

# Rectangle Groups

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## Introduction

We describe a class of groups which illustrate various themes in geometric group theory. Each of these groups contains a maximal free abelian subgroup. In fact each one contains infinitely many such subgroups. The actions of free abelian groups on  $\mathbf{R}$ -trees is well understood. If  $F$  is a free abelian group with generating set  $a_1, a_2, \dots, a_n$  then for any choice of real numbers  $r_1, r_2, \dots, r_n$  there is a homomorphism  $\theta : F \rightarrow \mathbf{R}$  in which  $\theta(a_i) = r_i$ . Here both groups are written additively. We can regard  $\mathbf{R}$  as an  $\mathbf{R}$ -tree on which  $\mathbf{R}$  acts via translation and  $F$  acts via the homomorphism  $\theta$ . Any action of  $F$  on an  $\mathbf{R}$ -tree is, essentially, like such an action. We will see that for each of our new groups  $G$  an action of  $G$  on an  $\mathbf{R}$ -tree corresponds uniquely to an action of the maximal free abelian subgroup  $F$  and there is an action of  $G$  for each such action of  $F$ .

In the theory of JSJ-decompositions of a finitely presented group  $G$  for splittings over slender subgroups the group  $G$  has a decomposition as a fundamental group of a graph of groups and incompatible decompositions of  $G$  over slender subgroups correspond to geometric splittings of vertex groups of this decomposition. If  $G$  has more than two mutually incompatible decompositions then the vertex group has a Euclidean structure and the vertex group has a homomorphic image which is a group of isometries of  $n$ -dimensional Euclidean space for  $n \geq 3$ . If one was attempting to generalize JSJ-theory to splittings over non-small subgroups then the groups above would have to occur as vertex groups in the decompositions. They do not map onto groups of isometries of Euclidean space, though they do have subgroups with this property.

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I am very grateful to Gilbert Levitt for correcting some of the wilder ideas I had about these groups.

## The groups

Let  $R$  be the group with presentation

$$R = \langle a, b, c, d \mid a^{n_a} = b^{n_b} = c^{n_c} = d^{n_d} = 1, ba^{-1} = dc^{-1}, ca^{-1} = db^{-1} \rangle.$$

Here  $n_a, n_b, n_c, n_d$  are integers  $\geq 2$ .

The form of this presentation that suggests calling  $R$  a *rectangle* group. I originally called these groups parallelogram groups but after further study it seems that the geometric description indicates that the angles are usually right angles.

First note that  $R$  has two incompatible decompositions as a free product with amalgamation,

$$R = \langle a, b \rangle *_C \langle c, d \rangle$$

where  $C = \langle ab^{-1}, b^{-1}a \rangle = \langle cd^{-1}, d^{-1}c \rangle$ .

or

$$R = \langle a, c \rangle *_C \langle b, d \rangle$$

where  $C = \langle ac^{-1}, c^{-1}a \rangle = \langle bd^{-1}, d^{-1}b \rangle$ .

In these decompositions  $C$  is free of rank two unless at least two of  $n_a, n_b, n_c, n_d$  are two.

Note that  $x = ab^{-1} = cd^{-1}$  commutes with  $y = ac^{-1} = bd^{-1}$  since  $xy = yx = ad^{-1}$ , and  $x, y$  generate a free abelian group of rank 2. This subgroup is denoted  $F$ .

If  $n_a = n_b = n_c = n_d = 2$ , then  $C = \langle ab \rangle = \langle cd \rangle$  is infinite cyclic and  $R$  is the free product with amalgamation of two infinite dihedral groups. In this case  $R$  has an action on the Euclidean plane. Let  $A, B, C, D$  be the vertices of a rectangle as in Fig 1. Let  $a, b, c, d$  be rotations through  $\pi$  at  $A, B, C, D$  respectively. Then this gives the action of  $R$  on the plane. Note that  $x = ab^{-1}$  will be a translation of two units in the horizontal direction and  $y = ac^{-1}$  will be translation by two units in a vertical direction. In the action of  $R$  there will be two orbits of rectangles giving a chequer board pattern as shown. There will be 4 orbits of both vertices and edges. One gets a transversal for the action on the 1-skeleton by taking the edges and vertices of a single rectangle.

