**Groups Acting on Real Trees**

by

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## 1. Introduction

Earlier versions of this paper have appeared on the Southampton Preprint Server and on my Home Page in 2003 and 2004 and 2006. In 2006 I revised the version of 2004 and added more examples and an extra section (Section 4) analysing the unstable actions of finitely presented groups on **R**-trees with cyclic arc stabilizers. It turns out that there is a nice classification of the finitely presented groups that have such actions. I thank Richard Weidmann, Ilya Kapovich and Vincent Guirardel for their comments on previous versions of this later version. They pointed out that my proof of Theorem 1 was inadequate. In this version I have attempted to address the points they raised.

I thank Gilbert Levitt for a helpful conversation.

Let $G$ be a finitely presented group and let $T$ be an **R**-tree on which $G$ acts. In [BF2] Bestvina and Feighn show how $T$ is *resolved* by a band complex. They show that in certain situations one can carry out moves on the band complex to put it into a normal form that gives information about the action on $T$. We show that there are advantages in taking the resolving complex to be a complex of groups which has $G$ as its fundamental group. This approach brings together that of Rips via the Rips Machine and the folding sequences of [D2].

We prove the following theorem.

**Theorem 1.** *Let $G$ be a finitely presented group and suppose $G$ admits an action on an* **R***-tree $T$ with slender arc stabilizers. Then $G$ admits a decomposition as the fundamental group of a graph of groups in which each edge group is either slender or is a finitely generated subgroup fixing a vertex of $T$, and vertex groups are of three types:-*

*T0 A finitely generated subgroup of $G$ which fixes a point of $T$.*

*T1 A group $H$ with a slender normal subgroup $N$ such that $H/N$ is isomorphic to a group of isometries of* **R**. *In fact there is a subtree $T_H$ of $T$ isometric to* **R** *invariant under $H$ such that $N$ is the subgroup of $H$ consisting of all elements which act trivially on $T_H$.*

*T2 A group $H$ which is the fundamental group of a complex of groups in which the underlying complex is a hyperbolic 2-orbifold and in which the groups associated with all 2-cells are trivial, the groups associatied with edges and vertices are slender and they stabilize arcs and points of $T$ repectively. If $N$ is the normal subgroup of $H$ generated by the edge groups of the complex, then there is a subtree $T_H$ of $T$ invariant under $H$ and there is an induced action of the 2-orbifold group $H/N$ on the* **R***-tree $T_H/N$ corresponding to a foliation on the 2-orbifold.*

Earlier versions of this theorem omitted the possibility that some edge groups in the decomposition need not be slender. This may happen for edges incident with a T2 vertex. If such an edge group occurs then the 2-orbifold has at least two cone points of infinite index. In Section 3 we give a number of examples of T2 actions which are unstable. In a stable T2 action $N$ itself is slender. In the added Section 4 a more detailed examination of the situation in T2 is given in the case when arc stabilizers are infinite cyclic. We obtain a classification of the groups that arise. We are able to give a positive answer to a question of Shalen by showing that if a finitely presented group admits a non-trivial action on an **R**-tree with cyclic arc stabilizers then it admits an action on a simplicial tree with small arc stabilizers. The small arc stabilizers are in fact always soluble Baumslag-Solitar groups. I showed in [D1] and [D3] that the finitely presented condition cannot be removed. In particular in [D3] I constructed an example of a finitely generated group which acts on an **R**-tree with cyclic arc stabilizers, but which does not split over a small subgroup. These actions are not stable.

For stable actions the finitely presented condition can be removed In Section 5 we generalize the results of Bestvina and Feighn [BF2] for stable actions from finitely presented groups to finitely generated groups. Our account of this avoids using complexes of groups. Instead we use arguments using groups acting on graphs and folding of graphs. The same approach could have been used for the first part of the paper. It sometimes helps to see arguments explained in more than one way. We prove the following.

**Theorem 2.** *Let $G$ be a finitely generated group and let $T$ be a $G$-tree. Then either $G$ has a subgroup $H$ which does not fix an arc of $T$ but for each finitely generated subgroup of $H$ there is an arc of $T$ fixed by that subgroup, or $G$ is the fundamental group of a graph of groups in which each edge group fixes an arc of $T$ and the vertex groups are of three types:-*

- *T0* *A finitely generated subgroup of $G$ which fixes a point of $T$.*
- *T1* *A group $H$ with a finitely generated normal subgroup $N$ which fixes an arc of $T$ such that $H/N$ is isomorphic to a group of isometries of $\mathbf{R}$. In fact there is a subtree $T_H$ of $T$ isometric to $\mathbf{R}$ invariant under $H$ such that $N$ is the subgroup of $H$ consisting of all elements which act trivially on $T_H$.*
- *T2′* *A group $H$ which is the extension of a finitely generated normal subgroup $N$ by a hyperbolic 2-orbifold group. There is a subtree $T_H$ of $T$ invariant under $H$ and fixed by $N$, and there is an induced action of the 2-orbifold group $H/N$ on the $\mathbf{R}$-tree $T_H/N$ corresponding to a foliation on the 2-orbifold.*

  *In particular if the action of $G$ is stable then $G$ has the above structure.*

As remarked above in [D1] and [D3] examples were given of finitely generated groups with unstable actions on **R**-trees with cyclic arc stabilizers for which the groups do not have graph of groups decompositions as above.

## 2. Finitely presented groups

Let $X$ be a finite CW 2-complex. We introduce the idea of a complex of groups $G(X)$ based on $X$. This is a slightly different notion to a special case of the complex of groups described by Haefliger [H]. Haefliger restricts $X$ to be a simplicial cell complex. One can get from our situation to that of Haefliger by triangulating each 2 cell. We are only concerned with the situation when each group assigned to a 2-cell is trivial.

Thus the 1-skeleton $X^1$ of $X$ is a graph. We take the edges to be oriented, and use Serre's notation, so that each edge $e$ has an initial vertex $\iota e$ and a terminal vertex $\tau e$ and $\bar{e}$ is $e$ with the opposite orientation. Let $G(X^1)$ be a graph of groups based on $X^1$ The attaching map of each 2-cell $\sigma$ is given by a closed path in $X^1$. Let $S$ be a spanning tree in

$X^1$. The fundamental group $\pi(G(X), S)$ of the complex of groups $G(X)$ is the fundamental group of the graph of groups $G(X^1)$ together with extra relations corresponding to the attaching maps of the 2-cells. Thus $\pi(G(X), S)$ is generated by the groups $G(v), v \in V(X^1)$ and the elements $e \in E(X^1)$. For each $e \in E(X^1), G(e)$ is a distinguished subgroup of $G(\iota e)$ and there are injective homomorphisms $t_e : G(e) \to G(\tau e), g \mapsto g^{\tau e}$. The relations of $\pi(G(X), S)$ are as follows:-

the relations for $G(v)$, for each $v \in V(X^1)$

$e^{-1}ge = g^{\tau e}$ for all $e \in E(X^1), g \in G(e) \leq G(\iota e)$,

$e = 1$ if $e \in E(S)$.

