HOMOLOGY OF HANTZSCHE-WENDT GROUPS

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ABSTRACT. An n-dimensional Hantzsche-Wendt group is an n-dimensional orientable Bieberbach group with holonomy group $\mathbb{Z}_2^{n-1}$. We develop an algorithm that computes the homology of any Hantzsche-Wendt group by constructing a practical free resolution induced from the crystallographic action of the group on $\mathbb{R}^n$. As applications we compute the homology of all five and seven dimensional Hantzsche-Wendt groups.

CONTENTS

1. INTRODUCTION

Let Γ be a discrete subgroup of the group of isometries of $\mathbb{R}^n$. When the quotient space of the action is compact, the group is said to be *crystallographic*. A *Bieberbach group* is a torsion free crystallographic group. It acts freely on $\mathbb{R}^n$, and the quotient M is a manifold with fundamental group Γ. Conversely, every Bieberbach group can be realized in this way as the fundamental group of a compact flat Riemannian manifold. By the classical Bieberbach theorems every n-dimensional crystallographic group is a finite extension of an integral lattice in $\mathbb{R}^n$. Also, up to affine isomorphisms, there are only finitely many crystallographic groups of a given dimension. A *Hantzsche-Wendt group (HW-group)* Γ is an n-dimensional Bieberbach group such that the corresponding *Hantzsche-Wendt*

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manifold M is orientable with holonomy group $\mathbb{Z}_2^{n-1}$. It is known that HW-manifolds only exist in odd dimensions (see [11] or page 4). A particularly interesting property of these manifolds is the fact that the rank of the holonomy group is maximal. It is also well known that any n-dimensional HW-manifold M is a rational homology n-sphere (see, for example, Lemma 2.8). This is somewhat counterintuitive, considering that all n-dimensional compact flat Riemannian manifolds are finitely covered by the n-torus.

In the classical case of dimension 3 there is only one HW-group. The 3-dimensional compact flat Riemannian manifold with this group as its fundamental group is called *didicosm*. It was initially studied by Hantzsche and Wendt (see [5]). The first homology group of this manifold is $\mathbb{Z}_4^2$. In [10] Putrycz showed that this is in fact an exceptional case and the first homology of any HW-group of dimension $n \geq 5$ is isomorphic to the holonomy group $\mathbb{Z}_2^{n-1}$. So, the dimension n completely determines the first homology group of a HW–group. Naturally, one can ask to what extent this generalizes to other homology groups of a HW-group.

As a consequence of our results it follows that there are many HW-groups that have different higher homology groups. In section 3 we introduce an algorithm that computes the homology of any Bieberbach group Γ with a diagonal integral holonomy representation. In particular, it computes the homology of any Hantzsche-Wendt group (see page 4). The basis of the algorithm stems from a practical free $\mathbb{Z}\Gamma$-resolution of $\mathbb{Z}$. This resolution arises from the geometric action of the group Γ on $\mathbb{R}^n$, where we decompose $\mathbb{R}^n$ into n-cubes and view it as a free-Γ-CW-complex.

In dimension 5, there are only two non-isomorphic HW-groups (see [8]). Our computations show that the homology of these groups are equal. In dimension 7, up to isomorphisms, there are exactly sixty two HW-groups. These groups are classified by Miatello and Rossetti in [9]. In section 5, we compute their homology and show that it is one of four possible types.

2. Facts and Preliminaries

The results in this section are well known. For convenience we discuss the necessaries from the theory of group cohomology and outline the proofs of the facts stated in the introduction about HW-groups. For a general reference on group cohomology we advise Brown's book [1], which is an excellent source.

Given a discrete group Γ, we can view the integers $\mathbb{Z}$ as a $\mathbb{Z}\Gamma$-module with a trivial Γ-action. There exists a projective resolution of $\mathbb{Z}$ over $\mathbb{Z}\Gamma$ given by an exact sequence

$$\longrightarrow P_k \xrightarrow{\partial_k} \cdots \xrightarrow{\partial_2} P_1 \xrightarrow{\partial_1} P_0 \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 0,$$

where each P_k is a projective $\mathbb{Z}\Gamma$-module and $ker\partial_k = Im\partial_{k+1}$ and $ker\varepsilon = Im\partial_1$. For example, let $E\Gamma$ be a contractible free-Γ-CW-complex. Then the associated chain groups are free-$\mathbb{Z}\Gamma$-modules, and there is a long exact sequence

$$\cdots \longrightarrow C_k(E\Gamma) \xrightarrow{\partial_k} \cdots \xrightarrow{\partial_2} C_1(E\Gamma) \xrightarrow{\partial_1} C_0(E\Gamma) \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 0.$$

For two projective resolutions P_* and $P_*^{'}$ of $\mathbb{Z}$ over $\mathbb{Z}\Gamma$, there is an augmentation-preserving chain homotopy equivalence $\phi : P_* \to P_*^{'}$, which is unique up to homotopy (see for instance [1], page 24).

Definition 2.1. Let P_* be a projective resolution of $\mathbb{Z}$ over $\mathbb{Z}\Gamma$ and let N be an arbitrary $\mathbb{Z}\Gamma$-module. Given an integer $k \geq 0$, the *kth-homology* of Γ with coefficients in N is defined by

$$\mathrm{H}_k(\Gamma, N) = \mathrm{H}_k(P_* \otimes_{\mathbb{Z}\Gamma} N).$$

We recall that every compact flat Riemannian manifold M has a contractible universal cover $\widetilde{M}$ (see [4]). It comes equipped with a natural CW-complex structure induced from that of M. Then $\widetilde{M}$ becomes a free-π-CW-complex, where π is the fundamental group of M. Hence, by the above remark $\mathrm{H}_k(M, \mathbb{Z}) \cong \mathrm{H}_k(\pi, \mathbb{Z})$ for all $k \geq 0$. Similar isomorphisms hold for the cohomology groups.

Definition 2.2. A discrete group Γ is an n-dimensional *orientable Poincaré Duality group (PD-group)* if there is a class $[\Gamma] \in \mathrm{H}_n(\Gamma, \mathbb{Z})$, such that the cap product map $- \cap [\Gamma] : \mathrm{H}^k(\Gamma, N) \to \mathrm{H}_{n-k}(\Gamma, N)$ is an isomorphism for all $k \geq 0$ and all $\mathbb{Z}\Gamma$-modules N.

It is a standard fact that for any orientable closed manifold, Poincaré Duality holds for arbitrary *local coefficients* (see [3]). Moreover, if this manifold is flat, then the fundamental group is a PD-group.

Definition 2.3. A *crystallographic group* Γ is a discrete subgroup of the group of isometries of $\mathbb{R}^n$ such that the quotient space $\mathbb{R}^n/\Gamma$ is compact. If Γ is also torsion-free, then it is said to be a *Bieberbach group*.

Any Bieberbach group Γ acts freely on $\mathbb{R}^n$ (see [2]), and the quotient $\mathbb{R}^n/\Gamma$ is a compact flat Riemannian manifold. It is said to be orientable, if $\mathbb{R}^n/\Gamma$ is an orientable manifold. Thus, we have the following lemma.

Lemma 2.4. Any n-dimensional orientable Bieberbach group is an n-dimensional orientable PD-group.

Given an orientable compact flat Riemannian manifold M, there is an epimorphism $\phi : \pi_1(M,p) \to G$, where G is the *holonomy group* of M. This map is defined by moving a vector of the tangent space TM_p at the point $p \in M$ by the parallel vector field along a loop that starts and ends at the point $p \in M$. Since parallel transports of a given vector along homotopic loops produce the same resulting vector, this gives us a well-defined homomorphism from $\pi_1(M,p)$ onto the quotient group. We can then identify the image of the map ϕ by the holonomy group G.