For the general situation in which  $n_a, n_b, n_c, n_d$  are not all two, we construct a simply connected 2-dimensional  $R$ -complex  $X$  in which again there are 4 orbits of vertices and edges and there is a sub-complex which is a plane on which  $x, y$  act as above. Let  $A, B, C, D$  be a transversal for the vertex set of  $X$ , and let the stabilizers of  $A, B, C, D$  be generated by  $a, b, c, d$  respectively. Let  $e = \vec{AB}, f = \vec{AC}, g = \vec{BD}, h = \vec{CD}$  be a transversal for the action on the directed edges of  $X$  each of which has trivial stabilizer. Clearly  $A, B, D, C$  are the vertices of a cycle with edges  $e, g, \bar{h}, \bar{f}$ . However so also are  $A, aB, cD, C$  since  $aB = ab^{-1}B$  and  $cD = cd^{-1}D$  and so the edge  $g = \vec{BD}$  acted on by  $ab^{-1} = cd^{-1}$  is the edge with vertices  $aB, cD$ . Similarly there is an edge joining  $aC$  and  $bD$  and so there is a cycle  $A, aC, bD, B$ . Note that we are assuming the 1-skeleton of  $X$  is a simple graph, so that there is only one directed edge with a given ordered pair of vertices. The three cycles  $(A, B, D, C)$ ,  $(A, aB, cD, C)$ ,  $(A, aC, bD, B)$  are in different  $R$ -orbits. In Fig 1 there is another cycle containing  $A$ , namely  $A, aB, aD, aC$ . Clearly this is in the same orbit as  $(A, B, D, C)$ . We can therefore attach three  $R$ -orbits of 2-cells and obtain the required complex  $X$ , in which the  $(x, y)$  plane is indeed a plane. To summarize, in Fig 1 the shaded rectangles containing  $A$  as a boundary point are in the same orbit, but the unshaded rectangles containing  $A$  as a boundary point are in different orbits. A vertical line in the  $(x, y)$ -plane  $E$  will be part of a track in  $X$  which will correspond to one of the splittings of  $R$  over a free rank 2 subgroup. This track will be a 3-regular tree and its stabilizer in  $R$  will have 2 orbits of vertices. Here we note that  $y$  is a translation by 2 vertical units. There is no element in  $R$  translating through one vertical unit. A horizontal line in the  $(x, y)$ -plane is part of a track giving the other splitting of  $R$ . The space  $X$  is simply connected. It is the universal cover of a developable complex of groups which is shown to be simply connected in [H]. One can also establish that  $X$  is simply connected by cutting it up along the simply connected tracks just discussed and considering the pieces that remain.

The link of the vertex  $A$  in  $X$  will be a 3-regular bipartite graph with  $2n_a$  vertices. The vertices of the link correspond to the edges  $(A, gB)$  for  $g \in \langle a \rangle$  and  $(A, g'C)$  for

$g' \in \langle a \rangle$  and the edges to the 2-cells which include the three vertices  $g'C, A, gB$ .

Since  $X$  is locally  $CAT(0)$  - since the link of any vertex has no cycles of length less than 4 - and it is simply connected, it will be  $CAT(0)$  by [BH, p206].

Another - probably better - way of constructing  $X$  is to use the cubing construction of [Sa]. This is the approach that can best be generalized in higher dimensional generalization. The two decompositions for  $R$  we described earlier give two multi-ended pairs  $(R, C_1), (R, C_2)$  where  $C_1 = \langle ab^{-1}, b^{-1}a \rangle = \langle cd^{-1}, d^{-1}c \rangle$  and  $C_2 = \langle ac^{-1}, c^{-1}a \rangle = \langle bd^{-1}, d^{-1}b \rangle$ . Using such data Sageev describes how to construct a 1-connected cube complex which will be our space  $X$ . The two decompositions are associated with hyperplanes in this complex.

Since  $X$  is simply connected 'tracks' separate (see [DD]). Here we take a track to be a connected subspace that intersects any 2-cell in finitely many parallel straight lines that do not intersect a vertex.