For each attaching closed path $e_1, e_2, \ldots, e_n$ in $X$ of a 2-cell, there is a relation

$$g_0 e_1 g_1 e_2 x_2 \ldots g_{n-1} e_n = 1,$$

where $g_i \in G_{\tau e_i} = G_{\iota e_{i+1}}$, called the *attaching word*. Such a word represents both a path $p$, called the *attaching path* in the Bass-Serre tree $T$ corresponding to the graph of groups $G(X^1)$, for which initial point $\iota p$ and end point $\tau p$ are in the same $\pi(G(X^1), S)$-orbit and an element $g \in \pi(G(X^1), S)$ for which $g\iota p = \tau p$. Adding the relation identifies the points $\iota p$ and $\tau p$ and puts $g = 1$. If we carry out all these identifications, we obtain a $G$-graph $\Gamma$ in which the attaching paths are all closed paths. We describe specifically how this path arises (as in [DD, p15]) We lift $S$ to an isomorphic subtree of $S_1$ of $\Gamma$. Thus the vertex set of $S_1$ is a transversal for the action of $G$ on $\Gamma$. For each edge $e$ in $X - S$ we can choose an edge $\tilde{e} \in T$ such that $\tilde{e}$ maps to $e$ in the natural projection, and $\iota\tilde{e}$ is a vertex of $S_1$. Let $\tilde{S}$ be the union of $S_1$ with these extra edges. Note that it will not normally be the case that $\tau\tilde{e} \in \tilde{S}$ and so $\tilde{S}$ is not usually a subtree of $T$, but there will be an element $c(e) \in G$ such that $c(e)^{-1}(\tau\tilde{e}) \in \tilde{S}$. These elements (called the connecting elements) together with the stabilizers of elements of $VS_1$ generate $G$. Clearly $\tilde{S}$ consists of a transversal for the action of $G$ on both the edges and vertices of $\Gamma$. Let $\iota p = v_0$ be the vertex of $\tilde{S}$ lying above $\iota e_1$, and put $x_0 = g_0$ Suppose we have constructed $v_i$ and $x_i \in G$ so that $v_i$ is the terminal vertex of the path corresponding to $g_0 e_1 g_1 e_2 g_2 \ldots g_{i-2} e_{i-1}$, and so that if $\tilde{v}_i$ is the element of $\tilde{S}$ in the orbit of $v_i$, then $v_i = x_i \tilde{v}_i$. This is certainly true when $i = 0$. To construct $v_{i+1}$ and $x_{i+1}$, put $x_{i+1} = x_i c(e_{i+1}) g_{i+1}$ where we put $c(e) = 1$ if $e \in S$. Then $x_{i+1}\tilde{v}_{i+1} = x_i c(e_{i+1})\tilde{v}_{i+1}$ is the terminal vertex of the edge $x_i\tilde{e}_{i+1}$ with initial vertex $v_i$. Note that $x_i$ is obtained from $g_0 e_1 g_1 e_2 g_2 \ldots g_{i-2} e_{i-1}$ by replacing each $e_i$ by $c(e_i)$.

We now foliate each 2-cell of $X$ in a particular way. Thus let

$$D = \{(x, y) | x, y \in \mathbf{R}, x^2 + y^2 \leq 1\}$$

be the unit disc. Give this the foliation in which leaves are the intersection of $D$ with the vertical lines $x = c$ where c is a constant in the interval $[-1, 1]$. Let $\sigma$ be a 2-cell of $X$ which is attached via the closed path $e_1, e_2, \ldots, e_n$. We map $D$ to $\sigma$ so that for some $j = 2, \ldots, n-1$ the upper semi-circle joining $(-1, 0)$ and $(1, 0)$ is mapped to the path $e_1, \ldots e_j$. Thus there are points $z_0 = (-1, 0), z_1, \ldots, z_j = (1, 0)$ on the upper semi-circle so that $z_i \mapsto \iota e_i, i = 1, 2, \ldots e_{j+1}$ and the map is continuous and injective on each segment $[z_i, z_{i+1}]$, except if $\iota e_i, = \iota e_{i+1}$ in which case the map is injective on the interior points of this segment. In a similar way the lower semi-circle is mapped to the path $\bar{e}_n, \ldots, \bar{e}_{j+1}$.

Let, then, $X$ be a 2-complex of groups in which each 2-cell is foliated as described above and let $T$ be a $G$-tree, i.e. $T$ is an $\mathbf{R}$-tree on which $G$ acts by isometries. We say that the $X$ *resolves* $T$ if there is an isomorphism $\theta : \pi(X, S) \to G$ which is injective on vertex groups (and hence on all groups $G_\sigma$ for all cells $\sigma$ of $X$). In this situation (see [H]) the complex of groups is developable, i.e. there is a cell complex $\tilde{X}$ on which $G$ acts and $G(X)$ is the complex of groups associated with this action. We also require that there be a $G$-map $\alpha : \tilde{X} \to T$ such that for each 1-cell $\gamma$ the restriction of $\alpha$ to $\gamma$ is injective and for each 2-cell $\sigma$ and each $t \in T$, the intersection of $\sigma$ with $\alpha^{-1}(t)$ is either empty or a leaf of the foliation described above.

We show that if $G$ is finitely presented then any $G$-tree has a resolution, i.e. there is a cell complex $X$ as above that resolves $T$.

Since $G$ is finitely presented, there is simplicial 2-complex $X$ such that $\pi(X, S) \cong G$. Here $S$ is a spanning tree in the 1-skeleton of $X$. Let $\tilde{X}$ be the universal cover of $X$. Clearly there is a $G$-map $\theta_0 : V\tilde{X} \to T$, which can be obtained by first mapping a representive of each $G$-orbit of vertices into $T$ and then extending so as to make the map commute with the $G$-action. Now extend this map to the 1-skeleton so that each 1-simplex $\gamma$ with vertices $u, v$ of $\tilde{X}$ is mapped injectively to the geodesic joining $\theta_0(u)$ and $\theta_0(v)$. It may be necessary to subdivide $X$ and choose the map $\theta_0$ to ensure that $\theta_0(u) \neq \theta_0(v)$ for every 1-simplex $\gamma$. We can extend the map to every 1-simplex so that it commutes with the $G$-action giving a $G$-map $\theta_1 : \tilde{X}^1 \to T$. Now we extend the map to the 2-simplices. Let $\sigma$ be a 2-simplex with vertices $u, v, w$. If $\theta_0(u)$ lies on the geodesic joining $\theta_0(v)$ and $\theta_0(w)$ then we can map $\sigma$ as indicated in Fig 1 (ii). Each vertical line is mapped to a point. If $\theta_0(u), \theta_0(v)$ and $\theta_0(w)$ are situated as in Fig 1 (i) so that no point is on the geodesic joining the other two, then we subdivide $\sigma$ as in Fig 1 (iii). The new vertex is mapped to the point $p$ of (i) and the three new simplexes now have the middle vertex mapped into the geodesic joining the images of the other two sides and are mapped as shown in (iii).