Let $\mathrm{Isom}(\mathbb{R}^n)$ denote the group of isometries of $\mathbb{R}^n$. It is well known that $\mathrm{Isom}(\mathbb{R}^n) \cong \mathbb{R}^n \rtimes \mathrm{O}(n,\mathbb{R})$. Thus, any element γ of an n-dimensional crystallographic group Γ acts on $\mathbb{R}^n$ by a rotation $L(\gamma) \in \mathrm{O}(n,\mathbb{R})$ and by a translation $T(\gamma) \in \mathbb{R}^n$ in a canonical way

$$\forall x \in \mathbb{R}^n, \forall \gamma \in \Gamma : \gamma x = L(\gamma)x + T(\gamma).$$

If we identify $\mathbb{R}^n$ with the hyperplane $\mathbb{R}^n \times \{1\} \subset \mathbb{R}^{n+1}$, then the natural representation $\theta : \Gamma \hookrightarrow \mathrm{GL}(n+1,\mathbb{R})$ defined by

$$\gamma \mapsto \begin{pmatrix} L(\gamma) & T(\gamma) \\ 0 & 1 \end{pmatrix}$$

induces the given action of Γ on $\mathbb{R}^n$.

By the classical Bieberbach theorems (see [2]) we know that for an n-dimensional Bieberbach group $\Gamma \subset \mathrm{Isom}(\mathbb{R}^n)$, $\mathbb{R}^n \cap \Gamma$ is a lattice isomorphic to $\mathbb{Z}^n$, the holonomy group G is finite, and there is an exact sequence

$$0 \longrightarrow \mathbb{Z}^n \xrightarrow{\iota} \Gamma \xrightarrow{\phi} G \longrightarrow 1.$$

The induced action of G on $\mathbb{Z}^n$ is given by the representation $\rho : G \to \mathrm{GL}(n,\mathbb{Z})$ defined by $\rho(g)(z) = \iota^{-1}(\gamma\iota(z)\gamma^{-1})$, $\forall g \in G$, $\phi(\gamma) = g$.

Definition 2.5. *A Hantzsche-Wendt group (HW-group)* Γ is an n-dimensional Bieberbach group such that the corresponding *Hantzsche-Wendt manifold* $\mathbb{R}^n/\Gamma$ is orientable with holonomy group $\mathbb{Z}_2^{n-1}$.

Given a HW-group Γ, by [11] there exists a $\mathbb{Z}$-module basis $\{e_i | 1 \le i \le n\}$ of $\mathbb{R}^n \cap \Gamma$ such that with respect to this basis $L(\gamma)$ is a diagonal matrix of $\mathrm{SL}(n,\mathbb{Z})$ and $T(\gamma)$ is a vector with coordinates $\frac{m}{2}$, $m \in \mathbb{Z}$, $\forall \gamma \in \Gamma$. We call this a *standard* presentation of Γ. If Γ is in standard form, then $\rho(G)$ is the diagonal subgroup $\{D \in \mathrm{SL}(n,\mathbb{Z}) | D = (\pm 1, \ldots, \pm 1)\}$. Since Γ is torsion free, $-I \notin \rho(G)$; otherwise we can choose $\gamma \in \Gamma$ such that $\phi(\gamma) = -I$. This would imply that $\gamma^2 = 1$, which is a contradiction. It follows that the dimension of any HW-group must be odd.

Next, we state the theorem that motivated our interest in the homology of HW-groups.

Theorem 2.6. (Putrycz, 2007, [10]) Let Γ be an n-dimensional HW-group with $n > 3$. Then, $[\Gamma, \Gamma] \cong \mathbb{Z}^n$ and $H_1(\Gamma) = \Gamma/[\Gamma, \Gamma] \cong \mathbb{Z}_2^{n-1}$.

The following fact is useful in the computations of the torsion in the homology of crystallographic groups.

Lemma 2.7. Let Γ be an n-dimensional crystallographic group with holonomy group G. The torsion subgroup of $H_*(\Gamma, \mathbb{Z})$ is annihilated by the order of G.

Proof. Let $\mathbb{Z}^n$ denote the lattice subgroup of Γ. Let $\mathrm{cor} : H_*(\mathbb{Z}^n, \mathbb{Z}) \to H_*(\Gamma, \mathbb{Z})$ be the map induced by the inclusion $\iota : \mathbb{Z}^n \to \Gamma$ and let $\mathrm{tr} : H_*(\Gamma, \mathbb{Z}) \to H_*(\mathbb{Z}^n, \mathbb{Z})$ be the transfer homomorphism. The composition $\mathrm{cor} \circ \mathrm{tr} : H_*(\Gamma, \mathbb{Z}) \to H_*(\Gamma, \mathbb{Z})$ is then the map defined by $x \mapsto |G| \cdot x$ for all $x \in H_*(\Gamma, \mathbb{Z})$, i.e. multiplication by the order of G. On the other hand, since $H_*(\mathbb{Z}^n, \mathbb{Z})$ is torsion free, the transfer and hence $\mathrm{cor} \circ \mathrm{tr}$ are trivial homomorphisms when restricted to the torsion subgroup of $H_*(\Gamma, \mathbb{Z})$. □

With a little work, one finds the following additional properties about the homology of HW-groups.

Lemma 2.8. Let Γ be an n-dimensional HW-group. Then,

(i) $\mathrm{H}_i(\Gamma, \mathbb{Z})$ is a torsion group annihilated by 2^{n-1} for $1 \leq i \leq n-1$,

(ii) $\mathrm{H}_i(\Gamma, \mathbb{Z}) \cong \mathrm{H}_{n-i-1}(\Gamma, \mathbb{Z})$ for $1 \leq i < n-1$,

(iii) $\mathrm{H}_{n-1}(\Gamma, \mathbb{Z}) = 0$, $\mathrm{H}_n(\Gamma, \mathbb{Z}) = \mathbb{Z}$.

Proof. We first assume $\mathrm{H}_i(\Gamma, \mathbb{Z})$ is a torsion group for all $1 \leq i \leq n-1$. Since Γ is a PD-group, $\mathrm{H}_i(\Gamma, \mathbb{Z}) \cong \mathrm{H}^{n-i}(\Gamma, \mathbb{Z})$, $\forall i \in \mathbb{Z}$. By the Universal Coefficient Theorem, we have the exact sequence

$$0 \longrightarrow \mathrm{Ext}(\mathrm{H}_{n-i-1}(\Gamma, \mathbb{Z}), \mathbb{Z}) \longrightarrow \mathrm{H}^{n-i}(\Gamma, \mathbb{Z}) \longrightarrow \mathrm{Hom}(\mathrm{H}_{n-i}(\Gamma, \mathbb{Z}), \mathbb{Z}) \longrightarrow 0.$$

If $1 \leq i < n-1$, then $\mathrm{Hom}(\mathrm{H}_{n-i}(\Gamma, \mathbb{Z}), \mathbb{Z}) = 0$. This implies $\mathrm{H}_i(\Gamma, \mathbb{Z}) \cong \mathrm{H}^{n-i}(\Gamma, \mathbb{Z}) \cong \mathrm{Ext}(\mathrm{H}_{n-i-1}(\Gamma, \mathbb{Z}), \mathbb{Z}) \cong \mathrm{H}_{n-i-1}(\Gamma, \mathbb{Z})$. From the exact sequence it also follows that $\mathrm{H}_{n-1}(\Gamma, \mathbb{Z}) \cong \mathrm{H}^1(\Gamma, \mathbb{Z}) \cong \mathrm{Hom}(\mathrm{H}_1(\Gamma, \mathbb{Z}), \mathbb{Z}) = 0$.

Let L denote the lattice subgroup $\mathbb{Z}^n$ of Γ and G be the holonomy group $\mathbb{Z}_2^{n-1}$. To show (i), consider the extension

$$0 \longrightarrow L \longrightarrow \Gamma \xrightarrow{\phi} G \longrightarrow 1,$$

and its associate Lyndon-Hochschild-Serre spectral sequence

$$\mathrm{E}_2^{p,q}(\mathbb{Q}) = \mathrm{H}^p(G, \mathrm{H}^q(L, \mathbb{Q})) \Longrightarrow \mathrm{H}^{p+q}(\Gamma, \mathbb{Q}).$$

We would like to show that $\mathrm{E}_2^{p,q}(\mathbb{Q}) = 0$ unless $(p, q) = (0, 0)$ or $(p, q) = (0, n)$. This would imply $\mathrm{H}^i(\Gamma, \mathbb{Q}) = 0$ for all $1 \leq i \leq n-1$. In view of the previous lemma, Poincaré Duality would give us the desired result.

