

VERTEX CUTS

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ABSTRACT. Given a connected graph, in many cases it is possible to construct a structure tree that provides information about the ends of the graph or its connectivity. For example Stallings' theorem on the structure of groups with more than one end can be proved by analyzing the action of the group on a structure tree and Tutte used a structure tree to investigate finite 2-connected graphs, that are not 3-connected. Most of these structure tree theories have been based on edge cuts, which are components of the graph obtained by removing finitely many edges. A new theory is described here using vertex cuts, components of the graph obtained by removing finitely many vertices. This generalizes Tutte's tree decomposition of 2-connected graphs to k -connected graphs for any k , in finite and infinite graphs. The theory can be applied to non-locally finite graphs with more than one vertex end, i.e. ends that can be separated by removing a finite number of vertices. This gives a decomposition for a group acting on such a graph, generalizing Stallings' theorem. Further applications include the classification of distance transitive graphs and k -CS-graphs.

1. INTRODUCTION

A connected simple graph $X = (VX, EX)$ is said to be n -connected if for every pair u, v of distinct vertices there are n paths joining u to v such that every vertex in $VX \setminus \{u, v\}$ lies on at most one of the paths. If X is not 2-connected then it has *cut-points*, i.e. vertices whose removal disconnects the graph. If this happens, and X has no disconnecting edges, then X decomposes into a collection of maximal 2-connected subgraphs and edges which connect two cut-points. Any two such so-called "2-blocks" intersect in at most one vertex and this vertex will be a cut-point. Every edge of X lies in exactly one 2-block. Associated with this decomposition is a *structure tree* T in which $VT = \mathcal{B} \cup \mathcal{S}$, where \mathcal{S} is the set of cut-points, \mathcal{B} is the set of 2-blocks and there is an edge joining $b \in \mathcal{B}$ with $s \in \mathcal{S}$ if and only if $s \in b$. If G is a group of automorphisms of X , then there is an induced action of G on T . If X is a finite graph then T will be a finite tree and any action on a finite tree is trivial, i.e. there is a vertex or an edge which is fixed by G . This is illustrated in Figure 1. The number next to a vertex of T indicates the order of the subgroup of the automorphism group fixing that vertex. Note the 2-colouring of T , in which white vertices are cut points and black vertices are blocks.

There is a similar decomposition if X is 2-connected but not 3-connected. This was described by Tutte [30] if X is finite and by Droms Servatius and Servatius [4] if X is infinite and locally finite. A somewhat different account is given in [8]. The decomposition gives a structure tree T , which again has a 2-coloring $VT = \mathcal{B} \cup \mathcal{S}$. In this case each vertex $s \in \mathcal{S}$ corresponds to a 2-separator, i.e. a pair of vertices whose removal disconnects the graph. Sets of at least three vertices which never

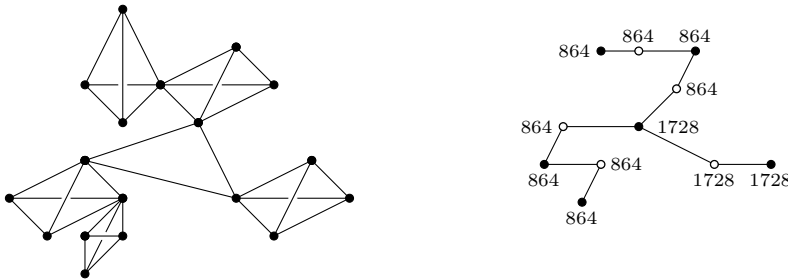


FIGURE 1. One-connected graph and structure tree

lie in different components after removing any two vertices from the graph are called *2-inseparable*. Maximal 2-inseparable sets correspond to vertices of T . But there are also other vertices of the tree which correspond to sets of vertices that, although they can be separated by removing two vertices of the graph, cannot be separated by removing a pair of vertices in the set \mathcal{S} of separators. An example is given in Figure 2. Here the black central vertex of T corresponds to the 4-cycle (x_1, x_2, x_3, x_4) one edge of which, shown dashed in Figure 2a, is what Tutte calls an *ideal edge*. This edge is not in the original graph, and joins the vertices of a 2-separator in \mathcal{S} . Here $\{x_1, x_3\}$ is a 2-separator, not in \mathcal{S} , that separates x_2 and x_4 . In Tutte's structure tree any vertices that do not correspond to maximal 2-inseparable sets, have this cyclic structure.

Note that we could add ideal edges so that all black vertices correspond to complete graphs as in Figure 2b. The corresponding structure tree is shown Figure 2c.

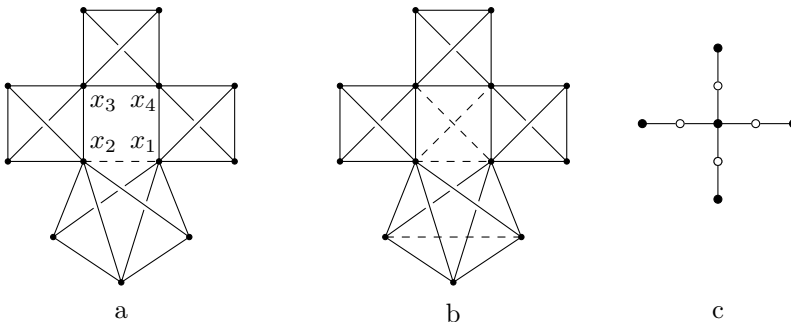


FIGURE 2. Decomposition of a 2-connected graph

In this paper we show that if a graph X has a finite set of vertices whose removal produces at least two components that are large in some sense and G is the automorphism group of X then there is a G -tree (or structure tree) T with a bipartition $(\mathcal{S}, \mathcal{B})$ of the set of vertices $VT = \mathcal{S} \cup \mathcal{B}$ so that the vertices in \mathcal{S} correspond to finite separating sets.

Such a structure tree had been known to exist in the case when X is an infinite graph and X can be disconnected into two infinite components by removing finitely many edges (see [3, 5]).

A *ray* is a sequence of distinct vertices v_0, v_1, \dots such that v_i and v_{i+1} are adjacent for each i . Let $E \subset VX$. The *coboundary* δE is the set of edges that have

one vertex in E and one in $VX \setminus E$. If δE is finite, then E is called an *edge cut*. If R is a ray, then all the terms from a certain point onwards will either lie in E or in $VX \setminus E$. We say that E *separates* rays R_1, R_2 if one of the rays eventually belongs to E and the other eventually belongs to $VX \setminus E$. We say two rays belong to the same edge end if they are not separated by any edge cut. It is easy to see that this is an equivalence relation on the set of rays, and so we can take an equivalence class to be an *edge end*.

In [5] it is shown that if a graph has more than one edge end and k is the smallest integer for which there is an edge cut E with $|\delta E| = k$ that separates two ends, then there is a structure tree in which the edges correspond to edge cuts with this property. In [3] there has been a substantial theory developed for edge cuts and edge ends starting from Stallings' Theorem on the structure of groups with infinitely many ends ([26, 28]).

A set of vertices $C \subset VX$ is said to be *connected* if any two vertices in C can be joined by a path all of whose vertices are in C . *Components* of a set of vertices are its maximal connected subsets.

In this paper we are concerned with vertex cuts and vertex ends. We say that $C \subset VX$ is a vertex cut if C is connected and VX can be partitioned $C \cup NC \cup C^*$, where NC is finite and consists of the vertices which are not in C , but which are adjacent to vertices in C . Note that generally C^* will not be connected. As for edge cuts any ray is eventually in C or in C^* . We say two rays belong to the same *vertex end* if they are not separated by any vertex cut. A finite set F of vertices is called a *separator* if $VX \setminus F$ has at least two components which contain an end. If C is an edge cut then it is also a vertex cut in which NC is the set of vertices of δC which are not in C . Thus if two rays belong to the same vertex end, then they belong to the same edge end. The converse is true if X is locally finite. However it is easy to construct examples of graphs which are not locally finite in which there are more vertex ends than edge ends. For example if K_∞ is the complete graph on a countably infinite set of vertices and X is the graph consisting of n copies of K_∞ , in which a single vertex from each copy is identified, then X has n vertex ends but only one edge end.

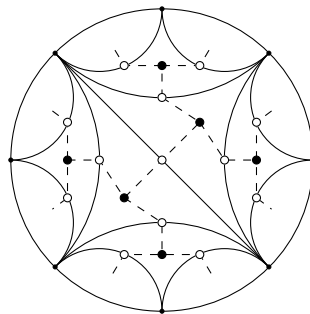


FIGURE 3. Farey graph with structure tree.

For the Farey graph vertex cuts yield a tree decomposition but edge cuts do not, see Figure 3. This example was pointed out to us by Hamish Short. This graph is obtained by taking an ideal triangle in the hyperbolic plane and then taking all translates of this triangle under the group of isometries generated by reflexions in the three sides. One obtains a graph, in which the vertices are the translates of the

vertices of the triangle. All of these will lie in the boundary of the plane, which will be a circle in the disc model. The edges of the graph will be the translates of the edges of the triangle. The vertices of any edge will form a 2-separator. In this graph every vertex has infinite valency. The structure tree is easy to see. There will be one orbit of vertices corresponding to the separating edges. The other orbit corresponds to the triangles. For each such triangle there will be three edges of the structure tree joining the vertex corresponding to the triangle to the three vertices corresponding to its boundary edges. The structure tree is essentially the dual graph to the tessellation of the hyperbolic plane. This dual graph is a tree since each edge of the original graph is separating.

In developing our theory we give a set of axioms that it is sufficient for a set of vertex cuts to satisfy in order that a structure tree can be constructed. In Figure 2 removing any two of the central four vertices will leave two components. The 12 components thus obtained satisfy the axioms of a *cut system*. The 12 cuts are not *nested* with each other. Thus if one takes two cuts C, D such that NC, ND are the two distinct diagonal pairs, then $C \cap D, C \cap D^*, C^* \cap D, C^* \cap D^*$ are all non-empty intersections. If we restrict to the components obtained by removing two adjacent vertices in the central cycle then we obtain a nested cut system, and the cuts in this system can be regarded as the directed edges of the structure tree.

If X is an infinite graph with more than one vertex end and k is the smallest integer for which there is a vertex cut C such that $|NC| = k$ and NC separates two ends then there is a set of such vertex cuts which satisfies the axioms.

One big advantage of using vertex cuts over edge cuts is that we can obtain a structure tree theory that applies to finite graphs, and gives information about the k -connectivity of the graph for any k . For a complete graph the structure tree is trivial, i.e. it only has one vertex. A finite graph X has a non-trivial structure tree if and only if for some integer k , there are k -separators. A k -separator is a set S of k vertices whose removal leaves at least two components C, D such that $C \cup S$ and $D \cup S$ each contain k -inseparable subsets. Here a set Y of vertices is k -inseparable if Y has at least $k + 1$ vertices and no two of the vertices will lie in distinct components of the graph when at most k vertices of X are removed. Let κ be the smallest value of k for which the above occurs. We show that a finite graph contains a unique nested set \mathcal{N} of κ -separators such that if two κ -inseparable subsets are separated by some κ -separator then they are separated by a set in \mathcal{N} . The set \mathcal{N} is invariant under the automorphism group of X and forms the directed edge set of a structure tree for X .

The authors first met at the conference on Totally Disconnected Groups, Graphs and Geometry at Blaubeuren in May 2007. This project grew from a problem raised at a problem session at that conference. The authors are very grateful to the organizers for inviting them to the meeting.

2. SYSTEMS OF CUTS AND SEPARATORS

The *boundary* NE of a set of vertices E is the set of vertices in $VX \setminus E$ which are adjacent to E . Set $E^* = VX \setminus (E \cup NE)$. We call E^* the **-complement* of E .

