PATTERNS AND TRACKS

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Abstract. Patterns in triangulated 2-spheres and 3-spheres are investigated. A new proof of a lemma in Abigail Thompson's proof of the Recognition Algorithm for 3-spheres is obtained.

1. Introduction

Patterns in 2-complexes were introduced in [2] and used later [1] to give a combinatorial approach to minimal surface theory in 3-manifolds. Recently I used them in [3].

The Recognition Algorithm of Hyam Rubinstein [5] gives a way of deciding if a compact triangulated 3-manifold M is a 3-sphere. Given such a 3-manifold one decomposes it by cutting it up along a maximal set of non-parallel, normal, embedded, separating 2-spheres. This will give a finite set of compact 3-manifolds M_0 with at least one boundary component, where each such component is a 2-sphere. Then the algorithm as adapted in [6] says that M is a 3-sphere if each M_0 with more than one boundary component is a punctured 3-ball, and every M_0 with one boundary component either contains a vertex of the triangulation or it contains an embedded almost normal 2-sphere. In [6] the first part of the algorithm, namely that M_0 is a punctured 3-ball if it is bounded by more than one 2-sphere, is proved under the condition that 2-spheres separate.

A proof of the result of [6] that M_0 is a punctured ball if it has more than one boundary component is given here using a proof that is the same as part of a proof of a corresponding result for patterns in a triangulated 2-sphere.

This paper, alongside the discussion in [3], also provides a full account of the role of the Recognition Algorithm in the study of closed simply connected 3-manifolds.

I am very grateful to Peter Kropholler for his interest in my research.

2. Patterns and Tracks

Recall from [1] the definition of a pattern. Let K be a finite 2-complex with polyhedron $|K|$. A pattern is a subset P of $|K|$ satisfying the following conditions:-

(i) For each 2-simplex σ of K, $P \cap |\sigma|$ is a union of finitely many disjoint straight lines joining distinct faces of σ.

(ii) For each 1-simplex γ of K, $P \cap |\gamma|$ consists of finitely many points in the interior of $|\gamma|$.

A track is a connected pattern. If two patterns P and Q intersect each 1-simplex in the same number of points then the patterns are said to be *equivalen*t. Two equivalent disjoint tracks in the same 2-complex are said to be *parallel*. We call a track in T an n-track if it has n intersections with edges (1-simplexes).

If a pattern in a tetrahedron T is as in Figure 1 then the tracks are all 3-tracks or 4-tracks. A pattern in a 3-manifold is called a normal pattern if the intersection with the boundary of every 3-simplex ρ is like this.

A track in T is a simple closed curve, which will bound a disc in $|\rho|$. If M is a 3-manifold and M is triangulated so that $M = |K|$ where K is a finite 3-complex, then a pattern P in $|K^2|$ determines a *patterned surface* S such that for each 3-simplex ρ, $S \cap |\rho|$ consists of disjoint properly embedded discs and $S \cap |K^2| = P$. A patterned surface is determined, up to isotopy, by the intersection $P \cap |K^1|$. If the pattern in $|K^2|$ is normal, then the patterned surface is a *normal surface.*

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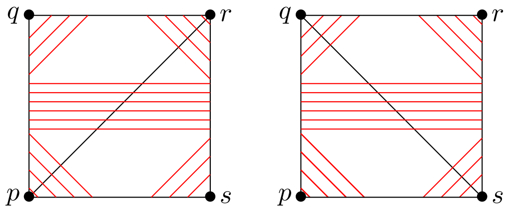


FIGURE 1. Normal Pattern

3. Patterns in 2-spheres

Let K be a 2-complex that is a triangulation of a 2-sphere. Let P be a pattern in K consisting of a maximal set of tracks in K no two of which are parallel. Such a maximal set exists since the number of non-parallel tracks in a finite 2-complex is bounded by the number of vertices plus the number of 2-simplexes. If the tracks in a pattern separate as they do in this situation, or more generally if $H^1(K, \mathbb{Z}_2) = 0$, then any pattern P in K^2 will determine a tree D_P in which the edge set is the set of tracks and the vertex set is the set of components of $K^2 - P$.

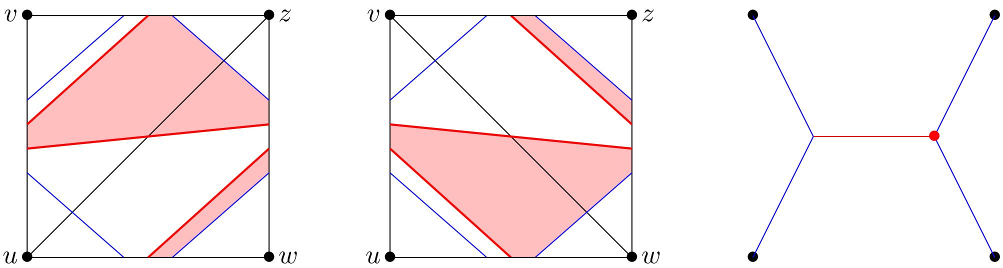


FIGURE 2.

