.)

Since

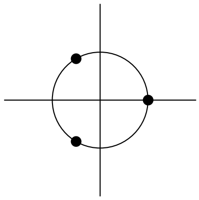
$$\cos\left(\frac{2k\pi}{n}\right) + i\sin\left(\frac{2k\pi}{n}\right) k = 0, 1, ...n-1,$$

are n distinct solutions of the equation $z^n - 1 = 0$, we conclude that they are all of the n^{th} roots of unity.

Example 2.2.3. For a specific example let us take $n = 3$. The roots of $z^3 = 1$ are

$$1, \quad \cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right) = \frac{-1+i\sqrt{3}}{2}, \text{ and } \cos\left(\frac{4\pi}{3}\right) + i\sin\left(\frac{4\pi}{3}\right) = \frac{-1-i\sqrt{3}}{2}$$

The three points lie at the vertices of an equilateral triangle on the unit circle in the complex plane:



In general the n-th roots of unity lie at the vertices of a regular n-gon inscribed in the unit circle, with one vertex at $(1, 0)$.

This technique can be applied to any complex number. Consider the following example.

Example 2.2.4. Given $z = -\frac{1}{2} + \frac{1}{2}i$, evaluate $z^{1/3}$.

For $z^{1/3}$, a third root, three solutions are expected.

We need the modulus r and the principal argument θ of z:

$$r^2 = \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}, \quad \rightarrow \quad r = \frac{1}{\sqrt{2}}.$$

To find the argument we begin with

$$\tan^{-1}\left(\frac{1/2}{-1/2}\right) = \tan^{-1}(-1) = -\frac{\pi}{4},$$

but in this case, crude application of the arctan function is misleading as z is in the second quadrant. Recall that

$$\tan\left(-\frac{\pi}{4}\right) = \tan\left(\frac{3\pi}{4}\right) = -1$$

and we need

$$\theta = \text{Arg}(z) = \frac{3\pi}{4}.$$

Therefore

$$z = -\frac{1}{2} + \frac{1}{2}i = \frac{1}{\sqrt{2}}\left\{\cos\left(\frac{3\pi}{4}\right) + i\sin\left(\frac{3\pi}{4}\right)\right\}$$

$$= \quad 2^{-1/2}\left\{\cos\left(\frac{3\pi}{4}+2n\pi\right)+i\sin\left(\frac{3\pi}{4}+2n\pi\right)\right\}$$

Then

$$z^{1/3} \quad = \quad \left(2^{-1/2}\right)^{1/3}\left\{\cos\left(\frac{\pi}{4}+\frac{2n\pi}{3}\right)+i\sin\left(\frac{\pi}{4}+\frac{2n\pi}{3}\right)\right\}, \qquad n=0,1,2$$

Hence the roots are

$$2^{-1/6}\left(\cos\tfrac{\pi}{4}+i\sin\tfrac{\pi}{4}\right), \quad 2^{-1/6}\left(\cos\tfrac{11\pi}{12}+i\sin\tfrac{11\pi}{12}\right), \quad 2^{-1/6}\left(\cos\tfrac{19\pi}{12}+i\sin\tfrac{19\pi}{12}\right)$$

The third argument lies outside the required range and so must be modified by adding or subtracting integral multiples of 2π. In this case it is necessary to subtract 2π making the third solution

$$2^{-1/6}\left\{\cos\left(-\frac{5\pi}{12}\right)+i\sin\left(-\frac{5\pi}{12}\right)\right\}.$$

2.3. Complex functions

So far we have studied functions that map a subset of $\mathbb{R}$ to a subset of $\mathbb{R}$. These were functions of one variable which we denoted by $f : D \subseteq \mathbb{R} \to \mathbb{R}$. More generally, we also considered functions which mapped a subset of $\mathbb{R}^2$ to a subset of $\mathbb{R}^2$, denoted $f : D \subseteq \mathbb{R}^2 \to \mathbb{R}^2$. There is a natural way of extending this study to the set of complex numbers.

Definition 2.3.1. A *complex function* $f : D \subseteq \mathbb{C} \to \mathbb{C}$ is an assignment of a unique complex number $f(z)$ for each complex number z in D. The set D is called the *domain* of the function.

Remark 2.3.2. As for functions from $\mathbb{R}^n$ to $\mathbb{R}^m$, we suppress the information about the domain and just write $f : \mathbb{C} \to \mathbb{C}$ with the understanding that the function may not be defined on all of the complex numbers.

There are some basic functions that we use extensively in complex analysis. The first consists of the polynomials

$$p(z) = a_n z^n + \cdots + a_1 z + a_0 = \sum_{k=0}^{n} a_k z^k$$

in the variable $z = x + iy$, where $a_0, \ldots, a_n \in \mathbb{C}$. Unless otherwise noted, we adopt the convention that the leading coefficient $a_n \neq 0$.

Since $p(z)$ takes its values in $\mathbb{C}$, we can express $p(z)$ in terms of its real and imaginary parts

$$p(z) = u(x, y) + iv(x, y),$$

where $u(x, y) = \mathrm{Re}(p(z))$ and $v(x, y) = \mathrm{Im}(p(z))$ are both real-valued functions of the real variables $x = \mathrm{Re}(z)$ and $y = \mathrm{Im}(z)$.

Example 2.3.3. Let $p(z) = z^2 + 3z - 2\overline{z}$. Express $p(z)$ in terms of its real and imaginary parts.

We start by substituting $z = x + iy$ and then calculating. So,

$$\begin{aligned} p(z) &= z^2 + 3z - 2\overline{z} \\ &= (x+iy)^2 + 3(x+iy) - 2(x-iy) \\ &= x^2 - y^2 + 2ixy + 3x + 3iy - 2x + 2iy \\ &= x^2 - y^2 + x + i(2xy + 5y) \end{aligned}$$

and so $p(z) = u(x,y) + iv(x,y)$ where $u(x,y) = \text{Re}(p(z)) = x^2 - y^2 + x$ and $v(x,y) = \text{Im}(z) = 2xy + 5y$.

Another basic function is the exponential function $f(z) = e^z = \exp(z)$. Calculating using Euler's formula, and assuming that the laws of exponents hold in this case, we see that

$$\exp(z) = e^{x+iy} = e^x e^{iy} = e^x(\cos(y) + i\sin(y)) = e^x\cos(y) + ie^x\sin(y),$$

and so

$$u(x,y) = \text{Re}(\exp(z)) = e^x\cos(y) \text{ and } v(x,y) = \text{Im}(\exp(z)) = e^x\sin(y).$$

We can build the standard trigonometric functions $\cos(z)$ and $\sin(z)$ from the exponential functions. Using Euler's formula, we see that since

$$e^{i\theta} = \cos(\theta) + i\sin(\theta),$$

we have that

$$e^{-i\theta} = \cos(-\theta) + i\sin(-\theta) = \cos(\theta) - i\sin(\theta).$$

Combining these two expressions, we see that

$$\cos(\theta) = \frac{1}{2}(e^{i\theta} + e^{-i\theta}) \quad \text{and} \quad \sin(\theta) = \frac{1}{2i}(e^{i\theta} - e^{-i\theta}).$$

So, for a complex number $z = x + iy$, let us define

$$\cos(z) = \frac{1}{2}(e^{iz} + e^{-iz}) \text{ and } \sin(z) = \frac{1}{2i}(e^{iz} - e^{-iz}).$$

It is straightforward to expand the expressions for $\cos(z)$ and $\sin(z)$ into their real and imaginary parts, and to verify that the standard trigonometric identities hold under this definition. Some of these are listed here, and you are invited to check that they do indeed hold.

(1) $\cos^2(z) + \sin^2(z) = 1$;

(2) $2\cos(z)\sin(z) = \sin(2z)$;

(3) $\cos^2(z) - \sin^2(z) = \cos(2z)$;

(4) $\sin(\frac{\pi}{2} + z) = \sin(\frac{\pi}{2} - z) = \cos(z)$.

2.4. THE DERIVATIVE OF A COMPLEX-VALUED FUNCTION AND THE CAUCHY-RIEMANN EQUATIONS

Let $f(z)$ be a function of the complex variable z. As a first attempt to make sense of the derivative of a complex valued function, we use the standard definition of the derivative and just see what happens and what sense we can make of it.

Definition 2.4.1. For $z_0 \in \mathbb{C}$, we say that $f(z)$ is *differentiable at* $z_0 \in \mathbb{C}$ if the limit

$$\lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h}$$

exists. If this limit exists, we set

$$f'(z_0) = \lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h},$$

and we refer to $f'(z_0)$ as the *derivative of* $f(z)$ *at* z_0.

For this to make sense we need the analogous definition of the limit but for complex valued functions.

Definition 2.4.2. Given a complex number L. We say that

$$\lim_{z \to a} g(z) = L$$

if for every $\varepsilon > 0$ there exists $\delta > 0$ so that $0 < |z - a| < \delta$ implies that $|g(z) - L| < \varepsilon$.

Note that we are using the same form of the definition of the limit as we have before, and all the terms in it make sense. In fact this definition the limit is similar to one for a function $g : \mathbb{R}^2 \to \mathbb{R}^2$ where the norm in $\mathbb{R}^2$ is replace by the norm of a complex number (compare with Definition 1.2.10). This flexibility and breadth of application is one of the reasons that this definition of limit is so powerful.

Looking at the limit in Definition 2.4.1, we make the observation that since this limit exists for all h approaching 0 where $h \in \mathbb{C}$, this limit then necessarily still exists if we restrict the values of h that we consider. Writing $h = \alpha + i\beta$ where α, $\beta \in \mathbb{R}$, we will evaluate this limit twice, once when we restrict to h being real and a second time when we restrict to h being purely imaginary.

Restricting to h being real, we have $h = \alpha$ and so $z_0 + h = x_0 + \alpha + iy_0$. Hence, we have that

$$\begin{aligned} f'(z_0) &= \lim_{h\to 0}\frac{f(z_0+h)-f(z_0)}{h} \\ &= \lim_{\alpha\to 0}\frac{u(x_0+\alpha,y_0)+iv(x_0+\alpha,y_0)-u(x_0,y_0)-iv(x_0,y_0)}{\alpha} \\ &= \lim_{\alpha\to 0}\frac{u(x_0+\alpha,y_0)-u(x_0,y_0)}{\alpha}+i\lim_{\alpha\to 0}\frac{v(x_0+\alpha,y_0)-v(x_0,y_0)}{\alpha} \\ &= \frac{\partial u}{\partial x}(x_0,y_0)+i\frac{\partial v}{\partial x}(x_0,y_0). \end{aligned}$$

This gives one representation for $f'(z_0)$.

Restricting to h being purely imaginary, we have $h = i\beta$ and so $z_0 + h = x_0 + i(y_0 + \beta)$. Hence, we have that

$$\begin{aligned} f'(z_0) &= \lim_{h\to 0}\frac{f(z_0+h)-f(z_0)}{h} \\ &= \lim_{\beta\to 0}\frac{u(x_0,y_0+\beta)+iv(x_0,y_0+\beta)-u(x_0,y_0)-iv(x_0,y_0)}{i\beta} \\ &= \lim_{\beta\to 0}\frac{u(x_0,y_0+\beta)-u(x_0,y_0)}{i\beta}+i\lim_{\beta\to 0}\frac{v(x_0,y_0+\beta)-v(x_0,y_0)}{i\beta} \\ &= -i\lim_{\beta\to 0}\frac{u(x_0,y_0+\beta)-u(x_0,y_0)}{\beta}+\lim_{\beta\to 0}\frac{v(x_0,y_0+\beta)-v(x_0,y_0)}{\beta} \\ &= -i\frac{\partial u}{\partial y}(x_0,y_0)+\frac{\partial v}{\partial y}(x_0,y_0). \end{aligned}$$

This gives a second representation for $f'(z_0)$. (We note that there is a bit of arithmetic involving i in the middle of this calculation. This is necessary because when evaluating the limits to produce the partial derivatives with respect to y, the incremental term we add to the second argument of the function must equal exactly the denominator.)

Setting these two representations for $f'(z_0)$ equal to one another, we see that the partial derivatives of $u(x, y)$ and $v(x, y)$ are linked by two equations. We refer to these as the Cauchy-Riemann equations.

The argument we gave above to derive the Cauchy-Riemann Equations above contains the proof of the following fundamental result.

Theorem 2.4.3 (Cauchy-Riemann Theorem). *If $f(z) = u(x, y) + i\,v(x, y)$ is differentiable at $z_0 = x_0 + iy_0$, then the Cauchy-Riemann equations associated to $f(z)$ hold at $z_0 = x_0 + iy_0$. That is,*

$$\frac{\partial u}{\partial x}(x_0,y_0)=\frac{\partial v}{\partial y}(x_0,y_0) \text{ and } \frac{\partial v}{\partial x}(x_0,y_0)=-\frac{\partial u}{\partial y}(x_0,y_0).$$

As we have hinted already, given a function $f : \mathbb{C} \to \mathbb{C}$ by forgetting for a moment that the complex numbers can be multiplied we can identify $\mathbb{C}$ with the real plane $\mathbb{R}^2$ and in this way we get $f : \mathbb{R}^2 \to \mathbb{R}^2$. As a consequence of complex differentiability we saw that both partial derivatives of $u(x, y)$ and $v(x, y)$ must exist at the point (x_0, y_0). In fact more is true. It is nontrivial exercise to check that the differentiability of f as complex function at $z_0 = x_0 + iy_0$ implies its differentiability as a function $f : \mathbb{R}^2 \to \mathbb{R}^2$ at (x_0, y_0) where the derivative is of course the Jacobian matrix:

$$J_f = \begin{pmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{pmatrix} \overset{\text{Th. 2.4.3}}{=} \begin{pmatrix} \dfrac{\partial u}{\partial x} & -\dfrac{\partial v}{\partial x} \\ \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial u}{\partial x} \end{pmatrix}.$$

evaluated at (x_0, y_0).

We have the following converse to the Cauchy-Riemann Theorem.