Let E and F be sets of vertices. The intersections $E \cap F$, $E^* \cap F$, $E \cap F^*$ and $E^* \cap F^*$ are called the *corners* of E and F , see Figure 4. The sets $E \cap NF$, $E^* \cap NF$, $F \cap NE$ and $F^* \cap NE$ are called the *links* and $NE \cap NF$ is the *centre*. A link and a corner are said to be *adjacent* if they are adjacent in Figure 4. We say that two

links are the links *of* their adjacent corner (or we say they are *its* links), and we say that two corners are the corners *of* their adjacent link (or *its* corners). Two links or two corners are said to be *adjacent* if they are adjacent to the same link or corner, respectively. Otherwise they are called *opposite*.

A set \mathcal{C} of non-empty connected sets of vertices with finite boundaries in a connected graph is a *cut system* if the $*$ -complement is an involution on \mathcal{C} and the following axioms are satisfied.

- (A1) If C is in \mathcal{C} then every component of C^* which contains an element of \mathcal{C} is in \mathcal{C} .
- (A2) If C and D are in \mathcal{C} then either a component of $C \cap D$ and a component of $C^* \cap D^*$ are in \mathcal{C} or a component of $C \cap D^*$ and a component of $C^* \cap D$ are in \mathcal{C} .

Elements of a cut-system are called *cuts*. *Cut components* of set are components of the set which are cuts. Note that Axiom (A2) for $C = D$ implies that the $*$ -complement of every cut has a cut-component.

Let C' denote the elements of NC which are not adjacent to some element in C^* . Then $(C^*)^* = C \cup C'$ and $((C^*)^*)^* = (C \cup C')^* = C^*$. Hence the $*$ -complement is not an involution in general, but it is an involution on sets which are $*$ -complements of other sets. Hence the assumption that the $*$ -complement is an involution is not a restriction, because if it is not an involution, we replace \mathcal{C} by $(\mathcal{C}^*)^*$ and then it is. Also note that then $(\mathcal{C}^*)^*$ will again consists of connected sets.

If \mathcal{C} is a cut system then the boundary of a cut is called a *separator* and we denote the system of separators by \mathcal{S} . Note that in general a given system of separators does not determine a cut system.

A set of vertices E is called *large* if E has a cut component and E^* contains a cut. The $*$ -complement of a cut is large but it is not necessarily a cut, because it is not necessarily connected. Axiom (A2) says that if C, D are cuts then there is a pair of large opposite corners of C and D .

Note that (A2) can equivalently be replaced by the following.

- (A2') If C and D are in \mathcal{C} then $C \setminus ND$ has a component which is in \mathcal{C} .

Let us consider some examples of cut systems before developing the theory.

Example 2.1. For infinite graphs with more than one (vertex) end, take cuts to be connected sets of vertices with finite boundary which contain a ray (i.e. an end) and whose complement also contains a ray. Axiom (A2) holds because if C, D, C^*, D^* all contain rays, then so do two opposite corners.

Example 2.2. For infinite graphs with more than one edge end, the separators would naturally be finite sets of edges which separate rays. But separators are by definition sets of vertices. Hence we replace every edge of the original graph by paths of length two. Let M be the set of new vertices, that is, the set of middle vertices of these paths of length two. Then we take cuts to be connected sets of vertices which contain a ray, whose $*$ -complement also contains a ray and whose boundary is a finite subset of M .

Examples 2.1 and 2.2 can be generalized as follows.

Example 2.3. Let X be a connected graph and let $M \subset VX$ be some set of vertices. Take cuts to be connected sets of vertices which contain a ray, whose complement also contains a ray and whose boundary is a finite subset of M .

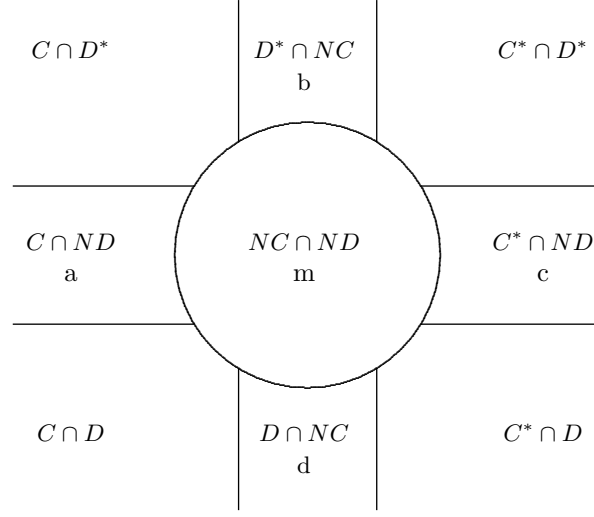


FIGURE 4. Corners, links and centre

We say that a cut C *separates* a set A from a set B if $A \subset C \cup NC$ and $B \subset C^* \cup NC$ or $B \subset C \cup NC$ and $A \subset C^* \cup NC$. Here we also require that neither A nor B is a subset of NC . A separator S is said to *separate* A from B if for some cut C with $NC = S$, C separates A and B .

Next we consider a cut system which also makes sense in finite graphs. Let k be a positive integer. A subset Y of VX is said to be k -*inseparable* if it has at least $k + 1$ elements and if for every set $E \subset VX$ with $|NE| \leq k$, either $Y \subset E \cup NE$ or $Y \subset E^* \cup NE$. Examples of k -inseparable subgraphs are the vertex set of a $(k + 1)$ -connected subgraph, or the vertex set of a subgraph which is complete on $k + 1$ vertices. The vertices of a separating edge form a maximal 1-inseparable set.

Example 2.4. Let κ be the smallest positive integer for which there are sets E , Y_1 and Y_2 such that $|NE| = \kappa$, Y_1 and Y_2 are κ -inseparable, $Y_1 \subset E \cup NE$ and $Y_2 \subset E^* \cup NE$. We will show shortly that the connected sets E with this property form a cut system.

Note that there are graphs where this cut system is empty. It is easy to see that this is the case for finite complete graphs or cycles. In fact, this holds for all finite transitive graphs, see Remark 9.1. We say that a cut C *separates* a set A from a set B if $A \subset C \cup NC$ and $B \subset C^* \cup NC$ or $B \subset C \cup NC$ and $A \subset C^* \cup NC$. Here we also require that neither A nor B is a subset of NC . A separator S is said to *separate* A from B if for some cut C with $NC = S$, C separates A and B .

Set $a = |C \cap ND|$, $b = |D^* \cap NC|$, $c = |C^* \cap ND|$, $d = |D \cap NC|$ and $m = |NC \cap ND|$, see Figure 4.

Proof for Example 2.4. It follows from the construction that (A1) is satisfied. We have show that (A2) is satisfied.

Let C, D be cuts. We want to show that opposite corners have components that are cuts. We know that there are κ -inseparable sets Y_1, Y_2 separated by NC , and κ -inseparable sets Y_3, Y_4 separated by ND . Each Y_i determines a unique corner A_i of C and D such that Y_i is contained in the union of A_i , the two links which are

adjacent to A_i and the center. Even though A_i is uniquely determined, we cannot rule out the possibility that $Y_i \cap A_i = \emptyset$ at this stage.

Two different Y_i may determine the same corner. There must be two Y_i 's which determine opposite corners. For if this is not the case then all four Y_i 's will determine one of a pair of adjacent corners. But then there will either be no Y_i 's separated by NC or no Y_i 's separated by ND . Suppose then, say, that

$$Y_1 \subset (C \cap D) \cup (C \cap ND) \cup (D \cap NC) \cup (NC \cap ND) \quad \text{and}$$

$$Y_2 \subset (C^* \cap D^*) \cup (C^* \cap ND) \cup (D^* \cap NC) \cup (NC \cap ND).$$

Consider Figure 4. We see that $|N(C \cap D)| \leq a + m + d$ and $|N(C^* \cap D^*)| \leq b + m + c$. But $\kappa = |NC| = b + m + d = |ND| = a + m + c$, and so $2\kappa = a + b + c + d + 2m$. Hence either $|N(C \cap D)| < \kappa$ or $|N(C^* \cap D^*)| < \kappa$ or $|N(C \cap D)| = |N(C^* \cap D^*)| = \kappa$. Whenever one of these boundaries has less or equal κ elements then it separates Y_1 from Y_2 . By the minimality of κ we get $|N(C \cap D)| = |N(C^* \cap D^*)| = \kappa$. It follows that the opposite corners $C \cap D$ and $C^* \cap D^*$ have components that are cuts and so (A2) is satisfied. \square

Example 2.5. We define X_n for $n \geq 3$ by $VX_n = \{a, b, c, d, 1, 2, \dots, n\}$. There is a path $1, 2, \dots, n$, a circle a, b, c, d, a and each of the vertices $1, 2, \dots, n$ is adjacent to each of the vertices a, b , see Figure 5. The graph is 2-connected but there are no 2-inseparable sets which are separated by 2-element sets of vertices. But there are 3-inseparable sets (the sets $\{k, k + 1, a, b\}$) which are pairwise separated from each other by 3-element sets. Hence $\kappa = 3$. Set $C_k = \{1, 2, \dots, k - 1\}$, $D_k = \{k + 1, k + 2, \dots, n\}$. The cut-system of Example 2.4 is

$$\mathcal{C}_n = \{C_k, D_k \mid 2 \leq k \leq n - 1\}.$$

A further discussion of this cut-system can be found in Example 7.3.

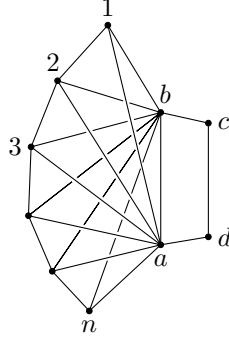


FIGURE 5. Finite 2-connected graph with 3-inseparable blocks

3. THIN SUBSYSTEMS

Let \mathcal{C} be a cut system in a connected graph X . Let κ be the smallest cardinality of a separator. A separator with κ elements is a *thin separator*. A cut C in \mathcal{C} is called *thin* if $|NC| = \kappa$.

Lemma 3.1. *Let C and D be thin cuts and assume that $C \cap D$ and $C^* \cap D^*$ are large. Then $N(C \cap D)$ and $N(C^* \cap D^*)$ are thin separators and $NC \cap ND = N(C \cap D) \cap N(C^* \cap D^*)$.*

In [15, Theorem 2] and [16, Proposition 2.1], Jung and Watkins prove a similar result.

Proof of Lemma 3.1. This is similar to the proof for Example 2.4 and we again consider Figure 4. From the diagram, $\kappa = a + m + c = b + m + d$ and hence

$$(1) \quad 2\kappa = a + b + c + d + 2m.$$

Also

$$\begin{aligned} |N(C \cap D)| &\leq a + d + m, \\ |N(C \cap D^*)| &\leq a + b + m, \\ |N(C^* \cap D^*)| &\leq b + c + m, \\ |N(C^* \cap D)| &\leq c + d + m, \end{aligned}$$

The intersections $C \cap D$ and $C^* \cap D^*$ are large. Hence $a + d + m \geq \kappa$ and $b + c + m \geq \kappa$. Now (1) implies that these inequalities are equalities and $|N(C \cap D)| = |N(C^* \cap D^*)| = \kappa$. If $a < b$ then $d < c$, otherwise $a + m + c < \kappa$, and hence $a + m + d < \kappa$ which is impossible. If $b < a$ we get a contradiction in the same way and hence $a = b$, $c = d$. We also get $NC \cap ND = N(C \cap D) \cap N(C^* \cap D^*)$. \square

Corollary 3.2. *The thin cuts in a cut system form a cut system.*

Proof. It is clear that the thin cuts and separators in some given cut system \mathcal{C} satisfy axiom (A1), while axiom (A2) follows from Lemma 3.1. \square

A corner of two sets of vertices is called *isolated* if it does not contain a cut and if its adjacent links are both empty. We call two sets of vertices *nested* if they have an isolated corner.

Lemma 3.3. *If $A \subset B$ then $A \cap B^*$ is an empty isolated corner.*

If a corner of two thin cuts does not contain any cut and one of its links is empty then both links are empty (and the corner is isolated).