Recall that the total homology $\mathrm{H}_*(L, \mathbb{Z})$ with the Pontryagin product forms an algebra isomorphic to the exterior algebra $\wedge^*(L)$ (see [1], Theorem 6.4 on page 123). Since $\mathrm{H}^p(G, \mathrm{H}^q(L, \mathbb{Q})) = 0$ for $p > 0$, the only elements that can survive past the second page of the spectral sequence lie in $\mathrm{E}_2^{0,q}(\mathbb{Q}) = \mathrm{H}^0(G, \mathrm{H}^q(L, \mathbb{Q})) \cong \mathrm{H}^q(L, \mathbb{Q})^G \cong \mathrm{Hom}(\mathrm{H}_q(L, \mathbb{Z}), \mathbb{Q})^G \cong (\wedge^q(L^*) \otimes \mathbb{Q})^G \cong \wedge^q(L^*)^G \otimes \mathbb{Q}$, where $\wedge^q(L^*)$ denotes the q-th exterior power of the dual $\mathbb{Z}G$-module $L^* = \mathrm{Hom}(L, \mathbb{Z})$. Let $a_1, a_2, \ldots, a_n$ be a set of generators of $\wedge^1(L^*) \otimes \mathbb{Q}$. Then, $\wedge^q(L^*) \otimes \mathbb{Q}$ has basis elements $a_{i_1 \ldots i_q} = a_{i_1} \wedge \cdots \wedge a_{i_q}$, where $1 \leq q \leq n-1$ and $1 \leq i_1 < \cdots < i_q \leq n$. We observe that $\wedge^1(L^*) \cong L^*$, and for an arbitrary element $g \in G$, $ga_i = \delta_i^g a_i$ where $\delta_i^g \in \{\pm 1\}$, $1 \leq i \leq n$. Suppose $x = \sum_{i=1}^q k_i a_{i_1 \ldots i_q} \in \wedge^q(L^*) \otimes \mathbb{Q}$ where $k_i \in \mathbb{Z}$, $k_i \neq 0$, and $Gx = x$. This means that for all $g \in G$, $\sum_{i=1}^q k_i a_{i_1 \ldots i_q} = g \sum_{i=1}^q k_i a_{i_1 \ldots i_q} = \sum_{i=1}^q k_i g a_{i_1 \ldots i_q} = \sum_{i=1}^q (\delta_{i_1}^g \cdots \delta_{i_q}^g) k_i a_{i_1 \ldots i_q}$. It follows that $\delta_{i_1}^g \cdots \delta_{i_q}^g = 1$ for all $g \in G$. Since $\rho(G)$ is the subgroup of $\mathrm{SL}(n, \mathbb{Z})$ consisting of all the diagonal matrices, for a given $g \in G$ and $1 \leq m \leq n$, we can always find $h \in G$ such that $\delta_i^h = -\delta_i^g$, $\forall i \neq m$ and $\delta_m^h = \delta_m^g$. By an appropriate choice of m this implies $\delta_{i_1}^h \cdots \delta_{i_q}^h = -1$, which is a contradiction. Therefore $\wedge^q(L^*)^G \otimes \mathbb{Q} = 0$ for all $0 < q < n$.

□

3. Algorithm for Computing Homology

Let Γ be an n-dimensional Bieberbach group. Let us assume that the canonical action of Γ on $\mathbb{R}^n$ is given by a diagonal matrix $L(\gamma) \in \mathrm{GL}(n, \mathbb{Z})$ and by a translation vector $T(\gamma) \in \mathbb{R}^n$ with coordinates $\frac{m}{2}$, $m \in \mathbb{Z}$, in a natural way

$$\forall x \in \mathbb{R}^n, \forall \gamma \in \Gamma : \gamma x = L(\gamma)x + T(\gamma).$$

We will now describe an algorithm for computing the homology of Bieberbach groups which can be presented as above. Note that this is the case when Γ is an HW-group. The reader may find it helpful to refer to section 4, where we explicitly illustrate the algorithm in the case of the 3-dimensional HW-group.

First, we define a CW-complex structure X on $\mathbb{R}^n$ given by the k-cubes $\widetilde{\sigma}^k$, $0 \leq k \leq n$, constructed by slicing $\mathbb{R}^n$ with the hyperplanes $x_i = \frac{j}{2}$, $\forall j \in \mathbb{Z}$. That is, the cubes are of the form

$$\widetilde{\sigma}^k = \left[\frac{p_1}{2}, \frac{q_1}{2}\right] \times \cdots \times \left[\frac{p_n}{2}, \frac{q_n}{2}\right],$$

where $p_1, \ldots, p_n \in \mathbb{Z}$ and $q_i = p_i$ or $q_i = p_i + 1$, for all $1 \leq i \leq n$. It is not difficult to observe that the dimension k of a given k-cube is the number of q_i-s such that $q_i \neq p_i$ for $1 \leq i \leq n$.

Any element of Γ acts by freely permuting all the k-dimensional cubes. Since X is a contractible CW-complex, we can form a free $\mathbb{Z}\Gamma$-resolution, $\epsilon : C_*(X) \to \mathbb{Z}$. The homology groups of Γ are then given by

$$\mathrm{H}_i(\Gamma) = \mathrm{H}_i(C_*(X) \otimes_{\mathbb{Z}\Gamma} \mathbb{Z}).$$

Let L be the lattice subgroup $\mathbb{Z}^n$ of Γ. Then the quotient space $\mathbb{R}^n/L$ is an n-dimensional torus $\mathbb{T}^n$. It is a CW-complex Y with the cell structure induced from X. Note that the holonomy group G of Γ acts freely on Y, and $C_*(X) \otimes_{\mathbb{Z}\Gamma} \mathbb{Z} \cong C_*(Y) \otimes_{\mathbb{Z}G} \mathbb{Z}$. This shows

$$\mathrm{H}_i(\Gamma) = \mathrm{H}_i(C_*(Y) \otimes_{\mathbb{Z}G} \mathbb{Z}).$$

We can view $\mathbb{T}^n$ as the unit n-cube $[0,1]^n$ with the opposite hyperplane segments identified, i.e. $(t_1, \ldots, t_{i-1}, 0, t_{i+1}, \ldots, t_n) \sim (t_1, \ldots, t_{i-1}, 1, t_{i+1}, \ldots, t_n)$ for all $1 \leq i \leq n$. Then Y is the complex consisting of 4^n cubes. A k-cube $\sigma^k \in Y$ is given by

$$\sigma^k = \left[\frac{p_1}{2}, \frac{q_1}{2}\right] \times \cdots \times \left[\frac{p_n}{2}, \frac{q_n}{2}\right],$$

where $p_i \in \{0, 1\}$ and $q_i \in \{p_i, p_i + 1\}$ for all $1 \leq i \leq n$. We observe that a k-cube σ^k is completely determined by its center point

$$\left(\frac{p_1 + q_1}{4}, \ldots, \frac{p_n + q_n}{4}\right).$$

The action by an element of G permutes two k-cubes and therefore moves the center point of one cube to the center point of the other. Thus, we can identify every k-cube σ^k by its center point

$$\sigma^k = \left(\frac{t_1}{4}, \ldots, \frac{t_n}{4}\right),$$

$t_i \in \{0, 1, 2, 3\}$ for any $1 \leq i \leq n$. With this identification, the dimension k of a k-cube is the number of t_i such that t_i is odd.

Let $p : X \to Y$ denote the cellular map induced by the quotient map $\mathbb{R}^n \to \mathbb{T}^n$. The action of the holonomy group on each cube σ^k can be defined by

$$\forall g \in G, \forall \sigma^k \in \mathbb{T}^n : g\sigma^k = p(\gamma\widetilde{\sigma}^k),$$

for any $\gamma \in \Gamma$ such that $\phi(\gamma) = g$, and for any cube $\widetilde{\sigma}^k$ in X over σ^k.