Theorem 2.4.4 (Sufficiency Theorem)**.** *Let $f(z) = u(x, y) + iv(x, y)$. If f is differentiable at (x_0, y_0) as a function $f : \mathbb{R}^2 \to \mathbb{R}^2$ where $z_0 = x_0 + iy_0$ and if the Cauchy-Riemann equations associated to $f(z)$ hold at z_0, then $f(z)$ is differentiable at $z_0 = x_0 + iy_0$. Moreover, the derivative of $f(z)$ at z_0 is $f'(z_0) = \frac{\partial u}{\partial x}(x_0, y_0) + i\frac{\partial v}{\partial x}(x_0, y_0)$.*

Remark 2.4.5. Recall that, by Theorem 1.8.6, the function $f : \mathbb{R}^2 \to \mathbb{R}^2$ would be differentiable at (x_0, y_0) if $u(x, y)$, $v(x, y)$, $\frac{\partial u}{\partial x}(x, y)$, $\frac{\partial u}{\partial y}(x, y)$, $\frac{\partial v}{\partial x}(x, y)$, and $\frac{\partial v}{\partial y}(x, y)$ are continuous on a neighbourhood of (x_0, y_0).

Proof of Theorem 2.4.4. We need to show that

$$\lim_{h\to 0} \frac{f(z_0 + h) - f(z_0)}{h} = \frac{\partial u}{\partial x}(x_0, y_0) + i\frac{\partial v}{\partial x}(x_0, y_0).$$

Viewing $\mathbb{C}$ as a vector spaces with basis 1 and i, denote $h = s + it = \begin{pmatrix} s \\ t \end{pmatrix}$ for $s, t \in \mathbb{R}$. Then

$$\begin{aligned} \lim_{h\to 0} \frac{f(z_0 + h) - f(z_0)}{h} &= \lim_{h\to 0} \frac{f(z_0 + h) - f(z_0) - J_f h + J_f h}{h} \\ &= \lim_{h\to 0} \frac{f(z_0 + h) - f(z_0) - J_f h}{h} + \lim_{h\to 0} \frac{J_f h}{h} \\ &= \lim_{h\to 0} \frac{\overline{h}}{|h|} \cdot \left(\frac{f(z_0 + h) - f(z_0) - J_f h}{|h|} \right) + \lim_{h\to 0} \frac{J_f h}{h} \\ &= \lim_{h\to 0} \frac{J_f h}{h} \end{aligned}$$

$$= \lim_{(s+it)\to 0} \frac{\begin{pmatrix} \frac{\partial u}{\partial x} & -\frac{\partial v}{\partial x} \\ \frac{\partial v}{\partial x} & \frac{\partial u}{\partial x} \end{pmatrix} \begin{pmatrix} s \\ t \end{pmatrix}}{(s+it)}$$

$$= \lim_{(s+it)\to 0} \frac{\begin{pmatrix} \frac{\partial u}{\partial x} s - \frac{\partial v}{\partial x} t \\ \frac{\partial v}{\partial x} s + \frac{\partial u}{\partial x} t \end{pmatrix}}{(s+it)}$$

$$= \lim_{(s+it)\to 0} \frac{\left(\frac{\partial u}{\partial x} s - \frac{\partial v}{\partial x} t\right) + i\left(\frac{\partial v}{\partial x} s + \frac{\partial u}{\partial x} t\right)}{(s+it)}$$
$$= \lim_{(s+it)\to 0} \frac{\frac{\partial u}{\partial x}(s+it) + i\frac{\partial v}{\partial x}(s+it)}{(s+it)}$$
$$= \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}.$$

Here, the fourth equality follows by differentiability of f at $z_0 = (x_0, y_0)$ as a function $f : \mathbb{R}^2 \to \mathbb{R}^2$. □

As we are examining things through the lens of calculus, one of the basic questions we ask is:

Given a function $f(z)$, at which points is $f(z)$ differentiable?

The basic strategy for determining where a complex-valued function $f(z) = u(x, y) + iv(x, y)$ is differentiable has two basic steps:

(1) We first determine the points at which the Cauchy-Riemann equations hold, as these are all the the points in $\mathbb{C}$ at which $f(z)$ might be differentiable;

(2) Verify that $f(z)$ is indeed differentiable at these points, normally using either the Sufficiency Theorem (Theorem 2.4.4), but perhaps using the definition of the derivative if that is what is required.

Example 2.4.6. Determine the points of $\mathbb{C}$ at which $f(z) = \overline{z}$ is differentiable.

We start by writing $f(z) = \overline{z} = x - iy$ in terms of its real and imaginary parts, so that if we have $f(z) = u(x, y) + iv(x, y)$, we have in this case that $u(x, y) = x$ and $v(x, y) = -y$. To check where $f(z)$ might be differentiable, we first calculate the partial derivatives of $u(x, y)$ and $v(x, y)$ with respect to x and y:

$$\frac{\partial u}{\partial x} = 1 \text{ and } \frac{\partial u}{\partial y} = 0,$$

and

$$\frac{\partial v}{\partial x} = 0 \text{ and } \frac{\partial v}{\partial y} = -1.$$

Applying the Cauchy-Riemann equations yields that $f(z)$ can only be differentiable at points $z_0 = x_0 + iy_0$ where

$$\frac{\partial u}{\partial x}(x_0, y_0) = \frac{\partial v}{\partial y}(x_0, y_0) \text{ and } \frac{\partial u}{\partial y}(x_0, y_0) = -\frac{\partial v}{\partial x}(x_0, y_0).$$

However, since

$$\frac{\partial u}{\partial x}(x_0, y_0) = 1 \neq -1 = \frac{\partial v}{\partial y}(x_0, y_0)$$

for all $z_0 = x_0 + iy_0$ for our function $f(z) = \overline{z}$, we see that $f(z) = \overline{z}$ is differentiable at no point of $\mathbb{C}$.

Example 2.4.7. Let $f(z) = u(x, y) + iv(x, y)$ denote the function defined by

$$f(z) = \begin{cases} \dfrac{\overline{z}^2}{z} & \text{if } z \neq 0, \\ 0 & \text{if } z = 0. \end{cases}$$

Show that $f(z)$ satisfies the Cauchy-Riemann equations at $z = 0$ but that $f(z)$ is not differentiable there.

To see that the Cauchy-Riemann equations hold at $z = 0$, we calculate the partial derivatives of $u(x, y)$ and $v(x, y)$ with respect to both x and y at $(x, y) = (0, 0)$. We begin by rewriting $f(z)$ (for $z \neq 0$) in terms of real and imaginary parts:

$$f(z) = \frac{\overline{z}^2}{z} = \frac{\overline{z}^3}{z\overline{z}} = \frac{x^3 - 3xy^2}{x^2 + y^2} + i\frac{y^3 - 3x^2y}{x^2 + y^2} = u(x, y) + iv(x, y).$$

To evaluate the partial derivatives at $z = 0$, we use the limit definition:

$$\frac{\partial u}{\partial x}(0, 0) = \lim_{x \to 0} \frac{u(x, 0) - u(0, 0)}{x} = \lim_{x \to 0} \frac{x^3}{x^3} = 1.$$

Similarly,

$$\frac{\partial u}{\partial y}(0, 0) = \lim_{y \to 0} \frac{u(0, y) - u(0, 0)}{y} = 0,$$

$$\frac{\partial v}{\partial x}(0, 0) = \lim_{x \to 0} \frac{v(x, 0) - v(0, 0)}{x} = 0,$$

and

$$\frac{\partial v}{\partial y}(0, 0) = \lim_{y \to 0} \frac{v(0, y) - v(0, 0)}{y} = 1.$$

So, we see that

$$\frac{\partial u}{\partial x}(0, 0) = \frac{\partial v}{\partial y}(0, 0) \text{ and } \frac{\partial v}{\partial x}(0, 0) = -\frac{\partial v}{\partial x}(0, 0),$$

and so the Cauchy-Riemann equations are satisfied at $z = 0$.

To see that $f(z)$ is not differentiable at $z = 0$, we consider the definition at $z = 0$:

$$\lim_{h\to 0} \frac{f(h) - f(0)}{h} = \lim_{h\to 0} \frac{\overline{h}^2}{h^2}.$$

If we take h to be real, so that we approach 0 along the real axis, we get

$$\lim_{h\to 0} \frac{f(h) - f(0)}{h} = \lim_{h\to 0} \frac{\overline{h}^2}{h^2} = 1.$$

If we take $h = \alpha \exp(i\pi/4)$, so that we approach 0 along the line making angle $\frac{\pi}{4}$ with the positive real axis, we get

$$\lim_{h\to 0} \frac{f(h) - f(0)}{h} = \lim_{\alpha\to 0} \frac{\alpha^2 \exp(-i\pi/2)}{\alpha^2 \exp(i\pi/2)} = \lim_{\alpha\to 0+} \frac{-i}{i} = -1.$$

Hence, the limit

$$\lim_{h\to 0} \frac{f(h) - f(0)}{h}$$

does not exist, and so $f(z)$ is not differentiable at $z = 0$.

Example 2.4.8. Let $f(z)$ be a real-valued function of the complex variable $z = x + iy$ that is differentiable at every point $z_0 \in \mathbb{C}$. Then, $f(z)$ is constant.

We start by writing $f(z)$ in terms of its real and imaginary parts:

$$f(z) = f(x + iy) = u(x, y) + iv(x, y).$$

The hypotheses that $f(z)$ is real-valued then translates to $v(x, y) = 0$. Since $f(z)$ is differentiable at every point of $\mathbb{C}$, we can apply the Cauchy-Riemann equations. Since $\frac{\partial v}{\partial x} = 0$ we see that $\frac{\partial u}{\partial y} = 0$, and since $\frac{\partial v}{\partial y} = 0$ we see that $\frac{\partial u}{\partial x} = 0$.

We now solve for $u(x, y)$. Since $\frac{\partial u}{\partial y} = 0$, we integrate with respect to y to see that $u(x, y) = \varphi(x)$, which is the constant of integration with respect to y. Applying that $\frac{\partial u}{\partial x} = 0$, we see that $\varphi'(x) = 0$ and hence that $\varphi(x) = K$ is constant. Hence, $f(z) = K$ is constant as well, and we are done.

Example 2.4.9. Let $f(z) = u(x, y) + i\, v(x, y)$ be a function of the complex variable $z = x + iy$. Suppose that both $f(z)$ and the square $|f(z)|^2$ of the norm of $f(z)$ are differentiable at all points of the complex plane $\mathbb{C}$. Show that $f(z)$ is constant.

Since $f(z) = u(x, y) + i\, v(x, y)$ is differentiable at all points $z = x + iy$ of $\mathbb{C}$, we know from the Cauchy-Riemann Theorem that

$$\frac{\partial u}{\partial x}(x, y) = \frac{\partial v}{\partial y}(x, y) \text{ and } \frac{\partial u}{\partial y}(x, y) = -\frac{\partial v}{\partial x}(x, y)$$

at all points $z = x + iy$ of $\mathbb{C}$.

Similarly, since $|f(z)|^2 = u^2(x,y) + v^2(x,y)$ is differentiable at all points $z = x + iy$ of $\mathbb{C}$, we know from the Cauchy-Riemann equations applied to the function $|f(z)|^2$ that

$$\frac{\partial}{\partial x}(u^2(x,y) + v^2(x,y)) = \frac{\partial}{\partial y}(0) = 0$$

and

$$\frac{\partial}{\partial y}(u^2(x,y) + v^2(x,y)) = -\frac{\partial}{\partial x}(0) = 0$$

at all points $z = x + iy$ of $\mathbb{C}$. Expanding by using the product rule, we see that

$$2u(x,y)\frac{\partial u}{\partial x}(x,y) + 2v(x,y)\frac{\partial v}{\partial x}(x,y) = 0$$

and

$$2u(x,y)\frac{\partial u}{\partial y}(x,y) + 2v(x,y)\frac{\partial v}{\partial y}(x,y) = 0.$$

Using the relations from the differentiability of $f(z) = u(x,y) + i\,v(x,y)$ coming from the Cauchy-Riemann equations, namely that

$$\frac{\partial u}{\partial x}(x,y) = \frac{\partial v}{\partial y}(x,y) \text{ and } \frac{\partial u}{\partial y}(x,y) = -\frac{\partial v}{\partial x}(x,y),$$

we can express everything in terms of the partial derivatives of $u(x,y)$.

We now have that

$$u(x,y)\frac{\partial u}{\partial x}(x,y) - v(x,y)\frac{\partial u}{\partial y}(x,y) = 0,$$

and

$$u(x,y)\frac{\partial u}{\partial y}(x,y) + v(x,y)\frac{\partial u}{\partial x}(x,y) = 0.$$

We multiply the first equation by $u(x,y)$ and the second equation by $v(x,y)$ to obtain

$$u^2(x,y)\frac{\partial u}{\partial x}(x,y) - u(x,y)v(x,y)\frac{\partial u}{\partial y}(x,y) = 0,$$

and

$$u(x,y)v(x,y)\frac{\partial u}{\partial y}(x,y) + v^2(x,y)\frac{\partial u}{\partial x}(x,y) = 0.$$

Combining these two equations, we see that

$$(u^2(x,y) + v^2(x,y))\,\frac{\partial u}{\partial x}(x,y) = 0.$$

From this, we have either that $u^2(x,y) + v^2(x,y) = 0$, which forces both $u(x,y) = 0$ and $v(x,y) = 0$ and hence that $f(z) = u(x,y) + iv(x,y)$ is constant (and equal to 0), or that $\frac{\partial u}{\partial x}(x,y) = 0$.

Similarly, if we multiply the first equation by $v(x,y)$ and the second equation by $u(x,y)$, we can then combine the equations to obtain that

$$(u^2(x,y) + v^2(x,y))\,\frac{\partial u}{\partial y}(x,y) = 0,$$

so that either $f(z)$ is constant or $\frac{\partial u}{\partial y}(x,y) = 0$.