Minimal cuts have either (i) no empty link or (ii) two non-empty links and two empty links which are adjacent to an isolated corner or (iii) all four links are empty, two opposite corners are large and at least one corner is empty. In case (i) the cuts are not nested, in the cases (ii) and (iii) they are.

Proof. If $A \subset B$ then $A \cap B^*$ and $A \cap NB$ are empty. A vertex x in $B^* \cap NA$ would have to be adjacent to a vertex in $(A \cap B^*) \cup (A \cap NB)$, because it cannot be adjacent to a vertex in $A \cap B$. Hence $B^* \cap NA$ is also empty and $A \cap B^*$ is an empty isolated corner.

Let C and D be cuts of a cut system. By (A2) two opposite corners are large. We saw in the proof of Lemma 3.1 that the links of a corner of two thin cuts must have the same number of elements. Thus if one link is empty then so is the other.

Also if two opposite corners are large then the links of each of the other two corners have the same number of elements. Thus it is not possible for thin cuts E and F to have exactly one or three empty links. If there are exactly two empty links then they are adjacent and the corresponding corner is not large. If (iii) all links are empty then one corner has to be empty, otherwise one of the cuts would not be connected. \square

A subset $E \subset VX$ is a *pre-cut* if it is a cut or a $*$ -complement of a cut. A pre-cut E is *thin* if E or E^* is a thin cut.

Lemma 3.4. *If E and F are thin pre-cuts then there is a large component of E which contains $E \cap NF$.*

Proof. Axiom (A2) implies that there is an A in $\{E \cap F, E \cap F^*\}$ which is large and which has a large opposite corner. This corner A has a large component C and Lemma 3.1 implies $|NA| = \kappa$. Now $NC \subset NA$ and $|NC| \geq \kappa$, because C is a cut. This implies $NC = NA$. Let E_0 be the component of E which contains C . Every vertex x in $E \cap NF$ is adjacent to C . Hence x is in E_0 and $E \cap NF \subset E_0$. Note that we have not excluded the case $E \cap NF = \emptyset$. \square

A thin cut is called an *A-cut* if it is nested with all other thin cuts. A cut C is called a *B-cut* if C^* has only one large component.

Theorem 3.5. *A thin cut is either an A-cut or a B-cut.*

Proof. Let C be a thin cut which is not an A-cut. Then there is a thin cut D which is not nested with C . By Lemma 3.4, there is a large component C_0^* of C^* which contains $C^* \cap ND$. In order to prove that C is a B-cut we have to show that C^* has no other large component.

Suppose there is another large component C_1^* of C^* . Then NC_1^* is a separator. Hence $NC_1^* = NC^*$, otherwise NC_1^* would have less than κ elements. Now $C_1^* \cap C^* \cap ND = C_1^* \cap ND = \emptyset$, because, $C^* \cap ND \subset C_0^*$. There is an element y in $D \cap NC^*$ and an element z in $D^* \cap NC^*$, because, by case (i) of Lemma 3.3, all links and corners are not empty. Since $NC_1^* = NC$ and C_1^* is connected, there is a path from y to z which is completely contained in C_1^* , except for its end-vertices y and z . This path has to intersect $C^* \cap ND$ which contradicts $C_1^* \cap ND = \emptyset$. \square

We define a *slice* to be a component of $VX \setminus S$ that is not a cut, where S is a separator. Note that every component of an isolated corner is a slice.

Lemma 3.6. *Let C be a thin cut system. A slice has empty intersection with each separator. Distinct slices are disjoint. If Q is a slice, then no pair of elements of NQ are separated by any separator.*

Proof. Let Q_1, Q_2 be distinct slice components of $VX \setminus NC$ and $VX \setminus ND$ respectively where C, D are thin cuts. By Lemma 3.4, the links $ND \cap C$ and $ND \cap C^*$ are contained in large components of $VX \setminus NC$. Hence they are disjoint from Q_1 . It follows that $ND \cap Q_1 = \emptyset$.

Consider corners and links of the pair Q_1, Q_2 . We have shown that the links $Q_1 \cap NQ_2, Q_2 \cap NQ_1$ are empty. By the connectedness of Q_1, Q_2 this means that either $Q_1 = Q_2$ or $Q_1 \cap Q_2 = \emptyset$.

Finally suppose $x, y \in NQ$ for some slice Q and x, y are separated by NC for some cut C . Now Q is connected and the path in Q joining x, y must intersect the separator NC , which we have already shown cannot occur. \square

Corollary 3.7. *If a group G acts transitively on a connected graph with a G -invariant thin cut system then there are no slices.*

Let X be a graph with a cut system of thin cuts. Lemma 3.6 enables us to replace X with another graph \hat{X} with essentially the same cut system, but in which there are no slices. In this new cut system two cuts are nested if and only if there is an empty corner with empty adjacent links.

Let, then, X be a graph with a cut system \mathcal{C} of thin cuts. We define \hat{X} as follows; $V\hat{X} \subset VX$ and $v \in V\hat{X}$ if and only if $v \notin Q$ for every slice Q . Note that if $v \in NC$ for any $C \in \mathcal{C}$, then by Lemma 3.6 $v \in V\hat{X}$. Two vertices $u, v \in V\hat{X}$ are joined by an edge in \hat{X} if and only if u, v are joined by an edge in X or $u, v \in NQ$ for some slice Q . We define $\hat{\mathcal{C}}$ to be the set of subsets \hat{C} of $V\hat{X}$, where $\hat{C} = C \cap V\hat{X}$ for some $C \in \mathcal{C}$.

Theorem 3.8. *The graph \hat{X} is connected and $\hat{\mathcal{C}}$ is a cut system of thin cuts in \hat{X} . There are no slices in \hat{X} with respect to $\hat{\mathcal{C}}$.*

Proof. Let C be a cut in \mathcal{C} . We need to show that $\hat{C} = C \cap V\hat{X}$ is connected in \hat{X} . Any two points of \hat{C} are joined by a path with vertices in C . If this path does not contain any vertices of a slice then it is a path in \hat{C} . If it does contain vertices that are in a slice then delete these vertices. The resulting sequence will be a path in \hat{C} .

Let \mathcal{S} be the set of separators with respect to \mathcal{C} in X . Note that they are all contained in $V\hat{X}$. Let $S \in \mathcal{S}$ and K be a component in $VX \setminus S$. If K is disjoint from all separators, then either K is a slice or $K = \hat{K}$. If K intersects a separator S' then Lemma 3.6 (applied to X with respect to \mathcal{C}) implies that \hat{K} contains $K \cap S'$ and hence \hat{K} is not empty. Thus no \hat{K} in $\hat{\mathcal{C}}$ is empty. It is also clear that K is not a slice with respect to $\hat{\mathcal{C}}$ and hence \hat{X} has no slices at all. The axioms of a cut system for $\hat{\mathcal{C}}$ follow from the arguments above and from the fact that they are satisfied for \mathcal{C} . The system $\hat{\mathcal{C}}$ is thin, because \mathcal{C} and $\hat{\mathcal{C}}$ have the same set of separators. \square

Let E, F be pre-cuts. We say that E is *almost contained* in F (notation: $E \subset^a F$) if $E \cap F^*$ is an isolated corner. Such a corner is a union of slices. Thus $E \subset^a F$ if and only if $\hat{E} \subset \hat{F}$ in \hat{X} . In \hat{X} we can say that sets of vertices E and F are nested if one of the conditions $E \subset F$, $E^* \subset F$, $E \subset F^*$ or $E^* \subset F^*$ holds.

We analyze the properties of the ordered set (\mathcal{C}, \subset^a) . This will be the same as the properties of $(\hat{\mathcal{C}}, \subset)$ and so we may as well assume that X has no slices and that the order relation is \subset . We know that $E \subset F$ is equivalent to $F^* \subset E^*$. Also \subset is transitive. It requires a certain amount of effort to prove directly that \subset^a is transitive.

Lemma 3.9. *Let C, D, E be thin cuts and let D and E be not nested.*

If C is nested with D then it is nested with each cut component of two adjacent corners either $D \cap E$ and $D \cap E^$ or $D^* \cap E$ and $D^* \cap E^*$. If C is nested with D and E , then it is nested with every cut that is a component of a corner of D and E .*

Proof. Suppose C is nested with D . We know that D, E are B -cuts. We can assume that there are no slices by replacing X with \hat{X} if necessary. Thus D^*, E^* are cuts. By changing D to D^* if necessary we can assume that $C \subset D^*$ or $C^* \subset D^*$. If $C \subset D^*$ then $C \subset D^* \cup E$ and $C \subset D^* \cup E^*$ and so C is nested with any cut in the corners $D \cap E^* = (D^* \cup E)^*$ or $D \cap E$. Similarly if $C^* \subset D^*$, then C^* is nested with any cut in the same two corners. But C is nested with a cut if and only if C^* is nested with the same cut.

If C is nested with both D and E , then by the above it will be nested with three of the four corners. In fact it will be nested with all four corners. Thus if say $C \subset D^*$ and $C \subset E^*$, then the above shows C is nested with the three corners $D \cap E^*$, $D^* \cap E$ or $D \cap E$. But $C \subset D^* \cap E^*$ and so it is nested with the fourth corner also. \square

4. SEPARATION BY FINITELY MANY CUTS

We recall some results concerning separating by removing sets of edges. A *minimal k -edge separator* is a set of k edges whose complement is disconnected and the complement of any proper subset of this separator is connected. Note that the complement of a minimal edge separator has exactly two components and each edge in the separator is adjacent to both components. Every finite set of edges with a disconnected complement contains a minimal edge separator. Let p be any edge of some connected graph and let k be any integer. Thomassen and Woess have given a short proof of the fact that there are only finitely many minimal k -edge separators containing p (see [29, Proposition 4.1]).

We call a set S of k vertices a *minimal k -separator*, if $VX \setminus S$ has at least two components which are adjacent to all elements of S . That is, S is minimal with respect to separating these two components. Note that this is a general graph-theoretic definition which does not refer to our axiomatic cut systems and their separators. Also note that a graph may have minimal k -separators for different values of k . We say that a minimal k -separator S *separates* vertices u and v *properly* if u and v lie in distinct components of $VX \setminus S$ which are adjacent to every element of S . When we think of our cut systems satisfying axioms (A1) and (A2) then we observe that thin separators S are κ -vertex separators which separate vertices in distinct cuts properly whenever the boundaries of these cuts are S .

Thomassen and Woess showed that in a locally finite graph, there are only finitely many minimal k -vertex separators containing a given vertex x (see [29, Proposition 4.2]). This does not hold in non-locally finite graphs. But we can prove the following two lemmas for finite vertex separators in non-locally finite graphs using similar arguments as in the proof of [29, Proposition 4.1].

Lemma 4.1. *Every pair of vertices in a connected graph is separated properly by only finitely many k -vertex separators, for any given k .*

Proof. We use induction on k . Any minimal 1-separator of vertices x and y is a vertex in any path joining x and y and so the lemma is true for $k = 1$.

Suppose the lemma holds for all minimal k -separators in all connected graphs. Let x and y be any vertices of a graph X . We choose a path π from x to y and assume that there are infinitely many minimal $(k + 1)$ -separators, $k + 1 \geq 2$, separating x from y properly. Then there is a vertex $z \in \pi \setminus \{x, y\}$ which is contained in infinitely many of these $(k + 1)$ -separators. If S_1 and S_2 are such $(k + 1)$ -separators then $S_1 \setminus \{z\}$ and $S_2 \setminus \{z\}$ are distinct k -separators of x and y in $X - \{z\}$, and they separate x from y properly in $X - \{z\}$. Hence there are infinitely many distinct k -separators in $X - \{z\}$ which separate x from y properly, in contradiction to the induction hypothesis. \square

Lemma 4.2. *A thin pre-cut is nested with all but finitely many thin pre-cuts.*

Proof. Let E and F be thin pre-cuts which are not nested. By Lemma 3.3 all links are not empty. Hence NF has to separate two elements x and y of NE , and NF separates these vertices properly as a κ -vertex separator. Lemma 4.1 says that there are only finitely many such separators NF . Since NE is finite and since, by Theorem 3.5, the complement $VX \setminus NF$ of each such set NF has exactly two large components (i.e. NF corresponds to two cuts), we obtain the statement of the lemma. \square

We conclude the section with another observation.