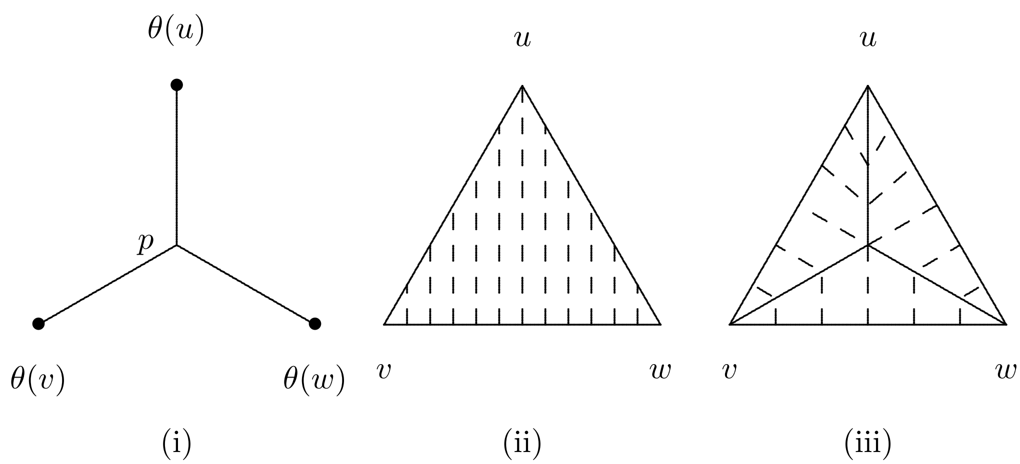



**Fig 1**

Again this map can be extended to every subdivided 2-simplex so that it commutes with the $G$-action. We change $X$ to be this subdivided complex. Regard $X$ as a 2-complex in which each cell is attached via a loop of length three. We can make a complex of groups in which each $G_\sigma$ is the trivial group. Since $G$ is the fundamental group of $X$ it is the fundamental group of this complex of groups. We have described a way of foliating the 2-cells which shows that this complex of groups resolves $T$.

We now describe some moves on a resolving 2-complex which will be used to convert the 2-complex to a standard form in the case when there are restrictions on the arc stabilizers of $T$.

**Move 1** Subdividing a 1-cell.

Let $\gamma$ be a 1-cell, with vertices $u, v$, which may be the same. This can be replaced by two 1-cells $\gamma_1, \gamma_2$ and a new vertex $w$, so that $\gamma_1$ has vertices $u, w$ and $\gamma_2$ has vertices $v, w$. The groups associated with $w, \gamma_1, \gamma_2$ in the new complex of groups are all $G(\gamma)$. The attaching maps of 2-cells are adjusted in the obvious way.

**Move 2** Folding the corner of a 2-cell.

Suppose that one end of a foliated 2-cell is as in Fig 2. Thus $v$ is the end vertex of the 2-cell and adjacent vertices are $x, y$ and $x, y$ are mapped to the same point of $T$, so that they lie on the same vertical line. Let the adjacent 1-cells to $v$ be $e_1$ and $e_2$, which conflicts with our earlier notation but is in line with that of [D2] and [BF1]. Let the groups associated with the cells (in the complex of groups) be denoted by the corresponding capital letters.

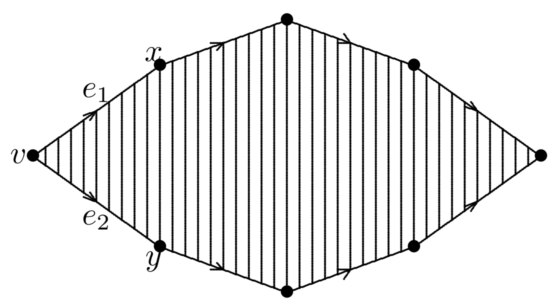



**Fig 2**

Folding the corner results in a fold of the graph of groups associated with the 1-skeleton of $X$. Such a fold is one of three types which are listed in [BF1] (as Type A folds) or in [D2]. They are shown in Fig 3 for the reader's convenience. As the group acting is always $G$ it is not necessary to carry out vertex morphisms (see [D2]) which are necessary when carrying out morphisms of trees rather than graphs.

The attaching word of the 2-cell, whose corner has been folded is changed by replacing the subword (in the cyclic word) $g_x \bar{e}_2 g_v e_1 g_y$ by $g_x g_v g_y$.

In any other attaching word of a 2-cell that involves $e_1$ or $e_2$, $\bar{e}_2$ is replaced by the folded edge element $< e_1, e_2 >$ and $e_1$ is replaced by $g_v^{-1} < e_1, e_2 >$. Let the new complex of groups be $X'$

Clearly there is a surjective homomophism $\phi : \pi(X, S) \to \pi(X', S')$ in which $g_v e_1 g_v^{-1}$ and $e_2$ are both mapped to $< e_1, e_2 >$. In fact this homomorphism is an isomorphism

