PROPER DECOMPOSITIONS OF FINITELY PRESENTED GROUPS

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ABSTRACT.

1. INTRODUCTION

In his seminal work [16] Stallings showed that a finitely generated group with more than one end splits over a finite subgroup. In [3] it was shown that a finitely presented group is accessible. This means that a finitely presented group G has a decomposition as the fundamental group of a graph of groups in which vertex groups are at most one ended and edge groups are finite. This decomposition provides information about every action of G on a simplicial tree with finite edge groups. Thus, let S be the Bass-Serre G-tree associated with the decomposition described and let T be an arbitrary G-tree with finite edge stabilizers, then there is a G-morphism $\theta : S \to T$. We say that any action is *resolved* by the action on S. In [4] and [5] examples are given of inaccessible groups. These are finitely generated groups - but not finitely presented - for which there is no such G-tree S. These groups do have actions on a special sort of $\mathbb{R}$-tree (a realization of a protree) but there appears to be no such action which resolves all the other actions.

The result -and its proof - on the accessibility of finitely presented groups can be seen as a generalization of a result by Kneser (see [7]) - and its proof - that a compact 3-manifold (without boundary) has a prime decomposition, i.e. it can be expressed as a connected sum of a finite number of prime factors. A compact 3-manifold M is prime if for every decomposition $M = M_1 \sharp M_2$ as a connected sum, either M_1 or M_2 is a 3-sphere. Expressed as a result about fundamental groups, it says that the fundamental group of a compact 3-manifold is a free product of finitely many factors, which, of course, is true for any finitely generated group by Grushko's Theorem. The JSJ-decomposition of a compact 3-manifold M, due to Jaco-Shalen [9] and Johannson [11] concerns the embeddings of tori in compact prime 3-manifolds. They show that there exists a finite collection of embedded 2-sided incompressible tori, such that the pieces obtained by cutting M along these tori are either Seifert fibered spaces or simple manifolds (acylindrical and atoroidal). The JSJ–decomposition provides information about the splittings of $\pi_1(M)$ over rank 2 free abelian subgroups. In a group theoretic setting JSJ-decompositions were discussed first by Kropholler [12] and subsequently by many authors. The first result for all finitely presented groups (over cyclic subgroups) was by Rips and Sela [14] In their result 2-orbifold groups appear as special vertex groups for the first time.

In this paper it is shown that a finitely presented group G has a decomposition as the fundamental group of a cube complex of groups which provides information about every action of G on an $\mathbb{R}$-tree.

For any group G a subgroup H of G is said to be *G-unsplittable* if, in any action of G on an $\mathbb{R}$-tree T, H fixes a point of T. We prove the following theorem.

Theorem 1.1. *Let G be a finitely presented group.*

Then there is a cubing $\tilde{C}$ with a G-action such that $G\backslash\tilde{C}$ is finite. Every edge and vertex stabilizer of $\tilde{C}$ is G-unsplittable. Every G-unsplittable subgroup of G fixes a vertex of $\tilde{C}$

If G has a non-trivial splitting then some hyperplane of $\tilde{C}$ is associated with a non-trivial splitting of G.

For any action of G on a simplicial tree T, there is a pattern in C so that the corresponding tree with its G-action resolves the action on T.

We also show that the decompositions of Theorem 1.1 can be computed. It may, however, not be possible to decide if the decomposition is non-trivial. In some cases this will provide a way of deciding if a group has a non-trivial action on a tree or if it has more than one end. Thus one will obtain a finite list of decompositions of G as a free product with amalgamation or as an HNN-group. The theory indicates that if G has a non-trivial decomposition then one of the decompositions in the list will be non-trivial and if G has more than one end then one of the decompositions in the list will be non-trivial and over a finite group. However it may not be possible to decide if a particular decomposition is non-trivial. It is known that there is a group H which has a presentation for which it cannot be decided if the group is non-trivial. One could use this presentation to construct a presentation for $H * H$. Clearly it will not be possible to decide it this decomposition is non-trivial. If the group G has a solvable membership algorithm then it will be possible to decide if a decomposition in the list is non-trivial.

In [6] the following theorem is proved.

Theorem 1.2. *Let G be a finitely presented group and let T be a G-tree, i.e. an $\mathbb{R}$-tree on which G acts by isometries.*

Then G is the fundamental group of a finite graph $(\mathcal{Y}, Y)$ of groups, in which every edge group is finitely generated and fixes a point of T. If $v \in VY$, then either $\mathcal{Y}(v)$ fixes a vertex of T or there is a homomorphism from $\mathcal{Y}(v)$ to a target group $Z(v)$ (an augmented parallelepiped group), which is the fundamental group of a cube complex of groups based on a single n-cube $c(v)$.

Every hyperplane of $c(v)$ is associated with a non-trivial splitting of G.

There is a marking of the cube $c(v)$ so that the corresponding $\mathbb{R}$-tree with its $Z(v)$-action is the image of a morphism from a $\mathcal{Y}(v)$-tree T_v and this tree is the minimal $\mathcal{Y}(v)$-subtree of T.

In this paper we strengthen this result by showing that that there is a decomposition as in the theorem that works for any G-tree.

Thus we prove the following.

Theorem 1.3. *Let G be a finitely presented group.*

Then G is the fundamental group of a finite graph $(\mathcal{Y}, Y)$ of groups, in which every edge group is finitely generated and G-unsplittable. If $v \in VY$, then either $\mathcal{Y}(v)$ is G-unsplittable or there is a homomorphism from $\mathcal{Y}(v)$ to a target group $Z(v)$ (an augmented parallelepiped group), which is the fundamental group of a cube complex of groups based on a single n-cube $c(v)$.

*Every hyperplane of $c(v)$ is associated with a non-trivial splitting of G.*

For any G-tree T there is a marking of the cube $c(v)$ so that the corresponding $\mathbb{R}$-tree with its $Z(v)$-action is the image of a morphism from a $\mathcal{Y}(v)$-tree T_v and this tree is the minimal $\mathcal{Y}(v)$-subtree of T.

It follows from the proof of this result that the list of splittings in Theorem 1.1 contains a list of compatible splittings of G over finite subgroups that together define an action of G on a tree with finite edge stabillizers and for which vertex stabilizers have at most one end. Thus if G is a finitely generated group with a solvable membership problem, and for which it is possible to decide if a finitely generated subgroup is finite, then it is possible to decide if the group has more than one end.