Since we have that $\frac{\partial u}{\partial x}(x,y) = 0$ and $\frac{\partial u}{\partial y}(x,y) = 0$ for all points $x + iy \in \mathbb{C}$, reasoning as in Example 2.4.8, we have that $u(x,y)$ and $v(x,y)$ are both constant.

Example 2.4.10. Let $u(x,y) = x^3 - 3xy^2 - 2x^2 + 2y^2 - 4$. Find a real valued function $v(x,y)$ so that $f(z) = u(x,y) + i\,v(x,y)$ is differentiable at all points $z \in \mathbb{C}$.

Since we are asked to find $v(x,y)$ so that $f(z) = u(x,y) + i\,v(x,y)$ is differentiable at all points $z \in \mathbb{C}$, we know that the Cauchy-Riemann equations need to hold at all points $z \in \mathbb{C}$. Calculating, we then see that

$$\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} = 3x^2 - 3y^2 - 4x$$

and

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = -(-6xy + 4y) = 6xy - 4y.$$

To find a function $v(x,y)$ that satisfies such conditions, we can start by taking the first equation and integrating with respect to y to get

$$v(x,y) = 3x^2y - y^3 - 4xy + \varphi(x),$$

where $\varphi(x)$ is the constant of integration with respect to y.

We now use the second equation to see that

$$6xy - 4y = \frac{\partial v}{\partial x} = 6xy - 4y + \varphi'(x).$$

Therefore, we have that $\varphi'(x) = 0$ and hence that $\varphi(x) = K$ is constant.

Putting everything together and setting $K = 0$, we then see that

$$f(z) = u(x,y) + iv(x,y) = x^3 - 3xy^2 - 2x^2 + 2y^2 - 4 + i(3x^2y - y^3 - 4xy)$$

is differentiable at all points $z \in \mathbb{C}$.

Example 2.4.11. Let $u(x,y) = x^2 + y^2$. Show that there does not exist a real valued function $v(x,y)$ so that $f(z) = u(x,y) + i\,v(x,y)$ is differentiable at all points $z \in \mathbb{C}$.

Since we are considering that $f(z) = u(x,y) + i\,v(x,y)$ is differentiable at all points $z \in \mathbb{C}$, we know that we need the Cauchy-Riemann Equations need to hold at all points $z \in \mathbb{C}$. Calculating, we then see that

$$\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} = 2x$$

and

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = -2y.$$

To find a function $v(x,y)$ that satisfies such conditions, we can start by taking the first equation and integrating with respect to y to get

$$v(x,y) = 2xy + \varphi(x),$$

where $\varphi(x)$ is the constant of integration with respect to y. As before, the second equation above then yields that

$$-2y = \frac{\partial v}{\partial x} = 2y + \varphi'(x).$$

However, in this case we then have that $-4y = \varphi'(x)$, which cannot be as $\varphi(x)$ is a function of x alone.

What this means is that the given function $u(x, y) = x^2 + y^2$ cannot be the real part of a function of a complex variable, as we are unable to find its corresponding imaginary part.

We close this section by noting that in addition to writing a function $f(z) = u(x, y) + iv(x, y)$ in terms of its real and imaginary parts, we can also write $f(z) = s(r, \theta) \exp(it(r, \theta))$ in polar form, where here $z = r \exp(i\theta)$. We have the analogue of the Cauchy-Riemann equations, namely,

$$r\frac{\partial u}{\partial r}(r_0, \theta_0) = \frac{\partial v}{\partial \theta}(r_0, \theta_0) \quad \text{and} \quad -r\frac{\partial v}{\partial r}(r_0, \theta_0) = -\frac{\partial u}{\partial \theta}(r_0, \theta_0),$$

and the Sufficiency Theorem for the polar form of $f(z)$.

Chapter 3. Integral calculus of functions of two or three variables

We were told that the area of the disc in $\mathbb{R}^2$ of radius $r > 0$ (and with any centre) is πr^2, that the volume of the ball in $\mathbb{R}^3$ of radius $r > 0$ (and with any centre) is $\frac{4}{3}\pi r^3$, that the volume of a pyramid of height h whose base has area b is $\frac{1}{3}bh$, et cetera, but we were rarely if ever told why. As a point of focus for the other side of calculus, integration, we will consider the why of such formulae.

As with differentiation, we integrate a function of several variables one variable at a time. This approach does leave us with some fundamental questions to consider, such as, to what extent, if any, does the order of integration matter or is the result independent of the order of integration. We will address these questions later. However, our focus in this chapter is on the mechanics of integration, the setting of limits, the different coordinate systems within which we can describe regions and perform the integration in question, and the change of variables formula. While integration, like differentiation, is defined in terms of limits, we do not focus on the more theoretical aspects here.

We focus in this chapter on definite integrals, which are those integrals for which we have limits at which we evaluate the integrated integrand. That is, using an example from one-variable calculus, we will work with the higher dimensional equivalent of integrals of the sort $\int_a^b f(x)\,\mathrm{d}x$ rather than indefinite integrals of the sort $\int f(x)\,\mathrm{d}x$.

3.1. Double integral over rectangle

We take as our starting point the most straightforward of cases, which is the integration of a continuous function $f(x,y)$ over a rectangle R in the plane, described in terms of the standard Cartesian coordinates x and y, where the sides of R are parallel to the coordinate axes. The first question we face is how to define the integral of $f(x,y)$ over the region R.

The answer is similar to what we do for functions of one variable. That is, we can approximate the function $f(x,y)$ by subdividing the rectangle R into smaller rectangles and restrict $f(x,y)$ to each of these rectangles.

Suppose $R = [a,b] \times [c,d]$ and we have the subdivisions of the two intervals

$$a = x_0 < x_1 < \cdots < x_m = b,$$
$$c = y_0 < y_1 < \cdots < y_n = d.$$

This gives us a *partition* of R into mn smaller rectangles R_{ij}, $0 \leq i \leq m$, $0 \leq j \leq n$ (see Figure 12) each of which has area A_{ij} given by $A_{ij} = (x_i - x_{i-1})(y_j - y_{j-1})$.

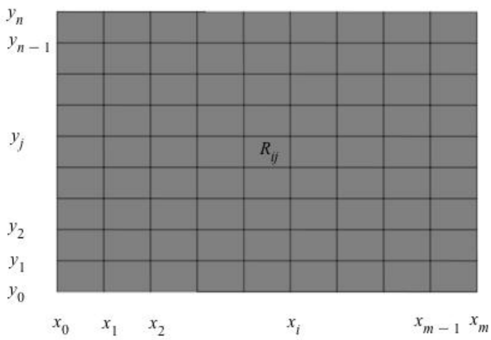


FIGURE 12. partition of R into smaller rectangles R_{ij} $(1 \leq i \leq m, 1 \leq j \leq n)$

Now, the volume under the graph of the function $f(x, y)$ (counting any volume below the xy-plane as negative) is then approximated by adding the areas of all the rectangular boxes

$$\sum_{i=1}^{m}\sum_{j=1}^{n} f(x'_i, y'_j)A_{ij}$$

where (x'_i, y'_j) is an arbitrary point in the interior of R_{ij}.

This double sum depends of course on the choice of the partition of R and on the choice of the points (x'_i, y'_j) is each smaller rectangle R_{ij}. But as we make the rectangles R_{ij} subdividing R smaller and smaller we hope that it will tend to a limit which we then call the integral of $f(x, y)$ on R.

Definition 3.1.1. A function $f : \mathbb{R}^2 \to \mathbb{R}$ is said to be *integrable* over the rectangular region $R = [a, b] \times [c, d]$ and has *double integral* denoted by $\int\int_R f(x, y)\, \mathrm{d}A$, if for every $\varepsilon > 0$, the exists a partition of R so that

$$|\sum_{i=1}^{m}\sum_{j=1}^{n} f(x'_i, y'_j)A_{ij} - \int\int_R f(x, y)\, \mathrm{d}A| < \varepsilon$$

for all choices of the points (x'_i, y'_j) in the interior of the sub-rectangles R_{ij} of R.

It is generally not so simple to check whether a given function is integrable. The following theorem allows us to circumvent this problem at least in the cases that will be of interest to us.

Theorem 3.1.2. *If $f : \mathbb{R}^2 \to \mathbb{R}$ is continuous on a rectangular region R, then it is in integrable over R.*

The next questions we face are, what notation do we use to express the integral of $f(x, y)$ over the region R, and how do we evaluate this integral?

We address the notational question first. We wish to include in our notation the information that we are integrating with respect to both variables x and y, which leaves us with the choices $\int\int_R f(x, y)\,\mathrm{d}x\,\mathrm{d}y$ and $\int\int_R f(x, y)\,\mathrm{d}y\,\mathrm{d}x$, and they are equally valid. (Here, we have in essence run into the basic issue that we do not want to yet have to specify an order of integration, as we have not yet chosen an order of integration, but given that writing is at its core a linear act, we have no choice but to put either $\mathrm{d}x$ or $\mathrm{d}y$ first.) So, the convention we use is that if we do not have limits on the individual integrals, then we accept that we have not yet chosen an order of integration. We use $\mathrm{d}A$ rather than $\mathrm{d}x\,\mathrm{d}y$ or $\mathrm{d}y\,\mathrm{d}x$ in order to decrease confusion, writing the integral as $\int\int_R f(x, y)\,\mathrm{d}A$, in those cases when we have not yet chosen an order of integration.

We now discuss the process of evaluation. Since we are focussing our attention on the mechanics of describing regions, we will by and large integrate functions which are straightforward to integrate, rather than functions for which the integration itself is the interesting or tricky part of the calculation. The *area* of a region R in the plane is just the integral over R of the constant function $f(x, y) = 1$. We will often evaluate areas of regions, rather than evaluating integrals with more complicated integrands than $f(x, y) = 1$, so that we do not confuse the issue by mixing choosing the order of integration and setting the limits, with the issue of the mechanics of integration.

By the description of R as having sides parallel to the coordinate axes, we can express R as the product $R = [a, b] \times [c, d]$, and we integrate $f(x, y)$ one variable at a time. Holding y constant and first integrating $f(x, y)$ with respect to x over the interval $[a, b]$, we obtain $\int_a^b f(x, y)\,\mathrm{d}x$, and the result of this integration is then a function of y only. (This is the integration analogue of partial differentiation.) We are now able to integrate this function $\int_a^b f(x, y)\,\mathrm{d}x$ of y with respect to y over the interval $[c, d]$ to obtain $\int_c^d \left(\int_a^b f(x, y)\,\mathrm{d}x\right)\,\mathrm{d}y$.

Alternatively, holding x constant and first integrating $f(x, y)$ with respect to y over the interval $[c, d]$, we obtain the partial integral $\int_c^d f(x, y)\,\mathrm{d}y$, which is a function of x only. We then integrate this function $\int_c^d f(x, y)\,\mathrm{d}y$ of x with respect to x over the interval $[a, b]$ to obtain $\int_a^b \left(\int_c^d f(x, y)\,\mathrm{d}y\right)\mathrm{d}x$.

Example 3.1.3. Let $R = [0, 1] \times [0, 2]$ and evaluate $\int\int_R xy\,\mathrm{d}A$.

Since there are two variables, we have two possible orders of integration: first with respect to x and then with respect to y, or first with respect to y and then with respect to x.

Start with integration first with respect to x and then with respect to y. The integral is then written and evaluated (working from the inside out) as follows:
(13)
$$\int_0^2 \int_0^1 xy\mathrm{d}x\mathrm{d}y = \int_0^2 \left(\int_0^1 xy\,\mathrm{d}x\right)\mathrm{d}y = \int_0^2 \left(\frac{1}{2}x^2y\Big|_{x=0}^{x=1}\right)\mathrm{d}y = \int_0^2 \frac{1}{2}y\mathrm{d}y = \frac{1}{4}y^2\Big|_0^2 = 1.$$
First integrating with respect to y and then with respect to x, the integral in then written and evaluated (again working from the inside out) as follows:

(14)
$$\int_0^1 \int_0^2 xy\mathrm{d}y\mathrm{d}x = \int_0^1 \left(\int_0^2 xy\,\mathrm{d}y\right)\mathrm{d}x = \int_0^1 \left(\frac{1}{2}xy^2\Big|_{y=0}^{y=2}\right)\mathrm{d}x = \int_0^1 2x\mathrm{d}x = x^2\big|_0^1 = 1.$$

When we first integrate with respect to x, as in (13) above, we are in effect breaking our rectangle up into horizontal slices. We fix a value of y satisfying $0 \leq y \leq 2$ (as these are the limits on y coming from the rectangle R), and we integrate $f(x, y)$ as a function of x along this horizontal slice, viewing y as a constant. The value of the integral along the horizontal slice depends on y, the parameter determining the horizontal slice along which we are integrating, and so we then integrate the resulting function of y. In (14), we proceed in exactly the same way, but we break the rectangle up into vertical slices instead of horizontal slices.

Not surprisingly, the evaluations of this integral in the 2 different orders of integration are the same. That this fact holds in full generality for continuous functions over rectangles is a result of Fubini.

Theorem 3.1.4. *(Fubini) Let $R = [a, b] \times [c, d]$ be a rectangle in $\mathbb{R}^2$ whose sides are parallel to the coordinate axes and let $f(x, y)$ be a function that is continuous on R. Then,*
$$\int_a^b \int_c^d f(x, y)\,\mathrm{d}y\,\mathrm{d}x = \int_c^d \int_a^b f(x, y)\,\mathrm{d}x\,\mathrm{d}y.$$

Caution: When we change the order of integration as in Example 3.1.3 above, we need also to ensure that the limits on x remain the limits on x and that the limits on y remain the limits of y.