Our next goal is to find appropriate basis elements for each free $\mathbb{Z}$-modules $C_k = C_k(Y) \otimes_{\mathbb{Z}G} \mathbb{Z}$. For this, we determine which k-cubes of Y are identified by the action of G by using the following looping process.

(i) Number all the cubes σ_i of Y for $1 \leq i \leq 4^n$ using the identification by the center points. Let $i = 1$.

(ii) Identify all the images of the cube σ_i under the action of G, and fix this cube as the representative of all of its images. Assign a sign δ (see description below) to each image and store this information.

(iii) Consider the next cube σ_{i+1} in the ordering. If there are no more cubes, stop the process.

(iv) If a given cube σ_{i+1} is equivalent to a stored cube go to step (iii). Otherwise, move to step (ii).

In order to compute the boundary maps of the chain complex C_* in step (ii) we also include data about the orientation of the action.

Let all the cubes of the complex X have orientations induced from a fixed orientation on $\mathbb{R}^n$. This gives us well defined orientations on the cubes of the complex Y. Suppose τ^k is a representative k-cube, and for some $g \in G$, $g\tau^k{=}\sigma^k$. The homeomorphism given by g is either orientation preserving or reversing. We then attach a positive or negative sign to σ^k respectively, i.e. $\delta^k \in \{\pm 1\}$ and $\delta^k \sigma^k$ is stored. To determine δ^k is rather easy. Let $\rho : G \to \mathrm{GL}(n, \mathbb{Z})$ be the holonomy representation and let $\rho(g) = L(\gamma)$ for $\gamma \in \Gamma$ such that $\phi(\gamma) = g$. The orientation is determined by the sign of the determinant of the submatrix of $\rho(g)$ corresponding to the odd numerators of the coordinate fractions of $\tau^k = \{\frac{a_1}{4}, \ldots, \frac{a_n}{4}\}$. It is positive if the number of (-1)-s that lie on the diagonal of the matrix $\rho(g)$ corresponding to the odd numerators is even, otherwise it is negative.

Our next task is to determine the images of the boundary maps $\partial_k : C_k \to C_{k-1}$ for the representative k-cubes, which form a basis of C_k. This homomorphism is defined by

$$\partial_k(\tau^k) = \sum_{j=1}^{k} (-1)^j \left[F_j(\tau^k) - B_j(\tau^k) \right],$$

where $F_j(\tau^k)$ is the front j-face and $B_j(\tau^k)$ is the back j-face of τ^k. If $\tau^k = \{\frac{a_1}{4}, \ldots, \frac{a_n}{4}\}$, with a_{p_j} odd, for $j = 1, \ldots, k$, then

$$F_j(\tau^k) = \left\{ \frac{a_1}{4}, \ldots, \frac{a_{p_j - 1}}{4}, \frac{a_{p_j} - 1}{4}, \frac{a_{p_j + 1}}{4}, \ldots, \frac{a_n}{4} \right\},$$

$$B_j(\tau^k) = \left\{ \frac{a_1}{4}, \dots, \frac{a_{p_j-1}}{4}, \frac{a_{p_j}+1}{4}, \frac{a_{p_j+1}}{4}, \dots, \frac{a_n}{4} \right\}.$$

We can identify these $(k-1)$-cubes by their representatives in the G-orbits with the appropriate sign depending on the orientation preserving or reversing of the action.

(v) For every $j = 1, \dots, k$, identify $F_j(\tau^k)$ and $B_j(\tau^k)$ with their respective representative $(k-1)$-cubes $\delta_j^k \tau_j^{k-1}$ and $\delta_j'^k \tau_j'^{k-1}$.

(vi) For every representative k-cube τ^k, compute the image

$$\partial_k(\tau^k) = \sum_{j=1}^{k} (-1)^j \left[\delta_j^k \tau_j^{k-1} - \delta_j'^k \tau_j'^{k-1} \right],$$

and assign the final coordinates to a row of a matrix A_k representing the boundary map $\partial_k : C_k \to C_{k-1}$ as a homomorphism of based free-$\mathbb{Z}$-modules.

To calculate $\mathrm{H}_k(\Gamma) = \ker \partial_k / \mathrm{Im}\partial_{k+1}$, we observe that $C_k / \ker \partial_k \cong \mathrm{Im}\partial_k$. Therefore, $C_k \cong \ker \partial_k \oplus \mathrm{Im}\partial_k$ (free $\mathbb{Z}$-modules of the same finite rank). Moreover, since $\mathrm{Im}\partial_{k+1} \subseteq \ker \partial_k$, we have that $C_k/\mathrm{Im}\partial_{k+1} \cong \ker \partial_k/\mathrm{Im}\partial_{k+1} \oplus \mathrm{Im}\partial_k \cong \mathrm{H}_k(\Gamma) \oplus \mathrm{Im}\partial_k$, where $\mathrm{Im}\partial_k$ is a free $\mathbb{Z}$-submodule of C_{k-1}.

After step (vi), we can compute the rank r_k of A_k and thus, find $\mathrm{Im}\partial_k = \mathbb{Z}^{r_k}$.

(vii) Compute the rank r_k of the matrix A_k representing $\partial_k : C_k \to C_{k-1}$.

Now it is only left to determine the quotient $C_k/\mathrm{Im}\partial_{k+1} \cong C_k/\mathrm{Im}(A_{k+1})$. For this we use an algorithm of reducing the matrix A_k to the Smith normal form. Since most mathematical software has this reduction algorithm, we only give a brief outline here and refer the reader to [12] for more detailed description.

Definition 3.1. An integer matrix S is in *Smith normal form*, if for some $t \geq 0$, the entries s_{jj} are positive for all $1 \leq j \leq t$, they are the only nonzero entries of S, and s_{jj} divides $s_{(j+1)(j+1)}$ for all $1 \leq j < t$.

The Smith normal matrix S_k is obtained from A_k by the following integer row and column operations.

(1) Interchange two rows(columns).

(2) Multiply a row(column) by -1.

(3) Add an integer multiple of a row(column) to another row(column).

Suppose A and B are arbitrary $p \times q$-integer matrices. Then they are said to be *equivalent over* $\mathbb{Z}$ if one can be obtained from the other by a sequence of row and column operations. If A is equivalent to a matrix B, then $\mathbb{Z}^q/\mathrm{Im}(A) \cong \mathbb{Z}^q/\mathrm{Im}(B)$.

Every integer matrix is equivalent to a unique matrix in a Smith normal form, and for any given $p \times q$-matrix S in Smith normal form it is trivial to see the structure of the cokernel $\mathbb{Z}^q/\mathrm{Im}(S)$.

One way to reduce a given $p \times q$-matrix A_{k+1} to the Smith normal form S_{k+1} is by the following steps. First use row and column operations to reduce A_{k+1} to a matrix where the entry a_{11} is positive and divides all entries in row 1 and column 1. Then the operation of type (3) can be used to make $a_{j1} = a_{1j} = 0$ for all $j > 1$. By repeating this process for the subsequent submatrices we get a matrix where the only nonzero entries are $d_j = a_{jj}$, for all $j = 1, \dots, t$, and $t \leq \min(p, q)$. Using integer row and column operations, d_j and d_{j+1} can be changed such that d_j divides d_{j+1} for all $j < t$. This gives us the Smith normal matrix S_{k+1}. The cokernel $C_k/\mathrm{Im}(S_{k+1})$ is then isomorphic to the direct sum $\mathbb{Z}_{d_1} \oplus \cdots \oplus \mathbb{Z}_{d_t} \oplus \mathbb{Z}^{q-t}$. If $d_j = 1$ for all $1 \leq j \leq s$, we have

$$\mathrm{H}_k(\Gamma) = \mathbb{Z}_{d_{s+1}} \oplus \cdots \oplus \mathbb{Z}_{d_t} \oplus \mathbb{Z}^{q-t-r_k},$$

where $q = \mathrm{rk}(C_k) = \dfrac{2^n}{|G|}\dbinom{n}{k}$ and $\{d_j | s < j \leq t\}$ is an increasing sequence of powers of 2.