2. Proof of Theorem 1.1

We illustrate the theory by repeated reference to a particular example.
The cell complex for the trefoil group $G = \langle c, d | c^3 = d^2 \rangle$
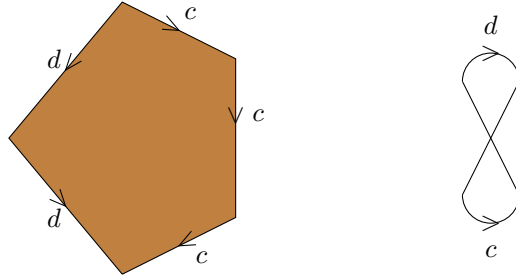
Attach the 5-sided disc to the figure eight as specified by the letters and arrows. The space X has $\pi_1(X) = G$.

A group presentation can be changed so that every relation has length at most three, giving a presentation complex with 2-cells having at most 3 edges.
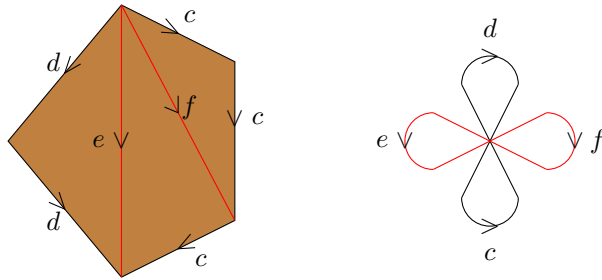

Thus $G = \langle c, d | c^3 = d^2 \rangle = \langle c, d, e, f | d^2 = e, e = fc, f = c^2 \rangle$.
The cell complex X consists of three 3-sided 2-cells attached to a 4-leaved rose.

Let X be a cell complex in which each 2-cell is 3-sided.
A *pattern* is a subset of X which intersects each 2-cell in a finite number of disjoint lines each of which intersects the boundary of the 2-cell in its two end points which lie in distinct edges.
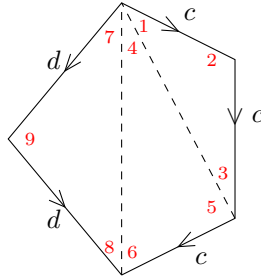A *track* is a connected pattern.
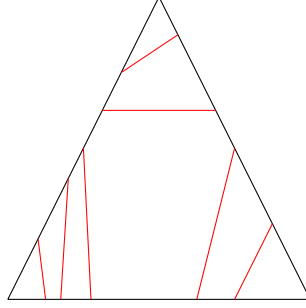
If X has m 2-cells then a pattern is specified (up to an obvious equivalence) by a $3m$-vector in which there are three coefficients for each 2-cell which record the number of lines joining the two edges at each corner.

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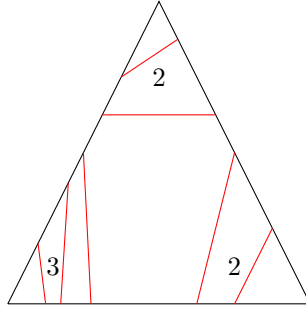
the coefficients $2, 2, 3$ record the intersection of the pattern with that particular 2-cell.

For the complex X for the trefoil group G a pattern is specified by a 9-vector, where the i-th coefficient corresponds to the number of lines crossing the i-th corner labelled i in red in the diagram below. In the trefoil complex a vector of non-

negative integers $x = (x_1, x_2, \ldots, x_9)$ is a pattern in if it satisfies the matching equations

$$x_1 + x_2 = x_2 + x_3 = x_5 + x_6$$

(number of intersection points with edge c)

$$x_1 + x_3 = x_4 + x_5$$

(number of intersection points with edge f)

$$x_4 + x_6 = x_7 + x_8$$

(number of intersection points with edge e)

$$x_7 + x_9 = x_8 + x_9$$

(number of intersection points with edge d)

In general a $3m$-vector corresponds to a pattern, if and only if

(i) Each entry is a non-negative integer.
(ii) It is a solution vector to a finite set of linear equations called the *matching equations*, where if an edge e lies in k 2-simplexes, then there are $k - 1$ matching equations corresponding to the intersection of the pattern with e.

In general a pattern P in a 2-complex X will lift to a pattern $\tilde{P}$ in $\tilde{X}$. Each track component of $\tilde{P}$ will separate and there is a G-tree T_P in which the edges correspond to the track components of $\tilde{P}$ (see [2], Chapter VI or [3] for details). If P consists of a single track then T_P will be the Bass-Serre tree for a decomposition of G as a free product with amalgamation, if the track is separating, and as an HNN-group if it is untwisted and non-separating. An *untwisted* track t is one which has a neighbourhood that is homeomorphic to $t \times I$ where I is a closed interval.
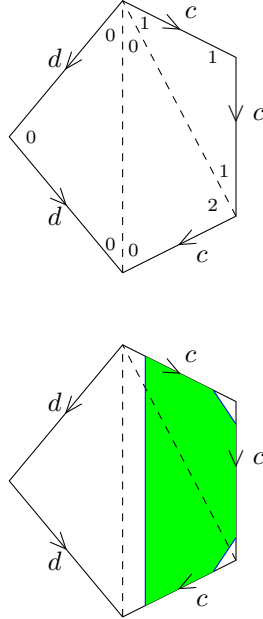
In the trefoil complex X an example of a pattern is as follows. The 9-vector
$t_1 = (1, 1, 1, 0, 2, 0, 0, 0, 0)$
corresponds to the pattern shown above. Thus there is one line crossing each of the corners labelled $1, 2$ and 3 and 2 lines crossing the corner labelled 5.

This pattern is in fact a separating track and corresponds to the decomposition of G.

$$G = \langle d \rangle *_{\langle d^2 = c^3 \rangle} \langle c \rangle.$$

The track separates into two regions one of which is coloured green.

A separating track is always untwisted. It t is twisted, then $2t$ is separating and hence untwisted.

The track t shown below in blue is twisted so the pattern $2t$ is also a track. The separating track $2t$ gives the trivial decomposition $G = G *_H H$ where H has index two in G
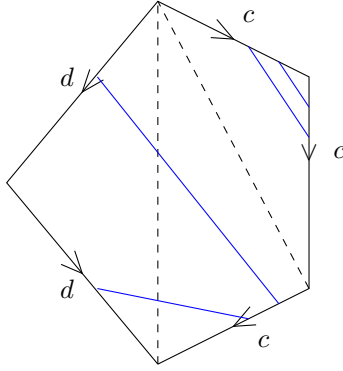
The track shown in red is non-separating and untwisted, and gives a decomposition of G as an HNN-group.
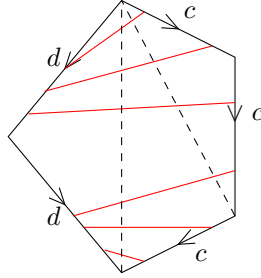
Such a track is always associated with a homomorphism $G \to \mathbb{Z}$. In this case $c \mapsto 2, d \mapsto 3$.