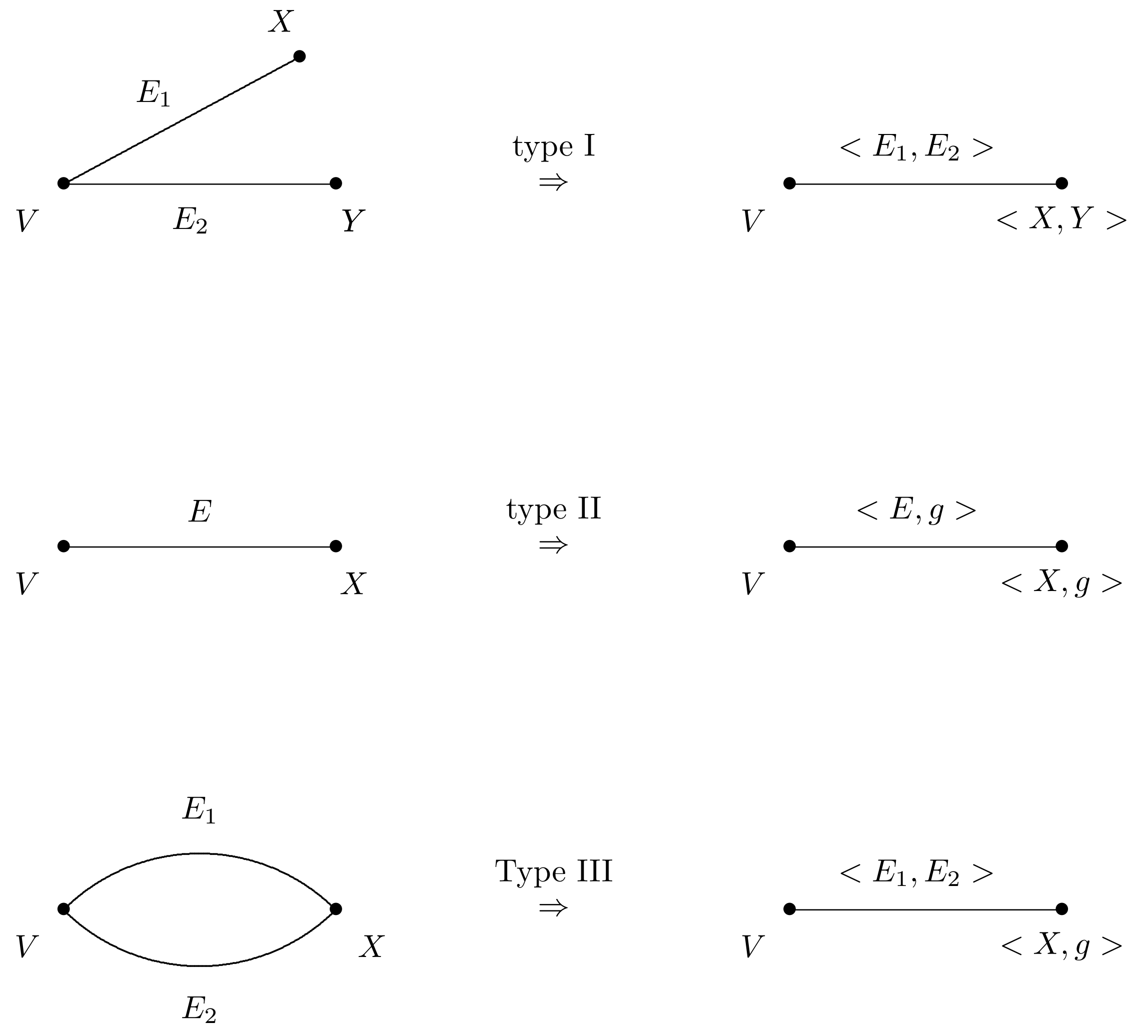



Fig 3

since the resolving isomorphism $\alpha : \pi(X, S) \to G$ factors through $\phi$. We conclude that $X'$ also resolves the $G$-tree $T$.

If both the upper semi-circle and the lower semi-circle consist of a single 1-cell then folding results in the elimination of a 2-cell.

**Move 3** Contracting a leaf.

Consider a foliated 2-cell . Let $\ell$ be a particular vertical line of the foliation. This will contain points $u, v$ of the upper semi-circle and lower semi-circle respectively. After subdividing the relevant 1-cells, it can be assumed that these points are vertices. Contracting the leaf $\ell$ results in the 2-cell $\sigma$ being replaced by two 2-cells $\sigma_1$ and $\sigma_2$. The vertices $u, v$ become a single vertex $w$ and its group $G_w$ is the subgroup of $G$ generated by $G_u$ and $G_v$ in $G$, except if $u, v$ belong to the same $G$-orbit, in which case $G_w$ is generated by $G_u$ and an element $g \in G$ such that $gv = u$. Let $g_u, g_v$ be the respective elements of $G_u$ and $G_v$ in the attaching word for $\sigma$. Let the edge after reaching $u$ in the attaching word end up in $\sigma_2$. This means that the edge after reaching $v$ ends up in $\sigma_1$. Suppose first that $u, v$ are in different orbits, then after the move the element for $w$ in $\sigma_2$ is $g_u$ and the element for $w$ in $\sigma_1$ is $g_v$. Note that an edge has to be removed from the spanning tree $S$. If $u, v$ are in the same orbit then the element for $w$ in $\sigma_2$ is $g_u g$ and the element for $w$ in $\sigma_1$ is $g_v g^{-1}$.

A similar argument to that for Move 2 shows that the complex we have created also resolves the $G$-tree $T$.

Let $\sigma$ be a 2-cell of $X$. We now examine what can happen as we repeatedly fold corners of $\sigma$, at each stage replacing $\sigma$ by the new 2-cell created. Since each 1-cell of $\tilde{X}$ injects into $T$ we can assign each 1-cell $\gamma$ of $X$ a length, namely the distance in $T$ between $\theta(u)$ and $\theta(v)$ where $u, v$ are the vertices of a lift of $\gamma$ in $\tilde{X}$.

As above let $x$ be the corner vertex and let $e_1, e_2$ be the incident edges.

If $e_1, e_2$ have the same length, then we can fold the corner of $\sigma$. If $e_1$ is shorter than $e_2$ then subdivide $e_2$ so that the initial part has the same length as $e_1$ and then fold the corner. If $e_2$ is shorter than $e_1$ then we subdivide $e_1$ and then fold the corner. Now repeat the process. This process may terminate when all the 2-cell is folded away.

However it may happen that the folding sequence is infinite i.e. it never terminates. We examine when this happens. Suppose this is the case and that the 2-complexes in the sequence are $X_n, n = 1, 2, \dots$ .

We can assign lengths to the edges (1-cells) of $X_n$. Traversing the top semi-circular boundary of the 2-cell $\sigma$ determines to a path (or rather walk) $w$ in the 1-skeleton of $X_1$. Let $w'$ be the path corresponding to the lower semi-circular boundary. These paths are usually not segments - they can even backtrack. Let $\ell_n$ be the total length of edges of $X_n$. It is clear that $\ell_n \geq \ell_{n+1} \geq 0$. We have $\ell_{n+1} = \ell_n$ if and only if the fold is a subdivision or a type II fold. In going from $X_n$ to $X_{n+1}$ an arc $[y_n, y_{n+1}]$ of the upper semicircular boundary of $\sigma$ is identified with an are $[y'_n, y'_{n+1}]$ of the lower semicircular boundary. Each such arc is identified with a 1-cell of $X_n$ and so has a length. In $X_n$ the folding has identified $[x, y_n]$ with $[x, y'_n]$. We assume that $y = \lim_n y_n$, and that $y' = \lim y'_n$. It is possible that $y = y'$ is the end point of $\sigma$ and we will see that this is often the case. Let $\lambda_n$ be the length of the arc $[y_n, y]$. Thus $\lambda_n$ is the length of the arc which remains to be folded.

We show that there can only be finitely many type II folds in our sequence. This is because there can only be a finite number of type II folds to start with as each such fold will use up the full length of an edge of $X_1$. In our sequence, a type I fold can only be followed by a type II fold if the type II fold is between edges in the same orbits as as the ones that were folded together in the type I fold. Thus there is a vertex in $[x, y]$ such that the adjacent edges are in the same orbit. Such a vertex must have been a vertex in the original path $[x, y]$ in $X_1$ and so this happens only finitely many times. Each type III fold decreases the first Betti number of the quotient graph and so there can only be a finite number of type III folds. In our sequence there are therefore only finitely many type II or type III folds. Assume then that all folds in the sequence are of type I. Consider the subspace of $X_1$ which is the union of the images of the paths corresponding to $[x, y]$ and $[x, y']$. If this is not a subgraph of $X_1$, then one of the paths corresponding to $[x, y], [x, y']$ in $X_1$ must end in part of an edge not visited by the other path. It is not hard to see that this will not produce an infinite folding sequence. Thus we assume that this subgraph is all of the 1-skeleton of $X_1$.