The final step of the algorithm can be stated as follows.

(viii) Reduce the matrix A_{k+1} to Smith normal form $S_{k+1} = (d_1, \dots, d_t)$. Store $\mathrm{H}_k(\Gamma) = \mathbb{Z}_{d_{s+1}} \oplus \cdots \oplus \mathbb{Z}_{d_t} \oplus \mathbb{Z}^{q-t-r_k}$.

Remark 3.2. Recall from Lemma 2.8, that when Γ is an n-dimensional HW-group, then $\mathrm{H}_k(\Gamma)$ is all torsion for every $1 \leq k \leq n-1$. This implies $q - t - r_k = 0$. Hence, in this case it is not necessary to compute the rank r_k when $1 \leq k \leq n-1$. Although we do not assume this information in our computation, the algorithm can be modified to take into account such facts about the homology of HW-groups. Alternatively, they can be computed as we have stated in the algorithm to check the validity of the results and to find manual errors.

In summary, we have proved the following theorem in this section.

Theorem 3.3. Let Γ be an n-dimensional HW-group with a standard presentation. Then, there is an algorithm that computes the integral homology of Γ from the given generators of Γ.

4. Example of Didicosm

Next, we implement the algorithm to compute the homology of the 3-dimensional HW-group Γ. This group is unique up to affine isomorphisms and it is the lowest dimensional

HW-group. Although, the homology of Γ is known, our computations illustrate how the algorithm works in such a simple yet characteristic case.

The group Γ fits into a short exact sequence

$$0 \longrightarrow \mathbb{Z}^3 \longrightarrow \Gamma \xrightarrow{\phi} \mathbb{Z}_2^2 \longrightarrow 1.$$

It can be represented as a discrete subgroup of $\mathrm{GL}(4,\mathbb{R})$, generated by the matrices

$$\alpha_1 = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & \frac{1}{2} \\ 0 & 0 & -1 & \frac{1}{2} \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad \alpha_2 = \begin{pmatrix} 1 & 0 & 0 & \frac{1}{2} \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$v_1 = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad v_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad v_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Γ acts crystallographically on $\mathbb{R}^3 \subset \mathbb{R}^4$ through its representation in $\mathrm{GL}(4,\mathbb{R})$, where $\mathbb{R}^3$ is the subspace defined by the hyperplane $\{(x_1,x_2,x_3,1)|x_1,x_2,x_3 \in \mathbb{R}\}$. The lattice subgroup $\mathbb{Z}^3$ is the free abelian group $\langle v_1,v_2,v_3\rangle$, and the quotient $\mathbb{R}^3/\mathbb{Z}^3$ is the 3-dimensional torus $\mathbb{T}^3$. It has a cell structure given by 64 cubes

$$\sigma^k = \left(\frac{t_1}{4},\frac{t_2}{4},\frac{t_3}{4}\right),$$

$t_i \in \{0,1,2,3\}$, $0 \le k \le 3$. Let G be the holonomy group $\mathbb{Z}_2^2$. This group is generated by the elements $\phi(\alpha_1)$ and $\phi(\alpha_2)$, and it acts on $\mathbb{T}^3$ by freely permuting all the k-cubes for every k. This action can be defined explicitly by

$$\forall g \in G, \forall \sigma^k \in \mathbb{T}^3 : g\sigma^k = p(\gamma\widetilde{\sigma}^k),$$

for all $\gamma \in \Gamma$ such that $\phi(\gamma) = g$, and for any cube $\widetilde{\sigma}^k$ in $\mathbb{R}^3$ over σ^k.

Let $C_*(\mathbb{T}^3)$ be the chain complex associated to the cubical complex defined by σ^k. Then each chain group is a free $\mathbb{Z}G$-module, and the homology of Γ can be computed by the formula

$$\mathrm{H}_k(\Gamma,\mathbb{Z}) = \mathrm{H}_k\left(C_*(\mathbb{T}^3) \otimes_{\mathbb{Z}G} \mathbb{Z}\right).$$

Steps i – iv of the algorithm are summarized by the table below that shows the representative cubes and their G-orbits. In the table we also identify the orientation δ of the induced homeomorphism $g : \tau \to g\tau$ defined by the action of an element $g \in G$.

G-ORBITS OF REPRESENTATIVE CUBES

$\delta g\tau$	1	$\phi(\alpha_1)$	$\phi(\alpha_2)$	$\phi(\alpha_1\alpha_2)$
τ_1^0	$\left(\frac{0}{4},\frac{0}{4},\frac{0}{4}\right)$	$\left(\frac{0}{4},\frac{2}{4},\frac{2}{4}\right)$	$\left(\frac{2}{4},\frac{0}{4},\frac{0}{4}\right)$	$\left(\frac{2}{4},\frac{2}{4},\frac{2}{4}\right)$
τ_2^0	$\left(\frac{0}{4},\frac{0}{4},\frac{2}{4}\right)$	$\left(\frac{0}{4},\frac{2}{4},\frac{0}{4}\right)$	$\left(\frac{2}{4},\frac{0}{4},\frac{2}{4}\right)$	$\left(\frac{2}{4},\frac{2}{4},\frac{0}{4}\right)$
τ_1^1	$\left(\frac{0}{4},\frac{0}{4},\frac{1}{4}\right)$	$-\left(\frac{0}{4},\frac{2}{4},\frac{1}{4}\right)$	$-\left(\frac{2}{4},\frac{0}{4},\frac{3}{4}\right)$	$\left(\frac{2}{4},\frac{2}{4},\frac{3}{4}\right)$
τ_2^1	$\left(\frac{0}{4},\frac{0}{4},\frac{3}{4}\right)$	$-\left(\frac{0}{4},\frac{2}{4},\frac{3}{4}\right)$	$-\left(\frac{2}{4},\frac{0}{4},\frac{1}{4}\right)$	$\left(\frac{2}{4},\frac{2}{4},\frac{1}{4}\right)$
τ_3^1	$\left(\frac{0}{4},\frac{1}{4},\frac{0}{4}\right)$	$\left(\frac{0}{4},\frac{3}{4},\frac{2}{4}\right)$	$-\left(\frac{2}{4},\frac{3}{4},\frac{0}{4}\right)$	$-\left(\frac{2}{4},\frac{1}{4},\frac{2}{4}\right)$
τ_4^1	$\left(\frac{0}{4},\frac{1}{4},\frac{2}{4}\right)$	$\left(\frac{0}{4},\frac{3}{4},\frac{0}{4}\right)$	$-\left(\frac{2}{4},\frac{3}{4},\frac{2}{4}\right)$	$-\left(\frac{2}{4},\frac{1}{4},\frac{0}{4}\right)$
τ_5^1	$\left(\frac{1}{4},\frac{0}{4},\frac{0}{4}\right)$	$-\left(\frac{3}{4},\frac{2}{4},\frac{2}{4}\right)$	$\left(\frac{3}{4},\frac{0}{4},\frac{0}{4}\right)$	$-\left(\frac{1}{4},\frac{2}{4},\frac{2}{4}\right)$
τ_6^1	$\left(\frac{1}{4},\frac{0}{4},\frac{2}{4}\right)$	$-\left(\frac{3}{4},\frac{2}{4},\frac{0}{4}\right)$	$\left(\frac{3}{4},\frac{0}{4},\frac{2}{4}\right)$	$-\left(\frac{1}{4},\frac{2}{4},\frac{0}{4}\right)$
τ_1^2	$\left(\frac{0}{4},\frac{1}{4},\frac{1}{4}\right)$	$-\left(\frac{0}{4},\frac{3}{4},\frac{1}{4}\right)$	$\left(\frac{2}{4},\frac{3}{4},\frac{3}{4}\right)$	$-\left(\frac{2}{4},\frac{1}{4},\frac{3}{4}\right)$
τ_2^2	$\left(\frac{0}{4},\frac{1}{4},\frac{3}{4}\right)$	$-\left(\frac{0}{4},\frac{3}{4},\frac{3}{4}\right)$	$\left(\frac{2}{4},\frac{3}{4},\frac{1}{4}\right)$	$-\left(\frac{2}{4},\frac{1}{4},\frac{1}{4}\right)$
τ_3^2	$\left(\frac{1}{4},\frac{0}{4},\frac{1}{4}\right)$	$\left(\frac{3}{4},\frac{2}{4},\frac{1}{4}\right)$	$-\left(\frac{3}{4},\frac{0}{4},\frac{3}{4}\right)$	$-\left(\frac{1}{4},\frac{2}{4},\frac{3}{4}\right)$
τ_4^2	$\left(\frac{1}{4},\frac{0}{4},\frac{3}{4}\right)$	$\left(\frac{3}{4},\frac{2}{4},\frac{3}{4}\right)$	$-\left(\frac{3}{4},\frac{0}{4},\frac{1}{4}\right)$	$-\left(\frac{1}{4},\frac{2}{4},\frac{1}{4}\right)$
τ_5^2	$\left(\frac{1}{4},\frac{1}{4},\frac{0}{4}\right)$	$-\left(\frac{3}{4},\frac{3}{4},\frac{2}{4}\right)$	$-\left(\frac{3}{4},\frac{3}{4},\frac{0}{4}\right)$	$\left(\frac{1}{4},\frac{1}{4},\frac{2}{4}\right)$
τ_6^2	$\left(\frac{1}{4},\frac{3}{4},\frac{0}{4}\right)$	$-\left(\frac{3}{4},\frac{1}{4},\frac{2}{4}\right)$	$-\left(\frac{3}{4},\frac{1}{4},\frac{0}{4}\right)$	$\left(\frac{1}{4},\frac{3}{4},\frac{2}{4}\right)$
τ_1^3	$\left(\frac{1}{4},\frac{1}{4},\frac{1}{4}\right)$	$\left(\frac{3}{4},\frac{3}{4},\frac{1}{4}\right)$	$\left(\frac{3}{4},\frac{3}{4},\frac{3}{4}\right)$	$\left(\frac{1}{4},\frac{1}{4},\frac{3}{4}\right)$
τ_2^3	$\left(\frac{1}{4},\frac{3}{4},\frac{1}{4}\right)$	$\left(\frac{3}{4},\frac{1}{4},\frac{1}{4}\right)$	$\left(\frac{3}{4},\frac{1}{4},\frac{3}{4}\right)$	$\left(\frac{1}{4},\frac{3}{4},\frac{3}{4}\right)$