If X has n 2-simplexes and m 1-simplexes (edges) then X_1 has $3n$ 2-cells and $3n+m$ 1-cells. A *marking* of X_1 is a solution to the matching equations. A marking will be any point of a compact, convex linear cell in $\mathbb{R}^{3n+m}$ called the projective solution space $\mathcal{P}$. This theory is a generalization of the theory of normal surfaces or patterned surfaces in 3-manifolds (see [8],[10] and [2], Chapter VI). The *extreme* or *vertex* solutions are the ones corresponding to vertices of the projective solution space. Jaco-Oertel [8] and Jaco-Tollefson [10] have shown that vertex solutions carry important information about normal surfaces in a 3-manifold. Thus in [10] it is shown that there is a face of $\mathcal{P}$ for which the vertex solutions give a set of 2-spheres giving a complete factorization of a closed 3-manifold. A solution

is a vertex solution $\mathbf{v}$ if it has integer coefficients and integer multiples of v are the only solutions to $n\mathbf{v} = \mathbf{v}_1 + \mathbf{v}_2$, where n is a positive integer and $\mathbf{v}_1, \mathbf{v}_2$ are non-zero vectors in $\mathcal{P}$ with non-negative integer coefficients. The first author, in his D.Phil. Thesis [1] investigated the solution space for a group presentation on a computer. It was hoped to show that at least one vertex solution gives a non-trivial decomposition if the group has such a decomposition. At the time we were unable to show that this was the case. Happily we are now able to show that that the theory and algorithm described here provide a more efficient way of finding a finite set of solutions that provide a non-trivial decomposition if such a decomposition exists.

Choose a basis of solutions $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$ for the solution space to the matching equations, with integer coefficients. We can, in fact, choose the $\mathbf{u}_i$'s so that they have non-negative integer coefficients. This is because there is a solution to the matching equations in which each entry is a positive integer. Thus we can give each edge in the original cell complex the value 2, and give a value of 1 to each of the edges created by subdividing a 2-cell into three new 2-cells. This gives a solution $\mathbf{o}$ in which all coefficients are positive and a multiple of this vector can be added to any vector solution to give a solution with non-negative coefficients.

Two patterns are *equivalent* if they have the same number of intersections with each edge, so that they determine the same vector $\mathbf{u}$. Two tracks t_1, t_2 are *compatible* if there is a pattern with two components which are equivalent to t_1 and t_2.

If all the $\mathbf{u}_i$'s are compatible, then when lifted to the universal cover $\tilde{X}_1$, the pattern of lifted tracks forms the edges of G-tree.

If the solution space is one dimensional, then G has no non-trivial action on an $\mathbb{R}$-tree. This is because the non-zero vector $\mathbf{o}$ corresponds to a trivial action.

If $\mathbf{u}$ is a twisted track, then $2\mathbf{u}$ is a separating track. By replacing any twisted $\mathbf{u}_i$ by $2\mathbf{u}_i$, we obtain a basis $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$ of the solution space consisting of untwisted tracks. Each separating track gives a decomposition of G as a free product with amalgamation (possibly trivial). Each non-separating track gives a decomposition of G as an HNN-group.

3. The Sageev Cubing

An untwisted track corresponds to a decomposition of the group G either as a free product with amalgamation or an HNN-group. Michah Sageev [15] described a cubing S associated with a finite number of such decompositions. The space S is a $CAT(0)$ cube complex with a G-action.

In fact we will be concerned with a G-sub-complex $\tilde{C}$ of the Sageev cubing. Let $t_1, t_2, \ldots, t_n$ be a set of untwisted tracks in a complex X. These tracks lift to a G-pattern $\tilde{P}$ in the universal cover $\tilde{X}$. Each track in $\tilde{P}$ is separating. If $\tilde{b} \in \tilde{P}$ then $\tilde{P} \setminus \tilde{b}$ has two components. Let Σ be the set of all such components. If $A \in \Sigma$ is associated with the track $\tilde{b}$, then let A^* be the other component of $\tilde{P} \setminus \tilde{b}$. A vertex V of the Sageev cubing S is a subset of Σ which satisfies the conditions

(i) For each $A \in \Sigma$ exactly one of A, A^* is in V.
(ii) If $A \in V, B \in \Sigma$ and $A \subset B$, then $B \in V$.

Two vertices in $U, V \in S$ are joined by an edge if as subsets of Σ they differ by exactly one element. If $v \in V\tilde{X}$ then the subset

$$V_v = \{A \in \Sigma | v \in A\}$$

is easily seen to be in VS. If u, v are joined by an edge e in $\tilde{X}$, then V_u, V_v differ on finitely many elements of Σ, each such element corresponding to a track in $\tilde{X}$ that intersects e. If there are n such tracks, then in S there are $n!$ geodesic paths joining V_u and V_v, one path for each permutation of the tracks. We take $\tilde{C}$ to be the sub-complex of S consisting of all vertices and edges in a geodesic path joining V_u, V_v where u, v are the vertices of an edge of $\tilde{X}$. An n-cell of S is in $\tilde{C}$ if all its vertices are in $\tilde{C}$.

Theorem 3.1. *The space $\tilde{C}$ is a simply connected $CAT(0)$-cube complex i.e. $\tilde{C}$ is a cubing. The action of G on $\tilde{X}$ induces a cocompact action on $\tilde{C}$.*

Proof. It follows from the fact that S is $CAT(0)$ that $\tilde{C}$ is $CAT(0)$. Let u, v be adjacent vertices in $\tilde{X}$. In $\tilde{C}$ any two geodesic paths between V_u and V_v are homotopic. This is because they are homotopic in S and the homotopy involves changes between paths in $\tilde{C}$. Let u, v, w be vertices of a 2-cell in $\tilde{X}$. The tracks that intersect the edge uv partition into two disjoint subsets, namely the ones that intersect uw and the ones that intersect vw. There is a geodesic from V_u to V_v in $\tilde{C}$ that first crosses the tracks that intersect uw and then those that intersect vw. Let V_p be the vertex it reaches after crossing the tracks that intersect uw. Then V_p is on a geodesic between any two of the vertices V_u, V_v, V_w. It follows easily that any loop in $\tilde{C}$ consisting of three geodesic paths going from u to v and then v to w and then w back to u must be null homotopic in $\tilde{C}$. It follows from this that $\tilde{C}$ is simply connected.