A surjective homomorphism from  $F$  to  $\mathbf{Z}$  the integers under addition is determined by a pair  $(p, q)$  of coprime integers. One way of getting tracks in the  $(x, y)$ -plane corresponding to such a homomorphism, in the case when  $p, q \neq 0$ , is to assign length  $p$  to the edge  $AB$  and length  $q$  to the edge  $AC$  and then foliate the plane by lines of gradient one (See Fig 2). Each leaf in  $E$  maps to one of two  $(p, q)$ -curves in the torus  $F \setminus E$ . Note that the action of  $x$  is translation through  $2p$  and the action of  $y$  is translation through  $2q$ . In the quotient space  $R \setminus X$  the leaves maps to a track which has  $2(p+q)$  intersections with the 1-skeleton. It has  $p$  intersection points with each of the edges  $AB$  and  $CD$  and  $q$  intersections with each of  $AC$  and  $BD$ . In  $X$  this track will lift to an  $R$ -pattern of tracks. Each component track will be a 3-regular tree. The stabilizer of this track will be a free group. The quotient graph has  $2(p+q)$  vertices and - since it is a 3-regular graph - it has  $3(p+q)$  edges. The rank of the stabilizer will therefore be  $p+q+1$ . Apart from the splittings described above, which can be regarded as corresponding to the pairs  $(1, 0), (0, 1)$ , there is a splitting of  $R$  for each pair of coprime positive integers  $(p, q)$ .

### The tree

If we assign positive lengths  $\alpha, \beta$  to the edges of  $X$  so that  $|AB| = |CD| = \alpha, |AC| = |BD| = \beta$ , then we obtain a metric on  $X$  in which each 2-cell has the topology of a rectangle in the Euclidean plane with sides  $\alpha, \beta$ . There will be a foliation on  $X$  invariant under  $R$  in which the  $(x, y)$  plane is foliated by straight lines with gradient one as in Fig 2. If  $\alpha, \beta$  are dependent over the rationals then the image of a leaf in  $X$  which does not intersect a vertex will be a track in  $R \setminus X$  and will correspond to one of the decompositions of  $R$  given earlier, where  $\alpha = p\delta, \beta = q\delta$  where  $p, q$  are coprime integers. We now investigate what happens if  $\alpha, \beta$  are independent over the rationals. The action of  $R$  on a simplicial tree as described above will determine an element of  $PLF(R)$  the projectivized length functions on  $R$ , which is a compact space (see [Sh]). By approximating  $\alpha, \beta$  by rationals we obtain a sequence of actions on simplicial trees which will tend to a non-simplicial action of  $R$  on an  $\mathbf{R}$ -tree  $T$ , which restricts to the obvious action of  $F$  on  $\mathbf{R}$ .

The action of  $R$  on  $T$  has non-trivial free arc stabilizers, though it restricts to a free action of  $F$  on  $T$ . To see this note the two ways of foliating the rectangle with sides  $\alpha, \beta$ . In one foliation the leaves have gradient 1, and in the other  $-1$ . In the orbit space  $R \setminus X$  there are three rectangular discs attached to the same 1-skeleton. The foliation on each is

one of the two foliations above and so two of them have the same foliation. This means that there will be pairs of points in the boundary which are joined by leaves in different 2-cells and so there will be loops in leaves in the orbit space. These will not lift to loops in  $X$  and so there will be non-trivial point stabilizers. The action of  $R$  on the  $\mathbf{R}$ -tree will not be stable. It will restrict to the obvious free action of  $F$  corresponding to a foliation by lines of gradient 1 on  $E$ .

The tree  $T$  has four orbits of branch points and the number of directions at a transversal of branch points will be  $n_a, n_b, n_c$  and  $n_d$ . If these integers are odd primes then the only morphisms in the category of  $\mathbf{R}$ -trees with domain  $T$  will be isomorphisms as there can be no folding at such a branch point. The tree corresponding to the foliation with dense leaves on the 2-sphere with 4 cone points will be another such  $\mathbf{R}$ -tree and there is a surjective homomorphism from the group of this orbifold to  $R$ . However there is no morphism of  $\mathbf{R}$ -trees which includes this homomorphism as it is not an isomorphism.

### Parallelepiped groups

We now generalize the construction by showing that there are  $n$ -dimensional parallelepiped groups, rectangle groups being the groups obtained when  $n = 2$ .

We first consider the case  $n = 3$ . Consider the cube as shown in Fig 3.