Lemma 4.3. *Let c be an integer. There is no strictly descending sequence of connected sets of vertices with non-empty intersection, whose boundaries have less than c elements.*

Proof. Suppose the intersection I of these sets is not empty. If NI contains a set A of $c+1$ elements then there is a set C in the sequence which is disjoint with A and then $|NC| > c$, a contradiction. If NI is finite, then there has to be a set C in that sequence such that $NC = NI$. Since C is connected this implies $C \subset I$, and hence $C = I$. But this contradicts the assumption that the sequence is strictly decreasing. \square

5. OPTIMALLY NESTED CUTS

Let \mathcal{C} be a cut system of thin cuts. Let C be a cut and let $M(C)$ be the set of cuts which are not nested with C . Set $\mu(C) = |M(C)|$. It follows from Lemma 4.2 that $\mu(C)$ is finite. Put $\mu(D^*) = \mu(C)$ where D is a cut and C is a cut component of D^* . There is no ambiguity in doing this as if there are two cuts C_1, C_2 which are large components of D^* then D, C_1, C_2 are all A -cuts and $\mu(C_1) = \mu(C_2) = 0$.

Lemma 5.1. *Let C and D be B -cuts which are not nested, and suppose $C \cap D$ and $C^* \cap D^*$ are large, then*

$$\mu(C \cap D) + \mu(C^* \cap D^*) < \mu(C) + \mu(D).$$

Proof. Let E be any thin cut. If E is in $M(C^* \cap D^*) \cap M(C \cap D)$ then, by Lemma 3.9, E is in $M(C)$ and in $M(D)$. Hence if E is counted twice on the left of the above inequality then it is also counted twice on the right.

If otherwise E is in $M(C \cap D) \setminus M(C^* \cap D^*)$ or in $M(C^* \cap D^*) \setminus M(C \cap D)$, that is E is counted exactly once on the left, then, again by Lemma 3.9, E is in $M(C)$ or in $M(D)$. Hence E is counted at least once on the right side of the inequality. We have now proved that $\mu(C \cap D) + \mu(C^* \cap D^*) \leq \mu(C) + \mu(D)$. Since $C \in M(D)$ and $D \in M(C)$, the cuts C and D are counted on the right side, but not on the left side, and we see that this inequality is a strict inequality. \square

Set $\mu_{\min} = \min\{\mu(C) \mid C \text{ is a thin cut.}\}$. This minimum exists, because the values $\mu(C)$ are all finite. A thin cut C with $\mu(C) = \mu_{\min}$ is called *optimally nested*. Every non-empty cut system contains an optimally nested thin cut.

Theorem 5.2. *Every optimally nested thin cut is nested with all other optimally nested thin cuts. The optimally nested thin cuts form a nested cut system.*

Proof. Suppose there are optimally nested thin cuts C and D which are not nested with each other. By (A2) there will be opposite corners that are large. By relabeling we can assume that these large corners are $C \cap D$ and $C^* \cap D^*$ and by Lemma 3.1, each of $C \cap D$ and $C^* \cap D^*$ has a component which is a thin cut. Now Lemma 5.1 says that

$$\mu(C \cap D) + \mu(C^* \cap D^*) < \mu(C) + \mu(D) = 2\mu_{\min}.$$

Thus one of the summands on the left side is less than μ , contradicting the minimality of m . \square

In fact it is often the case that $\mu_{\min} = 0$. If $\mu_{\min} = 0$ then there are thin cuts that are nested with every other thin cut. It can happen though that $\mu_{\min} \neq 0$ as we show in an example.

Example 5.3. Let X_n be the 2-ended graph which is the full subgraph of the integer lattice in the plane in which $VX = \{(i, j) | i \in \mathbb{Z}, j = 1, 2, \dots, n\}$. There is a cut system for X as in Example 2.1. The set $C = \{(i, j) \in VX | i < 0\}$ is an optimally nested cut. However if $n \geq 3$ then there are thin cuts with which it is not nested. Thus for $n = 5$ there is a cut D with $ND = \{(-2, 1), (-1, 2), (0, 3), (1, 4), (2, 5)\}$, and this is not nested with C .

Another graph for which $\mu_{\min} \neq 0$ is given in Figure 8.

Theorem 5.4. *Let X be a graph with a thin cut system \mathcal{C} for which $\mu_{\min} \neq 0$. Then X is an infinite graph such that each cut contains an end.*

Proof. We know that for every $C \in \mathcal{C}$ there is another cut D with which it is not nested, and for every such pair C, D there are at least two cut corners. Thus for any $C \in \mathcal{C}$ we can form a sequence of distinct cuts $C = C_1, C_2, \dots$, where $C_{i+1} \subset C_i$ and C_{i+1} is a corner of C_i and another cut in \mathcal{C} . This sequence will determine an end of X , because its intersection is empty by Lemma 4.3. There will be another such sequence starting with C^* and so X has more than one end. \square

6. CUTS AND TREES

We will show that a cut system of thin nested cuts can be regarded as the edge set of a directed tree. First we consider again the structure tree given by cut-points and 2-blocks described in the introduction. In the discussion here we do allow disconnecting edges. The vertices of a disconnecting edge form a 2-block.

Example 6.1. Let X be a connected graph. If X is not 2-connected then it has *cut-points*, i.e. vertices whose removal disconnects the graph. If this happens, then VX decomposes into a collection of maximal 2-inseparable subsets or *2-blocks*. Any two 2-blocks intersect in at most one vertex and this vertex will be a cut-point. Every edge of X joins vertices in exactly one 2-block. Note that the boundary of a 2-block in X consists of cut-points. Let \mathcal{C} be the set of connected sets of vertices C such that $NC = \{x\}$ and x is a cut-point. Then \mathcal{C} is a system of thin nested cuts.

As noted in the introduction, associated with this decomposition is a tree T in which $VT = \mathcal{B} \cup \mathcal{S}$, where \mathcal{S} is the set of cut-points, \mathcal{B} is the set of 2-blocks and there is an edge joining $b \in \mathcal{B}$ with $s \in \mathcal{S}$ if and only if $s \in b$. But note that the edges of this tree can be regarded as the set \mathcal{C} . Thus for $C \in \mathcal{C}$, let $t(C) = NC$ and let $o(C)$ be the block that is contained in C that has the cut-point NC in its boundary. If we direct the edges of T so that an arrow points from $o(C)$ to $t(C)$, then the vertices in \mathcal{S} have every adjacent edge pointing towards it and every vertex in \mathcal{B} has every adjacent arrow pointing away from it. Then any path will have alternating arrows as one proceeds along it.

In general let \mathcal{C} be a nested cut system of thin cuts. By replacing X by \hat{X} if necessary, we can assume that there are no slices and so \subset^a is the same as \subset for pre-cuts. Let \mathcal{S} be the set of separators. We define \mathcal{B} as the set of equivalence classes for a particular equivalence relation \sim on \mathcal{C} . Thus for $C, D \in \mathcal{C}$ put $C \sim D$ if either

- (i) $C = D$ or
- (ii) $NC \neq ND$ and $C^* \subset D$ and if $C^* \subsetneq E \subset D$, for $E \in \mathcal{C}$, then $E = D$.

We prove that \sim is an equivalence relation. Clearly \sim is reflexive. It is symmetric, because $C^* \subset D$ if and only if $D^* \subset C$, and $C \subset D$ if and only if $C^* \subset D^*$ for $C, D \in \mathcal{C}$.

Suppose $E \sim C$ and $C \sim D$ and suppose that C, D, E are distinct elements of \mathcal{C} , so that $E^* \subset C$ and $C^* \subset D$. We know that D, E are nested, so that there is one of four possibilities. If (1) $E \subset D^*$ then $C^* \subset E$ implies $C^* \subset D^*$. Since $D^* \subset C$ we get $C^* \subset C$, which is impossible. If (2) $E \subset D$ then, $D^* \subset E^* \subset C$ which implies $D^* = E^*$, and hence $D = E$, which we have excluded. If (3) $E^* \subset D^*$ then $E^* \subset D^* \subset C$ and either so $E = D$ or $D^* = C$. We have excluded $E = D$ and if $D^* = C$ then $ND = NC$ and so $C = D$ which is also excluded. If (4) $E^* \subset D$ then either $E \sim D$ or there is an $A \in \mathcal{C} \cup \mathcal{C}^*$, $A \neq D, A \neq E^*$ such that

$$E^* \subset A \subset D.$$

We have again four cases, because A and C are nested. If (1) $C \subset A$ then $C \subset D$ contradicting $C^* \subset D$. If (2) $C^* \subset A$ then $C^* \subset A \subset D$ and $C^* = A$, by the definition of \sim . Then $E^* \subset C^*$ contradicting $E^* \subset C$. If (3) $C \subset A^*$ then $A \subset C^* \subset E$, which implies $E^* \subset A^*$, in contradiction to $E^* \subset A$. If (4) $C^* \subset A^*$ then $E^* \subset A \subset C$ and so $A = C$. This implies $C \subset D$, contradicting $D \subset C^*$.

We obtain a directed graph $T' = T'(\mathcal{C})$

$$VT' = \mathcal{S} \cup \mathcal{B} \quad \text{and} \quad ET' = \mathcal{C}$$

where \mathcal{S} is the set of separators and $\mathcal{B} = \mathcal{C}/\sim$, where $o(C) = NC$ of $C \in \mathcal{C}$ and $t(C)$ is the \sim -class which contains C . Clearly T' is a bipartite. In particular there are no loops. Each vertex in \mathcal{B} has every adjacent edge pointing towards it and each vertex in \mathcal{S} has every adjacent arrow pointing away from it. Any path in T' will have alternating arrows as one proceeds along it.

Next we add the $*$ -complements of cuts C as edges to T' such that $o(C) = t(C^*)$ and $t(C) = o(C^*)$. We obtain a graph T'' where $ET'' = \mathcal{C} \cup \mathcal{C}^*$. Let $E_1, E_2, E_3 \dots$ be the edges of an oriented path in T'' . Then $E_1 \supset E_2 \supset E_3 \dots$ and either $E_{2n} \in \mathcal{C}$ and $E_{2n+1} \in \mathcal{C}^*$, for all n , or vice versa. Let T denote the undirected graph which corresponds to T' and T'' .

Lemma 6.2. *The graph $T = T(\mathcal{C})$ is a tree.*

Proof. If T contains a cycle then it has at least length four, because T is bipartite. Let us consider this cycle with orientation. Then it corresponds to a sequence of distinct edges in the disjoint union $ET'' = \mathcal{C} \sqcup \mathcal{C}^*$ (where sets which are cuts and $*$ -complements of cuts are considered twice) such that $E_1 \supset E_2 \supset \dots \supset E_{2n} \supset E_1$, for some $n \geq 2$, a contradiction.

Let E and F be elements of \mathcal{C} . If $E \subset F$, then Lemma 4.1 implies that there is a finite path in T connecting $o(E)$ and $t(F)$. Hence T is connected. \square

The following example, illustrated in Figure 6, shows that it is necessary to use \subset^a rather than \subset when defining nestedness if there are slices. It is similar to Examples 2.5 and 7.3.