Let $C_k = C_k(\mathbb{T}^3) \otimes_{\mathbb{Z}G} \mathbb{Z}$, $0 \leq k \leq 3$. The representative k-cubes τ_*^k form a basis for the chain groups C_k. To compute the matrices A_k representing the boundary maps

$\partial_k : C_k \to C_{k-1}$ (steps v and vi), we first compute $\partial_k(\tau^k_*)$.

$$\partial_1(\tau^1_1) = -F_1(\tau^1_1) + B_1(\tau^1_1) = -\left(\frac{0}{4}, \frac{0}{4}, \frac{0}{4}\right) + \left(\frac{0}{4}, \frac{0}{4}, \frac{2}{4}\right) = -\tau^0_1 + \tau^0_2,$$

$$\partial_1(\tau^1_2) = -F_1(\tau^1_2) + B_1(\tau^1_2) = -\left(\frac{0}{4}, \frac{0}{4}, \frac{2}{4}\right) + \left(\frac{0}{4}, \frac{0}{4}, \frac{0}{4}\right) = -\tau^0_2 + \tau^0_1,$$

$$\partial_1(\tau^1_3) = -F_1(\tau^1_3) + B_1(\tau^1_3) = -\left(\frac{0}{4}, \frac{0}{4}, \frac{0}{4}\right) + \left(\frac{0}{4}, \frac{2}{4}, \frac{0}{4}\right) \equiv -\tau^0_1 + \tau^0_2,$$

$$\partial_1(\tau^1_4) = -F_1(\tau^1_4) + B_1(\tau^1_4) = -\left(\frac{0}{4}, \frac{0}{4}, \frac{2}{4}\right) + \left(\frac{0}{4}, \frac{2}{4}, \frac{2}{4}\right) \equiv -\tau^0_2 + \tau^0_1,$$

$$\partial_1(\tau^1_5) = -F_1(\tau^1_5) + B_1(\tau^1_5) = -\left(\frac{0}{4}, \frac{0}{4}, \frac{0}{4}\right) + \left(\frac{2}{4}, \frac{0}{4}, \frac{0}{4}\right) \equiv 0,$$

$$\partial_1(\tau^1_6) = -F_1(\tau^1_6) + B_1(\tau^1_6) = -\left(\frac{0}{4}, \frac{0}{4}, \frac{2}{4}\right) + \left(\frac{2}{4}, \frac{0}{4}, \frac{2}{4}\right) \equiv 0,$$

$$\begin{aligned}\partial_1(\tau^2_1) &= -F_1(\tau^2_1) + B_1(\tau^2_1) + F_2(\tau^2_1) - B_2(\tau^2_1)\\ &= -\left(\frac{0}{4}, \frac{0}{4}, \frac{1}{4}\right) + \left(\frac{0}{4}, \frac{2}{4}, \frac{1}{4}\right) + \left(\frac{0}{4}, \frac{1}{4}, \frac{0}{4}\right) - \left(\frac{0}{4}, \frac{1}{4}, \frac{2}{4}\right)\\ &\equiv -2\tau^1_1 + \tau^1_3 - \tau^1_4,\end{aligned}$$

$$\begin{aligned}\partial_1(\tau^2_2) &= -F_1(\tau^2_2) + B_1(\tau^2_2) + F_2(\tau^2_2) - B_2(\tau^2_2)\\ &= -\left(\frac{0}{4}, \frac{0}{4}, \frac{3}{4}\right) + \left(\frac{0}{4}, \frac{2}{4}, \frac{3}{4}\right) + \left(\frac{0}{4}, \frac{1}{4}, \frac{2}{4}\right) - \left(\frac{0}{4}, \frac{1}{4}, \frac{0}{4}\right)\\ &\equiv -2\tau^1_2 + \tau^1_4 - \tau^1_3,\end{aligned}$$

$$\begin{aligned}\partial_1(\tau^2_3) &= -F_1(\tau^2_3) + B_1(\tau^2_3) + F_2(\tau^2_3) - B_2(\tau^2_3)\\ &= -\left(\frac{0}{4}, \frac{0}{4}, \frac{1}{4}\right) + \left(\frac{2}{4}, \frac{0}{4}, \frac{1}{4}\right) + \left(\frac{1}{4}, \frac{0}{4}, \frac{0}{4}\right) - \left(\frac{1}{4}, \frac{0}{4}, \frac{2}{4}\right)\\ &\equiv -\tau^1_1 - \tau^1_2 + \tau^1_5 - \tau^1_6,\end{aligned}$$

$$\begin{aligned}\partial_1(\tau^2_4) &= -F_1(\tau^2_4) + B_1(\tau^2_4) + F_2(\tau^2_4) - B_2(\tau^2_4)\\ &= -\left(\frac{0}{4}, \frac{0}{4}, \frac{3}{4}\right) + \left(\frac{2}{4}, \frac{0}{4}, \frac{3}{4}\right) + \left(\frac{1}{4}, \frac{0}{4}, \frac{2}{4}\right) - \left(\frac{1}{4}, \frac{0}{4}, \frac{0}{4}\right)\\ &\equiv -\tau^1_2 - \tau^1_1 + \tau^1_6 - \tau^1_5,\end{aligned}$$

$$\begin{aligned}\partial_1(\tau_5^2) &= -F_1(\tau_5^2) + B_1(\tau_5^2) + F_2(\tau_5^2) - B_2(\tau_5^2)\\ &= -\left(\frac{0}{4},\frac{1}{4},\frac{0}{4}\right) + \left(\frac{2}{4},\frac{1}{4},\frac{0}{4}\right) + \left(\frac{1}{4},\frac{0}{4},\frac{0}{4}\right) - \left(\frac{1}{4},\frac{2}{4},\frac{0}{4}\right)\\ &\equiv -\tau_3^1 - \tau_4^1 + \tau_5^1 + \tau_6^1,\end{aligned}$$