An example is shown in Figure 2 for the tetrahedron T. The maximal pattern P consists of four blue 3-tracks and one red 8-track. The tree D_P will have five edges corresponding to the tracks of P and six vertices, two of degree 3 and four of degree 1, as shown. The component corresponding to the right hand vertex of degree 3 is shown shaded. The situation is as in Figure 5(i).

Theorem 3.1. *Every vertex in the tree D_P has degree (valency) one or three. A component of degree one contains one vertex. A component of degree three contains no vertex and is a disc with two smaller discs removed.*

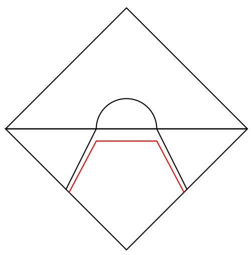
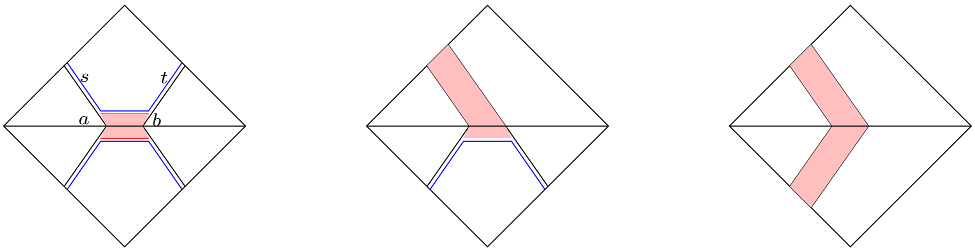


FIGURE 3.

Proof. Let γ be a 1-simplex. Let a, b be adjacent points of $\gamma \cap P$. Removing small open neighbourhoods of a, b and replacing them with lines parallel to γ joining the end points of the open neighbourhoods will create either one simple closed curve (scc) or two simple closed curves. If a, b lie in distinct tracks s, t then just one scc U is created. This scc will not usually be a track (after straightening the intersection lines in 2-simplexes). It will be a track if and only if there are no returning arcs. A returning arc is one as in Figure 3 for which the end points are in the same edge of the triangulation. Returning arcs can be removed as shown, but a new returning arc may be created. Removing a returning arc in the 2-simplex σ joining two points a, b will create a returning arc in the other 2-simplex containing a, b if a, b are joined to points in the same edge. Thus it will be a track if and only the situation in the two 2-simplexes containing γ is as in Figure 4(a). If the situation is as Figure 4(b) then one returning arc is created, and if it is as in Figure 4(c) then two returning arcs are created.



(a) (b) (c)

Figure 4.

If the points a, b lie in the same track u, then the process of removing small neighbourhoods of a, b and joining the ends with lines parallel to γ will create two simple closed curves S, T. This can be seen from Figure 5(i). Now removing any returning arcs from S, T will create two tracks,, which must be parallel to tracks s, t in P. In neither case can the removal of returning arcs end with the total removal of the scc. If this happens then the last returning arc removed must have been a returning arc in the original track u, and so it would not have been a track.

If the points a, b lie in different tracks the situation will be as Figure 5 (ii). It is only necessary to consider this case if in the vertex region there are no instances of adjacent points in a 1-simplex lying in the same track. If this were the case then we could use the argument that has just concluded. If we cannot use the previous case, then removing small neighbourhoods of a, b and joining the ends with lines parallel to γ will create one simple closed curves U. The process of removing returning arcs must eventually stop as the number of intersection points with the 1-skeleton is reduced by two at each removal. In the end U has become a track. It cannot end with U being removed entirely as this would mean that s, t were parallel. Since we are dealing with a maximal set of tracks, this track will be parallel to a track u in P. It can be seen that there is a vertex v of D_P with incident edges s, t, u. The region corresponding to v will consist of triangular regions in two 2-simplexes joined by three bands

Even if there are adjacent points a, b lying in the same track and a different pair a', b' lying in different tracks, then working with a', b' and removing returning arcs, one will get the same triple of boundary tracks but it will be more complicated to describe. Thus in Figure 2 it is easy to see what happens after removing a pair of points in the same track, but less easy for a pair in different tracks.