Consider the integral
$$\int_0^2 \int_0^1 x^2y\,\mathrm{d}x\,\mathrm{d}y = \int_0^2 \left(\frac{1}{3}x^2y\Big|_{x=0}^{1}\right)\,\mathrm{d}y = \int_0^2 \frac{1}{3}y\,\mathrm{d}y = \frac{1}{6}y^2\Big|_0^2 = \frac{2}{3}.$$
If we change the order of integration, integrating first with respect to x but forget to change the limits correspondingly, we get
$$\int_0^2 \int_0^1 x^2y\,\mathrm{d}y\,\mathrm{d}x = \int_0^2 \left(\frac{1}{2}x^2y^2\Big|_{y=0}^{1}\right)\,\mathrm{d}x = \int_0^2 \frac{1}{2}x^2\,\mathrm{d}x = \frac{1}{6}x^3\Big|_0^2 = \frac{4}{3}.$$

We can be a bit more expansive in our statement of Theorem 3.1.4. Let a *generalised interval* be an interval in $\mathbb{R}$ of the form $[a,b]$, $[a,\infty)$, $(\infty,b]$ or all of $\mathbb{R}$, expressed as an interval as $(-\infty,\infty)$. We can generalise Theorem 3.1.4 as follows (assuming that you the reader have some familiarity with improper integrals).

Theorem 3.1.5. *Let $R = I \times J$ be a region in $\mathbb{R}^2$, where I and J are both generalised intervals in $\mathbb{R}$. Let $f(x,y)$ be a function that is continuous on R. If either*

$$\int_I \left(\int_J |f(x,y)|\ \mathrm{d}y \right)\ \mathrm{d}x < \infty \ \text{or} \ \int_J \left(\int_I |f(x,y)|\ \mathrm{d}x \right)\ \mathrm{d}y < \infty,$$

then

$$\int_I \left(\int_J f(x,y)\,\mathrm{d}y \right)\ \mathrm{d}x = \int_J \left(\int_I f(x,y)\,\mathrm{d}x \right)\ \mathrm{d}y = \int_{I\times J} f(x,y)\,\mathrm{d}x\,\mathrm{d}y.$$

Corollary 3.1.6. *Let $R = I \times J$ be a region in $\mathbb{R}^2$, where I and J are both generalised intervals in $\mathbb{R}$. Let $f(x,y) = g(x)h(y)$ be a function that is continuous on R. If either*

$$\int_I \left(\int_J |f(x,y)|\ \mathrm{d}y \right)\ \mathrm{d}x < \infty \ \text{or} \ \int_J \left(\int_I |f(x,y)|\ \mathrm{d}x \right)\ \mathrm{d}y < \infty,$$

then

$$\left(\int_I g(x)\,\mathrm{d}x \right) \left(\int_J h(y)\,\mathrm{d}y \right) = \int_{I\times J} f(x,y)\,\mathrm{d}x\,\mathrm{d}y = \int_{I\times J} g(x)h(y)\,\mathrm{d}x\,\mathrm{d}y.$$

3.2. Double integrals over more complicated regions

In this section, we consider integration over more complicated regions in the plane than those discussed in the previous Section 3.1. The main technical issue to be addressed is the determination of the endpoints of the vertical and horizontal slices as once these are determined, the integration itself becomes a mechanical act. As there are no new technical results that we need, we dive directly into an example.

Example 3.2.1. Let R be the triangular region in the plane $\mathbb{R}^2$ with vertices $(0,0)$, $(2,0)$ and $(1,1)$ (see Figure 13) and evaluate the integral $\int\int_R xy\,\mathrm{d}A$.

There are two possible orders of integration and we will tackle both. We start with integration first respect to x and then with respect to y. Since we will integrate with respect to y last, we have that the limits of integration for y need to be constants. They are the smallest and largest values of y for any point (x,y) in R, and so we see that $0 \le y \le 1$.

For a fixed value of y satisfying $0 \le y \le 1$, we now need to determine the limits on x. That is, for this fixed value of y, we need to know the smallest and largest values of x for points with this fixed y coordinate that lie in R.

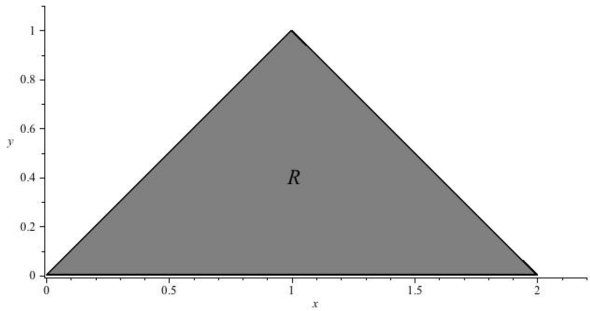


FIGURE 13. triangular region R

The slice through R at height y is a single interval, with its left endpoint on the left-hand side of R (which has equation $y = x$) and with its right-hand endpoint on the right-hand side of R (which has equation $y = 2 - x$, or equivalently, $x = 2 - y$). Hence, for this fixed value of y, the smallest value of x so that (x, y) lies in R is $x = y$, and the largest value of x so that (x, y) lies in R is $x = 2 - y$, and so $y \leq x \leq 2 - y$.

We do note here that, since we first integrate with respect to x, the limits on x can involve y, which is the other variable in the question. However, since we are last integrating with respect to y, the limits on y cannot involve the variable x but rather can involve only constants. Similarly, the integrand when we integrate with respect to x can involve both x and y in addition to constants, but the integrand of the integral with respect to y can involve only y and constants but not the variable x.

Hence, we can write the integral as

$$\begin{aligned}
\int\int_R xy \, \mathrm{d}x \, \mathrm{d}y &= \int_0^1 \int_y^{2-y} xy \, \mathrm{d}x \, \mathrm{d}y \\
&= \int_0^1 \left(\frac{1}{2} x^2 y \Big|_{x=y}^{x=2-y} \right) \mathrm{d}y \\
&= \int_0^1 \frac{1}{2} y \left((2-y)^2 - y^2 \right) \mathrm{d}y \\
&= \int_0^1 (2y - 2y^2) \, \mathrm{d}y \\
&= \left(y^2 - \frac{2}{3} y^3 \right) \Big|_0^1 = \frac{1}{3}.
\end{aligned}$$

We can now consider this integral but with the other order of integration, namely first integrate with respect to y and then integrate with respect to x.

Since we integrate with respect to x last, the limits on x are the smallest and largest values of x for any point (x, y) in R, and so $0 \le x \le 2$.

Now fix a value of x satisfying $0 \le x \le 2$ and consider the vertical slice through R passing through x on the real axis. Something different happens here than happened with the other order of integration. Namely, for $0 \le x \le 1$, the bottom of the vertical slice through R lies on the x-axis, which is the line $y = 0$, while the top of the vertical slice lies on the line $y = x$. However, for $1 \le x \le 2$, while the bottom of the vertical slice lies again on the line $y = 0$, the top now lies on the line $y = 2 - x$. That is, there are two types of vertical slices and hence we have to break up the integral in this case into two integrals. To whit,

$$\begin{aligned}
\int\int_R xy\,\mathrm{d}y\,\mathrm{d}x &= \int_0^1 \int_0^x xy\,\mathrm{d}y\,\mathrm{d}x + \int_1^2 \int_0^{2-x} xy\,\mathrm{d}y\,\mathrm{d}x \\
&= \int_0^1 \left(\frac{1}{2}xy^2 \Big|_{y=0}^{y=x} \right) \mathrm{d}x + \int_1^2 \left(\frac{1}{2}xy^2 \Big|_{y=0}^{y=2-x} \right) \mathrm{d}x \\
&= \frac{1}{2}\int_0^1 x^3\,\mathrm{d}x + \frac{1}{2}\int_1^2 x(2-x)^2\,\mathrm{d}x \\
&= \frac{1}{8}\,x^4\Big|_0^1 + \frac{1}{2}\int_1^2 (4x - 4x^2 + x^3)\,\mathrm{d}x \\
&= \frac{1}{8} + \frac{1}{2}\left(2x^2 - \frac{4}{3}x^3 + \frac{1}{4}x^4 \right)\Big|_1^2 \\
&= \frac{1}{8} + \frac{1}{2}\left(8 - \frac{32}{3} + 4 - 2 + \frac{4}{3} - \frac{1}{4} \right) = \frac{1}{3}
\end{aligned}$$

We can encapsulate the behaviour of the region R in the previous example by saying that R is *horizontally simple* but *not vertically simple.*

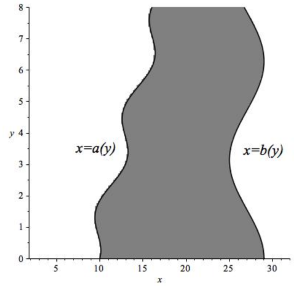


FIGURE 14. horizontally simple region

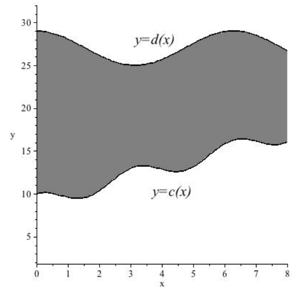


FIGURE 15. vertically simple region

Here, *horizontally simple* means that all of the horizontal slices through the region in question have the property that their left endpoints all are given by a single equation, and all of their right endpoints are given by a (different) single equation (see Figure 14).

And *vertically simple* means that all of the vertical slices through the region in question have the property that their top endpoints all are given by a single equation, and all of their bottom endpoints are given by a (different) single equation (see Figure 15).

Let us explain the general principle which allows us to compute double integrals over horizontally or vertically simple regions.

If a given double integral of a function $f(x, y)$ is over a horizontally simple region R, we can first subdivide the region into horizontal slices. Then we integrate $f(x, y)$ as a function of x along a horizontal slice, viewing y as a constant. This is the inner integral

$$\int_{a(y)}^{b(y)} f(x, y)\mathrm{d}x$$

whose limits are continuous functions of y. The resulting expression (the inner integral) becomes a function of y and we integrate now with respect to y to get

$$\int_c^d \left(\int_{a(y)}^{b(y)} f(x, y)\mathrm{d}x \right) \mathrm{d}y.$$

The limits for the this outer integral are the bounds for the horizontal slices, that is, they are the minimum and the maximum values of y on the region R: $c \leq y \leq d$. We state this as a theorem.

Theorem 3.2.2. *Suppose $f : R \to \mathbb{R}$ is a continuous function on a region $R \subseteq \mathbb{R}^2$.*

(a) If R horizontally simple region defined by by $a(y) \leq x \leq b(y)$ and $c \leq y \leq d$, then

$$\int\int_R f(x, y)\mathrm{d}A = \int_c^d \left(\int_{a(y)}^{b(y)} f(x, y)\mathrm{d}x \right) \mathrm{d}y.$$

(b) If R vertically simple region defined by $c(x) \leq y \leq d(x)$ and $a \leq x \leq b$, then

$$\int\int_R f(x, y)\mathrm{d}A = \int_a^b \left(\int_{c(x)}^{d(x)} f(x, y)\mathrm{d}y \right) \mathrm{d}x.$$

To simply notation, when there is no confusion, will the drop the parenthesis from the double integral expressions such as in Theorem 3.2.2.

A region can be both horizontally simple and vertically simple (such as a rectangle with sides parallel to the coordinate axes), horizontally simple but not vertically simple (as the region R in Example 3.2.1), vertically simple but not horizontally simple, or neither horizontally simple nor vertically simple. When integrating over a region R that is one of horizontally or vertically simple but not the other, we typically start by choosing the order of integration that respects the direction of simplicity of R, whereas if R is neither horizontally nor vertically simple, then we consider both orders of integration, write out the resulting (sums of) integrals, and choose the one that seems easier to integrate. In general there the following properties of the double integral that can assist us with computations.

Theorem 3.2.3 (Properties of the Double Integral). *Let $f : R \to \mathbb{R}$ and $g : R \to \mathbb{R}$ be continuous functions on a bounded region R of $\mathbb{R}^2$. Suppose M and N are constants. Then*

(a)

$$\int\int_R 1\mathrm{d}A = \text{ area of } R.$$

(b) More generally, if $f(x,y) \geq 0$ on R, then

$$\int\int_R f(x,y)\mathrm{d}A = \text{ volume of } S$$

where S is the solid formed below the graph of $f(x,y)$ and directly above the region R.