In our sequence of subdivision and type I folds the number of edges in the quotient graph does not increase, since any subdivision which increases the number of edges by one is immediately followed by a type I fold which reduces it by one. Clearly there can only be a finite number of type I folds which are not preceded by a subdivision, since the number of such folds is bounded by the number of edges of $X_1$. It may happen that a fold at the $n$-th stage produces an edge which is not in the subgraph $X'_{n+1}$ determined by the remaining folding sequence. This can happen for only a finite number of folds. Since if this happens $X'_{n+1}$ has fewer edges than $X_n$. Thus we assume that each folded edge is in the subgraph determined by the remaining folding sequence.

For a type I fold $\ell_n$ and $\lambda_n$ are reduced by the same amount.

Since we are assuming that each folded edge is in the subgraph determined by the remaining folding sequence, it is clear that $\ell_n$ tends to zero as $\lambda_n$ tends to zero. Since $\ell_n - \lambda_n$ is constant, it follows that $\ell_n = \lambda_n$.

Let $w_y, w'_{y'}$ be the directed paths in $X_1$ which are the images of $[x, y], [x, y']$ respectively. Clearly they are initial parts of the paths $w, w'$, so they begin at the same point. In fact we can assume that they end at the same point by using a Move 3 to contract the leaf that contains the points $y$ and $y'$. In fact we will show that $y$ and $y'$ are always vertices in the original graph.

From length considerations every edge of $X_1$ occurs exactly twice in $w \cup w'$ or at least one edge occurs only once. If the latter occurs we will arrive at a contradiction by showing that the folding sequence must have been finite. Let $e$ the edge which occurs only once in $w \cup w'$. Without loss of generality suppose it is in $w$. In fact we can assume that it is the first edge of $w$, since we can fold away any edges which precede it. This folding will not affect the edge $e$. There is also a folding sequence starting at the other end of $\sigma$. It is not hard to see that this must also be an infinite sequence and in the limit all of $w \cup w'$ is folded away. Folding away those edges which occur before $e$ in this sequence and after $e$ in the original sequence, we arrive at a new 2-cell in which the entire path $w$ consists of a single edge $e$. But such a folding sequence must be finite - it will just fold $e$ onto the path $w'$. We have the desired contradiction.

An infinite folding sequence therefore occurs when there is a 2-cell in which the attaching map contains every edge exactly twice.

We want to show that we can carry out folding on the different 2-cells and end up with a complex in which each cell is attached via a quadratic word.

After carrying out a finite number of Type 3 moves we can assume that each leaf of the foliation intersects the top and bottom of each 2-cell in at most one vertex. The argument above shows that the limit points $y, y'$ of an infinite folding sequence must be vertex points on the same leaf of a foliation and so $y = y'$ will be an end point of the 2-cell. If one considers the folding sequence starting from the other end of the 2-cell, we see that the first point reached where the attaching word becomes quadratic must also correspond to a leaf of the foliation which, if it was different from an end point of the 2-cell, would contain two vertices. Thus every 2-cell corresponds to a finite folding sequence or it corresponds to an infinite folding sequence given by a quadratic attaching word.

We now show that if two 2-cells corresponding to quadratic words, have an edge in common then they belong to a homogenous or toral component (in the terminology of [C] or [BF2] respectively).

First we associate a system of isometries with one of the 2-cells. Let $D$ be an interval obtained by identifying $w$ and $w'$, so that $x$ the initial point of both $w$ and $w'$ is the left hand end point of $D$. We can also regard $D$ as a segment of $T$ in the following way. Lift $w$ and $w'$ to $\tilde{X}$ so that the initial points lift to the same point of $\tilde{X}$. The image of these lifted paths map to the same segment of $T$. We take this segment to be $D$. We define a system $S$ of isometries of subintervals of $D$ as follows. Each edge of $X$ occurs twice in $w \cup w'$. Then there will be a corresponding pair of subintervals of $D$ for which there is an element $g \in G$ taking one subinterval to its pair. The system $S$ consists of an isometry for each such pair. Thus $S$ contains the isometry between the subintervals determined by $g$. Let us call the intervals involved in the isometries of $S$ bases, i.e. a base is a domain or a codomain of one of the isometries. If $b$ is a base then it has a dual base $dual(b)$ which is the codomain or domain of the corresponding isometry. Move 2 - folding from a corner - results in a change in the system $S$. Let $b_1, b_2$ be the two bases containing the initial point $x$. Let $x_1, x_2$ be the end points of $b_1, b_2$ respectively. Assume $x_2 \geq x_1$.

Let $g_1, g_2$ be the isometries involving $b_1, b_2$ respectively, so $g_1, g_2$ have respective domains $b_1, b_2$. Consider the new system $S_1$ in which $D = [x, y]$ is replaced by $D_1 = [x_1, y]$, $g_1$ is replaced by $g_2 g_1^{-1}$ with domain $dual(b_1)$ and $g_2$ is replaced by $g_2$ restricted to $[x_1, x_2]$. In the new system every point of $D_1$ again lies in two bases.

Suppose that there is another 2-cell - corresponding to an infinite folding sequence which shares an edge with the one we are already considering. This will also give rise to a system of isometries as for the first 2-cell. By folding from either end of the new 2-cell, the new system of isometries can be replaced by a system for which the interval $D'$ is a subset of the shared edge or base. Combining these two systems of isometries we obtain a system $S$ of isometries based on subintervals of $D$ in which every point is in at least two bases and some points are in 4 bases.

In the notation of [C] the length $m(S)$ of $D$ is $y - x$. The sum $\ell(S)$ of the length of the domains of the generators is more than $m(S)$ and so $\ell(S) - m(S)$ (called the *excess* of $S$) will be positive. This means by [C] Chapter 6, Proposition 3.10, that $S$ has a homogeneous component.

We will adapt an argument from [BF2] to show that $S$ itself is homogeneous and that all elements of $G$ inducing the isometries in $S$ have the same axis. Every point of $D$, which is not the end point of an interval, is in two or four bases. Starting at the left hand end point of $D$, choose the longest base $b$ containing that point. This is the domain of an isometry $\iota : b \to dual(b)$. For each other base $b'$ containing that point, and corresponding isometry $\iota' : b' \to dual(b')$ there is a new isometry $\iota' \iota^{-1} : \iota(b') \to dual(b')$. This moves the base $b'$ to the right leaving $dual(b)$ unchanged. After replacing each such isometry $b'$ of $S$ in this way, the base $b$ will then have an initial segment $[0, y]$ which lies in no other base. We then alter $S$ by removing the initial interval $[0, a)$ from both $b$ and $D$. A corresponding interval is removed from $dual(b)$. Let the new system be denoted $S'$. It is easy to see that $\ell(S') = \ell(S) - a$ and $m(S') = m(S) - a$ and so the new system has the same excess as the old. If we keep repeating this process (Process II of [BF2]) the bases will reach a limiting position.