$\square$

Let K be a 2-dimensional cube complex. We extend the idea of patterns in a complex where the 2-cells are three sided to patterns in a complex where the 2-cells are four sided. We define a *pattern* in K to be a subset such that its intersection with each 2-cell is specified by a pair of integers p, q and the lines join points on the boundary so that there are p points of intersection on each of one pair of opposite sides and q points on each of the other two sides. Thus the intersection corresponding to the pair $(2, 3)$ is as below. Note that for a given set of intersections with the boundary, there are two different ways of joining up the boundary points. A *track* is a connected pattern. As in the case for tracks in a simplicial 2-complex, a track in a simply connected 2-dimensional cube complex will separate.
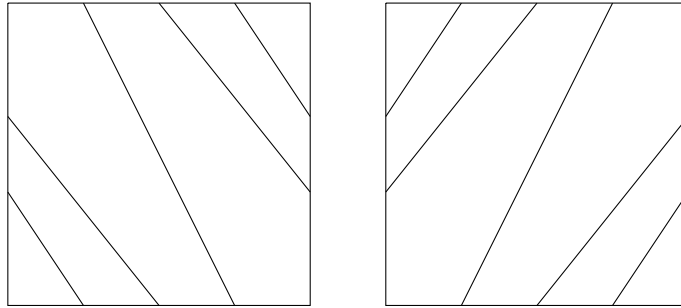
We now describe how a pattern p in X is associated in a natural way with a pattern in the cube complex C associated with a basis, consisting of untwisted tracks, for the solution space.

The G-pattern $\tilde{p}$ is such that all the component tracks are untwisted. Suppose that $t_1, t_2, \ldots, t_n$ is a basis for the solution space, consisting of untwisted tracks. Then for some positive integer β, $\beta p = \beta_1 t_1 + \beta_2 t_2 + \cdots + \beta_n t_n$ where the β_i's are integers. Choose an embedding of the tracks t_i in X so that the t_i's intersect transversely, specifically (for $i \neq j$) so that if σ is a 2-cell then a component of $t_i \cap \sigma$ and a component of $t_j \cap \sigma$ intersect transversely in at most one point in the interior of σ.

We can then choose a small closed neighbourhood b_i of each t_i so that each component of $b_i \cap b_j$ is a 4-sided disc containing exactly one point of intersection.
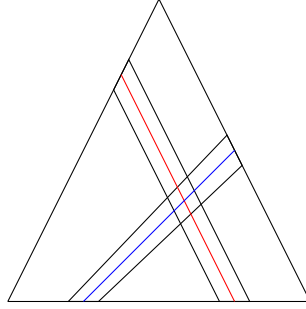

Now replace each t_i by β_i parallel copies lying within a b_i.
Below $\beta_i = 2, \beta_j = 3$.
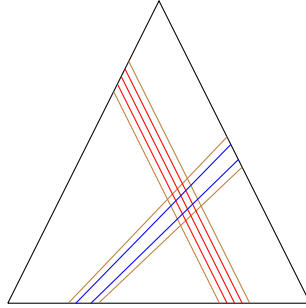
If in each intersection of b_i with b_j is replaced by non intersecting lines as below then the pattern $p = \beta_i t_i + \beta_j t_j$ is obtained. If in the first of the above diagrams, we had transposed the two intersection points at the bottom edge, then t_i and t_j would not intersect in σ. In this case replacing t_i and t_j with non-parallel copies will give the same intersection with σ as in the case when t_i and t_j do intersect in σ, and we carry out the above intersection replacement in $b_i \cap b_j$ and straighten lines. If all the β_i's are non-negative, and we carry out the above process at each intersection in each σ then the pattern we end up with will be βp.

Let Q be the subspace of X that is the union of all the t_i. There are many possibilities for the intersection of Q with σ. If $\gamma_1, \gamma_2, \gamma_3$ are the three sides of σ. then permuting the intersection of the t_i's with any one of the sides will produce a different possibility for $Q \cap \sigma$. If $\pi_j, j = 1, 2, 3$ are permutations of $\gamma_j \cap Q$, then each
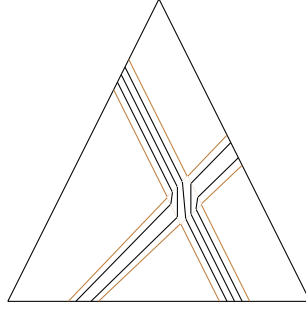
triple of permutations (π_1, π_2, π_3) corresponds to an embedding of σ in $C = G \backslash \tilde{C}$. In fact every cell of C is obtained in at least one such embedding. The image of βp will be a pattern in C. In this embedding the image of a pattern in X maps into a pattern in C. As one varies the triple of permutations for every 2-cell in X the whole pattern in C is obtained. To see how this happens we show that there is a contraction $\rho : X \to D \subset C$ which restricts to a contraction on p. This lifts to a contraction $\tilde{\rho}; \tilde{X} \to \tilde{D} \subset \tilde{C}$ and the pattern $\tilde{\rho}(\tilde{p})$ in $\tilde{D}$ has a dual graph which can be identified with the resolving tree corresponding to $\tilde{p}$.

To define ρ consider the union of the bands b_i in X. Each point of X lies in either zero, one or two bands.
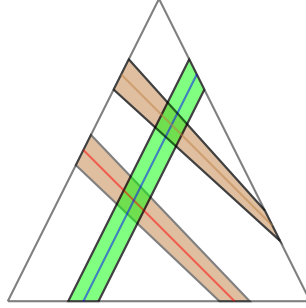

Let x be a point in zero bands. In $\tilde{X}$ a point $\tilde{x}$ lying above x determines a vertex of $\tilde{C}$, since, for each component track t of the pattern $\tilde{t}_i$, it lies in one side of t. We define $\tilde{\rho}(\tilde{x})$ to be this vertex. Let now x be a point which lies in a single band, and let $\tilde{x}$ be a point lying above x. Then $\tilde{x}$ will lie in a region as below. The top (or bottom) side will border a region of points belonging to no bands. This side is mapped by $\tilde{\rho}$ to the vertex already assigned to that region. The vertices assigned to the two sides will be the vertices of an edge in $\tilde{C}$, since there is exactly one track that separates the points in $\tilde{X}$. The region is contracted by $\tilde{\rho}$ to that edge, so that any horizontal line is mapped to the same point of the edge.
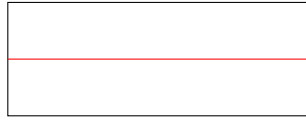
Finally the intersection of two bands has component regions as below. We have already defined $\tilde{\rho}$ on the boundary of this region, and the image of this boundary

is the boundary of a unique 2-cube in $\tilde{C}$. We map the region in $\tilde{X}$ to this 2-cube in the obvious way.
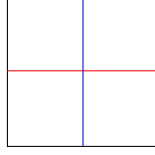
The map $\tilde{\rho}$ is a G-map and so it induces a map $\rho : X \to C$. Note that this map will not usually be surjective. A different embedding of the tracks in X will produce a different image in C.