(c)

$$\int\int_R Mf(x,y) + Ng(x,y)\mathrm{d}A = M\int\int_R f(x,y)\mathrm{d}A + N\int\int_R f(x,y)\mathrm{d}A.$$

(d) If $R = R_1 \cup \cdots \cup R_k$ and any two R_i and R_j do not intersect for $i \neq j$, then

$$\int\int_R f(x,y)\mathrm{d}A = \sum_{i=1}^{k}\int\int_{R_i} f(x,y)\mathrm{d}A.$$

Example 3.2.4. Suppose $R = \{(x,y) \mid x^2 + y^2 \leq 1\}$ is the disc of radius 1 centred at the origin. Compute the integral

$$\int\int_R 3\sqrt{1 - x^2 - y^2}\mathrm{d}A.$$

Note that the graph of the function

$$f(x,y) = \sqrt{1 - x^2 - y^2}$$

on R is the upper hemisphere of radius 1. Since the volume of the sphere of radius 1 is $\frac{4\pi}{3}$, the hemisphere has volume $\frac{2\pi}{3}$. Now, we see that

$$\int\int_R 3\sqrt{1 - x^2 - y^2}\mathrm{d}A = 3\int\int_R \sqrt{1 - x^2 - y^2}\mathrm{d}A = 3 \cdot \frac{2\pi}{3} = 2\pi.$$

Example 3.2.5. Let R be the square bounded by the lines $x + y = 1$, $x + y = -1$, $x - y = 1$, and $x - y = -1$. Evaluate

$$\int\int_R (3xy + 2x)\mathrm{d}A.$$

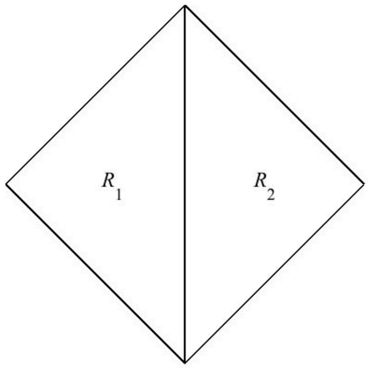


FIGURE 16. square region R subdivided into two triangles

Note that we can write R as the union of of the two triangles R_1 and R_2 bounded respectively by the lines $x-y=-1$, $x+y=-1$, $x=0$ and $x+y=1$, $x-y=1$, $x=0$ (see Figure 16). Since the intersection of these triangles is a line segment on $x=0$, it contributes zero to the integral. So, we can write

$$\int\int_R (3xy+2x)\mathrm{d}A = \int\int_{R_1} (3xy+2x)\mathrm{d}A + \int\int_{R_2} (3xy+2x)\mathrm{d}A.$$

Note that both triangles are vertically simple regions where R_1 is bounded above by $x-y=-1$ and below by $x+y=-1$ and R_2 bounded above by $x+y=1$ and below by $x-y=1$. This give us

$$\int\int_R (3xy+2x)\mathrm{d}A = \int_{x=-1}^{x=0}\int_{y=-x-1}^{y=x+1} (3xy+2x)\mathrm{d}y\ \mathrm{d}x + \int_{x=0}^{x=1}\int_{y=x-1}^{y=-x+1} (3xy+2x)\mathrm{d}y\ \mathrm{d}x.$$

Now, we get

$$\int_{x=-1}^{x=0}\int_{y=-x-1}^{y=x+1} (3xy+2x)\mathrm{d}y\ \mathrm{d}x =$$

$$= \int_{x=-1}^{x=0}\int_{y=-x-1}^{y=x+1} 3xy\mathrm{d}y\ \mathrm{d}x + \int_{x=-1}^{x=0}\int_{y=-x-1}^{y=x+1} 2x\mathrm{d}y\ \mathrm{d}x$$

$$= \int_{x=-1}^{x=0} \frac{3}{2}xy^2\Big|_{y=-x-1}^{y=x+1} \mathrm{d}x + \int_{x=-1}^{x=0} 2xy\Big|_{y=-x-1}^{y=x+1} \mathrm{d}x$$

$$= 0 + \int_{x=-1}^{x=0} (4x^2+4x)\ \mathrm{d}x$$

$$= (4x^2+4x)\Big|_{x=-1}^{x=0} = 0.$$

Similarly, we have

$$\int_{x=0}^{x=1}\int_{y=x-1}^{y=-x+1} (3xy+2x)\mathrm{d}y\ \mathrm{d}x =$$

$$= \int_{x=0}^{x=1} \int_{y=x-1}^{y=-x+1} 3xy\mathrm{d}y\ \mathrm{d}x + \int_{x=0}^{x=1} \int_{y=x-1}^{y=-x+1} 2x\mathrm{d}y\ \mathrm{d}x$$

$$= \int_{x=0}^{x=1} \frac{3}{2}xy^2\Big|_{y=x-1}^{y=-x+1} \mathrm{d}x + \int_{x=0}^{x=1} 2xy\Big|_{y=x-1}^{y=-x+1} \mathrm{d}x$$

$$= 0 + \int_{x=0}^{x=1} (-4x^2 + 4x)\ \mathrm{d}x$$

$$= (-4x^2 + 4x)\Big|_{x=0}^{x=1} = 0.$$

This shows that $\int\int_R (3xy + 2x)\mathrm{d}A = 0$.

There are also situations in which we need to change the order of integration in order to be able to evaluate the integral at all. For this consider the following example.

Example 3.2.6. Change the order of integration and evaluate the integral $\int_0^{\sqrt{\pi}} \int_y^{\sqrt{\pi}} \sin(x^2)\ \mathrm{d}x\ \mathrm{d}y$.

The triangular region over which we are integrating is bounded by the lines $y = 0$, $x = \sqrt{\pi}$ and $y = x$. With the order of integration $\mathrm{d}x\ \mathrm{d}y$, we are using horizontal slices. Moving to vertical slices, we get the integral

$$\begin{aligned} \int_0^{\sqrt{\pi}} \int_y^{\sqrt{\pi}} \sin(x^2)\ \mathrm{d}x\ \mathrm{d}y &= \int_0^{\sqrt{\pi}} \int_0^{x} \sin(x^2)\ \mathrm{d}y\ \mathrm{d}x \\ &= \int_0^{\sqrt{\pi}} x\sin(x^2)\ \mathrm{d}x \\ &= -\frac{1}{2}\cos(x^2)\Big|_0^{\sqrt{\pi}} = 1. \end{aligned}$$

3.3. Triple integration

One of the points that we made above for regions in $\mathbb{R}^2$, particularly with Fubini's theorem under our belt, is that the actual integration possesses the same sorts of difficulties as the integration over intervals in $\mathbb{R}$. Hence, the main difficulty in integration over regions in the plane is setting up the integral in the first place and determining the limits of integration.

The same holds in $\mathbb{R}^3$. The essential difficulties do not arise in actually evaluating the integrals, but rather in setting them up, and in particular in determining the limits of integration.

As with the notation of $\mathrm{d}A$ used for integration over regions in the plane, we sometimes use $\mathrm{d}V$ as the volume form for integration over regions in $\mathbb{R}^3$

where we have not yet chosen an order of integration. We note here, and will note later as well, that implicit in our use of dA or dV is the understanding of which set of coordinates we are using.

As with integration over regions in the plane, we begin this section with an example.

Example 3.3.1. Let T be the tetrahedron in $\mathbb{R}^3$ bounded by the planes $\{x = 0\}$, $\{y = 0\}$, $\{z = 0\}$ and $\{x + y + z = 1\}$ (see Figure 17). Evaluate $\int\int\int_T \mathrm{d}V$.

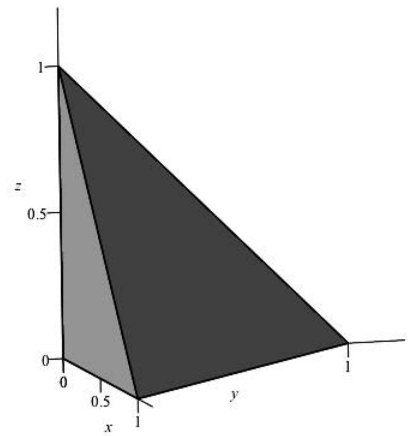


FIGURE 17. tetrahedron T

The first thing to do is to choose an order of integration. Of the 6 possible orders of integration with 3 variables, we choose to first integrate with respect to x, then z, and then y. So we need to find the appropriate limits.

We start by looking at the last variable with respect to which we are integrating, for the following reason. The limits of integration corresponding to a variable can contain only constants and the variables which get integrated after. So, since we have chosen to integrate first with respect to x, the limits of integration on x can involve constants and both y and z. Since we integrate with respect to z after we have integrated with respect to x, the limits on z can involve constants and y, but they cannot involve x, as we have already integrated with respect to x. Finally, we integrate with respect to y, and the limits on y can involve only constants and not the variables x and z, as we have already integrated out all terms involving on x and z.

So, we start from the outside and work inwards, and so we start with the limits on y. Since T is bounded by the plane $\{y = 0\}$, which is the xz-plane, we see that either $y \geq 0$ or $y \leq 0$ for all points (x, y, z) in T. The geometry of the situation dictates that $y \geq 0$ for all points in T. Similarly, we can see that $x \geq 0$ and $z \geq 0$ for all points in T. Hence, the largest possible value of y for any point in T is $y = 1$, which occurs at the point $(0, 1, 0)$.

We now fix an arbitrary value of y satisfying $0 \leq y \leq 1$ and we slice T with the plane parallel to the xz-plane corresponding to this value of y. This yields a triangle whose sides lie on the planes $\{x = 0\}$, $\{z = 0\}$, and $\{x + z = 1 - y\}$. For this last plane, we have moved y to the right-hand side, to remind ourselves that at this point in the argument, we are considering y to be a constant.

Draw the triangle with the line $x = 0$ along the left hand side, the line $z = 0$ along the bottom, and the hypotenuse is $x + z = 1 - y$. Since we are integrating with respect to x and then with respect to z, we are allowing x to vary and we are holding z constant, and this yields that we are working with horizontal slices of our triangle. (And we have thus reduced this question to the case of integrating over regions in the plane considered in the previous section.)

Viewing z as fixed, we can see that $0 \leq x \leq 1 - y - z$, and that the largest value that z can take, at the upper vertex of the triangle, occurs where the lines $x = 0$ and $x + z = 1 - y$ intersect, which occurs at $z = 1 - y$. Therefore, we see that

$$\begin{aligned}
\int\int\int_T \mathrm{d}V &= \int_0^1 \int_0^{1-y} \int_0^{1-y-z} \mathrm{d}x\,\mathrm{d}z\,\mathrm{d}y \\
&= \int_0^1 \int_0^{1-y} \left(x\Big|_{x=0}^{1-y-z} \right) \mathrm{d}z\,\mathrm{d}y \\
&= \int_0^1 \int_0^{1-y} (1 - y - z)\,\mathrm{d}z\,\mathrm{d}y \\
&= \int_0^1 \left(\left(z(1-y) - \frac{1}{2}z^2 \right) \Bigg|_{z=0}^{z=1-y} \right) \mathrm{d}y \\
&= \int_0^1 \frac{1}{2}(1-y)^2\,\mathrm{d}y = -\frac{1}{6}\,(1-y)^3\Big|_0^1 = \frac{1}{6}.
\end{aligned}$$

Example 3.3.2. Let T be the tetrahedron bounded by the planes $\{x = 1\}$, $\{y = 1\}$, $\{z = 1\}$ and $\{x + y + z = 4\}$. Evaluate $\int\int\int_T x\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$.

We first choose an order of integration. There is an inherent symmetry in the question, in that we have three of the bounding planes of T each being defined in terms of a single variable being constant, and the fourth plane being defined using all of the variables. Particularly given that the integrand is a polynomial in x, y and z, all choices of order of integration have equal levels of difficulty.

So, we integrate with respect to z last. The minimum value of z over all points (x, y, z) in T is 1 and the maximum value of z is 2, because we have that $x \geq 1$ and $y \geq 1$ for all points in T and hence we have that $z = 4 - x - y \leq 2$. So, we have that $1 \leq z \leq 2$.

Consider a horizontal slice of T for some constant value of z. In this slice, we have that $x \geq 1$ and $y \geq 1$ and that $x + y \leq 4 - z$. We integrate with respect to x first and with respect to y second. So, we choose a value of y in the range $1 \leq y \leq 3 - z$, and we finally have that $1 \leq x \leq 4 - y - z$. Therefore, we have that

$$\begin{aligned}\int\int\int x \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z &= \int_1^2 \int_1^{3-z} \int_1^{4-y-z} x \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \\ &= \int_1^2 \int_1^{3-z} \int_1^{4-y-z} x \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\end{aligned}$$

Evaluating, we see that

$$\begin{aligned}\int_1^2 \int_1^{3-z} \int_1^{4-y-z} x \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z &= \int_1^2 \int_1^{3-z} \left.\frac{x^2}{2}\right|_{x=1}^{x=4-y-z} \mathrm{d}y\,\mathrm{d}z \\ &= \frac{1}{2}\int_1^2 \int_1^{3-z} \left((4-y-z)^2 - 1\right) \mathrm{d}y\,\mathrm{d}z \\ &= -\frac{1}{6}\int_1^2 \left((4-y-z)^3 + 3y\right)\Big|_{y=1}^{y=3-z} \mathrm{d}z \\ &= -\frac{1}{6}\int_1^2 \left(-2 + 3(3-z) - (3-z)^3\right) \mathrm{d}z = \frac{5}{24}.\end{aligned}$$

All the properties of double integrals stated in Sections 3.1 and 3.2 have analogues for triple integrals. There are straightforward generalisations of Fubini's Theorem 3.1.4 to triple integrals over rectangular boxes, of Theorem 3.2.3 (a), namely, the volume formula:

$$\int\int\int_R 1 \mathrm{d}V = \text{ volume of } R,$$

and of the other properties (b), (c), and (d).

3.4. Change of variables and Jacobians

We have already seen for integrals of functions of one variable what is the effect of changing variables. This is integration by substitution. To take an example, let $[a, b]$ be a closed interval in the real line $\mathbb{R}$ and let $f : [a, b] \to \mathbb{R}$ be a continuous function (this is a stronger condition than we actually need), then $\int_a^b f(x)\,\mathrm{d}x$ exists. Let $[c, d]$ be another interval in $\mathbb{R}$ and let $g : [c, d] \to [a, b]$ be a bijective continuous function that is differentiable on (c, d).

Let t be the coordinate on $[c, d]$ and x the coordinate on $[a, b]$, so that we can make the substitution $x = g(t)$. Since $g : [c, d] \to [a, b]$ is injective, we understand the behaviour of g on the endpoints of these 2 intervals and on the interior. One possibility is that $g(c) = a$ and $g(d) = b$, in which case

$g'(t) \geq 0$ for all $c < t < d$; the other possibility is that $g(c) = b$ and $g(d) = a$, in which case $g'(t) \leq 0$ for all $c < t < d$. Here, we make use of the observation that if the derivative $g'(t)$ changes sign anywhere in (a, b), then $g : [a, b] \to [c, d]$ cannot be injective.

In the former case, we make the substitution $x = g(t)$ and rewrite $\int_a^b f(x)\,\mathrm{d}x$ as

$$\int_a^b f(x)\,\mathrm{d}x = \int_c^d f(g(t))\,g'(t)\,\mathrm{d}t = \int_c^d f(g(t))\,|g'(t)|\,\mathrm{d}t$$

since $g'(t) = |g'(t)|$ in this case.