As in [BF2] it follows that there is a base $b$, whose measure does not tend to 0, such that

$b$ and $dual(b)$ tend to the same limiting position. Indeed this is the case for any base that participates infinitely often in this process. For such a sequence of bases the translation lengths of the corresponding isometries tends to zero. It may happen that an isometry is orientation reversing, in which case it does not have a translation length. Howver there will be an infinite subsequence of bases with the same orientation and, taking isometries beween pairs of these, will give the required sequence of isometries whose translation lengths tend to zero. Also as in [BF2] there will be isometries $g, h$ that have translation lengths that are independent over the rationals and for which these lengths are small compared with the overlap $b \cap dual(b) \cap b' \cap dual(b')$, where $b, b'$ are the bases of the two isometries $g, h$ respectively. Since $b \cup dual(b)$ is in the axis $A_g$ of $g$ and $b' \cup dual(b')$ is in the axis $A_h$ of $h$ we can assume $\Delta(g,h) > \ell(g) + \ell(h)$. Using some results from [C] we show that this implies $A_g = A_h$. Chiswell's results are all for the more general case of actions on $\Lambda$-trees. We are only really interested in the case when $\Lambda = \mathbf{R}$, but the proof works for the more general case.

Thus in the following $(X, d)$ is a $\Lambda$-tree on which $G =< g, h >$ acts by isometries, where $g, h$ are hyperbolic elements. The axes of $g, h$ are denoted $A_g, A_h$. We say $g, h$ *meet* if $A_g \cap A_h \neq \emptyset$. We say that $g, h$ meet *coherently* if they meet and $g, h$ translate in the same direction along $A_g \cap A_h$. The hyperbolic length of an element $x$ is denoted $\ell(x)$. Let $\Delta(g, h)$ denote the diameter of $A_g \cap A_h$, if it is bounded. If it is unbounded put $\Delta(g, h) = \infty$.

The following statements are proved in [C].

C1.[C, Chapter 3, Lemma 3.4]

Let $g, h$ meet coherently, and $\ell(g) \geq \ell(h)$ and suppose $\Delta(g,h) > \ell(h)\}$, then $gh^{-1}$ meets $g, h$ coherently. Also $\ell(gh^{-1}) = \ell(g) - \ell(h)$ and $\Delta(gh^{-1}, h) = \Delta(g, h) - \ell(h)$.

C2. [C, Chapter 3, Lemma 3.9] If $g, h$ are as in C1, then

(i)

$$\ell(ghg^{-1}h^{-1}) = 2(\ell(g) + \ell(h) - \Delta(g,h),$$

(ii) if $g, h$ commute, then $A_g = A_h$.

Let $g, h$ be hyperbolic elements of $G$. If $A_g \cap A_h \neq \emptyset$ define $\mathcal{E}(g,h) = \ell(g) + \ell(h) - \Delta(g,h)$. If $A_g \cap A_h = \emptyset$ define $\mathcal{E}(g,h) = \ell(g) + \ell(h) + 2d(A_g, A_h)$. It can be seen from the above that if $g, h$ meet coherently, then $\mathcal{E}(g,h) = \mathcal{E}(gh^{-1}, h)$.

**Proposition 1.** *Let $(X, d)$ be a $G$-tree. Let $g, h \in G$ be hyperbolic elements meeting coherently and suppose $\ell(g), \ell(h)$ are independent over the rationals. Suppose $\Delta(g,h) > \ell(g) + \ell(h)$. Then either $A_g = A_h$, or $G$ contains a subgroup which is not finitely generated which fixes an arc of $X$. In particular if arc stabilizers are slender then $A_g = A_h$. Also if the action of $G$ on $X$ is stable then $A_g = A_h$.*

**Proof**. Let $\mathcal{E}(g,h) = \Delta(g,h) - \ell(g) - \ell(h)$. By hypothesis $\mathcal{E}(g,h) > 0$

By C1, $gh^{-1}$ meets $g, h$ coherently and $\ell(gh^{-1}) = \ell(g) - \ell(h)$ and $\Delta(gh^{-1}, h) = \Delta(g,h) - \ell(h)$. Thus $\mathcal{E}(g,h) = \mathcal{E}(gh^{-1}, h) = \mathcal{E}$ say. By examining the proof in [C] it can be seen that if $A_g \cap A_h$ has a right-hand end point $r$ then this is also the right hand end point of $A_{gh^{-1}} \cap A_h$. and so $A_{gh^{-1}} \cap A_h \subset A_g \cap A_h$. By performing a sequence of Nielsen operations of the above type we obtain a pair of generators $g^*, h^*$ for which $\ell(g^*), \ell(h^*)$ are arbitrarily small and $A_{g^*} \cap A_{h^*}$ contains the segment $\sigma = [l, r]$of $A_g$ of length $\mathcal{E}$ with right-hand endpoint $r$. Choose $\epsilon > 0$ so that $\sigma' = [l', r']$ is a closed subinterval of $\sigma$ of positive length, where $d(r, r') = d(l, l') = \epsilon$. Choose $N > 0$ so that $1/N < \epsilon$. If $n > N$ let $\sigma_n = [l' - 1/n, r' + 1/n]$. Thus $\sigma' \subset \sigma_{n+1} \subset \sigma_n \subset \sigma$ for all $n > N$. Let $G_n$ be the stabilizer of $\sigma_n$ and let $G'$ be the stabilizer of $\sigma'$. Then $G_n \leq G_{n+1} \leq G'$. This means that $G_n = G_{n+1}$ for $n$ sufficiently large or the union of the $G_n$'s is not finitely generated and fixes an arc of $T$. Also if the action is stable then the sequence is eventually constant. In particular $G_n = G_{n+1}$ for $n$ sufficiently large if arc stabilizers are slender. It also follows that for such an $n$ the stabilizer of $[l' - \frac{1}{n}, r' + \frac{1}{n+1}]$ is the same as the stabilizer of $[l' - \frac{1}{n+1}, r' + \frac{1}{n}]$. Thus there is a subsegment of $\sigma$ whose stabilizer $M$ is unchanged after a translation of length $\delta > 0$. Note that $M$ will contain the commutator of $g$ and $h$. If we choose the generators $g^*, h^*$ so that $\ell(g^*) < \delta$ and $\ell(h^*) < \delta$ then $g^*, h^*$ normalize $M$ which is therefore a normal subgroup of $< g, h >$ and must fix $A_g$ and $A_h$. Since the factor group of $< g, h >$ by $M$ is abelian, it follows from C2 that $A_g = A_h$.

Let $A = A_g = A_h$. If $k \in G$ and $A \cap A_k$ has diameter greater than $\ell(k)$, then $A = A_k$. This is easy to see, since by the above argument $< g, h >$ must contain an element $x$ such that $\ell(x) < d - \ell(k)$. But then $A = A_x = A_k$.

In another paper [D5] I prove the stronger result the in the situation of Proposition 1 either $A_g = A_h$ or there is an arc stabilizer which contains a non-cyclic free group, i.e. if the action is small, then $A_g = A_h$.

In the above description, which is our version of Process II of [BF2], any base which slides over the bases of $g$ or $h$ or its iterates must belong to an isometry which has an axis which intersects more of $A_g = A_h$ than its translation length. By the above, this means it must also have the same axis.