The image of a 1-cell under $\tilde{\rho}$ will be a geodesic in $\tilde{C}$. In Sageev's paper, he shows that for any two geodesics between two points in a cubing one can get from one to the other by a finite number of moves that involve changing two consecutive edges for two edges that run on opposite sides of some 2-cube.

We now show that if some of the coefficients in the equation $\beta p = \beta_1 t_1 + \beta_2 t_2 + \cdots + \beta_n t_n$ are negative, then we can still get a mapping of X to C so that the pattern p maps into a pattern in C.

Clearly we can write $\beta p = p_1 - p_2$, where p_1 is the sum of those terms in which the β_i's are positive and p_2 is the sum of those terms in which the β_i's are negative. Now p_1 and p_2 are patterns for which the above analysis applies, so that there are two mappings of X into C which take these patterns into patterns in C. We now show that we can combine these mappings into one mapping. The key point is that since $p_1 - p_2$ is a pattern, each edge of X intersects p_1 in at least as many points as it intersects p_2.

We proceed as before and thicken each track of p_1 and p_2 to a band b_i so that for any point of intersection is contained in a component of $b_i \cap b_j$ which only contains one intersection point. Now replace each t_i by β_i parallel copies lying within a b_i. For $\beta_i = 2, \beta_j = -3$ the process is now as follows.

In each intersection of b_i with b_j replace by non intersecting lines as indicated below. With this sort of intersection one will create a "pattern" in which there are line segments that begin and end at the same edge. However it is possible to remove such lines by moving the intersection across the edge as also indicated below. If we think of the pattern p_1 as being coloured red and p_2 as coloured blue, then this move cancels a red intersection with a blue intersection. It is possible to carry out a succession of these moves so that there are no lines joining points on the same edge. We can delete any simple closed curves lying inside a 2-cell.

We can assume that the points of intersection of the patterns with a particular 1-cell e are permuted so that the red and blue intersection points occur in any order. Choose a particular 2-cell σ containing e. In σ we can arrange that each blue line intersecting e is paired with a red line that it intersects.. No two blue lines are paired with the same red edge. If one has such an arrangement then when all intersections are replaced by non-intersecting lines as above, then all blue intersection will be removed by the moves in Fig . Suppose the blue intersection points in order along the edge are $x_1, x_2, \ldots x_r$. We can permute the red crossing points so that the first point y_1 is before x_1 or between x_1 and x_2. We assume the intersection points in all other edges stay the same. For exactly one of the two cases the line in σ in p_1 to x_1 will cross the line in p_2 to y_1. Fix that position for

y_1. Carry out the same procedure to determine the position of $y_2, y_3, \ldots, y_r$. Thus each such y_i lies between x_{i-1} and x_{i+1} and its position relative to x_i is determined by ensuring that the line in σ in p_1 to x_i will cross the line in p_2 to y_i. If we carry out the moves above for this arrangement of intersection points, then all blue intersection points will be cancelled with red intersection points.

If we work through all the edges of X in this way, we see that there is a pattern in C that will give rise to the pattern p in X.
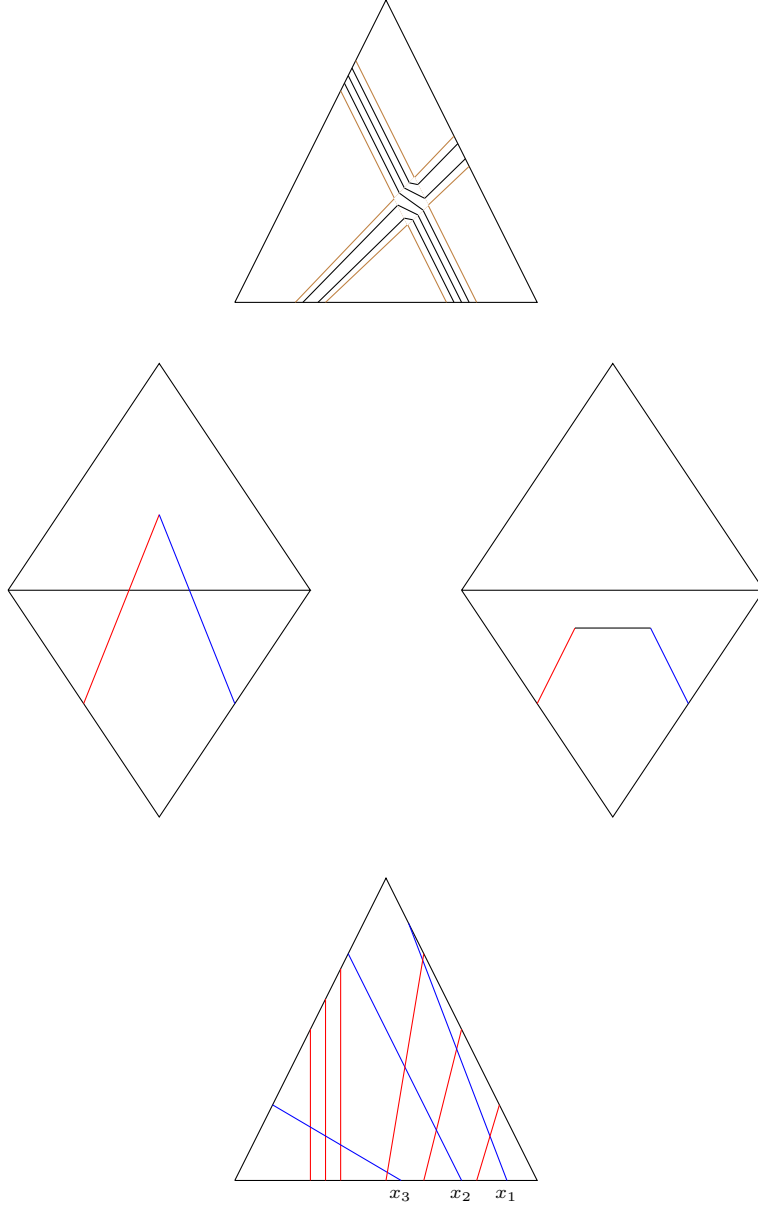
We now describe explicitly given X and a track basis, how to construct C and a pattern in C corresponding to a given pattern in X

Let $t_1, t_2, \ldots t_n$ be the track basis.