In the latter case, we make the substitution $x = g(t)$ and rewrite $\int_a^b f(x)\,\mathrm{d}x$ as

$$\int_a^b f(x)\,\mathrm{d}x = \int_d^c f(g(t))\,g'(t)\,\mathrm{d}t = \int_c^d f(g(t))\,|g'(t)|\,\mathrm{d}t$$

using that $|g'(t)| = -g'(t)$ and $\int_c^d h(t)\,\mathrm{d}t = -\int_d^c h(t)\,\mathrm{d}t$ in this case. Note that in both cases, we get the same relationship, namely

$$\boxed{\int_a^b f(x)\,\mathrm{d}x = \int_c^d f(g(t))\,|g'(t)|\,\mathrm{d}t}$$

The change of variables formula gives us the equivalent statement for changes of variables, or synonymously for changes of coordinates, for integrals over regions in $\mathbb{R}^2$ and $\mathbb{R}^3$.

As before, we start in the plane and give the general statement of the change of variables formula. Let $\mathbb{R}^2_{(x,y)}$ be the plane $\mathbb{R}^2$ with coordinates (x, y), and let $\mathbb{R}^2_{(s,t)}$ be the plane with coordinates (s, t). For the moment, we make no assumption of what these coordinates are, only that (as is the case with both cartesian and polar coordinates on the plane) we are able to (almost uniquely) locate a point by an ordered pair of coordinates and that the coordinates each take values in $\mathbb{R}$. We would like to impose the condition that points are determined by unique ordered pairs of coordinates, which is the case for cartesian coordinates, but not for polar coordinates, as it is not possible to locate the origin uniquely in polar coordinates.

Next, we would like a map $F : \mathbb{R}^2_{(s,t)} \to \mathbb{R}^2_{(x,y)}$ given by $F(s, t) = (x(s, t), y(s, t))$ that allows us to relate the two systems of coordinates from a region S in the (s, t)-plane to the region $R = F(S)$ in the (x, y)-plane.

We now bring the Jacobian matrix back into the picture. The Jacobian matrix is the natural notion of the derivative for the function $F : \mathbb{R}^2_{(s,t)} \to \mathbb{R}^2_{(x,y)}$, but the integrand of the integrals need to be functions into $\mathbb{R}$ and not functions into $\mathbb{R}^2$. We rectify this by taking the determinant. As a standard piece of notation, we write $F(s, t) = (x(s, t), y(s, t)) = (x, y)$ and set

$$\frac{\partial(x, y)}{\partial(s, t)} = \det(J_F(s, t)).$$

Definition 3.4.1. A map $F : \mathbb{R}^2_{(s,t)} \to \mathbb{R}^2_{(x,y)}$ given by $F(s,t) = (x(s,t), y(s,t))$ is called a *change of coordinates map* if it is injective in the interior $\text{int}(S)$ of S (that is, we allow the possibility that F identifies points on the boundary ∂S of S but not in the interior $\text{int}(S)$ of S) and if the Jacobian matrix of $F(s,t)$ is continuous and $\det(J_F(s,t)) \neq 0$ in S.

Theorem 3.4.2 (Change of Variables planar case)**.** *Let $F : \mathbb{R}^2_{(s,t)} \to \mathbb{R}^2_{(x,y)}$ given by $F(s,t) = (x(s,t), y(s,t))$ be a change of coordinates map between $\mathbb{R}^2_{(s,t)}$ and $\mathbb{R}^2_{(x,y)}$ on S. Let $R = F(S)$ and suppose $f : R \to \mathbb{R}$ is an integrable function. We then have that $f \circ F : S \to \mathbb{R}$ is an integrable function and*

$$\int\int_R f(x,y)\,\mathrm{d}x\,\mathrm{d}y = \int\int_S (f \circ F)(s,t) \left|\frac{\partial(x,y)}{\partial(s,t)}\right| \mathrm{d}s\,\mathrm{d}t.$$

Sketch of Proof. Let us assume for simplicity that the regions of integration are rectangles R and S and both (x,y)- and (s,t)-coordinate systems are rectangular.

The double integral $\int\int_R f(x,y)\,\mathrm{d}A$ is a limit of summations of the volumes of the rectangular boxes over partitions of R (see Definition 3.1.1). So, we can make the approximation

$$\int\int_R f(x,y)\,\mathrm{d}A \approx \sum_{i=1}^{m}\sum_{j=1}^{n} f(x_i', y_j')A_{ij} = \sum_{i=1}^{m}\sum_{j=1}^{n} f(x_i', y_j')\Delta x\,\Delta y. \tag{15}$$

On the other hand, we can approximate the area $\Delta x\,\Delta y$ of the sub-rectangle R_{ij} in (s,t)-coordinates using the derivative J_F of the function F because it approximates F near the value (x_i', y_j'). Since $J_F : \mathbb{R}^2_{(s,t)} \to \mathbb{R}^2_{(x,y)}$ is a linear transformation, one has

$$\Delta x\,\Delta y \approx \left|\frac{\partial(x,y)}{\partial(s,t)}\right| (s_i', t_j')\,\Delta s\,\Delta t$$

where $F(s_i', t_j') = (x_i', y_j')$. Substituting into (15), we get

$$\int\int_R f(x,y)\,\mathrm{d}A \approx \sum_{i=1}^{m}\sum_{j=1}^{n} (f \circ F)(s_i', t_j') \left|\frac{\partial(x,y)}{\partial(s,t)}\right| (s_i', t_j')\,\Delta s\,\Delta t. \tag{16}$$

Note that, by Definition 3.1.1, the right side of (16) also approximates the double integral:

$$\int\int_S (f \circ F)(s,t) \left|\frac{\partial(x,y)}{\partial(s,t)}\right| \mathrm{d}s\,\mathrm{d}t \approx \sum_{i=1}^{m}\sum_{j=1}^{n} (f \circ F)(s_i', t_j') \left|\frac{\partial(x,y)}{\partial(s,t)}\right| (s_i', t_j')\,\Delta s\,\Delta t.$$

Therefore, we conclude that

$$\int\int_R f(x,y)\,\mathrm{d}x\,\mathrm{d}y = \int\int_R f(x,y)\,\mathrm{d}A = \int\int_S (f \circ F)(s,t) \left|\frac{\partial(x,y)}{\partial(s,t)}\right| \mathrm{d}s\,\mathrm{d}t.$$

□

We can generalise the change of variables formula to three dimensions, and to all higher dimensions. For $n \geq 1$, let $F : \mathbb{R}^n_{(s_1,\ldots,s_n)} \to \mathbb{R}^n_{(x_1,\ldots,x_n)}$ given by

$$F(s_1, \ldots, s_n) = (x_1(s_1, \ldots, s_n), \ldots, x_n(s_1, \ldots, s_n))$$

be a *change of coordinates map* (the definition is analogous to the two dimensional case). As above, we set

$$\frac{\partial(x_1, \ldots, x_n)}{\partial(s_1, \ldots, s_n)} = \det(J_F(s_1, \ldots, s_n)).$$

The general version of Theorem 3.4.2 then becomes

Theorem 3.4.3 (Change of Variables general case)**.** *Let* $F : \mathbb{R}^n_{(s_1,\ldots,s_n)} \to \mathbb{R}^n_{(x_1,\ldots,x_n)}$ *given by*

$$F(s_1, \ldots, s_n) = (x_1(s_1, \ldots, s_n), \ldots, x_n(s_1, \ldots, s_n))$$

be a change of coordinates map between $\mathbb{R}^n_{(s_1,\ldots,s_n)}$ *and* $\mathbb{R}^n_{(x_1,\ldots,x_n)}$ *on* S*. Let* $R = F(S)$ *and suppose* $f : R \to \mathbb{R}$ *is an integrable function. We then have that* $f \circ F : S \to \mathbb{R}$ *is an integrable function and*

$$\int\int_R f(x_1, \ldots, x_n)\mathrm{d}x_1 \cdots \mathrm{d}x_n = \int\int_S (f \circ F)(s_1, \ldots, s_n) \left| \frac{\partial(x_1, \ldots, x_n)}{\partial(s_1, \ldots, s_n)} \right| \mathrm{d}s_1 \cdots \mathrm{d}s_n.$$

Example 3.4.4. Let R be the region in the plane bounded by the lines $\{x + y = 1\}$, $\{x + y = 2\}$, $\{2x - 3y = 2\}$, and $\{2x - 3y = 5\}$. Use the change of variables formula to determine the area of R.

Admittedly, this is the mathematical equivalent of using a hammer to kill a fly, but we use it as an illustrative example. Even if we do not have the formula for the area of a parallelogram (which we do), we could just as easily use the method of Section 3.2. However, we are asked to use the change of variables formula, and so use the change of variables formula we shall.

We need to evaluate $\int\int_R \mathrm{d}x\, \mathrm{d}y$. So, to make use of the change of variables formula, we need to introduce coordinates (s, t) on $\mathbb{R}^2$ and a map $F(s, t) = (x(s, t), y(s, t))$ relating (s, t) to (x, y). Let $s = x + y$ and $t = 2x - 3y$ and solve for x and y in terms of s and t. Setting this up as a system of linear equations, we have that

$$\begin{pmatrix} s \\ t \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 2 & -3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix},$$

and hence that

$$\begin{pmatrix} x \\ y \end{pmatrix} = -\frac{1}{5} \begin{pmatrix} -3 & -1 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} s \\ t \end{pmatrix}.$$

Therefore, $F(s, t) = (x(s, t), y(s, t)) = (\frac{3}{5}s + \frac{1}{5}t, \frac{2}{5}s - \frac{1}{5}t)$, and so

$$\det(J_F(s, t)) = \frac{\partial(x, y)}{\partial(s, t)} = -\frac{1}{5}.$$

In order to complete the calculation, we need determine the region S in the (s, t)-plane for which $R = F(S)$. Given how we defined s and t in terms of x

and y, we have that $1 \leq s \leq 2$ and $2 \leq t \leq 5$, so that $S = [1,2] \times [2,5]$. Note that $F(s,t)$ is indeed a change of coordinates map on S. Moreover, since we are integrating the function $f(x,y) = 1$ over R, we have that the function we integrate over S is also $f(s,t) = 1$. Therefore, we complete the calculation by evaluating

$$\int\int_R \mathrm{d}x\,\mathrm{d}y = \int\int_S \left|\frac{\partial(x,y)}{\partial(s,t)}\right| \mathrm{d}s\,\mathrm{d}t = \frac{1}{5}\int_2^5\int_1^2 \mathrm{d}s\,\mathrm{d}t = \frac{3}{5}.$$

Even in the case that the region R is described in terms that can be interpreted as non-linear expressions for s and t as functions of x and y, we can use the same logic as in the Example 3.4.4.

Example 3.4.5. Let R be the region in the first quadrant bounded by the curves $\{x^2 - y^2 = 1\}$, $\{x^2 - y^2 = 4\}$, $\{xy = 1\}$, and $\{xy = 3\}$. Use the change of variables formula to evaluate $\int\int_R (x^2 + y^2)\,\mathrm{d}x\,\mathrm{d}y$.

In this example, we will need to be cleverer in our use of the change of variables formula. As in Example 3.4.4, we set $s = x^2 - y^2$ and $t = xy$, so that the region S in the st-plane corresponding to the region R in the xy-plane is $S = [1,4] \times [1,3]$.

Note that even if we set up the change of variables map $F(s,t) = (x(s,t), y(s,t))$, we are not able to determine the functions $x(s,t)$ and $y(s,t)$ explicitly, given the expressions above for s and t in terms of x and y. Therefore, we will need to work through this example without knowing $F(s,t)$ explicitly. However, in order to create the list of what things we need to determine before moving forward, we invoke the change of variables formula to observe that

$$\int\int_R (x^2 + y^2)\,\mathrm{d}x\,\mathrm{d}y = \int\int_S (f \circ F)(s,t) \left|\frac{\partial(x,y)}{\partial(s,t)}\right| \mathrm{d}s\,\mathrm{d}t.$$

We do know that the composition

$$(f \circ F)(s,t) = f(x(s,t), y(s,t)) = x^2 + y^2.$$

Therefore, we need to express $x^2 + y^2$ in terms of s and t, using that $s = x^2 - y^2$ and $t = xy$. Playing around, we see that

$$(x^2 + y^2)^2 = x^4 + 2x^2y^2 + y^4 = x^4 - 2x^2y^2 + y^4 + 4x^2y^2 = s^2 + 4t^2,$$

so that $x^2 + y^2 = \sqrt{s^2 + 4t^2}$.

Since we have expressions for s and t in terms of x and y, we are able to determine $\frac{\partial(s,t)}{\partial(x,y)}$, namely

$$\frac{\partial(s,t)}{\partial(x,y)} = \det\begin{pmatrix} \frac{\partial s}{\partial x} & \frac{\partial s}{\partial y} \\ \frac{\partial t}{\partial x} & \frac{\partial t}{\partial y} \end{pmatrix} = \det\begin{pmatrix} 2x & -2y \\ y & x \end{pmatrix} = 2(x^2 + y^2).$$

We now need to relate $\frac{\partial(s,t)}{\partial(x,y)}$ to $\frac{\partial(x,y)}{\partial(s,t)}$, because we need the latter but have the former.

Recall that we have set up the function $F(s,t)$ but only implicitly. Suppose, in general, that there is a function going the other direction, a function $G(x,y)$ undoing $F(s,t)$, so that

$$F\circ G(x,y)=(x,y) \quad \text{and} \quad G\circ F(s,t)=(s,t),$$

or in other words, the function $G(x,y)$ is the *inverse* of $F(s,t)$.

We differentiate the latter composition using the Chain Rule (see Theorem 1.6.1) to see that

$$J_G(F(s,t))\cdot J_F(s,t)=I$$

and hence, taking determinants, that

$$\det(J_G(F(s,t)))\cdot\det(J_F(s,t))=1.$$

It remains only to reinterpret $\det(J_F(s,t))=\frac{\partial(x,y)}{\partial(s,t)}$ and $\det(J_G(F(s,t)))=\frac{\partial(s,t)}{\partial(x,y)}$. This leads to formula:

$$\boxed{\frac{\partial(x,y)}{\partial(s,t)}=\frac{1}{\dfrac{\partial(s,t)}{\partial(x,y)}}}$$

This shows that the function $F(s,t)$ is a change of coordinates map if and only if its inverse $G(x,y)$ is a change of coordinates map.