$$\begin{aligned}\partial_1(\tau_6^2) &= -F_1(\tau_6^2) + B_1(\tau_6^2) + F_2(\tau_6^2) - B_2(\tau_6^2)\\ &= -\left(\frac{0}{4},\frac{3}{4},\frac{0}{4}\right) + \left(\frac{2}{4},\frac{3}{4},\frac{0}{4}\right) + \left(\frac{1}{4},\frac{2}{4},\frac{0}{4}\right) - \left(\frac{1}{4},\frac{0}{4},\frac{0}{4}\right)\\ &\equiv -\tau_4^1 - \tau_3^1 - \tau_6^1 - \tau_5^1,\end{aligned}$$

$$\begin{aligned}\partial_1(\tau_1^3) &= -F_1(\tau_1^3) + B_1(\tau_1^3) + F_2(\tau_1^3) - B_2(\tau_1^3) - F_3(\tau_1^3) + B_3(\tau_1^3)\\ &= -\left(\frac{0}{4},\frac{1}{4},\frac{1}{4}\right) + \left(\frac{2}{4},\frac{1}{4},\frac{1}{4}\right) + \left(\frac{1}{4},\frac{0}{4},\frac{1}{4}\right) - \left(\frac{1}{4},\frac{2}{4},\frac{1}{4}\right) - \left(\frac{1}{4},\frac{1}{4},\frac{0}{4}\right) + \left(\frac{1}{4},\frac{1}{4},\frac{2}{4}\right)\\ &\equiv -\tau_1^2 - \tau_2^2 + \tau_3^2 + \tau_4^2,\end{aligned}$$

$$\begin{aligned}\partial_1(\tau_2^3) &= -F_1(\tau_2^3) + B_1(\tau_2^3) + F_2(\tau_2^3) - B_2(\tau_2^3) - F_3(\tau_2^3) + B_3(\tau_2^3)\\ &= -\left(\frac{0}{4},\frac{3}{4},\frac{1}{4}\right) + \left(\frac{2}{4},\frac{3}{4},\frac{1}{4}\right) + \left(\frac{1}{4},\frac{2}{4},\frac{1}{4}\right) - \left(\frac{1}{4},\frac{0}{4},\frac{1}{4}\right) - \left(\frac{1}{4},\frac{3}{4},\frac{0}{4}\right) + \left(\frac{1}{4},\frac{3}{4},\frac{2}{4}\right)\\ &\equiv \tau_1^2 + \tau_2^2 - \tau_4^2 - \tau_3^2.\end{aligned}$$

Following the final two steps of the algorithm, from the above computations we determine A_k, $0 \le k \le 3$.

$$A_1 = \begin{pmatrix} -1 & 1\\ 1 & -1\\ -1 & 1\\ 1 & -1\\ 0 & 0\\ 0 & 0 \end{pmatrix} A_2 = \begin{pmatrix} -2 & 0 & 1 & -1 & 0 & 0\\ 0 & -2 & -1 & 1 & 0 & 0\\ -1 & -1 & 0 & 0 & 1 & -1\\ -1 & -1 & 0 & 0 & -1 & 1\\ 0 & 0 & -1 & -1 & 1 & 1\\ 0 & 0 & -1 & -1 & -1 & -1 \end{pmatrix} A_3 = \begin{pmatrix} -1 & -1 & 1 & 1 & 0 & 0\\ 1 & 1 & -1 & -1 & 0 & 0 \end{pmatrix}$$

Next, we reduce each matrix to Smith normal form.

$$S_1 = \begin{pmatrix} 1 & 0\\ 0 & 0\\ 0 & 0\\ 0 & 0\\ 0 & 0\\ 0 & 0 \end{pmatrix} S_2 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 4 & 0 & 0\\ 0 & 0 & 0 & 0 & 4 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} S_3 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Recall that for the Smith normal matrix $S_{k+1} = (d_1, \dots, d_t)$ of A_{k+1}, we have $\mathrm{H}_k(\Gamma) = \mathbb{Z}_{d_{s+1}} \oplus \cdots \oplus \mathbb{Z}_{d_t} \oplus \mathbb{Z}^{q-t-r_k}$, where $q = \mathrm{rk}(C_k) = 2\binom{3}{k}$ and r_k is the rank of S_k. The homology groups can now be computed as follows.

$$
\begin{aligned}
\mathrm{H}_0(\Gamma,\mathbb{Z}) &= \mathbb{Z}^{2-1-0} = \mathbb{Z},\\
\mathrm{H}_1(\Gamma,\mathbb{Z}) &= \mathbb{Z}_4 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}^{6-5-1} = \mathbb{Z}_4^2,\\
\mathrm{H}_2(\Gamma,\mathbb{Z}) &= \mathbb{Z}^{6-1-5} = 0,\\
\mathrm{H}_3(\Gamma,\mathbb{Z}) &= \mathbb{Z}^{2-0-1} = \mathbb{Z}.
\end{aligned}
$$

5. Applications to Low Dimensions

In dimension 5, up to affine isomorphisms, there are only two HW-groups. Each of these groups fits into a short exact sequence

$$0 \longrightarrow \mathbb{Z}^5 \longrightarrow \Gamma_i \xrightarrow{\phi_i} G_i \cong \mathbb{Z}_2^4 \longrightarrow 1,$$

$1 \leq i \leq 2$. We consider their standard representations in $\mathrm{GL}(6,\mathbb{R})$ generated by the matrices

$$
\alpha_1 = \begin{pmatrix} 1&0&0&0&0&0\\ 0&1&0&0&0&0\\ 0&0&1&0&0&\frac12\\ 0&0&0&-1&0&\frac12\\ 0&0&0&0&-1&0\\ 0&0&0&0&0&1 \end{pmatrix}
\alpha_2 = \begin{pmatrix} 1&0&0&0&0&0\\ 0&1&0&0&0&\frac12\\ 0&0&-1&0&0&0\\ 0&0&0&-1&0&0\\ 0&0&0&0&1&0\\ 0&0&0&0&0&1 \end{pmatrix}
\beta_2 = \begin{pmatrix} 1&0&0&0&0&0\\ 0&1&0&0&0&\frac12\\ 0&0&-1&0&0&0\\ 0&0&0&-1&0&\frac12\\ 0&0&0&0&1&\frac12\\ 0&0&0&0&0&1 \end{pmatrix}
$$

$$
\alpha_3 = \begin{pmatrix} -1&0&0&0&0&0\\ 0&1&0&0&0&0\\ 0&0&1&0&0&0\\ 0&0&0&-1&0&0\\ 0&0&0&0&1&\frac12\\ 0&0&0&0&0&1 \end{pmatrix}
\alpha_4 = \begin{pmatrix} 1&0&0&0&0&\frac12\\ 0&-1&0&0&0&0\\ 0&0&-1&0&0&0\\ 0&0&0&1&0&0\\ 0&0&0&0&1&0\\ 0&0&0&0&0&1 \end{pmatrix}
$$

$$
v_1 = \begin{pmatrix} 1&0&0&0&0&1\\ 0&1&0&0&0&0\\ 0&0&1&0&0&0\\ 0&0&0&1&0&0\\ 0&0&0&0&1&0\\ 0&0&0&0&0&1 \end{pmatrix}
v_2 = \begin{pmatrix} 1&0&0&0&0&0\\ 0&1&0&0&0&1\\ 0&0&1&0&0&0\\ 0&0&0&1&0&0\\ 0&0&0&0&1&0\\ 0&0&0&0&0&1 \end{pmatrix}
v_3 = \begin{pmatrix} 1&0&0&0&0&0\\ 0&1&0&0&0&0\\ 0&0&1&0&0&1\\ 0&0&0&1&0&0\\ 0&0&0&0&1&0\\ 0&0&0&0&0&1 \end{pmatrix}
$$

$$v_4 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} v_5 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