Example 6.3. Set $VX = \mathbb{Z} \cup \{o, i\}$ and

$$EX = \{\{x, x+1\} \mid x \in \mathbb{Z}\} \cup \{\{x, o\} \mid x \in \mathbb{Z}\} \cup \{\{o, i\}\}.$$

The minimal number of vertices needed to separate the two ends of X is 2. The (connected) thin cuts are of the form $C_k^+ = \{k + 1, k + 2, \dots\}$ or $C_k^- = \{k - 1, k - 2, \dots\}$, where $NC_k^+ = NC_k^- = \{k, o\}$. The group of automorphisms acts transitively on the cut system $\mathcal{C} = \{C_k^+, C_k^- \mid k \in \mathbb{Z}\}$. Also \mathcal{C} is the only thin end separating automorphism invariant cut system. If $l > r$ then C_l^- and C_r^+ do not have an empty corner but they are nested because $\{i\}$ is their isolated corner. The set $\{i\}$ is the only slice. The graph \hat{X} is obtained by deleting i .

The tree $T(\mathcal{C})$ is a double ray.

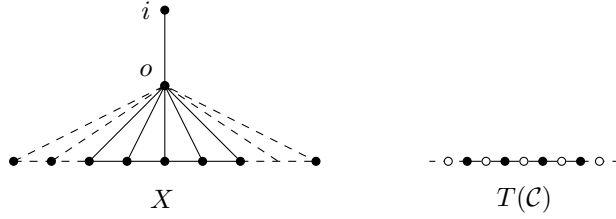


FIGURE 6. Structure tree for a two-ended graph

Let \mathcal{N} be the nested sub-system of an optimally nested thin cuts in a cut system as described in Section 5.

Let $T = T(\mathcal{N})$ be the structure tree for \mathcal{N} . If \mathcal{C} is invariant under G then T is a G -tree, as it is uniquely determined by \mathcal{C} .

It many cases an optimally nested thin cut is in fact nested with every thin cut. Thus the number μ_{\min} of thin cuts not nested with an optimally nested cut is zero. We will show that this means that there is more information about the blocks of the structure tree in this case.

7. BLOCKS

It is possible to define the set \mathcal{B} in a different - possibly better - way. Let \mathcal{C} be a thin cut system. For the moment we do not assume \mathcal{C} is nested. A subset Y of VX is said to be \mathcal{C} -inseparable if for every $C \in \mathcal{C}$ either $Y \subset C \cup NC$ or $Y \subset C^* \cup NC$ but not both. It follows from Zorn's Lemma that every \mathcal{C} -inseparable set is contained in a maximal \mathcal{C} -inseparable set. Thus if one has an increasing sequence of \mathcal{C} -inseparable sets $Y_1 \subset Y_2 \subset \dots$ and $Y = \bigcup_n Y_n$, then Y is \mathcal{C} -inseparable, since if $C \in \mathcal{C}$ and n is a positive integer such that Y_n has more than $|NC|$ elements then for all $m \geq n$ either $Y_m \subset C \cup NC$ or $Y_m \subset C^* \cup NC$ and the same is true for Y . A maximal \mathcal{C} -inseparable set which is disjoint with all slices is called a \mathcal{C} -block. The \mathcal{C} -blocks are the $\hat{\mathcal{C}}$ -blocks in \hat{X} . For the rest of this section we will again assume that \mathcal{C} is a nested system. For $b \in \mathcal{C}/\sim$ we define

$$B(b) = \bigcap_{C \in b} (C \cup NC).$$

Lemma 7.1. *An edge of \hat{X} which is not contained in any separator is contained in exactly one block. The set of all blocks is $\{B(b) \mid b \in \mathcal{C}/\sim\}$. If $b \in \mathcal{C}/\sim$ then*

$$(2) \quad \bigcup_{C \in b} NC \subset B(b).$$

If $C \in b$ then $B(b)$ is the only block B such that $NC \subset B \subset C \cup NC$. Moreover, $B(b) \setminus NC \neq \emptyset$.

Proof. The first statement follows from the fact that the vertices of an edge cannot be separated by a separator.

Let $C \in b$ and suppose two vertices $x, y \in B(b)$ are separated by some separator S . Then $S \subset C \cup NC$, because C is nested with the cuts D for which $ND = S$. One of these cuts D contains C^* . Let D' be the cut such that $C^* \subset D' \subset D$ and $D' \sim C$. Either x or y is not in D' and hence not in $B(b)$, a contradiction. Thus $B(b)$ is inseparable. On the other hand, any vertex which is not in $B(b)$ can be separated by some separator from $B(b)$. Hence $B(b)$ is a block.

If $D \sim C$ then $C^* \subset D$, $C^* \cup NC \subset D \cup ND$. This implies $NC \subset B$ and (2). The inclusion $B \subset C \cup NC$ follows from the definition of $B(b)$.

Suppose there is a block B' , $B' \neq B$, such that $NC \subset B' \subset C \cup NC$. There is a separator which separates B from B' . Since both B and B' contain NC , this separator is NC . But then one of these two blocks has to be in $C \cup NC$ and the other in $C^* \cup NC$, a contradiction.

If b contains two different cuts C, D , then $NC \neq ND$, by the definition of \sim , and (2) implies $B(b) \setminus NC \neq \emptyset$. If b only contains one cut C then $B(b) = C \cup NC$ and again $B(b) \setminus NC \neq \emptyset$. \square

Corollary 7.2. *If \mathcal{C} is thin then every block has at least $\kappa + 1$ elements.*

We saw that $b \mapsto B(b)$ defines a bijection from \mathcal{C}/\sim to the set of all blocks. If there are no ambiguities from the context we may now consider the set of black vertices \mathcal{B} as the set of \mathcal{C} -blocks instead of \mathcal{C}/\sim . For $C \in \mathcal{C}$ let $b(C)$ denote the \sim -class which contains C . In the tree $T(\mathcal{C})$ we now have $t(C) = B(b(C))$.

As mentioned in the introduction, we can join any two vertices by an (ideal) edge if they are not separated by any cut in some given nested cut system. The resulting cut system remains the same, but then every block spans a complete subgraph. This is illustrated in Figure 2b.

Example 7.3. Consider the graph X_n from Example 2.5. There are $n - 2$ separators $\{2, a, b\}, \{3, a, b\}, \dots, \{n - 1, a, b\}$. The set of \mathcal{C}_n -blocks is

$$\mathcal{B}_n = \{\{1, 2, a, b\}, \{2, 3, a, b\}, \dots, \{n - 1, n, a, b\}\}.$$

Instead of the blocks we could consider the \sim -classes

$$\mathcal{B}_n = \mathcal{C}_n / \sim = \{\{C_2\}, \{D_2, C_3\}, \{D_3, C_4\}, \dots, \{D_{n-2}, C_{n-1}\}, \{D_{n-1}\}\}.$$

The corresponding tree $T(\mathcal{C}_n)$ is an alternating path of length $2n - 2$ with $n - 2$ white and $n - 1$ black vertices. The system \mathcal{C}_n is thin.

For $l < k$ the intersection $C_k^* \cap D_l^* = \{c, d\}$ is a non-empty isolated corner, it is the only slice of \mathcal{C}_n . For other pairs of cuts we have an empty isolated corner. Hence \mathcal{C}_n is nested. The graph \hat{X}_n is obtained by deleting the vertices c, d .

8. STRUCTURE TREES AND EXTENSIONS OF CUT-SYSTEMS

We have shown the following main theorem.

Theorem 8.1. *Let X be a connected graph with a cut system \mathcal{C} invariant under a group G of automorphisms of X . The set \mathcal{N} of optimally nested cuts in \mathcal{C} forms the edges set of a G -tree $T(\mathcal{C})$.*

Let us now investigate further properties of structure trees. Let G be a group acting on a connected graph X and let \mathcal{N} be a nested system of thin cuts, invariant

under G . The action of G on \mathcal{N} induces an action on $T = T(\mathcal{N})$ and hence T is a G -tree.

A directed edge (u, v) of a tree is said to *point to* an edge, a vertex or an end, if a path from u to the edge, the vertex or the end, respectively, goes through v . We now show how a ray R in X determines either a unique end or a unique block vertex of T . We modify the tree T'' from Section 6, with $ET'' = \mathcal{N} \cup \mathcal{N}^*$, by deleting all edges which do not contain R eventually. Hence from each pair E, E^* we keep one edge and delete the other. In this new orientation, if E points to F then $F \subset E$ and both F and E contain R eventually. The out-degree with respect to this orientation is always 1 except for possibly one vertex whose out degree is 0. Hence all the edges point to a single vertex whose out degree is 0 or, if all vertices have out-degree 1, all edges point to a single end of T . If they point to a vertex, then this vertex cannot be a separator, since the vertices of a ray are distinct, and so it can only visit a separator S at most $|S|$ times. The ray determines the block $B \in \mathcal{B}$ if and only if it contains infinitely many vertices of B . We say that the ray R *belongs to* the block vertex or end of T .

Lemma 8.2. *If two rays belong to the same end of T then they belong to the same end of X .*

Proof. Suppose two rays R_1, R_2 belong to the same end of T . Then there is a sequence $C_1 \supset C_2 \supset C_3 \dots$ of cuts which all contain both R_1 and R_2 eventually. Lemma 4.3 implies that this sequence has empty intersection. If the rays are not in the same end of X there is a finite set S which separates them in X . From some index i on, the sets C_i are disjoint with S , contradicting the assumption that both rays are eventually contained in C_i . \square

Example 8.3. Consider Figure 2a. The set $B = \{x_1, x_2, x_3, x_4\}$ is not a \mathcal{C} -block for the cut system \mathcal{C} of all sets whose boundary is a 2-element subset of B . The nested system \mathcal{N} of optimally nested cuts consist of those cuts whose boundaries are one of the four separators $\{x_1, x_2\}, \{x_3, x_4\}, \{x_1, x_3\}, \{x_2, x_4\}$. Hence B is an \mathcal{N} -block.

We would like to show that for any thin cut system \mathcal{C} in a graph X , that there is a structure tree invariant under the automorphism group of X , in which distinct \mathcal{C} -blocks determine distinct vertices of the structure tree. However this is not always the case. For an example in which there is no structure tree separating all pairs of distinct \mathcal{C} -blocks we take \mathcal{C} to be the cut system for which the separators are $\{x_1, x_2\}, \{x_3, x_4\}, \{x_1, x_3\}, \{x_2, x_4\}$. Then there are four \mathcal{C} blocks, but two of these blocks are not separated by the optimally nested cuts corresponding to the separators $\{x_1, x_2\}, \{x_3, x_4\}$.

One could also construct a similar example in which there are two ends that are separated by thin cuts but which are not separated by optimally nested cuts.

Our goal is to find large G -invariant nested sub-systems of a given G -invariant cut system \mathcal{C} that separate as many structures from each other as possible. Cut-systems as in Examples 2.1, 2.2 and 2.4 are invariant under the action of the full automorphism group, because they are defined by separating geometric objects. These objects (κ -inseparable sets or ends) are large enough such that they are inseparable with respect to the cut-system they define. Such cut systems have G -invariant nested subsystems which separate all these objects, as the following two theorems will show.

First consider the thin cut system \mathcal{I} of Example 2.4 in which each cut $C \in \mathcal{I}$ has $|NC| = \kappa$ and C separates κ -inseparable sets.

Theorem 8.4. *Let X be a connected graph with automorphism group G . There is a nested cut system \mathcal{N} invariant under G which is a subsystem of \mathcal{I} with the following properties. If two κ -inseparable sets Y_1, Y_2 are separated by a cut in \mathcal{I} then they are separated by a cut in \mathcal{N} . Let $T = T(\mathcal{N})$ be the structure tree corresponding \mathcal{N} . Two distinct maximal κ -inseparable sets in X will belong to distinct vertices of T .*

Example 8.5. It may happen that κ -inseparable sets are not separated by optimally nested cuts. In Figure 7 there are four 4-inseparable sets (each of which is the vertex set of a complete subgraph on 5 vertices). There are two thin separators, shown in black, that correspond to optimally nested cuts C with $\mu(C) = 0$. The two central 4-inseparable sets Y_1, Y_2 are not separated by any cut for which $\mu(C) = 0$. The two separators, shown in grey, which separate Y_1, Y_2 correspond to cuts C with $\mu(C) = 16$. Any cut D which separates Y_1, Y_2 has $\mu(D) \geq 16$.