Each crossing point of tracks t_i, t_j in X corresponds to a 2-cell σ of C. Let s_1, s_2 be the line segments of the two tracks that intersect in a three-sided two cell in X. The 2-cell of C corresponding to this intersection has vertices corresponding to the four regions into which σ is divided by the segments. One of the three sides contains 2 points of $s_1 \cup s_2$. In this case the bottom side.
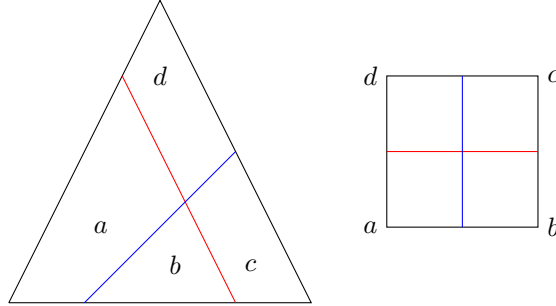
Now mark the corner (b) in the square in C corresponding to that side.
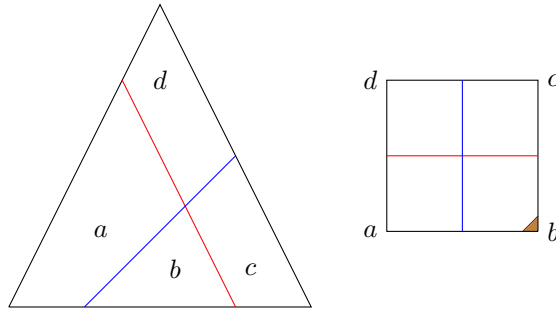
In the track corresponding to $2t_1 + 3t_2$ avoid lines crossing the marked corner.
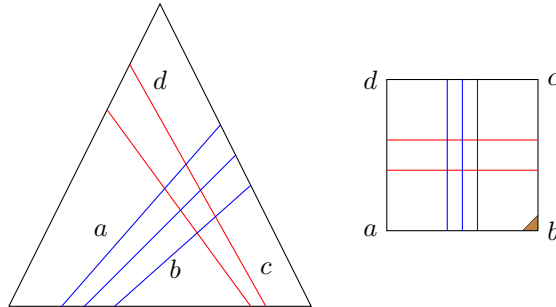
In the track corresponding to $2t_1 - 3t_2$ or $-2t_1 + 3t_2$ the lines should cross the the marked corner.

An untwisted track in C will determine a decomposition of G, i.e. an action on a tree. This is because $\tilde{C}$ is simply connected and so a track in $\tilde{C}$ separates and again the dual graph will be a tree. Let $\tilde{C}$ be the cubing corresponding to the basis of untwisted tracks. The advantage of using tracks in C rather than tracks in X
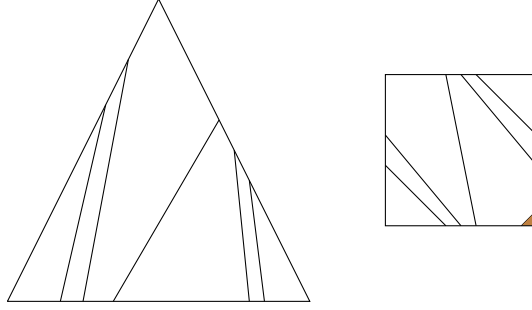
is that the action of G on $\tilde{C}$ is not usually free. Thus we will show that if all the tracks listed give trivial decompositions, then G fixes a vertex of $\tilde{C}$ which means that any track in C will give a trivial action.

Theorem 3.2. *Let $t_1, t_2, \ldots, t_n$ be a set of tracks in X and let $C = G \backslash \tilde{C}$ be as above. If each t_i is a separating track corresponding to a trivial decomposition then there is a vertex of $\tilde{C}$ fixed by G.*

Proof. Since each track is separating and corresponds to a trivial decomposition, one of the two components of $X - t_i$ lifts to a component of $\tilde{X} - G\tilde{t}_i$ fixed by G. Let c_i be the component of $X - t_i$ with this property. If it is the case that $c_1 \cap c_2 \cap \cdots \cap c_n$ is not empty then this intersection will correspond to a vertex of C which lifts to vertex of $\tilde{C}$ with the right property. If there is a 2-cell σ intersected by all the t_i's then it is fairly easy to see that we can arrange the intersections of the tracks so that this is the case. Thus one can choose a point x in the interior of σ and then move each track so that x is the right side of it. Suppose then that there is no 2-cell that intersects all the tracks. Choose a 2-cell σ that intersects the largest number of tracks. As before we can arrange that there is an $x \in \sigma$ that lies the right side of all the tracks that σ intersects. Let t_i be a track that σ does not intersect. We will show that all of σ is the right side of t_i. We can assume that X has one 0-cell v_0. Each edge of σ corresponds to a generator, and this generator will fix the vertex group corresponding to the side of t_i containing σ. Clearly any generator involved in a track that does intersect σ will fix the vertex of $\tilde{C}$ corresponding to x. Thus every generator fixes this vertex and we are done.

$\square$

Let now $t_1, t_2, \ldots, t_n$ be a track basis for the matching equations of X. If p is a pattern with untwisted track components in X, then for some positive integer β, βp will correspond to a pattern in $C = G \backslash \tilde{C}$. If all the t_i's correspond to trivial actions, then the G-tree associated with this pattern will have a trivial action. Thus p will give a trivial action. But any action of G on a simplicial tree is resolved by an action associated with a pattern in X.