where $\Gamma_1 = \langle \alpha_1, \alpha_2, \alpha_3, \alpha_4, v_1, v_2, v_3, v_4, v_5 \rangle$ and $\Gamma_2 = \langle \alpha_1, \beta_2, \alpha_3, \alpha_4, v_1, v_2, v_3, v_4, v_5 \rangle$. The subgroup $\mathbb{Z}^5 = \langle v_1, v_2, v_3, v_4, v_5 \rangle$ is the lattice subgroup of both Γ_1 and Γ_2. The holonomy groups G_1 and G_2 are generated respectively by $\{\phi_1(\alpha_1), \phi_1(\alpha_2), \phi_1(\alpha_3), \phi_1(\alpha_4)\}$ and $\{\phi_2(\alpha_1), \phi_2(\beta_2), \phi_2(\alpha_3), \phi_2(\alpha_4)\}$. Γ_i acts crystallographically on $\mathbb{R}^5 \cong \mathbb{R}^5 \times \{1\} \subset \mathbb{R}^6$ through its linear representation, and the quotient $M = \mathbb{R}^5/\Gamma_i$ is a Hantzsche-Wendt manifold. $\mathbb{T}^5 = \mathbb{R}^5/\mathbb{Z}^5$ is a 16-fold cover of M. It has a cell structure given by 1024 cubes

$$\sigma^k = \left(\frac{t_1}{4}, \frac{t_2}{4}, \frac{t_3}{4}, \frac{t_4}{4}, \frac{t_5}{4} \right),$$

$t_i \in \{0, 1, 2, 3\}$, $0 \le k \le 5$. G_i acts on $\mathbb{T}^5$ by freely permuting all the k-cubes. Using the algorithm, we compute the homology of Γ_1 and Γ_2, which turn out to be equal.

Homology of 5-dimensional HW-Groups

$\mathrm{H}_k(\Gamma)$	$k=0$	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$
I	$\mathbb{Z}$	$\mathbb{Z}_2^4$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^4$	0	$\mathbb{Z}$

In dimension 7, HW-groups have been classified by Miatello and Rosetti [9]. Up to affine isomorphisms, there are exactly 62 HW-groups. On p. $386 - 388$ of [9] there is a list of 62 7-dimensional square matrices with entries 0 or 1. Each matrix corresponds to a 7-dimensional HW-group as follows. Let $T_j^i \in \mathbb{R}^n$ be the vector given by the j-th row of the i-th matrix G_i in the list. For $1 \le q \le 7$, let L_q be the diagonal matrix with a 1 on the q-th place of the diagonal and (-1)-s elsewhere on the diagonal. Then, $\Gamma_i = \langle (e_1, I), \ldots, (e_7, I), (T_1^i, L_1), \ldots, (T_6^i, L_6) \rangle \subset \mathbb{R}^n \rtimes \mathrm{O}(n, \mathbb{R})$, $1 \le i \le 62$. For every group Γ_i we compute $\mathrm{H}_*(\Gamma_i, \mathbb{Z})$. The two tables below show all possible homology groups and the homology type of every HW-group corresponding to the list of matrices in [9].

Homology of 7-dimensional HW-Groups

$H_k(\Gamma)$	$k=0$	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$	$k=7$
I	$\mathbb{Z}$	$\mathbb{Z}_2^6$	$\mathbb{Z}_2^8$	$\mathbb{Z}_4^6$	$\mathbb{Z}_2^8$	${\mathbb{Z}_2}^6$	0	$\mathbb{Z}$
II	$\mathbb{Z}$	$\mathbb{Z}_2^6$	$\mathbb{Z}_2^9$	$\mathbb{Z}_2^{10} \oplus \mathbb{Z}_4^2$	$\mathbb{Z}_2^9$	$\mathbb{Z}_2^6$	0	$\mathbb{Z}$
III	$\mathbb{Z}$	$\mathbb{Z}_2^6$	$\mathbb{Z}_2^8$	$\mathbb{Z}_2^4 \oplus \mathbb{Z}_4^4$	$\mathbb{Z}_2^8$	$\mathbb{Z}_2^6$	0	$\mathbb{Z}$
IV	$\mathbb{Z}$	$\mathbb{Z}_2^6$	$\mathbb{Z}_2^8$	$\mathbb{Z}_2^8 \oplus \mathbb{Z}_4^2$	$\mathbb{Z}_2^8$	$\mathbb{Z}_2^6$	0	$\mathbb{Z}$

Homological Types

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
I	II	II	III	II	II	III	IV	IV	III	II	IV	II	IV	II	III	II	IV	III	III
21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40
IV	III	II	II	III	IV	III	IV	IV	I	III	III	III	III	II	III	III	III	II	III
41	42	43	44	45	46	47	48	49	50	51	52	53	54	55	56	57	58	59	60
IV	III	III	I	IV	IV	III	III	IV	III	IV	III	III	IV	III	III	III	III	III	I
61	62																		
III	III																		

6. Further Developments

Our computations show that already in dimension 7 the homology of HW-groups can be different. Yet, for all sixty two HW-groups it is only one of four types. As an interesting generalization one can consider not only the homology groups, but also the ring structure on the homology induced by the *intersection product.*

Let Γ be an n-dimensional HW-group and suppose $M = \mathbb{R}^n/\Gamma$ is the corresponding HW-manifold. Since M is a closed orientable manifold it has a product structure in homology which is dual to the cup product in the cohomology of M. For every $0 \leq k, l \leq n$, the product

$$H_k(M,\mathbb{Z}) \otimes H_l(M,\mathbb{Z}) \longrightarrow H_{k+l-n}(M,\mathbb{Z})$$

is defined by $x \cdot y = (D(x) \cup D(y)) \cap [M]$, where $x \in H_k(M,\mathbb{Z})$, $y \in H_l(M,\mathbb{Z})$, and D is the inverse of the Poincaré Duality $- \cap [M] : H^k(M,\mathbb{Z}) \to H_{n-k}(M,\mathbb{Z})$, $\forall k \geq 0$.

Intersection product was first considered by J. W. Alexander and especially S. Lefschetz long before the introduction of cohomology theory and hence of the cup product. In [6] and [7] Lefschetz defines a method for explicit computations of this product.

Roughly speaking, it can be described as follows. Suppose X is an n-dimensional orientable finite simplicial complex embedded in $\mathbb{R}^m$ for some $m > 0$, and let c^k, c^l be cycles of the chain complex $C_*(X)$. Let c'^k, c'^l be homologous cycles to c^k, c^l, respectively such that c'^k, c'^l are componentwise transverse and they have no common cells on their boundaries. Then $c^k \cdot c^l$ is a $(k + l - n)$-cycle defined by the intersections of the cells of c'^k with the cells of c'^l.

When Γ is an HW-group Γ, there is a CW-complex on M induced by the cubical complex of $\mathbb{R}^n$ as described in section 3. This complex may also be suitable for defining an algorithm for computing the intersection product on $\mathrm{H}_*(\Gamma, \mathbb{Z})$. For any given cycles $c^k, c^l \in C_*(M)$, let $\widetilde{c}^k, \widetilde{c}^l$ be chains in $C_*(\mathbb{R}^n)$ over c^k, c^l, respectively. The problem is to define the product $c^k \cdot c^l$ using the geometric intersection of the cubes in $\mathbb{R}^n$ forming $\widetilde{c}^k$ and $\widetilde{c}^l$.

Problem 6.1. Let Γ be an n-dimensional HW-group with a standard presentation. Find an algorithm that computes the homology ring $\mathrm{H}_*(\Gamma, \mathbb{Z})$ from the generators of Γ.

Note that the homology of didicosm has trivial intersection. In general, the intersection pairing also vanishes for $k + l \leq n$. When $n > 3$ and $k = l = \frac{n+1}{2}$, the product becomes

$$\mathrm{H}_k(M, \mathbb{Z}) \otimes \mathrm{H}_k(M, \mathbb{Z}) \longrightarrow \mathbb{Z}_2^{n-1}.$$

In dimension 5, all the remaining pairings produce trivial products. It is interesting to determine whether this particular product is nonzero and different for the two 5-dimensional HW-groups.

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