In both cases the vertex component is a disc with two discs removed. If a vertex region corresponds to a vertex of D_P of degree one, then it will be bounded by a track intersecting each edge at most once. The region must contain a single vertex of the triangulation. Conversely every vertex of the triangulation determines a track surrounding it, and the region inside this track will correspond to a vertex of D_P of degree one. A 1-simplex of K determines a path in D_P starting and ending in a vertex of degree one. Every other vertex visited has degree three. The path will backtrack at a vertex if and only if the corresponding part of the 1-simplex joins two points in the same track. □

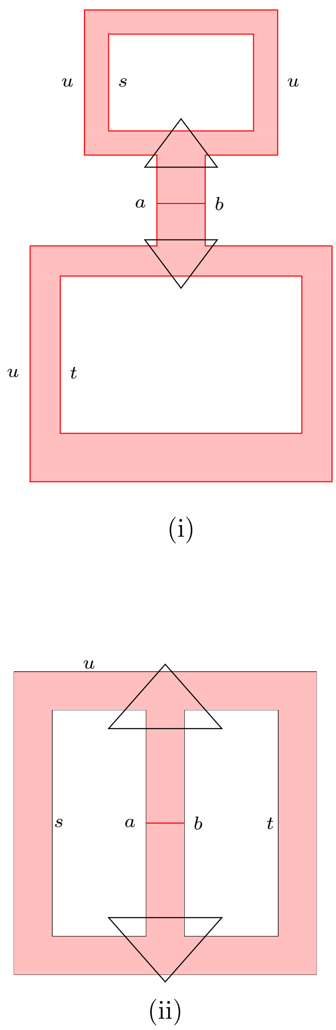


FIGURE 5.

Let v be the number of vertices of the triangulation, e the number of edges (1-simplexes) and f the number of faces (2-simplexes), then $3f = 2e$ since every face has 3 edges and each edge belongs to two faces. Thus f is even. Every track determines a proper decomposition of the set of vertices, i.e. there are no tracks bounding a region with no vertices. Hence the tree D_P has v vertices of degree 1. If v_P is the number of vertices of D_P, there are $v_P - v$ vertices of degree 3. But our argument above has shown that every vertex of degree 3 involves exactly two faces. Hence $2v_P - 2v = f$. The number of vertices of a finite tree is the number of edges plus one. Hence $v_P = e_P + 1$. Thus $e_P = v_P - 1 = \frac{1}{2}f + v - 1$, which shows the number of tracks in a maximal pattern does not vary with the pattern. For example if $K = T$ the faces of a tetrahedron, so that $v = f = 4$, then $e_P = 5$. This was also proved in [3], where it was shown that there are infinitely many possibilities for the track that is not a 3-track.

4. Patterns in 3-manifolds

Let M be a 3-manifold. Let $f : S^2 \to M$ be an injective general position map (see Hempel [4], Chapter 1), in which f is in general position with respect to a triangulation K of M. An i-piece of f is defined to be a component of $f^{-1}(\sigma)$ where σ is an $(i+1)$-simplex of K. Thus a 0-piece is a point of S^2. A 1-piece is either an scc (simple closed curve) or an arc joining two 0-pieces. If there are no 1-pieces that are scc's, then each 2-piece has boundary that is is a union of 1-pieces. One can use surgery along simple closed curves to change f to a map in which there are no 1-pieces that are scc's, and in which every 2-piece is a disc. The 2-pieces will then give a cell decomposition (tessellation) of the 2-sphere.

If R is a 1-piece with end points u, v whose images under f are in the same 1-simplex, then the restriction of f to R is called a returning arc.

If $f : S^2 \to M$ has no 1-pieces that are returning arcs or scc's, then the intersection of $f(S^2)$ with the 2-skeleton of M is a pattern P. In the case in which we are interested, there is a homotopy from f to $f' : S^2 \to M$ in which the image is the patterned surface determined by P.

Let γ be a 1-simplex of M. Two points $p, q \in \gamma \cap f(S^2)$ are said to be removable if there is a homotopy from f to a map $f' : S^2 \to M$ such that $f(x) = f'(x)$ for every x that is not in the interior of a simplex with γ as a face and $\gamma \cap f'(S^2)$ is the same as $\gamma \cap f(S^2)$ but with p, q removed.