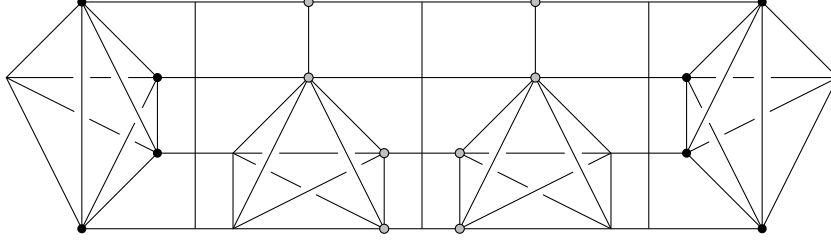


FIGURE 7. Graph in which 4-inseparable sets are not separated by optimally nested cuts

Proof of Theorem 8.4. From Theorem 8.1 we know that the optimally nested cuts form a nested subsystem \mathcal{N}_o of \mathcal{I} invariant under G . If this cut system does not have the required properties of the theorem then there is a vertex of $T_o = T(\mathcal{N}_o)$ that contains two κ -inseparable sets that are separated by a cut in \mathcal{I} , but which are not separated by any cuts in \mathcal{N}_o . There will be a cut $C \in \mathcal{I}$ that separates inseparable sets Y_1, Y_2 that are not separated by any cut in \mathcal{N}_o . Let

$$\mu'(C) = |\{D \mid D \in \mathcal{N}_o, C, D \text{ not nested}\}|.$$

We show that Y_1, Y_2 are separated by a cut C' for which $\mu'(C') = 0$. Let C separate Y_1, Y_2 and among all such C choose the one for which $\mu'(C)$ is smallest. If $\mu'(C) \neq 0$ then there is a $D \in \mathcal{N}_o$ which is not nested with C . Thus C, D have opposite corners $C \cap D, C^* \cap D^*$ with cut components. It is easy to show that if $E \in \mathcal{N}_o$ is not nested with C then it is not nested with at most one of the two corners and if it is nested with C then it is nested with both corners, see Lemma 3.9. Thus $\mu'(C \cap D) + \mu'(C^* \cap D^*) \leq \mu'(C)$. In fact the inequality is strict since D is not nested with C but it is nested with both corners. If C separates inseparable sets that are not separated by D , then the two sets will lie in distinct adjacent corners of C, D .

A thin cut with $\mu'(C) = 0$ but which separates inseparable sets will separate vertices of exactly one \mathcal{N}_o -block B . The cuts with $\mu'(C) = 0$ which separate inseparable sets in a particular block B will form a cut system, \mathcal{I}_B , which is a subsystem of \mathcal{I} . This does not happen with the above Example 8.3. The four cuts

corresponding to the separators $\{x_1, x_3\}, \{x_2, x_4\}$ do not form a cut system. The optimally nested cuts in this subsystem will be nested with each other, and with the cuts of \mathcal{N} . We get a larger nested subsystem than \mathcal{N} invariant under G by adding all the optimally nested cuts in \mathcal{I}_{gB} for each block gB in the orbit of gB . \square

Let \mathcal{E} be the cut system for vertex ends from Example 2.1.

Theorem 8.6. *Let X be a connected graph with automorphism group G . There is a nested cut system \mathcal{N} invariant under G which is a subsystem of \mathcal{E} with the following property. If two rays R_1, R_2 are separated by a thin cut in \mathcal{E} then they are separated by a cut in \mathcal{N} . Let $T = T(\mathcal{N})$ be the structure tree corresponding to \mathcal{N} . Then two rays which belong to different ends of X will belong to the same end or the same vertex of T if and only if they cannot be separated by a cut $C \in \mathcal{C}$.*

The proof is exactly the same as the proof of Theorem 8.4 except that we replace inseparable sets by rays.

Theorem 8.7. *Every cut-system \mathcal{C} in a connected graph has a nested subsystem which separates all \mathcal{C} -blocks.*

Proof. By Zorn's lemma there is a nested cut-system \mathcal{N} which is maximal with respect to inclusion. Suppose there is a \mathcal{N} -block B which is not a \mathcal{C} -block. Define

$$\mu'(C) = |\{D \mid D \in \mathcal{N}, C, D \text{ not nested}\}|.$$

Among all cuts which separate vertices in B , let C_o a cut for which μ' is minimal. Then $\mu'(C_o) = 0$ in contradiction to \mathcal{N} being maximal. \square

The proof of Theorem 8.4 fails under the general assumptions of Theorem 8.7, because the cuts which separate vertices in B will in general not form a subsystem (see Example 8.3) and hence we cannot extend \mathcal{N} in a G -invariant way. For this to be true, the cut system \mathcal{C} has to be defined by the separation of sufficiently large sets as in Theorems 8.4 and 8.6.

In Figure 8 an example is given of a 4-ended graph in which $\mu_{\min} = 4$. The vertices of four 3-element thin separators are drawn fat, corresponding sets $C \cup NC$ for cuts C are shown in light grey. The best way to work out $\mu(C)$ for a particular cut C is to count the number s of thin separators that have points in both C and C^* and then $\mu(C) = 2s$. For this graph any two rays that lie in distinct ends are separated by a cut with $\mu(C) = \mu_{\min} = 4$. The central block for the cut system of optimally nested cuts is shown in dark grey.

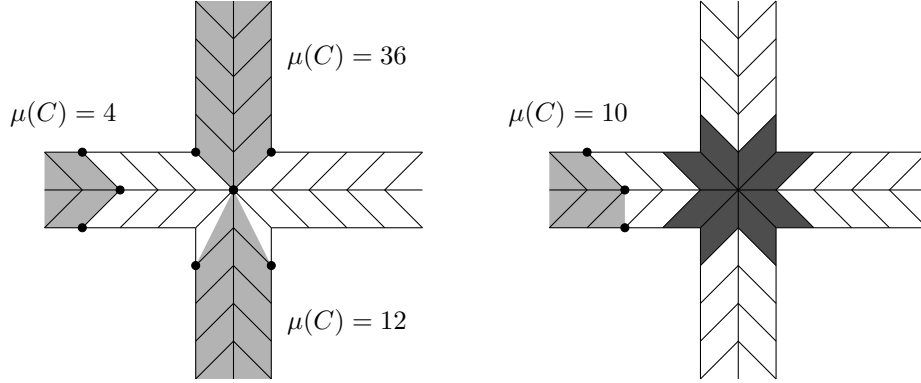
It is possible to change the graph of Figure 7 so that it gives an example of a graph in which there are ends that are separated by a thin cut but which are not separated by an optimally nested thin cut.

A result can be obtained for cut systems of edge cuts as in [3] in which the cut system and the nested sub-system do not only contain thin cuts. It is not possible to get such a strong general result for vertex cuts as the following example shows.

Example 8.8. Set $VX = \{v_i, u_j \mid i \in \mathbb{Z}, j \in \mathbb{N}\}$ and

$$EX = \{\{v_i, v_{i+1}\} \mid i \in \mathbb{Z}\} \cup \{\{v_i, u_1\} \mid i \in \mathbb{Z}\} \cup \{\{u_j, u_{j+1}\} \mid j \in \mathbb{N}\}.$$

This graph is shown in Figure 9. The cut point tree $T(\mathcal{C})$ is as in the figure. There is a block B which consists of the full subgraph on the vertices $\{v_i \mid i \in \mathbb{Z}\} \cup \{u_1\}$. This has a structure tree T_B for $\kappa = 2$ as also shown in the figure.

FIGURE 8. Graph with 4 ends with $\mu_{\min} \neq 0$

An automorphism of X restricts to an automorphism of both B and the full subgraph on the set $\{v_i \mid i \in \mathbb{Z}\}$. Any automorphism of these subgraphs is induced by an automorphism of X which fixes each u_i . Thus the automorphism group of X is an infinite dihedral group. It is not possible to expand the vertex B of $T(\mathcal{C})$ so that becomes T_B in such a way that the expanded tree has an infinite automorphism group.

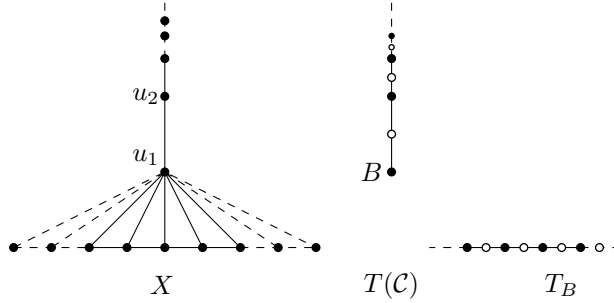


FIGURE 9. Structure trees for a three-ended graph

In general let X be a κ -connected graph with cut system \mathcal{C} and let \mathcal{N} be an automorphism invariant nested cut system of thin cuts. From a block B we obtain a connected graph X_B by taking the complete subgraph on B together with ideal edges joining any pair of vertices in the same \mathcal{N} -separator that are not already joined by an edge. We can then investigate cut systems for this graph X_B .

If \mathcal{C} does not just consist of thin cuts, then there may be \mathcal{C} -blocks in X_B that are not separated by a cut in \mathcal{N} . They will be separated by a cut in \mathcal{C} which is nested with every cut in \mathcal{N} . Such cuts will form an induced cut system \mathcal{C}_B on X_B , and we can get a structure tree for X_B for the nested thin cut system \mathcal{N}_B , in which we have $\kappa_B > \kappa$. We can expand the vertex v_B corresponding to B of the structure tree for \mathcal{N} and get a new expanded structure tree, in which the vertex v_B is replaced by a subtree isomorphic to $T(\mathcal{N}_B)$. The expansion involves choosing, for each edge e of $T(\mathcal{N})$ incident with v_B , a vertex of the tree T_B to which e will be attached in the expanded tree. There is not usually a canonical way of doing this. In the case when B is finite this can be done so that the expanded tree is still G -invariant.

This is illustrated in the next example. In the situation of an infinite graph such as in Figure 9 it may not be possible to carry out the expansion and get a G -tree.

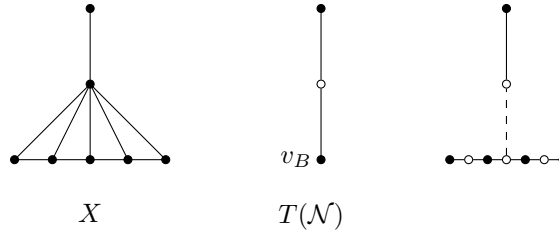


FIGURE 10. Structure trees for a finite graph

What happens for a finite graph is illustrated in Figure 10. The natural cut system for a finite graph consists only of thin cuts. However if we can still carry out the process described above and find a connected graph X_B corresponding to a block then there will be a new cut system for X_B . Again the constant κ_B for X_B (if it has a non-trivial cut system) will be larger than κ for X . In our example the graph X shown has automorphism group G of order two, and has one cut point. This gives rise to a cut system \mathcal{N} with two cuts one of which is the top vertex and the other the bottom 5 vertices. These cuts are nested and the corresponding tree $T(\mathcal{N})$ has three vertices, the middle vertex corresponds to the cut point and the two edges correspond to the cuts. There are two blocks, corresponding to the two other vertices of T . The vertices of one block are the vertices of the separating edge of X and the other block B corresponds to X with the top vertex removed. Now B has a nested system of cuts in which there are three separators each with two elements. The tree T_B corresponding to this cut system is shown at the bottom right of the diagram. It has 7 vertices, three of which correspond to these separators and the remaining four vertices correspond to the four 3-cycles in X . We can expand the vertex corresponding to B in $T(\mathcal{N})$ so that we get a structure tree for X , by attaching the dashed edge as shown. This is the only way to attach the edge to preserve the symmetry of the tree. Note though that it involves joining vertices corresponding to separators in the different cut systems, and so one cannot always expand blocks so that a 2-coloring of the expanded tree gives a natural way of getting separators and blocks. The reason one can always expand the tree in the case of a finite block and obtain a tree on which the automorphism group G acts is because a group action on a finite tree is trivial. There is always a vertex that is fixed by G . Thus one can always attach edges to the vertex in the structure tree for the block that is fixed by the automorphism group of the block, and this will mean that the expanded tree admits the automorphism group G of X .