Returning to our example, we see that

$$\frac{\partial(x,y)}{\partial(s,t)}=\frac{1}{\frac{\partial(s,t)}{\partial(x,y)}}=\frac{1}{2(x^2+y^2)}=\frac{1}{2\sqrt{s^2+4t^2}}.$$

The last equality in the previous line follows because we need to express $\frac{\partial(s,t)}{\partial(x,y)}$ as a function in x and y, rather than as a function in s and t.

As stated earlier, to see that $F(s,t)$ is a change of coordinate map on S it suffices to show that its inverse $G(x,y)$ is such a map on $D=F(S)$. But $G(x,y)$ is given explicitly by

$$G(x,y)=(s,t)=(x^2-y^2,xy)$$

and we already know that its first partial derivatives are continuous on D and the Jacobian matrix is nontrivial on D. So, it remains to see that $D=R$ and $G(x,y)$ is one-to-one on D. We leave this part as an exercise. It follows that $G(x,y)$ and hence $F(s,t)$ are coordinate maps. So, we can apply the Change of Variables Formula.

Bringing all of this together, we see that

$$\int\!\!\int_R (x^2+y^2)\,\mathrm{d}x\,\mathrm{d}y=\int\!\!\int_S \sqrt{s^2+4t^2}\left|\frac{\partial(x,y)}{\partial(s,t)}\right|\,\mathrm{d}s\,\mathrm{d}t=$$

$$=\int\!\!\int_S \sqrt{s^2+4t^2}\frac{1}{2\sqrt{s^2+4t^2}}\,\mathrm{d}s\,\mathrm{d}t=\frac{1}{2}\int\!\!\int_S \mathrm{d}s\,\mathrm{d}t=3.$$

3.5. Polar coordinates on $\mathbb{R}^2$

We normally use the Cartesian coordinates x and y as our base set of coordinates on the plane. However, we can use other coordinate systems as well. One very common set of coordinates on $\mathbb{R}^2$ are *polar coordinates.* Cartesian coordinates determine the location of a point with respect to two intersecting set of parallel lines intersecting at right angles, where one set of lines is parallel to the x-axis and the other is parallel to the y-axis. So, we use left-right and up-down as our directions in interpreting the location of a point given at an ordered pair (a, b) of real numbers.

For polar coordinates, we determine the location of a point P relative to the positive x-axis from the origin $\mathbf{0}$, where one coordinate of P is the distance from $\mathbf{0}$ and the other coordinate is the angle of the ray from $\mathbf{0}$ to P, measured counterclockwise from the positive x-axis. There is a standard conversion from the cartesian coordinates (x, y) of a point to its polar coordinates (r, θ) given by

$$x = r\cos(\theta) \quad \text{and} \quad y = r\sin(\theta),$$

which is a direct consequence of basic trigonometry. Let $P : [0, \infty) \times [0, 2\pi] \to \mathbb{R}^2$ be the map defined by

$$P(r, \theta) = (x(r, \theta), y(r, \theta)) = (r\cos(\theta), r\sin(\theta)).$$

This is the change of coordinates map from polar to Cartesian coordinates.

We note here that there is no clearly defined change of coordinates map from Cartesian to polar coordinates. Given the Cartesian coordinates (x, y) of a point in the plane $\mathbb{R}^2$, we can almost find its polar coordinates by setting $r = \sqrt{x^2 + y^2}$ and $\theta = \arctan(\frac{y}{x})$, but this does not determine θ completely; rather, this only determines θ up to the possible addition of π, depending on the quadrant in which the point lies.

Consider for instance $z = 1 + i$ and $w = -1 - i$. For both, the calculation of θ in terms of arctan given above yields $\theta = \frac{\pi}{4}$, but this is not the correct angle for w. The correct angle for w is $\frac{\pi}{4} + \pi = \frac{5\pi}{4}$.

So, let R be a region in $\mathbb{R}^2$, described in terms of the Cartesian coordinates, and let $f : \mathbb{R}^2 \to \mathbb{R}$ be a continuous function. Let T be the region R described in terms of polar coordinates. Then $R = P(T) = P(R)$. In order to use the change of variables formula in this case, we need to calculate the Jacobian matrix $J_P(r, \theta)$ of the change of coordinates map, which is

$$J_P(r, \theta) = \begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{pmatrix} = \begin{pmatrix} \cos(\theta) & -r\sin(\theta) \\ \sin(\theta) & r\cos(\theta) \end{pmatrix},$$

from which we see that

$$\boxed{\det(J_P(r, \theta)) = \frac{\partial(x, y)}{\partial(r, \theta)} = r}$$

and so

$$\boxed{\int\int_R f(x,y)\,\mathrm{d}x\,\mathrm{d}y = \int\int_R (f\circ P)(r,\theta)\,r\,\mathrm{d}r\,\mathrm{d}\theta}.$$

Before evaluating some example integrals, we consider the issue of describing regions in terms of polar coordinates.

Example 3.5.1. Let $R = \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 < 1\}$ be the open unit disc in $\mathbb{R}^2$, with radius 1 and centre the origin $\mathbf{0}$. Describe R in terms of both cartesian and polar coordinates.

In terms of the Cartesian coordinates (x,y) on $\mathbb{R}^2$, we can describe R as the set of points $R = \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 < 1\}$. If we wish to take horizontal slices, then

$$R = \{(x,y) \in \mathbb{R}^2 \mid -1 < y < 1 \text{ and } -\sqrt{1-y^2} < x < \sqrt{1-y^2}\}.$$

If we wish to take vertical slices, then

$$R = \{(x,y) \in \mathbb{R}^2 \mid -1 < x < 1 \text{ and } -\sqrt{1-x^2} < y < \sqrt{1-x^2}\}.$$

We use these latter two descriptions when we are integrating a function over R using Cartesian coordinates.

To describe R in terms of polar coordinates, we see that the points in the disc are those for which $0 \leq r < 1$ and $0 \leq \theta \leq 2\pi$, and so

$$R = \{(r,\theta) \in \mathbb{R}^2 \mid 0 \leq r < 1 \text{ and } 0 \leq \theta \leq 2\pi\}.$$

Note that in the inequality for θ, we should properly say that $0 \leq \theta < 2\pi$ so that we are not double counting any points. However, as the set of points we double count by letting $0 \leq \theta \leq 2\pi$ is only a line, it contributes nothing to the integral and is just easier to deal with. The formal justification of this sort of issue, which occurs rather frequently but which will brush under the carpet for now, comes from measure theory.

Therefore a disc (centred at the origin) in Cartesian coordinates becomes a rectangle when described in terms of polar coordinates. This is one of the main reasons we consider changes of coordinates, so that the region of integration becomes much simpler to integrate over. Ideally, a change of coordinates will produce a region of integration that is as simple as possible. There is of course a trade off. We may make the region of integration simpler but make the integrand more complicated, and we need to balance where we would prefer the difficulty to lie.

Example 3.5.2. Let S be the square in cartesian coordinates with vertices $(0,0)$, $(2,0)$, $(0,2)$, and $(2,2)$. Describe S in polar coordinates.

The most difficult aspect of describing a region in terms of coordinates which are not suited for the purpose, for instance describing this square in terms of polar coordinates, is determining which coordinate(s) to describe in terms of other coordinate(s) and which coordinate(s) to bound by constants. In this example, we attempt both.

Over all points in S, it is easy to see that $0 \leq \theta \leq \frac{\pi}{2}$. We can now ask the question, for each value of θ satisfying $0 \leq \theta \leq \frac{\pi}{2}$, what are the largest and smallest values of r so that the point (r, θ) in polar coordinates lies in the square. As sometimes happens, there are two cases.

For $0 \leq \theta \leq \frac{\pi}{4}$, the point in the square that lies on the ray making angle θ with the positive x-axis (measured anti-clockwise, as always) lies on the line $x = 2$. Using our standard conversion between cartesian and polar coordinates, we have that $r\cos(\theta) = 2$ and hence (since we're viewing θ as fixed) that $r = 2\sec(\theta)$.

For $\frac{\pi}{4} \leq \theta \leq \frac{\pi}{2}$, the point in the square that lies on the ray making angle θ with the positive x-axis (measured anti-clockwise, as always) lies on the line $y = 2$. Using our standard conversion between cartesian and polar coordinates, we have that $r\sin(\theta) = 2$ and hence (since we're viewing θ as fixed) that $r = 2\csc(\theta)$.

Hence, one description of this square in polar coordinates is as the union

$$S = \left\{(r,\theta) \ \mid 0 \leq \theta \leq \frac{\pi}{4} \text{ and } 0 \leq r \leq 2\sec(\theta)\right\} \cup \left\{(r,\theta) \ \mid \frac{\pi}{4} \leq \theta \leq \frac{\pi}{2} \text{ and } 0 \leq r \leq 2\csc(\theta)\right\}.$$

If we attempt to first determine the limits on r and from this place bounds on the possible values of θ, we will again have a description of S as a union. Finding the limits on r is not an issue. The point in S farthest from the origin is the point $(2, 2)$ which has distance $2\sqrt{2}$ from $\mathbf{0}$, from which we see that $0 \leq r \leq 2\sqrt{2}$. For $0 \leq r \leq 1$, the limits on θ are straightforwardly $0 \leq \theta \leq \frac{\pi}{2}$.

For $1 \leq r \leq 2\sqrt{2}$, we see that $\arccos(\frac{2}{r}) \leq \theta \leq \arcsin(\frac{2}{r})$, and so in this case the description of S is

$$\left\{(r,\theta) \ \mid 0 \leq r \leq 2 \text{ and } 0 \leq \theta \leq \frac{\pi}{2}\right\} \cup \left\{(r,\theta) \ \mid 2 \leq r \leq 2\sqrt{2}, \arccos\left(\frac{2}{r}\right) \leq \theta \leq \arcsin\left(\frac{2}{r}\right)\right\}.$$

We are now able to answer one of the questions raised at the beginning of this chapter.

Example 3.5.3. Let D be a disc in the plane with centre (a, b) and radius $R > 0$. The area of D is then πR^2.

We start with the Cartesian coordinates. The order of integration does not matter in this case, and so we will integrate first with respect to y and then with respect to x. The limits on x are $a - R \leq x \leq a + R$, since the x-coordinate of the centre of the disc is a and the radius of the disc is R. For a given value of x in this range, the limits on y come from the fact that the boundary circle of the disc has the equation

$$(x - a)^2 + (y - b)^2 = R^2.$$

Solving for y, we see that $y = b \pm \sqrt{R^2 - (x-a)^2}$.

Therefore, the integral for the area of R becomes

$$\int_{a-R}^{a+R} \int_{b-\sqrt{R^2-(x-a)^2}}^{b+\sqrt{r^2-(x-a)^2}} \mathrm{d}y\, \mathrm{d}x =$$

$$\begin{aligned}
&= \int_{a-R}^{a+R} 2\sqrt{R^2-(x-a)^2}\, \mathrm{d}x \\
&= -2\int_{\pi}^{0} R^2\sqrt{1-\cos^2(\theta)}\sin(\theta)\, \mathrm{d}\theta \text{ (with substitution } x = a + R\cos(\theta)) \\
&= -2R^2 \int_{\pi}^{0} \sin^2(\theta)\, \mathrm{d}\theta \\
&= -2R^2 \int_{\pi}^{0} \frac{1}{2}(1-\cos(2\theta))\, \mathrm{d}\theta = -R^2 \left(\theta - \frac{1}{2}\sin(2\theta)\right)\Big|_{\pi}^{0} = \pi R^2.
\end{aligned}$$

Using polar coordinates, we start with the observation that describing a disc with an arbitrary centre in polar coordinates is potentially complicated. So, we take this calculation in two parts. The first part is to work in the case that the disc R has centre $(0,0)$ and radius $R > 0$. The description of this disc in polar coordinates is

$$D = \{(r,\theta) \mid 0 \le r \le R,\ 0 \le \theta \le 2\pi\}.$$

Hence, the area of this disc using polar coordinates is

$$\int_0^{2\pi} \int_0^R r\, \mathrm{d}r\, \mathrm{d}\theta = \int_0^{2\pi} \left(\frac{1}{2}r^2\right)\Big|_0^R \mathrm{d}\theta = \pi R^2.$$

To handle the general disc with centre (a,b) and radius $R > 0$, let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the affine transformation given by

$$T(s,t) = (s+a, t+b).$$

The map T takes the disc with centre $(0,0)$ and radius R bijectively to the disc with centre (a,b) and radius R. Moreover, since the Jacobian matrix of T is

$$J_T(s,t) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

so that

$$\det(J_T(s,t)) = 1,$$

so that T is a change of coordinates map. In fact, it just moves the set coordinates lines crossing at $(0,0)$ to the ones crossing the point (a,b). So, we can apply the Change of Variables Formula to obtain

$$\int\int_D \mathrm{d}x\, \mathrm{d}y = \int\int_{T(D)} \mathrm{d}s\, \mathrm{d}t.$$

Since D has area πR^2, we see that $T(D)$ also has area πR^2.

We close this section with the following example.

Example 3.5.4. As an application of Corollary 3.1.6, we evaluate $\int_{\mathbb{R}} \exp(-x^2)\,\mathrm{d}x$.

This calculation uses several tricks that are potentially useful in other contexts.