Let  $S$  be the group with presentation

$$S = \langle a, b, c, d, e, f, g, h \mid ab^{-1} = cd^{-1} = ef^{-1} = gh^{-1}, ac^{-1} = bd^{-1} = eg^{-1} = fh^{-1}, \\ ae^{-1} = bf^{-1} = cg^{-1} = dh^{-1} \rangle.$$

The form of this presentation suggests calling  $S$  a *cube* or even a *rectangular brick* group. Note that there are 6 rectangle subgroups corresponding to the 6 faces of the cube  $\langle a, b, c, d \rangle, \langle e, f, g, h \rangle, \langle a, b, e, f \rangle, \langle a, c, e, g \rangle, \langle b, d, f, h \rangle, \langle c, d, g, h \rangle$  and there are free abelian rank 3 subgroups generated by  $x = ab^{-1}, y = ac^{-1}, z = ae^{-1}$ , and  $x' = b^{-1}a, y' = c^{-1}a, z' = e^{-1}a$ .

The group  $S$  has a decomposition as a free product with amalgamation

$$C = \langle a, b, c, d \rangle *_D \langle e, f, g, h \rangle$$

where  $D = \langle ab^{-1}, b^{-1}a, ac^{-1}, c^{-1}a \rangle = \langle ef^{-1}, f^{-1}e, eg^{-1}, g^{-1}e \rangle$ . Here  $ab^{-1} = x$  and  $ac^{-1} = y$  commute and  $b^{-1}a = x'$  and  $c^{-1}a = y'$  commute, and in fact,  $D$  is a free product of two free abelian rank 2 subgroups.

There are two other such decompositions corresponding to the two vertical hyperplanes.

The vertex groups in the above decompositions are rectangle groups.

We construct a simply connected 3-dimensional  $S$ -complex  $X$  in which again there are 8 orbits of vertices and edges and there is a sub-complex which is a three-dimensional Euclidean space on which  $x, y, z$  act by translation by 2 units in the corresponding direction. Let  $J = \langle x, y, z \rangle$ . The easiest way to construct the space  $X$  is as the cubing (see [Sa])

associated with the pairs  $(C, D_1), (C, D_2), (C, D_3)$  where  $D_1, D_2, D_3$  are the three splitting subgroups described above. The space  $X$  will contain subcomplexes invariant under the parallelogram groups associated with any two of these pairs and there will be three orbits of hyperplanes corresponding to the three splittings.

Let  $A, B, C, D, E, F, G, H$  be a transversal for the vertex set of  $X$ , and let the stabilizers of the vertices be generated by  $a, b, c, d, e, f, g, h$  respectively.

Let  $E$  be the subcomplex consisting of the 8  $J$ -orbits of a  $2 \times 2$ -cube. There are 20  $J$ -orbits of 2-cells and 8 orbits of 3-cells. Under the action of  $S$  two of the 8 3-cells are in the same orbit and there are no other cases in which  $J$ -orbits are identified giving 7  $S$ -orbits of 3-cells in all. The two 3-cells that are in the same orbit only share a single vertex  $A$  and are a 3-cell and its translate by  $a$ . Also under the action of  $S$  six pairs of 2-cells are identified giving 14  $S$ -orbits of 2-cells. If all the cubes containing a transversal of faces lay in different orbits there would be 42 orbits of 2-cells. In fact there are 14. Thus every face lies in 3 cubes.

The above description - in which  $S \backslash X$  is obtained from  $J \backslash E$  by identifying one pair of 3-cells - also enables us to understand the decompositions corresponding to the hyperplanes geometrically. In  $J \backslash E$  a hyperplane gives two 2-dimensional tori corresponding to  $x = -1/2$  and  $x = 1/2$ . Each of the tori is a  $(2 \times 2)$ -rectangle in which opposite sides are identified. In  $S \backslash X$  a  $(1 \times 1)$  rectangle in one torus is identified with a  $(1 \times 1)$ -rectangle in the other torus. The splitting group is the fundamental group of two tori with a rectangular disc identified, i.e. it is the free product of two free abelian groups of rank two.