The pair of end points of a returning arc R are removable by the following homotopy. Let σ be the 2-simplex of K such that $f(R) \subset \sigma$. Let V be a regular neighbourhood of R in S^2. Let V° be the interior of V regarded as a subspace of V. Let βV be the boundary of V regarded as a subspace of S^2, so that $\beta V = V - V^\circ$. Let γ be the 1-simplex containing the end points of R. The regular neighbourhood V is a disc and $\beta V = \delta V$ is a simple closed curve in S^2. The union of all the 3-simplexes that contain γ is a closed ball B and $f(\beta V) \subset B^\circ - \sigma$, which is contractible. Define $f' : S^2 \to M$ so that f' is continuous, f' and f are the same when restricted to $S^2 - V^\circ$, and $f'(V) \subset B^\circ - \sigma$. Note that removing p may create more 1-pieces that are returning arcs or sccs, but the size of the intersection with the 1-skeleton goes down by two.

Let $M = |K|$ be a 3-manifold, where K is a finite 3-complex. Let $H^1(K, \mathbb{Z}_2) = 0$ so that tracks in K^2 separate. In this situation any pattern P in K^2 will determine a finite tree D_P in which the edge set is the set of tracks and the vertex set is set of components of $K^2 - P$. In the situation when the patterned surface corresponding to P is a maximal set of normal 2-spheres, it is proved in [6] that a vertex of D_P of degree (valency) greater than one corresponds to a component which is a punctured 3-ball. In fact a slightly stronger result is true.

Theorem 4.1. *Let P be a pattern in K^2, in which no two tracks are parallel, for which the patterned surface corresponding to P is a maximal set of 2-spheres, then in the tree D_P every vertex has degree one, two or three. Each vertex of degree two or three is a component of $K^2 - P$ that is a punctured 3-ball. The same is true if the 2-spheres are restricted to normal spheres.*

Proof. Part of the previous proof works with some adjustments.

Let V be the closure of a component of $K^2 - P$ corresponding to a vertex of D_P of degree at least two. The following argument from [6] shows that there will be a 1-simplex of K, intersecting two distinct tracks s, t of P in the boundary of V at adjacent points a, b on the 1-simplex γ where $a \in s \cap \gamma, b \in t \cap \gamma$. Suppose this does not happen. Remove a small open neighbourhood of all intersections of V with the 1-skeleton. All segments join points in the same track and so the new space V' will still have more than one boundary component. Then if we cut V up along the faces of every 2-simplex that it intersects, we will be cutting along discs, where each disc in contained in the interior of a 2-simplex, and its boundary circle will be contained in one boundary 2-sphere of V'. Cutting along such discs will disconnect V', but there will be a component with more than one boundary component. Cutting along the boundary of all the 3-simplexes yields 3-balls with connected boundary 2-spheres, which is a contradiction.

There will now be a 2-sphere S which is the union of the patterned surface corresponding to s with a small neighbourhood of a removed and the patterned surface corresponding to t with small neighbourhoods of b removed together with a tube joining the boundary circles. The 2-sphere S will usually not be a patterned 2-sphere, but it will become one by removing returning arcs, as described above. It you have a surface in general position in a 3-manifold, then one can get a patterned

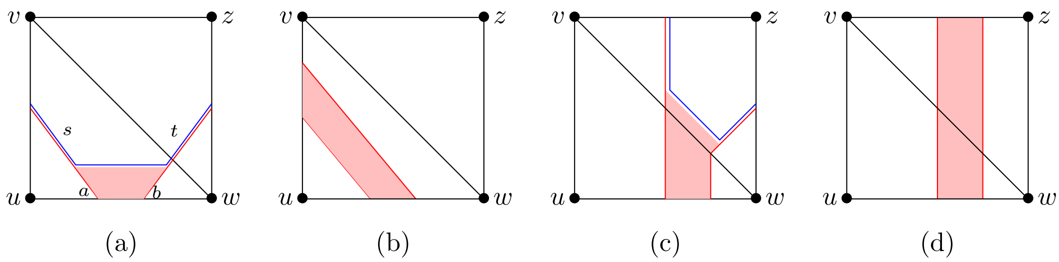


FIGURE 6.

surface by removing any simple closed curves in 2-simplexes and all returning arcs, unless the whole manifold is obtained.