A toral component corresponds to a line $L$ in the **R**-tree $T$ and a subgroup $H$ of $G$ stabilizing that line. There is a normal subgroup $N$ of $H$ consisting of elements which act trivially on$L$. By hypothesis $N$ is slender. The quotient $H/N$ is an isometry group of the reals. It is either finitely generated free abelian and every element acts by translation or $H/N$ has a subgroup of index 2 like that. There will be at least 2 translations which are rationally independent (otherwise the action is simplicial ).

A 1-cell of $X$ when lifted to $\tilde{X}$ and then mapped into $T$, maps injectively into $T$. The discussion above where we used the 1-cells to define a system of isometries shows that the lifted 1-cell maps either into a translate of $L$ or intersects a translate of $L$ in at most an end-point. Suppose a 1-cell in the top of a 2-cell lifts to a 1-cell mapping to a translate of $L$ then so will any 1-cell in the bottom whose projection on the $x$-axis shares more than a single point with the projection of the 1-cell in the top. Thus if the image of the lift of the boundary of a 2-cell intersects a translate of $L$ in a non-trivial closed interval, then there will be matching intervals in the top and bottom of the 2-cell. These intervals correspond to sub edge-paths of w and w'. Suppose we fix a particular L and look at all the corresponding subintervals in any boundary of our finitely many 2-cells. Every edge which occurs in an interval must have another occurrence (possibly in the same interval) and at least one will occur more than once. One can carry out Type 3 moves at the beginning and end of each pair of intervals. One will end up with 2-cells in which each edge corresponds to $L$ or no edge corresponds to $L$.

At this stage in the process we have 2-cells which belong to three types:-

(i) toral components,

(ii) 2-cells attached by quadratic words corresponding to infinite folding sequences

(iii) 2-cells corresponding to finite folding sequences.

The cells of type (ii) and (iii) cannot share a 1-cell with one of type (i) and no two cells of type (ii) can share a 1-cell. It is possible that cells of type (ii) and (iii) share an edge.

We now want to fold away the 2-cells corresponding to finite folding sequences. As we do this we may force two cells of type (ii) to share an edge and this will create a new toral component. Eventually we obtain a complex in which there are no 2-cells of type (iii). The process terminates as we do not create any cells of type (ii) in the process. Note that after this folding there may be 2-cells in T1 components which may not be attached by quadratic words.

We now show that by putting in some extra edges we can assume every vertex is in at most one T2-component and if a vertex is in such a component, then its group is generated by its incident edge groups and a single connecting element. Thus let $C$ be a 2-component with a single 2-cell so that the attaching word $w = x_0e_1x_1e_2 \dots x_{n-1}e_n$ is quadratic. The structure of $C$ is that of a 2-orbifold in which - possibly - some vertices have been identified. Thus there is a 2-orbifold $\hat{C}$ and a map $\hat{C} \to C$ which produces this identification. In the 2-orbifold $\hat{C}$ the word $w$ represents a path visiting vertices $v_0, v_1, \dots v_n = v_0$, where $v_i = \tau e_i = \iota e_{i+1}$ and $v_i = v_j$ if and only if this is a consequence of these equations and the fact that each $e_i$ occurs twice (as $e_i$ or $\bar{e}_i$) in $w$. Each $x_i$ corresponds to a choice of representative edge in an orbit of edges. These will vary as one varies the lift of the spanning tree $S$ in $C^1$ to the complex $\tilde{C}$. The corresponding $x_i$ is then such that $x_i\tilde{e}_{i+1} = \tilde{e}_j$, where $e_j$ is the other occurrence of $e_{i+1}$ or $\bar{e}_{i+1}$ in the quadratic word and $\tilde{e}_j$ is the chosen lift of $e_j$ in $\tilde{X}$. Here $\iota e_j$ will be the first visit to the vertex $v$ corresponding to $x_i$ as one follows the path in $X^1$ corresponding to the word. If $C = \hat{C}$ we can choose this lift and corresponding transversal of edges so that each $x_i = 1$ unless the corresponding vertex is visited for the last time as one proceeds along $w$. See Examples 5 and 6 in the next section. The index of $v = \iota e_{i+1}$ is the smallest power $n_i$ of $x_i$ (possibly 1 or $\infty$) such that $x_i^{n_i}\tilde{v} = \tilde{v}$. The index of $v$ will divide the order of $x_i$ and $x_i^{n_i}$ will fix all the directions at the image of $\tilde{v}$ in $T$. In general if $C \neq \hat{C}$ then in the complex of groups we replace $C$ by $\hat{C}$ together with an extra vertex for each vertex $v$ of $C$ which is not the image of a single vertex in $\hat{C}$. This vertex will also be denoted $v$ in the new complex, and it has the same group that it had previously. We add edges joining $v$ to each vertex of $\hat{C}$ of which it is an image, and we assign to each such edge the group generated by the single non-trivial vertex element in $\hat{C}$ together with the incident edge groups of $\hat{C}$.

Note that the edge created is compressible (in the sense of [DD, p90]. After we have done this for every vertex in any component, we may still have vertices which lie in more than one component. Suppose there is such a vertex $v$ incident with $n > 1$ components. For each of these components, $G(v)$ is generated by the vertex elements of attaching maps and the edge groups incident with $v$ in that component. In this case we expand the vertex to a tree with $n$ vertices joined to a single central vertex with group $G(v)$ and a vertex for each component containing $v$ for which the corresponding group is generated by the relevant vertex element in that component together with the incident edge groups in that component. This will also be the group of the edge connecting that vertex to the central vertex. and so this new edge will again be compressible. Once we have done this we will have a complex in which each edge and vertex group is in at most one component. The fundamental group of the new complex of groups is still $G$, since all we have done is create some compressible edges.

Using a similar process we can arrange that no two T1 components have a vertex in common and that no T1 component has a vertex in common with a T2 component.

If we remove the interior of 1-cells which do not lie in any 2-cells we obtain finitely many subcomplexes. The components corresponding to a single 2-cell of type (ii) is a T2 component. The single vertices remaining are the T0 components. The other components are the T1 components. This indicates that the group $G$ has most of the structure given in Theorem 1. It remains to discuss the group of a T2 component.

Let $X$ be a T2 complex which resolves an action of $G$ on an $\mathbf{R}$-tree $T$, where $G$ is the fundamental group of $X$. Thus $X$ has one 2-cell and the attaching word $x_0e_1x_1e_2\dots x_{n-1}e_n$ is quadratic. This means that each 1-cell $e$ occurs exactly twice (possibly as $\bar{e}$). Each $x_i$ corresponds to a choice of representative edge in an orbit of edges. These will vary as one varies the lift of the spanning tree $S$ in $X^1$ to the complex $\tilde{X}$. We can choose this lift and corresponding transversal of edges so that each $x_i = 1$ unless the corresponding vertex $V$ is visited for the last time as one proceeds along the word. The corresponding $x_i = v$ is then such that $x_i\tilde{e}_{i+1} = \tilde{e}_j$, where $e_j$ is the other occurrence of $e_{i+1}$ or $\bar{e}_{i+1}$ in the quadratic word and $\tilde{e}_j$ is the chosen lift of $e_j$ in $\tilde{X}$. Here $\iota e_j$ will be the first visit to the vertex $V$ corresponding to $x_i$ as one follows the path in $X^1$ corresponding to the word. See Examples 5 and 6 in the next section.