Note that $\exp(-x^2)$ has no anti-derivative and hence we are not able to evaluate this integral directly. Let

$$K = \int_{\mathbb{R}} \exp(-x^2)\,\mathrm{d}x$$

and note that $\mathbb{R}$ is a generalised interval. We can write the square K^2 of K as

$$\begin{aligned}
K^2 &= K \cdot K \\
&= \left(\int_{\mathbb{R}} \exp(-x^2)\,\mathrm{d}x\right)^2 \\
&= \left(\int_{\mathbb{R}} \exp(-x^2)\,\mathrm{d}x\right)\left(\int_{\mathbb{R}} \exp(-y^2)\,\mathrm{d}y\right) \\
&= \int\int_{\mathbb{R}^2} \exp(-x^2-y^2)\,\mathrm{d}x\,\mathrm{d}y \\
&= \int\int_{\mathbb{R}^2} \exp(-r^2) r\,\mathrm{d}r\,\mathrm{d}\theta \\
&= \int_0^\infty \int_0^{2\pi} r\exp(-r^2)\,\mathrm{d}\theta\,\mathrm{d}r \\
&= 2\pi \int_0^\infty r\exp(-r^2)\,\mathrm{d}r \\
&= 2\pi \lim_{M\to\infty} \int_0^M r\exp(-r^2)\,\mathrm{d}r = 2\pi\left(\lim_{M\to\infty}\left(-\frac{1}{2}\exp(-M^2)\right)+\frac{1}{2}\right) = \pi,
\end{aligned}$$

and so $K = \int_{\mathbb{R}} \exp(-x^2)\,\mathrm{d}x = \sqrt{\pi}$.

3.6. Cylindrical coordinates on $\mathbb{R}^3$

Cylindrical coordinates are one way of generalising polar coordinates to $\mathbb{R}^3$. In essence, we use polar coordinates on the xy-plane and then adjoin the standard cartesian z-coordinate. So, cylindrical coordinates are (r, θ, z), where the conversion from cartesian coordinates to cylindrical coordinates is

$$x = r\cos(\theta),\ y = r\sin(\theta),\ z = z.$$

In preparation for using the Change of Variables Formula, we first calculate the Jacobian matrix of the change of variables function

$$C(r,\theta,z) = (x(r,\theta,z), y(r,\theta,z), z(r,\theta,z)) = (r\cos(\theta), r\sin(\theta), z).$$

The Jacobian matrix is

$$\begin{aligned} J_C(r,\theta,z) &= \begin{pmatrix} \frac{\partial x}{\partial r}(r,\theta,z) & \frac{\partial x}{\partial \theta}(r,\theta,z) & \frac{\partial x}{\partial z}(r,\theta,z) \\ \frac{\partial y}{\partial r}(r,\theta,z) & \frac{\partial y}{\partial \theta}(r,\theta,z) & \frac{\partial y}{\partial z}(r,\theta,z) \\ \frac{\partial z}{\partial r}(r,\theta,z) & \frac{\partial z}{\partial \theta}(r,\theta,z) & \frac{\partial z}{\partial z}(r,\theta,z) \end{pmatrix} \\ &= \begin{pmatrix} \cos(\theta) & -r\sin(\theta) & 0 \\ \sin(\theta) & r\cos(\theta) & 0 \\ 0 & 0 & 1 \end{pmatrix}. \end{aligned}$$

Note that the upper left-hand 2×2-sub-matrix of $J_C(r,\theta,z)$ is the same as the Jacobian matrix for the change of coordinates from polar coordinates to the Cartesian coordinates in the plane. This should not be a surprise, as cylindrical coordinates in $\mathbb{R}^3$ are just polar coordinates in the xy-plane $\mathbb{R}^2$ with the same z-coordinate as the Cartesian coordinates.

An easy calculation then yields that the determinant of the Jacobian matrix of the change from cylindrical coordinates to cartesian coordinates in $\mathbb{R}^3$ is

$$\boxed{\det(J_C(r,\theta,z)) = r}.$$

Example 3.6.1. Let $B_K = \{(x,y,z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 \leq K^2\}$ be the closed ball of radius $K > 0$ centred at the origin $\mathbf{0}$. For $0 < k < K$, let $C_k = \{(x,y,z) \in \mathbb{R}^3 \mid x^2 + y^2 < k^2\}$ be the open cylindrical region of radius k centred on the z-axis. Remove C_k from B_K to form the region X (see Figure 18). Determine the volume of X.

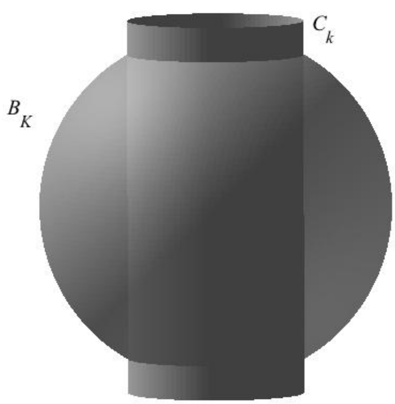


FIGURE 18. the region X formed by removing the solid cylinder C_k from the ball B_K

There are two ways to proceed. We can either use that we know the volume of the ball (which is $\frac{4}{3}\pi K^3$), calculate the volume of the portion of the cylindrical region C_k that lies inside B_K and subtract one from the other, using that $X \cup C_k = B_K$ and $X \cap C_k = \emptyset$, or we can directly evaluate the volume of X. Which way we proceed depends on which regions are easier to parametrise using the chosen coordinates.

In this example, we proceed in the former way.

We start by parametrising the portion $C_k \cap B_K$ of C_k that lies inside B_K. Given how C_k is defined, we can see that any point (x, y, z) in $C_k \cap B_K$ satisfies $x^2 + y^2 \leq k^2$, be definition of C_k. Converting into cylindrical coordinates, we see that $0 \leq r \leq k$ and $0 \leq \theta \leq 2\pi$.

It remains only to set the limits on z. Since the point (x, y, z) lies inside B_K, we see that since $x^2 + y^2 + z^2 \leq K^2$, we have that $z^2 \leq K^2 - x^2 - y^2 = K^2 - r^2$. Therefore, we have that $-\sqrt{K^2 - r^2} \leq z \leq \sqrt{K^2 - r^2}$. Hence, the integral for the volume of $B_K \cap C_k$ in cylindrical coordinates is

$$
\begin{aligned}
\mathrm{vol}(C_k \cap B_K) &= \int\int\int r \, \mathrm{d}z \, \mathrm{d}r \, \mathrm{d}\theta \\
&= \int_0^{2\pi} \int_0^k \int_{-\sqrt{K^2-r^2}}^{\sqrt{K^2-r^2}} r \, \mathrm{d}z \, \mathrm{d}r \, \mathrm{d}\theta \\
&= \int_0^{2\pi} \int_0^k rz\Big|_{z=-\sqrt{K^2-r^2}}^{\sqrt{K^2-r^2}} \, \mathrm{d}r \, \mathrm{d}\theta \\
&= \int_0^{2\pi} \int_0^k 2r\sqrt{K^2 - r^2} \, \mathrm{d}r \, \mathrm{d}\theta \\
&= -\int_0^{2\pi} \frac{2}{3} \, (K^2 - r^2)^{3/2}\Big|_{r=0}^{k} \, \mathrm{d}\theta \\
&= \frac{4}{3}\pi(K^3 - (K^2 - k^2)^{3/2})
\end{aligned}
$$

The volume of X is then $\frac{4}{3}\pi K^3 - \frac{4}{3}\pi(K^3 - (K^2 - k^2)^{3/2}) = \frac{4}{3}\pi(K^2 - k^2)^{3/2}$.

3.7. Spherical coordinates on $\mathbb{R}^3$

Spherical coordinates are a different generalisation of polar coordinates to $\mathbb{R}^3$. For a point (x, y, z) in $\mathbb{R}^3$, we let ρ be the coordinate giving the distance from the origin, so that $\rho = \sqrt{x^2 + y^2 + z^2}$. We use the same coordinate θ as in cylindrical coordinates, giving the angle around the z-axis. We let φ be the coordinate giving the angle between (x, y, z) and the positive z-axis. So, the limits on these coordinates are

$$\rho \geq 0, \; 0 \leq \theta \leq 2\pi \quad \text{and} \quad 0 \leq \varphi \leq \pi.$$

To determine the appropriate factor to use in the Change of Variables Formula, we first calculate the Jacobian matrix of the change of variables function

$$
\begin{gathered}
S(\rho, \varphi, \theta) = \\
= (x(\rho, \theta, \varphi), y(\rho, \theta, \varphi), z(\rho, \theta, \varphi)) = (\rho\cos(\theta)\sin(\varphi), \rho\sin(\theta)\sin(\varphi), \rho\cos(\varphi)).
\end{gathered}
$$

The Jacobian matrix of $S(\rho, \theta, \varphi)$ is

$$J_S(\rho,\theta,\varphi) = \begin{pmatrix} \frac{\partial x}{\partial \rho}(\rho,\theta,\varphi) & \frac{\partial x}{\partial \theta}(\rho,\theta,\varphi) & \frac{\partial x}{\partial \varphi}(\rho,\theta,\varphi) \\ \frac{\partial y}{\partial \rho}(\rho,\theta,\varphi) & \frac{\partial y}{\partial \theta}(\rho,\theta,\varphi) & \frac{\partial y}{\partial \varphi}(\rho,\theta,\varphi) \\ \frac{\partial z}{\partial \rho}(\rho,\theta,\varphi) & \frac{\partial z}{\partial \theta}(\rho,\theta,\varphi) & \frac{\partial z}{\partial \varphi}(\rho,\theta,\varphi) \end{pmatrix}$$

$$= \begin{pmatrix} \cos(\theta)\sin(\varphi) & -\rho\sin(\theta)\sin(\varphi) & \rho\cos(\theta)\cos(\varphi) \\ \sin(\theta)\sin(\varphi) & \rho\cos(\theta)\sin(\varphi) & \rho\sin(\theta)\cos(\varphi) \\ \cos(\varphi) & 0 & -\rho\sin(\varphi) \end{pmatrix}$$

A straightforward calculation yields that the determinant of the Jacobian matrix of the change of coordinates map is then

$$\boxed{\det(J_S(\rho,\theta,\varphi)) = \rho^2\sin(\varphi)}.$$

Example 3.7.1. We repeat the Example 3.6.1, but this time we parametrise X directly. We see that $0 \le \theta \le 2\pi$, since X is symmetric about the z-axis. The smallest value of φ occurs when the ray from the origin passes through the intersection of the boundary of C_k with the boundary of B_K. Letting φ_0 be this value of φ, we see that $0 < \varphi_0 < \frac{\pi}{2}$ and that $\sin(\varphi_0) = \frac{k}{K}$. Using basic trigonometric identities, we see that

$$\cos(\varphi_0) = \sqrt{1 - \left(\frac{k}{K}\right)^2} = \frac{1}{K}\sqrt{K^2 - k^2}.$$

We will need the expression for $\cot(\varphi_0) = \frac{1}{k}\sqrt{K^2 - k^2}$ and $\cot(\pi - \varphi_0) = -\frac{1}{k}\sqrt{K^2 - k^2}$.

As φ increases, we see that the smallest value of ρ satisfies $\sin(\varphi) = \frac{k}{\rho}$, so that $\rho = \frac{k}{\sin(\varphi)}$. Therefore, the range of ρ satisfies $\frac{k}{\sin(\varphi)} \le \rho \le K$. So, the volume of X, calculating using spherical coordinates, is

$$\begin{aligned}
\int\int\int_X \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z &= \int_0^{2\pi}\int_{\varphi_0}^{\pi-\varphi_0}\int_{\frac{k}{\sin(\varphi)}}^{K} \rho^2 \sin(\varphi)\,\mathrm{d}\rho\,\mathrm{d}\varphi\,\mathrm{d}\theta \\
&= \frac{1}{3}\int_0^{2\pi}\int_{\varphi_0}^{\pi-\varphi_0} \left(K^3 - \left(\frac{k}{\sin(\varphi)}\right)^3\right)\sin(\varphi)\,\mathrm{d}\varphi\,\mathrm{d}\theta \\
&= \frac{1}{3}\int_0^{2\pi}\int_{\varphi_0}^{\pi-\varphi_0} \left(K^3\sin(\varphi) - k^3\csc^2(\varphi)\right)\,\mathrm{d}\varphi\,\mathrm{d}\theta \\
&= \frac{1}{3}\int_0^{2\pi} \left(-K^3\cos(\varphi) + k^3\cot(\varphi)\right)\Big|_{\varphi_0}^{\pi-\varphi_0}\,\mathrm{d}\theta \\
&= \frac{2\pi}{3}(2K^3\cos(\varphi_0) - 2k^3\cot(\varphi_0)) \\
&= \frac{4\pi}{3}\left(K^2\sqrt{K^2-k^2} - k^2\sqrt{K^2-k^2}\right) = \frac{4\pi}{3}(K^2-k^2)^{3/2}.
\end{aligned}$$

Appendix: Further reading

Here is a selection of books that you may find helpful. Not all the material is covered by a single text, but the closest to the syllabus for this module is:

- ADAMS R. A. *Calculus - A Complete Course*, (Addison-Wesley) which also covers revisions of differentiation and integration.
- SPIVAK M., *Calculus* (CUP) nicely covers fundamental ideas of calculus.
- LARSON R. and EDWARDS B. E., *Calculus*, 10th edition, (Cengage Learning) also has a good introduction to calculus and its 10th edition covers the topics on multivariable calculus.
- BROWN J. W. and CHURCHILL R. V., *Complex variables and Applications*, (McGraw-Hill) has the material on complex numbers, complex functions and complex differentiation.

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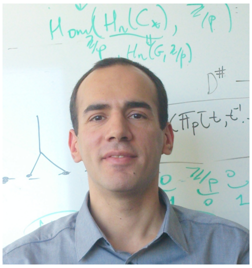

***Calculus** is one of the core modules taken by first-year students in all the Mathematics programmes at the University of Southampton. The module provides the theoretical basis for calculus and studies generalisations of calculus to functions of several variables and to complex analysis.*

Calculus and analysis form the foundation of much of modern mathematics and physics. Newton and Leibniz's discovery of the derivative and integral was fundamental to the understanding of classical mechanics. Nowadays calculus is pervasive throughout mathematics and science, from general relativity and quantum mechanics, through modelling population and the spread of diseases, to options pricing and the Black-Scholes equation.

In the second volume of this calculus text, we study functions of several variables and of a complex variable. We extend much of the theory from single variable calculus with a strong emphasis on applications such as determining extreme values, finding roots of complex numbers and calculating double and triple integrals over simple regions.

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