As for rectangle groups we can determine all the simplicial  $\mathbf{R}$ -trees on which  $S$  acts. Let  $(p, q, r)$  be a triple of integers for which 1 is the greatest common divisor. There is a surjective homomorphism  $\theta : J \rightarrow \mathbf{Z}$  in which  $\theta(x) = p, \theta(y) = q$  and  $\theta(z) = r$ . We assign lengths  $p, q, r$  to the edges of  $X$  in the orbits of  $AB, AC, AE$  respectively and foliate  $X$  so that  $E$  is foliated by planes given by  $x + y + z = c$  for a constant  $c$ . If we let  $c$  range over all constants with fractional part  $1/2$  then we obtain representatives of all non-parallel tracks in  $E$ . In  $J \backslash E$  these planes give 2 non-parallel tracks, since 2 is the greatest common divisor of  $2p, 2q, 2r$ . Each three cell of  $E$  will intersect the transversal of non-parallel planes in  $E$  in  $p + q + r - 1$  distinct discs. There is one such 3-cell which intersects the planes for which  $c$  takes the values  $1/2, 3/2, \dots, p + q + r - 1/2$ . In  $S \backslash X$  the two tracks in  $J \backslash E$  become a single track by identifying  $p + q + r - 1$  distinct pairs of discs. The splitting group will be a free product of two free abelian rank two subgroups and a free group of rank  $p + q + r - 2$ .

We have discussed the case  $n = 2$  and  $n = 3$ . In general, corresponding to an  $n$ -dimensional parallelepiped there will be a group  $P_n$  that has  $n$  decompositions over a subgroup that is a free product of two rank  $n - 1$  free abelian groups. Corresponding to these decompositions there is a 1-connected  $n$ -dimensional cubing  $X_n$ . Let  $E_n$  be  $n$ -dimensional Euclidean space and let  $J_n$  be generated by translations of 2 units in each of the coordinate directions. The space  $P_n \backslash X_n$  is obtained from  $J_n \backslash E_n$  by identifying an antipodal pair of  $n$ -cells in  $J_n \backslash E_n$ .

Suppose now that we give positive real lengths  $\alpha_1, \alpha_2, \dots, \alpha_n$  of translation to the generators of  $J_n$ . We choose these lengths so that they are independent over the rationals. We also foliate  $E_n$  by  $n - 1$ -dimensional affine subspaces which intersect each 2-dimensional coordinate hyperplane in a straight line of gradient one.

The leaves of this foliation are the points of an  $\mathbf{R}$ -tree on which  $P_n$  acts with free arc stabilizers.

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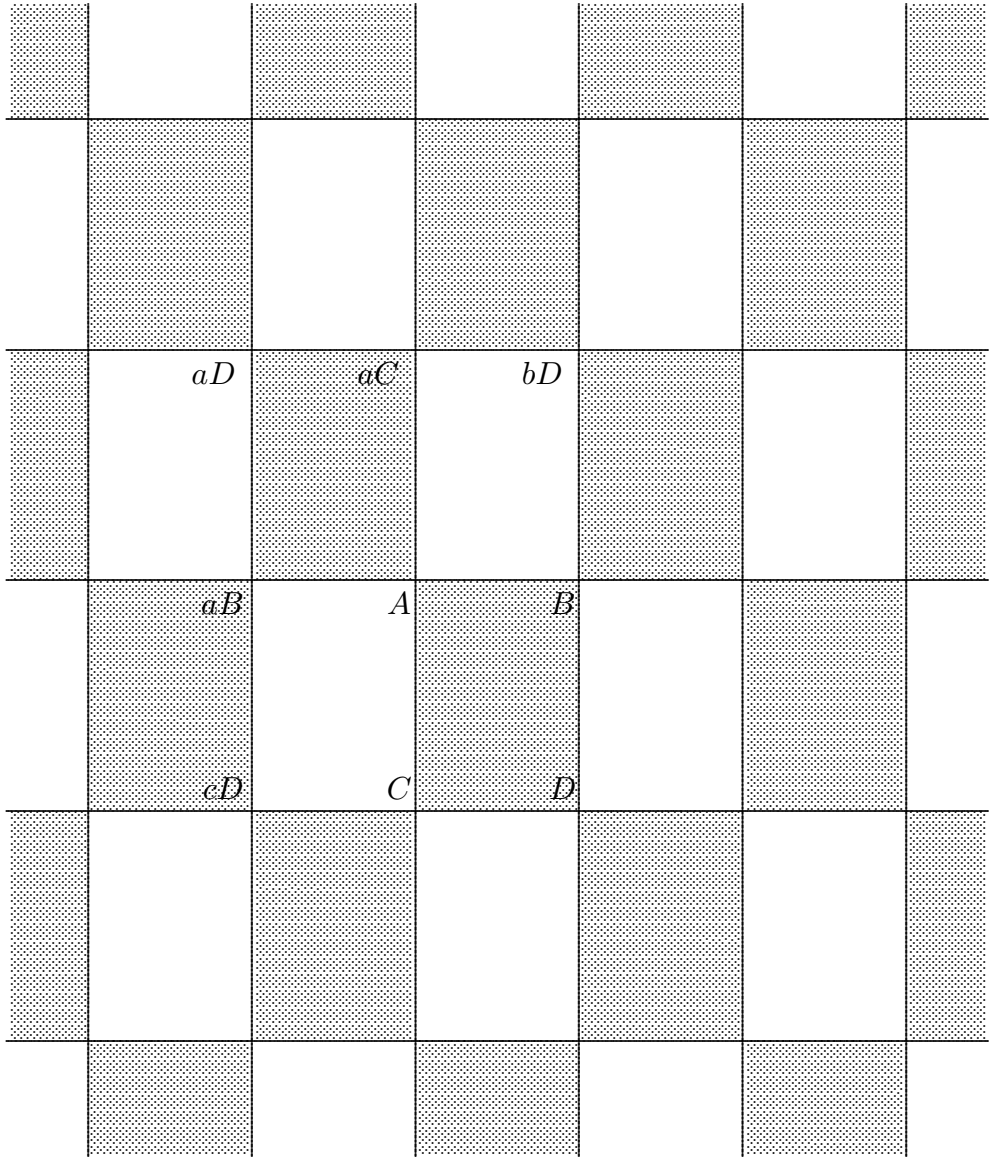