In this example there is only one vertex in T_B that is fixed by the automorphism group (of order two) but this will not always be the case. Thus the automorphism group may be trivial, in which case any vertex of T_B can be chosen for the expansion, which will then not be canonical.

9. APPLICATIONS

9.1. Structure trees for finite graphs. The natural cut system of finite graphs is the cut system of Example 2.4.

Remark 9.1. The cut system of Example 2.4 is empty or finite transitive graphs.

Proof. Suppose this cut system is not empty. Then it contains the non-empty nested automorphism invariant system of optimally nested cuts, see Theorems 8.1 and 5.2. The automorphisms act transitively on the set of separators and on the set of blocks. Since the graph is finite, the structure tree has to be a star with a separator (white vertex) in the middle. This separator has κ vertices, in total there are more than κ vertices. Transitivity implies that each vertex is contained in one of the separators, hence there has to be more than one separator, a contradiction. \square

We illustrate the theory in three more examples. From Theorem 8.4 we know that $\mu_{\min} = 0$. The set of cuts C with $\mu(C) = 0$ form a nested subsystem \mathcal{N} and in these examples each maximal κ -inseparable set is an \mathcal{N} -block. This is not always the case, as can be seen from Figure 7.

Example 9.2. The following example illustrates a tree decomposition of a 3-connected graph. We take X to be the graph shown in Figure 11 (a) and (b). Let \mathcal{N} be the sub-system of cuts nested with all cuts in the cut system \mathcal{I} from Example 2.4. In fact in this case $\mathcal{N} = \mathcal{I}$. We have $\kappa = 3$. There are ten (3-inseparable) blocks. Three of them are shown shaded dark in (a). The three corresponding vertices of the structure tree all have valency one. Six blocks each consists of a 3-cycle, which are all shaded light in (a), together with p . There is another block which has valency 3 in the structure tree. This is shown in (b) with dotted ideal edges. The structure tree is shown in (c).

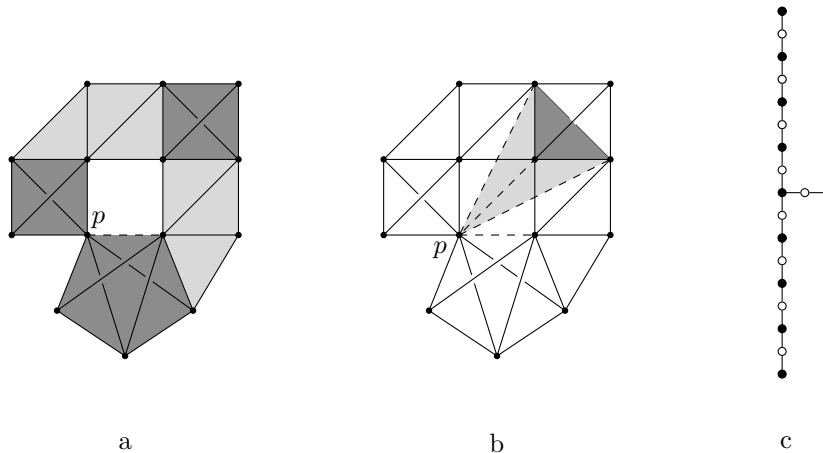


FIGURE 11. Decomposition of a 3-connected graph

Example 9.3. In the next example there is an \mathcal{N} -block which is also an \mathcal{I} -block and which is not connected. It consists of the three vertices on the top together with the three vertices at the bottom. In the tree it corresponds to the central vertex. The three “vertical sides” of the graph, which are unions of three tines of the dragon’s neck, correspond to a \mathcal{N} -block which is not an \mathcal{I} -block.

Example 9.4. The third example is the graph in Figure 13 which is similar to the example of Figure IV.3.4 from Tutte’s book [30]. In contrast to Tutte, we are not

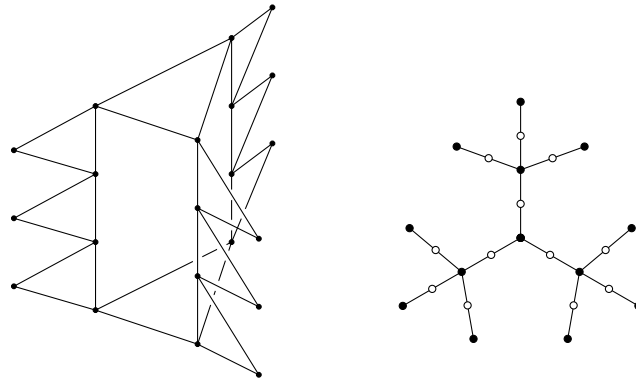


FIGURE 12. Dragon's Neck Graph

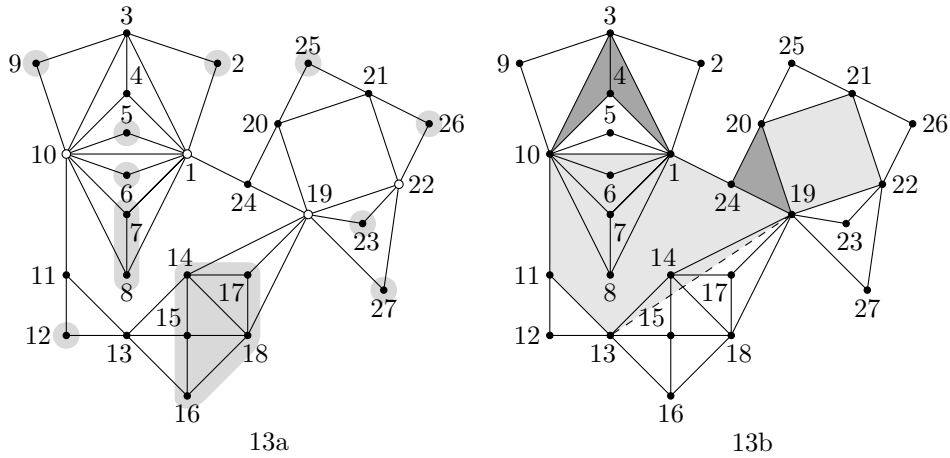


FIGURE 13. Tree decomposition of a 2-connected graph

considering multiple edges, because multiple and single edges are indistinguishable for vertex cuts.

The separators are $s_1 = \{1, 10\}$, $s_2 = \{3, 10\}$, $s_3 = \{1, 3\}$, $s_4 = \{11, 13\}$, $s_5 = \{13, 19\}$, $s_6 = \{19, 24\}$, $s_7 = \{19, 20\}$, $s_8 = \{20, 21\}$, $s_9 = \{21, 22\}$, $s_{10} = \{19, 22\}$. Every component in the complement of a separator is one of the cuts in \mathcal{N} . There are no slices. All separators have two components in their complement, except for s_1 and s_{10} , which are drawn white in Figure 13a.

The cuts which are only \sim -equivalent to themselves are $\{2\}$ (b_5), $\{5\}$ (b_1), $\{6\}$ (b_2), $\{7, 8\}$ (b_3), $\{9\}$ (b_4), $\{12\}$ (b_7), $\{14, 15, 16, 17, 18\}$ (b_8), $\{23\}$ (b_{13}), $\{25\}$ (b_{10}), $\{26\}$ (b_{11}), $\{27\}$ (b_{12}). The corresponding blocks b_i (in parentheses) are the union of cut and separator. These cuts are shaded grey in Figure 13a. The blocks are the leaves of the structure tree in Figure 14.

The block $b_8 = \{13, 14, \dots, 19\}$ is 2-inseparable (within the whole graph) but not 3-connected, but becomes 3-connected after adding an ideal edge joining 13 and 19.

The blocks $b_6 = \{1, 3, 4, 10\}$ and $b_9 = \{19, 20, 24\}$ are shaded in dark grey in Figure 13b. The corresponding \sim -classes consist of the cuts which are minimal with respect to containing these blocks.

The sets $b_{14} = \{1, 10, 11, 13, 19, 24\}$ and $b_{15} = \{19, 20, 21, 22\}$ are shaded in light grey in Figure 13b. They are blocks with respect to \mathcal{N} but not in the system of all cuts from Example 2.4 for $\kappa = 2$. Each separator in one of these two blocks has exactly one component C in its complement, such that $C \cup NC$ contains the block. These are the cuts which form the corresponding \sim -classes. The $*$ -complements of these cuts are arranged in a cycle.

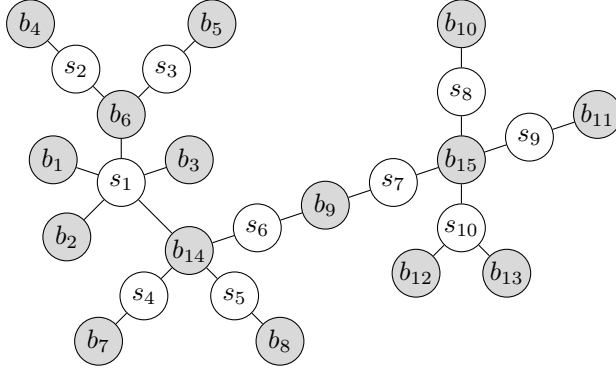


FIGURE 14. Structure tree for the graph in Figure 13

The number of cut components of a separator is the degree of the corresponding white vertex in the structure tree, see Figure 14. The degree of a black vertex is the cardinality of the corresponding \sim -class.

As mentioned earlier, if we connect any pair of vertices in a block, which are not adjacent, by an edge then this does not affect the cut system. Hence we obtain complete subgraphs which are arranged in a tree-like way. This works for k -connected graphs for any k .

It is a consequence of our main result (Theorem 8.1) that if a G -graph has a non-trivial cut system, then there is a homomorphism of G to the automorphism group of a tree. Thus there is a G -tree $T = T(\mathcal{N})$ associated with a nested sub-system \mathcal{N} of \mathcal{C} . The actions of groups on trees are completely described in the theory of Bass and Serre (see [3, 24, 25]). The action of a group G on a tree T is said to be *trivial* if G fixes a vertex. If T has finite diameter - in particular if T is finite - then this diameter is even and the action is trivial. The quotient graph $G \backslash T = Q$ is again a tree (possibly with loops).

Since X is finite $\mu_{\min} = 0$ by Theorem 5.4 and \mathcal{I} contains a unique nested sub-system \mathcal{N}_0 consisting of those cuts C for which $\mu(C) = 0$. This will be the directed edge set of a structure tree $T(\mathcal{N}_0)$. It is not always the case that any two κ -inseparable sets that are separated by a cut in \mathcal{I} are separated by a cut in \mathcal{N}_0 . An example was given in Figure 7. When this is not the case then by Theorem 8.4 we can get a larger nested cut system invariant under G which does have this property. We summarize this in the following theorem.

Theorem 9.5. *Let X be a finite graph with automorphism group G . Let κ be the smallest value of k for which there are a pair of k -inseparable subsets of VX*

which can be separated by a k -separator. There is a nested cut system \mathcal{N} of κ -separators, invariant under G with the following property. If two κ -inseparable sets are separated by a set of κ vertices in VX , then they are separated by a cut in \mathcal{N} .

The G -tree associated with the nested cut system of Theorem 9.5 is called the *structure tree for X* . If we join each pair of vertices by an ideal edge if the vertices lie in the same block, then we obtain a G -graph with the same structure tree in which the blocks are complete subgraphs. However one can obtain information about the connectivity of G for values of k bigger than κ if we only join two vertices by an ideal edge if they belong to the same separator (and not if they only belong to the same block). For a particular block B this graph is denoted X_B .