Assume then that a, b are as above in different tracks s, t. Now S may have more than two returning arcs, since γ will belong to more than two 2-simplexes. The initial 2-sphere S bounds a 3-ball consisting of two 3-balls bounded by s and t joined together by the filled tube. In M s and t might not bound 3-balls, but we assume they do for this argument. Removing a returning arc is done by expanding this 3-ball, and the possible expansions are shown in Figure 6. In each case only what happens in two of the faces of the 3-simplex are shown. In Figure 6(a) no returning arc has been created. In Figure 6(b) returning arcs are created in simplexes with face uv or uz and the 3-ball is expanded to include a 3-ball bounded by an annulus and two discs each bounded by a 3-track. In Figure 6(c) returning arcs may be created in simplexes with faces uz or vw and the 3-ball is expanded to include a 3-ball bounded by the shaded region and a disc bounded by the 3-track shown in blue. In Figure 6(d) returning arcs may be created in simplexes with a face in four of the six edges of T and the 3-ball is expanded to include a 3-ball bounded by an annulus and two discs each bounded by a 4-track. Note that in the process of expanding the 3-ball in the cases of Figure 6(b) and Figure 6(d) as just described, two returning arcs will occur in the same 2-simplex but s, t are not parallel. Once all the returning arcs have been removed the boundary of the expanded 3-ball will be a patterned 2-sphere U unless we have removed all of s and t which would mean that V had just two boundary 2spheres. If V had more than two boundary 2-spheres, then since we had a maximal set of tracks corresponding to patterned 2-spheres, U must be parallel to u, a track in P, and V has s, t, u as boundary 2-spheres.

Each 2-piece of u will be parallel to a 2-piece of s or t apart from ones that arise from the Figure 4(a) or the Figure 4(c) situation, which are 4-sided and 3-sided respectively. Thus if s and t are normal patterns then so is u. If we start with a pattern P in which the tracks are a maximal set of non-parallel, normal 2-spheres, then the proof works fine. □

The situation in a 3-sphere is different from that of a 2-sphere in that there will be vertices of D_P of degree one that do not contain a vertex of the triangulation. In the theory of the Recognition Algorithm it is shown that if P corresponds to a maximal set of normal 2-spheres in a 3-sphere, then a component of $K^2 - P$ of degree one that does not contain a vertex must contain a 2-sphere that is almost normal. A 2-sphere is almost normal if intersects the boundary of every 3-simplex in 3-tracks and 4-tracks apart from one occurrence of an 8-track. If in pattern P we start by including tracks corresponding to a maximal set of normal 2-spheres and then add in extra tracks corresponding to almost normal 2-spheres, then each extra track is a boundary to two vertex regions one of which has degree two, and the other has degree one. As is shown in the Recognition Algorithm there are two chains of isotopies starting with the almost normal 2-sphere, one ending at a 2-sphere parallel to the boundary normal 2-sphere and the other being a null sequence. In the case of the vertex region bounded by the normal sphere and the almost normal sphere, there will be a 1-simplex intersecting the two 2-spheres in adjacent points but when we carry out the process described above, we end up with the removal of both 2-spheres.

5. Programmes

Let X be a 2-complex in which tracks separate. A programme is described that produces a 'maximal' pattern P, In the case when X is a triangulation of a 2-sphere the pattern will be a maximal set of tracks no two of which are parallel as described earlier. When X is the 2-skeleton of a 3-manifold the tracks will correspond to a maximal set normal 2-spheres.

The programme proceeds as follows:-

1 For each vertex v there is a pattern $P(v)$ which contains exactly one line for each 2-simplex having vertex v, joining the two edges containing v. Include all such patterns in P. In the case of a manifold each $P(v)$ is a track.

2 For each adjacent pair a, b of intersections that lie in different tracks there is a uniquely determined pattern $P(a, b)$, as described in Figures 4 and 5. Check if the pattern is already included in P. If not add it to the list. If for every adjacent pair a, b we have $P(a, b) \subset P$, we are done.

The process of determining the pattern $P(a, b)$ for a, b adjacent points in the 1-simplex γ involves forming the "prepattern" consisting of the track s containing a, the track t containing b, removing small neighbourhoods of a and b, joining up with lines parallel to γ. One obtains the pattern $P(a, b)$ by removing all returning arcs. It does not matter in which order this is done. The same pattern will be obtained. This process must terminate as removing a returning arc will remove a pair of intersection points as in Figure 7. If X is the 2-skeleton of a 3-manifold $P(a, b)$ will be connected and so it is a track. In general removing a returning arc may disconnect the pattern.

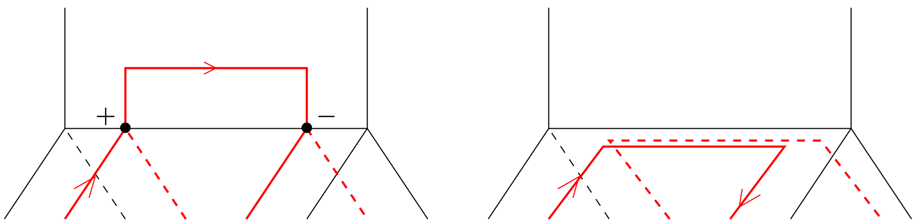


Figure 7.

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