Fig 1

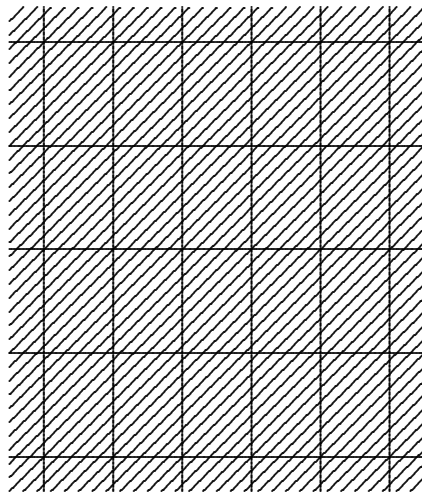


Fig 2

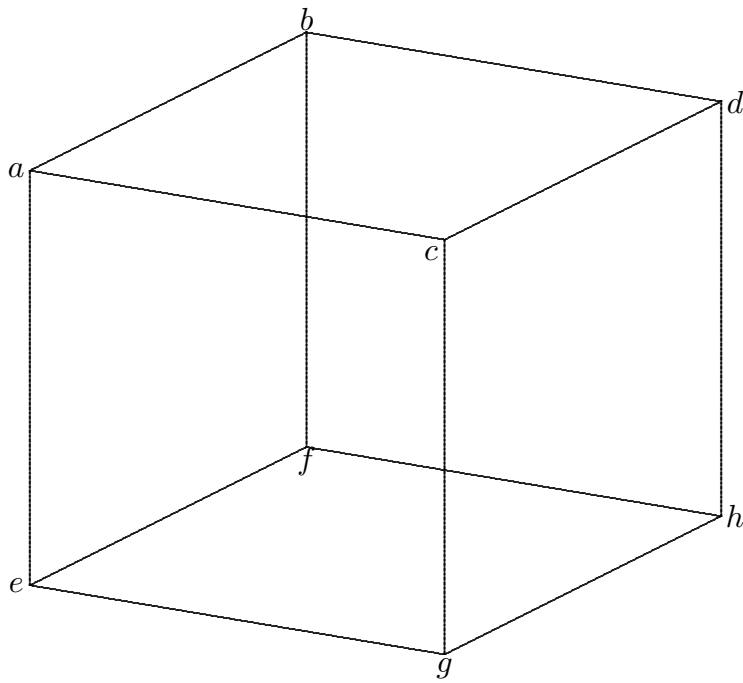


Fig 3