The graph X_B has at most one maximal κ -inseparable set. It may contain no κ -inseparable set. In this case if $\kappa = 2$, then the graph X_B is a cycle as noted by Tutte [30]. If $\kappa > 2$ the graph X_B may be quite complicated. In the graph of Figure 15 there is a central block shaded grey that contains no 3-inseparable set and four other blocks that span K_4 subgraphs.

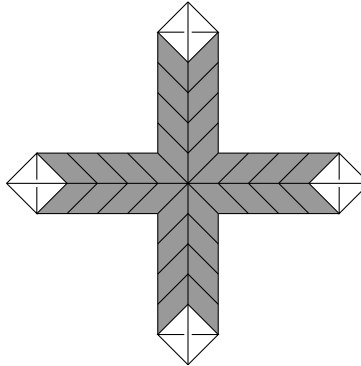


FIGURE 15. Graph with a block containing no 3-inseparable set

If there are k -inseparable sets that are separated by a set of k elements but are not separated by a separator in \mathcal{N} , then these sets must belong to the same \mathcal{N} -block. They will still be separated by the same set of elements after adding ideal edges between vertices in the same separator. Thus the k -inseparable sets will be separated in the graph X_B . Let κ' be the smallest value of k for which such sets exist. Then $\kappa' > \kappa$. We can obtain a structure tree $T(X_B)$ for X_B , and then expand the vertex of $T(\mathcal{N})$ in the way described earlier, and this can be done so that the automorphism group G of X acts on the expanded tree. If we repeat this process then one eventually obtains a structure tree T for X invariant under G in which if Y_1, Y_2 are k -inseparable and they are separated in X by a set with at most k elements then they correspond to distinct vertices of T .

9.2. Generalizations of Stallings' Theorem. If a structure tree has infinite diameter, then G may induce a non-trivial action on the structure tree. We briefly describe the relevant results from Bass-Serre theory.

A group is said to *split* over a subgroup H if it is either a free product of two groups with amalgamation over H , where these two groups contain H as a subgroup of index at least two, or if G is an HNN-extension of some group over H .

Suppose a group G acts transitively and without inversion on the set of edges of a tree T . Then either the quotient $G \backslash T$ is a loop and G is an HNN-extension of

the stabilizer of some vertex of T over the stabilizer of an incident edge. Or G has two orbits on VT , the quotient $G \backslash T$ is a graph with two vertices connected by an edge (called a *segment*), and G is a free product of the stabilizers of two adjacent vertices in T with amalgamation over the stabilizer of the edge which connects them. This decomposition is trivial if and only if the stabilizer of an edge is the same as the stabilizer of one of its vertices and the whole group stabilizes the other vertex v . If this happens then the tree T has diameter two, with central vertex v . The action is non-trivial if and only if for each edge $e \in ET$ both components of $T \setminus \{e\}$ contain at least one edge (or equivalently, at least two vertices). In fact if the action is non-trivial, then both components of $T \setminus \{e\}$ are infinite. Thus if G acts transitively without inversion on the set of edges of T then either T has diameter two or G splits over the stabilizer of an edge. The latter happens if and only if for some edge e both components of $T \setminus \{e\}$ intersect the orbit of e .

More generally, the action (without inversion) of a group G on a G -tree T is non-trivial if and only if either G splits over an edge stabilizer or it is a strictly ascending union

$$G = \bigcup_n G_n,$$

where $G_1 \subset G_2 \subset \dots$ is an infinite sequence of proper subgroups of G each of which stabilizes an edge of T .

If G is a group, a Cayley graph for G is a connected G -graph with one orbit of vertices and on which G acts freely. The edge orbits will correspond to a set of generators for G . There is a locally finite Cayley graph if and only if G is finitely generated. Different locally finite Cayley graphs of a finitely generated group are quasi-isometric. The number of ends of a locally finite graph is a quasi-isometry invariant and hence it does not depend on the finite set of generators. Thus we define the *number of ends of a finitely generated group* as the number of ends of its locally finite connected Cayley graphs.

The following was proved by Stallings in a series of papers (see [26, 27, 28]).

Theorem 9.6 (Stallings' Structure Theorem [5, 2]). *A finitely generated group has more than one end if and only if it splits over a finite subgroup.*

The first author proved Stallings' theorem in [5] by showing that the cut system of edge cuts, see Example 2.2, has a nested subsystem. We have proved that any cut system has a nested subsystem which separates ends. Hence we have a new proof of the main result in [5], and we get a new and relatively simple proof of Stallings' Structure Theorem. This is presented in detail in [20].

There are different ways of generalizing Stallings' theorem. One option is to drop the assumption of G being finitely generated. Another option is to consider splittings of finitely generated groups over groups which are not necessarily finite.

There are several ways of how to define ends for non-locally finite graphs (see [17]). The same holds for infinitely generated groups, where we have the further difficulty that without additional assumptions the Cayley graphs are not necessarily quasi-isometric. But whenever one defines ends of non-locally finite graphs then in locally finite graphs this definition should yield Freudenthal's end compactification for locally compact space (see [9, 10, 11]).

One way goes back to Freudenthal and D.E. Cohen [2] and says that G has more than one end if there is a subset A for which A and the complement $G \setminus A$ are both infinite and the symmetric difference of A and Ag is finite for all g in G .

It follows from the Almost Stability Theorem [3] that a group G has more than one end in this sense if and only if G splits over a finite subgroup or G is countably infinite and locally finite. This is a generalization of Stallings' structure theorem, because in the finitely generated case the definition above is equivalent to all other definitions of ends of graphs and groups. A more revealing way of stating this result follows from the Bass-Serre theory discussed above. Thus a group has more than one end if and only if it has a non-trivial action on a tree with finite edge stabilizers.

For a group that is not finitely generated there is no obvious way to choose a generating set to construct a Cayley graph. If we take the whole group as a set of generators, then the Cayley graph is essentially a complete graph which will have one end in any definition.

Stallings' theorem can be formulated as "A finitely generated group has a Cayley graph with more than one end if and only if it splits over a finite subgroup." Here we can just drop the assumption that the group is finitely generated.

Theorem 9.7. *A group has a Cayley graph with more than one end if and only if it splits over a finite subgroup.*

Proof. Suppose G splits over a finite group H . There are two possibilities. Let $\text{Cay}(G, S)$ denote the Cayley graph of G with respect to generating set S . Suppose $G = G_1 *_H G_2$ and $[G_i : H] \geq 2$, for $i = 1, 2$. If S_i is a set of generators for G_i then the graph $X = \text{Cay}(G, S_1 \cup H \cup S_2)$ has more than one end. Moreover, if $[G_1 : H] = [G_2 : H] = 2$ then X has two ends, otherwise X has infinitely many ends. If G is an HNN extension $G = G_1 *_H t$, so that G_1 is a subgroup of G with isomorphic finite subgroups H and $t^{-1}Ht$, then the Cayley graph $X = \text{Cay}(G, S_1 \cup \{t\})$ has more than one end.

If G has a Cayley graph X with more than one end then the cut system in Example 2.1 is not trivial and we can apply Theorem 8.1 to get a group action of G on a tree T . Then G splits over stabilizers of elements of a cut system (i.e. stabilizers of the edges of T). The splitting is non trivial as the graph X is vertex transitive and removing any separator in the cut system will leave at least two infinite components. The stabilizers of a cut A is finite, since it is a subgroup of the stabilizer of the finite set NA and the action of G on X is free. \square

Our results also provide a generalization of Stallings' Theorem to cases when the splitting group is not finite. Thus if a G -graph X with $G \backslash X$ finite has more than one vertex end then G has a non-trivial action on a tree T , which is a structure tree corresponding to the cut system of Example 2.1. The quotient $G \backslash T$ will be finite if $G \backslash X$ is finite and as T has infinite diameter, at least one orbit of edges of T will give a splitting of G . The splitting group will be the stabilizer of an edge in the selected orbit. This edge is a cut E in the cut system. Its stabilizer G_E will also stabilize NE . Since NE is finite, G_E will contain a subgroup of finite index which fixes each vertex in NE . Thus G_E has a subgroup of finite index which fixes a vertex of X . This subgroup may well be a proper subgroup of the stabilizer of this vertex. We summarize this in a theorem.

Theorem 9.8. *Let X be a G -graph with more than one vertex end and suppose $G \backslash X$ is finite, then G splits over a subgroup H such that a finite index subgroup of H fixes a vertex of X .*

Another possible application of vertex cuts is to the Kropholler Conjecture [23]. This arose out of work of Kropholler [21] on algebraic versions of the torus theorem for 3-manifolds.

Let H be a subgroup of the finitely generated group G . A subset A of a G -set is called *H -finite* if it is contained in the union of finitely many H -orbits, otherwise A is called *H -infinite*. We regard G as a G -set via the action of G on the left.

Conjecture 9.9 (Kropholler). *Let A be a subset of a finitely generated group G and let H be a subgroup of G such that $AH = A$. Let A and $G \setminus A$ be H -infinite and let $Ag \setminus A$ be H -finite, for all $g \in G$. Then G admits a non-trivial splitting over a group which is commensurable with a subgroup of H .*

If one could construct a G -graph X in which VX is the set of left cosets of H , which has more than one vertex end, then this conjecture would follow from the last theorem. One can get quite a long way in this direction. There will be a graph X in which VX is the set of left cosets of H , and such that $G \setminus X$ is finite. The set A will then determine a set E of vertices of this graph. The set NE is contained in finitely many H -orbits and since H fixes a vertex of X , NE has finite diameter in X . Both E and E^* have infinite diameter. This implies that both E and E^* contain rays. For more details we refer to [18, Theorem 3.5]. We have not been able to show that such a graph X can be constructed in which NE is finite, rather than just of finite diameter. If G is the commensurizer of H then one can construct X so that it is locally finite. A subset of VX will then be finite if and only if it has finite diameter. Thus the conjecture is true in this case. This was well known [7].

9.3. Applications in infinite graph theory. There are several applications of vertex cuts in infinite graph theory, in particular when classical structure tree theory is used and results are generalized from locally finite to non-locally finite graphs.

In [13] Hamann shows that an almost transitive end-transitive graph is quasi-isometric to a tree. Woess conjectured in [31] that infinitely ended graphs are quasi-isometric to a tree if the stabilizer of some end acts transitively on the set of vertices. This was proved by Möller for locally finite graphs in [22] and generalized to non-locally finite graphs with infinitely many edge-ends in [19]. Hamann uses vertex cuts to show that this also holds for the (more general) case of vertex ends.

A graph is called *connected-homogeneous* if every isomorphism between two finite connected subgraphs extends to an automorphism of the whole graph. In [14] Hamann and Hundertmark use the theory of the present paper to classify connected-homogeneous digraphs and show that if their underlying undirected graph is not connected-homogeneous then they are highly arc transitive.

A graph is *k -CS-transitive* if for any two connected isomorphic subgraphs of order k there is an automorphism between them which extends to the whole graph. Hamann and Pott [12] use vertex cuts in order to classify k -CS-transitive graphs for all k and non-locally finite distance transitive graphs which have more than one end.

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Carmesin and Fröhlich have pointed out in a seminar-paper [1] in Büsum that instead of considering minimal cut system one could choose a more general approach by considering so-called *weakly minimal* cuts. A cut is *weakly minimal* if its separator does not contain another separator. The set of weakly minimal cuts forms a cut system. One can replace the assumption of NC being finite by the assumption that each corner of a pair of cuts has only finitely many cut components. Or one assumes that the $*$ -complements of the cuts only have finitely many cut components. They call a cut a *weak B-cut* if its $*$ -complements of the cuts only have finitely many cut components. As conclusion they remark that either one has to assume the boundaries of cuts are finite or if not then one has to assume that cuts are either A-cuts or weak B-cuts which are nested with all but finitely many other cuts.

The function μ in the present paper counts cuts which are not nested with a given cut. An open question is, if there are interesting examples where it makes a significant difference to count separators instead of cuts.

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