More generally, if we no longer assume that all the t_i's correspond to trivial actions, and we take H to be a G-unsplittable subgroup of G, then we can show that there is a vertex of $\tilde{C}$ fixed by H. In this case, we consider the action of H on $\tilde{X}$ and put $X_H = H \backslash \tilde{X}$. There is a pattern $p_i = H \backslash G\tilde{t}_i$ for each track t_i in X. Each track in p_i corresponds to a trivial decomposition, since otherwise H would be G-splittable. There is a component of $X_H - p_i$ that lifts to a component of $\tilde{X} - G\tilde{t}_i$ that is fixed by H. This will map to a component c_i of $X - t_i$, which may be all

of $X - st_i$ if t_i in non-separating. Again we have to show that we can choose the position of the t_i's in X so that the intersection of all the c_i's is non-empty. This is done as in the last proof.

This completes the proof of Theorem 1.1.

In our example where X is the presentation complex of a trefoil group, the diagram shows the cube complex $C = G\backslash\tilde{C}$ for the two decompositions given by the tracks t_1 shown in red, and t_2 shown in blue.
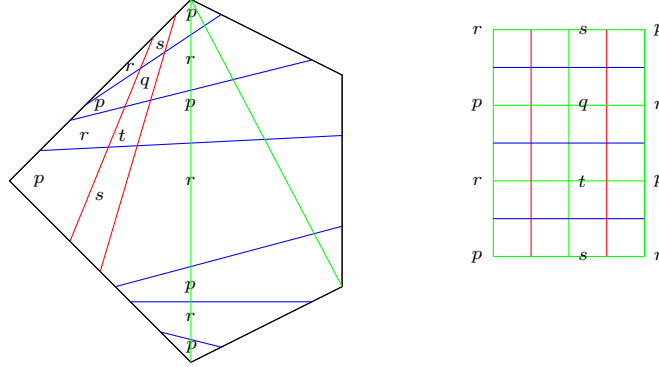
Now note that the track $t_1 + t_2$ also corresponds to a "track" in C. A pattern $\beta t_1 + \beta t_2$ where β_1, β_2 are integers will correspond to a pattern in C

ing to all the minimal untwisted tracks, then we obtain

Now note that the track $t_1 + t_2$ also corresponds to a "track" in C. A pattern $\beta t_1 + \beta t_2$ where β_1, β_2 are positive integers will correspond to a pattern in C, but only those with non-negative coefficients correspond to a pattern in X.
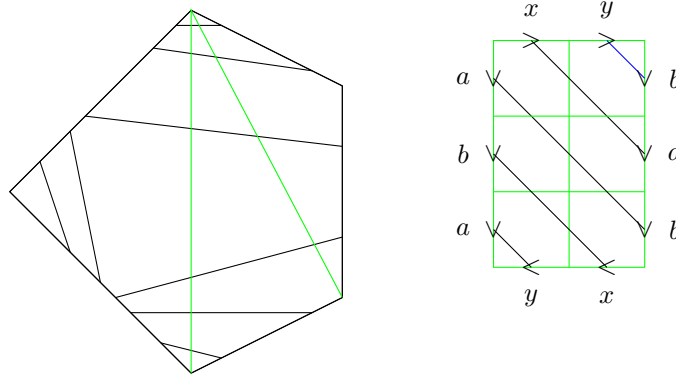
4. Proof of Theorem 1.3

Choose a basis of solutions $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$ for the solution space to the matching equations, with non-negative integer coefficients.

Let $\mathbf{p} = \alpha_1 \mathbf{u}_1 + \alpha_2 \mathbf{u}_2 + \ldots \alpha_n \mathbf{u}_n$ where $\alpha_1 = 1, \alpha_i = \sqrt{p}_{i-1}$, where $p_1, p_2 \ldots, p_{n-1}$ are distinct primes.

In [6] it is shown how a marking (a non-negative solution of the matching equations) gives rise to a folding sequence of a complexes of groups with fundamental

group G. In [6] the marking is given by a resolution of a particular G-tree. The marking $\mathbf{p}$ may not correspond to such a resolution. It will, however, determine a folding sequence. Each complex in the sequence has a marking induced by $\mathbf{p}$ which results in a foliation of each 2-cell as below. The attaching map of the 2-cell is given by a word $w \cup w'$ where w, w' are words in the 1-cells, corresponding to the top and bottom of the 2-cell. The marking $\mathbf{p}$ gives a length for each 1-cell and the lengths of the words w, w' are the same.
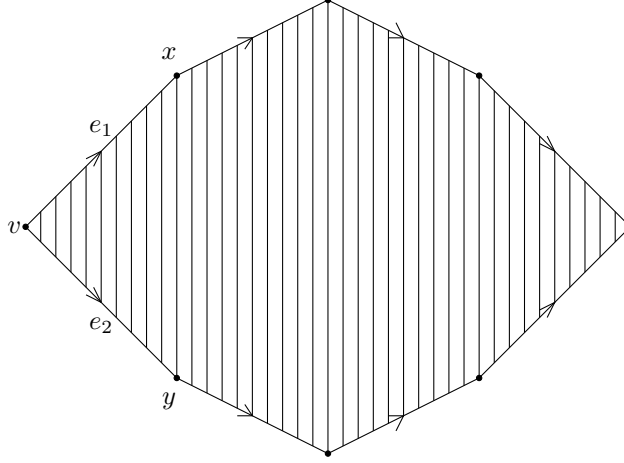
Let M be the matrix in which the first n rows are the $\mathbf{u}_i$'s. Each column corresponds to a 1-cell. Let M have an extra row corresponding to $\mathbf{p}$.

Consider the effect of folding on $\mathbf{p}$ and the $\mathbf{u}_i$'s, and thus on M. In a folding sequence we now only allow subdivision if it is immediately followed by a Type I fold involving one of the subdivided edges. A subdivision followed by a Type I fold results in an elementary column operation on M. If x, y are the lengths of the edges at the relevant corner and $x < y$, then after the subdivision and fold the edge of length y has been replaced by one of length $y - x$.

All the information about this operation can be obtained from the row of M corresponding to $\mathbf{p}$ as the coefficients of the sum for $\mathbf{p}$ are linearly independent over the rationals. If a Type III fold can be made, then two columns must be equal as this happens if and only if the two column entries in the row corresponding to $\mathbf{p}$ are the same. Note that this means that in any other linear combination of the $\mathbf{u}_i$'s the two column entries will be the same. This means that if instead of $\mathbf{p}$ we started with a marking corresponding to another internal point of the solution space then we would reach a Type III move at precisely the same point of the folding process. Deleting one of the two equal columns will induce a linear bijection on the solution space.

A Type II fold has no effect on the matrix M

In the case of a marking that does correspond to a resolving of a tree, it was deduced that if in the attaching word $w \cup w'$ either w or w' contained a subword $e\bar{e}$ then there had to be a non-trivial joining element g_v in G_v (modulo G_e) where $v = \tau e$. Thus to correspond to G-tree, $g_v \notin G_e$. If we start with our marking $\mathbf{p}$, which may not correspond to a G-tree, then it may happen that we have $g_v \in G_e$. If this happens then we delete any row $\mathbf{u}_i$ that contributes a non-zero entry to the

column corresponding to e, as we are only interested in finding decompositions that resolve actions on trees. Thus we delete all such $\mathbf{u}_i$'s from our list of tracks and remove the corresponding terms from the marking $\mathbf{p}$, and start a folding process with the new $\mathbf{p}$. One also deletes the column corresponding to e. If we continue with this process we eventually obtain a matrix corresponding to a complex that resolves any action of G on a tree. The marking $\mathbf{p}$ will then determine an action on an $\mathbb{R}$-tree. To see why this is the case we again follow the argument of [6].

Let X be a complex obtained in the above process with the smallest number of 1-cells. This corresponds to a matrix M with the smallest number of columns. There is a natural graph of groups decomposition for $G = \pi(\mathcal{Y}, Y)$ corresponding to X in which the edges of Y are the 1-cells of X that do not occur in the boundary of any 2-cell. Each vertex group of $\mathcal{G}$ is the group of a sub-complex in which every 1-cell lies in the boundary of at least one 2-cell.

As in [6], it follows that if $v \in VY$, then there is a homomorphism from each vertex group $Y(v)$ to a target group $\mathcal{Z}(v)$ which is the fundamental group of a cube complex associated with a single n-cube $c(v)$. The marking $\mathbf{p}$ will induce a marking on $c(v)$ which corresponds to a an action of $\mathcal{Z}(v)$ on an $\mathbb{R}$-tree. Thus each vertex group $\mathcal{Y}(v)$ will have an action on an $\mathbb{R}$-tree.

We now show that each vertex group $\mathcal{Y}(v)$ has at most one end.

This follows from the fact that if a group G acts on a cubing It follows that any action of G on a simplicial tree with finite edge stabilizers, is resolved by a combination of the $\mathbf{u}_i$'s which give splittings over finite subgroups. These decompositions correspond to a mutually compatible set of tracks.

5. Computing decompositions

A programme is available on the first author's website that calculates a basis of untwisted tracks for the presentation complex of a finitely presented group G. The programme then calculates the decomposition corresponding to each such track and identifies those that are clearly trivial. The remaining decompositions are left for manual inspection.

See http://www.layer8.co.uk/maths/tracks.htm

Here is some output for the Higman group.

$$H = \langle a, b, c, d | aba^{-1} = b^2, bcb^{-1} = c^2, cdc^{-1} = d^2, dad^{-1} = a^2 \rangle.$$

For this group presentation, the track basis contains 12 tracks. Of these there are three tracks that give non-trivial decompositions. In the following only the decompositions for the first four basis tracks is included.

Group presentation:
a b c d : ab-a-b-b bc-b-c-c cd-c-d-d da-d-a-a
Jobname: higman4
Triangular 2-complex comprises: 1 0-cells, 12 1-cells and 12 2-cells.
Track basis (size 12 x 36)
1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1
1 3 1 0 2 2 1 1 3 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2
2 0 4 5 1 1 3 3 1 3 1 1 1 3 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1
3 1 5 6 2 2 4 4 2 2 4 0 0 2 4 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2

check track basis element 0 The track

(1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1)
is untwisted and separating

check track basis element 1 The track (1, 3, 1, 0, 2, 2, 1, 1, 3, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2) is untwisted and separating

check track basis element 2 The track (2, 0, 4, 5, 1, 1, 3, 3, 1, 3, 1, 1, 1, 3, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1) is untwisted and separating

check track basis element 3 The track (3, 1, 5, 6, 2, 2, 4, 4, 2, 2, 4, 0, 0, 2, 4, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2) is untwisted and separating

prune list of 12 separating tracks:

track basis element 0

The separating track

(1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1)

Gives a trivial decomposition.

Edge stabilizer generators. ab-a-bc-b-c-c ab-a-bc-b-cd-c-da-d-a-a

First vertex stabilizer generators. ab-a-bc-b-c-c ab-a-bc-b-cd-c-da-d-a-a

Second vertex stabilizer. G

track basis element 1

The separating track

(1, 3, 1, 0, 2, 2, 1, 1, 3, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2)

Gives a trivial decomposition.

Edge stabilizer generators. -b-a b-a cd-c-da-d-a-a bc-b-cd-c-d-d

First vertex stabilizer. G

Second vertex stabilizer generators. ab b-a aad-adc-d-c ddc-dcb-c-b

track basis element 2

The separating track

(2, 0, 4, 5, 1, 1, 3, 3, 1, 3, 1, 1, 1, 3, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1)

Edge stabilizer generators. b-a-b-aba-b ab-a-b-a-b-ababa-b-a b-a-b-cba-b ab-a-b-a-b-cbaba-b-a ab-a-bc-b-c-c ab-a-bc-b-cd-c-da-d-a-a

First vertex stabilizer generators. ab-a-bc-b-c-c ab-a-b-a-bababa-b-a ab-a-b-a-b-cbaba-b-a ab-a-bc-b-cd-c-da-d-a-a b-a-b-aba-b b-a-bcba-b

Second vertex stabilizer generators. a b-a-b-aba-b b-a-b-a-b bba-b b-a-bcba-b c b-cba-b d

track basis element 3

The separating track

(3, 1, 5, 6, 2, 2, 4, 4, 2, 2, 4, 0, 0, 2, 4, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2)

Edge stabilizer generators. -cba-b-a-cba-b-a, -cba-b-cba-b-a, -c-a-b-b-c-c, -c-ac, ccb-a-b-c-c, -c-b-cba-b-a, ccbcbc, -c-b-c-c-cba-b-a, cd-c-da-d-a-a

First vertex stabilizer generators. ab-a-bc-b-c-c, cd-c-da-d-a-a, -c-a-b-b-c-c, -c-b-cba-b-a, ab-a-bcb-a-bc, ab-a-bc-a-cba-b-a, ab-a-bcab-a-bc, ab-a-bccbc, -cac, ab-a-bcccbc, ccba-b-c-c, -c-b-c-b-c-c, ab-a-bcd-cba-b-a

Second vertex stabilizer generators. a, c, b, d-adc-d

In fact tracks 0.1.2 all give trivial decompositions, but track 3 gives a non-trivial decomposition.

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