As we have seen, there is an infinite folding sequence of complexes $X_n$ each with fundamental group $G$ and each term in the sequence resolves the action of $G$ on $T$. Recall that each vertex group $V$ is generated by a vertex element $v$ and the incident edge groups. We can assign an index to $V$ as follows. Let $i_n$ be the smallest power of $v$ which lies in the subgroup generated by the incident edge groups at $V$ in $X_n$. The sequence $\{i_n\}$ is a decreasing sequence of positive integers and so it is eventually constant. The index of $V$ is taken to be this constant. It is possible that the index is infinity if no positive power of $v$ lies in appropriate subgroup. Note that if the underlying orbifold is a 2-sphere with cone points, then there must be at least 4 cone points with index more than 1, as there will be at least 4 occurrences of $\bar{e}ve$ in the attaching word and in this case $v$ cannot fix the direction in $T$ at the image of $V$ determined by the image of $e$. Here we use the fact that the top and bottom of a 2-cell is mapped injectively into $T$. We now construct a complex of groups which encodes all the information about the folding sequence. If $v$ is the vertex element corresponding to $V$ then the index $i(v)$ of $V$ divides the order $o(v)$ of $v$. It may be helpful to study some of the examples in the next section to understand the construction described here.

Let $Y$ be a complex of groups which is the same as $X$ except that all edge groups are trivial, and vertex groups are cyclic and a vertex element has order which is the order of that element in $G$ and let $Y_1$ be the same complex of groups except that each vertex element $v$ has order $i(v)$. Thus $Y$ and $Y_1$ are 2-orbifolds with cone points and $Y$ is an orbifold cover of $Y_1$. Let $H, H_1$ be the fundamental groups of $Y, Y_1$ respectively. Clearly there is a homomorphism from $\theta : H \to G$, and so there is an action of $H$ on $T$. Let $\tilde{Y}$ be the universal orbifold cover of $Y$. This will give a tessellation of the hyperbolic plane, and $H$ acts on $\tilde{Y}$. Note that vertex elements with infinite order will fix ideal points (on the boundary) of the tessellation. They will be parabolic elements. We now describe a way of attaching groups to the cells of $\tilde{Y}$ to get a complex of groups $Z$. A fundamental region for the action of $H$ on $\tilde{Y}$ consists of a single 2-cell and its incident 0-cells and 1-cells. Choose a transversal $F$ for the $H$-action from the edges and vertices of this region. Attach the same groups to the 1-cells of this fundamental region as are attached to the corresponding one cells of $X$. For each other 1-cell $\gamma \in \tilde{Y}$ there is an element $h \in H$ such that $h\gamma \in F$. Since $H$ acts freely on the 1-cells of $\tilde{Y}$ this $h$ is unique. If $E$ is attached to $h\gamma$ and $\theta(h) = g \in G$ then attach $g^{-1}Eg$ to $\gamma$. For each 0-cell of $\tilde{Y}$ we attach the subgroup of $G$ generated by the incident edge groups in $\tilde{Y}$. In $\tilde{Y}$ each 2-cell already has an attaching map, and we use this same attaching map in the new complex of groups, i.e. all vertex elements are trivial. Let $N_1$ be the fundamental group of the complex of groups $Z$. By our construction, there is a homomorphism $\phi : N_1 \to G$ which is injective on vertex groups and edge groups. Our aim is to show that the image $N$ of $\phi$ is a normal subgroup of $G$ which does not include a vertex element for which $i(v) > 1$ and in fact $G/N \cong H_1$. If $i(v) = o(v)$ for each vertex element , then $\phi$ is injective and in this case $N_1 \cong N$. The foliation on $X$ obtained from the action on $T$ will give a foliation on both $Y$ and $\tilde{Y}$, which corresponds to the action of $N_1$ on $T$. Consider the leaf intersecting a vertex $U \in V\tilde{Y}$. If the vertex element $u$ has order $n$, then the leaf will have a branch point with $n$ branches and $n$ infinite rays starting at $u$. The stabilizer of the image of $u$ in $T$ will contain all the edge groups of edges intersected by the leaf. Note that from the theory of complexes of groups, there is a simply connected 2-complex $\tilde{Z}$ on which $N_1$ acts so that $N_1\backslash\tilde{Z} = \tilde{Y}$. The foliation on $Z$ pulls back to a foliation on $\tilde{Z}$ and the leaf space, made Hausdorff, will be an $\mathbf{R}$-tree $T_1$ on which $N_1$ acts, and the action resolves the action on $T$. In $T_1$ there is a branch point of index $n$ at the leaf of $u$. Note that the stabilizer of this point is generated by the edge groups of edges intersected by the leaf. A point in the stabilizer of one branch must fix the other branches, since the branches are in the quotient space of $\tilde{Z}$ by $N_1$ and $N_1$ includes all edge groups. It follows that the stabilizer of the image of $U$ in $T_1$ is the same as the stabilizer of each branch. A direction at $u$ is stabilized by the edge groups which intersect two adjacent branches in the cyclic ordering around $u$. Clearly this will also have the same stabilizer. Consider the map from $T_1$ to $T$ at $U$, The image of $T_1$ will have $i(u)$ branches at $U$ and the map of branches will be reducing modulo $i(u)$ as on proceeds cyclically around $u$ in $\tilde{Y}$. Since the vertex element $u$ permutes the directions at the image of $U$ it must normalize the stabilizer of a direction. It follows that $Im\phi$ is normalized by each vertex element, since in the fundamental group of a graph of groups, the normal closure of the edge groups intersects each vertex group in the normal closure of the incident edge groups. The index $i(v)$ is the smallest power of $v$ that lies in $N$. Clearly $G/N \cong H$.

One might think that one could make exotic actions on an $\mathbf{R}$-tree by constructing a graph of groups as for a T2 complex in which vertex groups just satisfy the condition that they are generated by incident edge groups and the vertex element. However this graph of groups may not be developable, i.e. the vertex group may not inject into the fundamental group of the graph of groups. If the fundamental group resolves an action then it must be developable.

Theorem 1 follows immediately.

We now consider the 2-orbifold components - the T2 vertices - of the decomposition. The discussion is facilitated by considering some examples.

## 3. The examples

The first example that I thought of exhibiting an unstable action of a finitely presented group with cyclic arc stabilizers is discussed extensively in [D4]. Some other examples are presented here.

**Example 1** Let the complex $X$ have one vertex $U$ and two oriented edges $a, b$ which necessarily are loops at $U$. Let the group of $U$ be $< z, u | uz = zu, u^2 = 1 >$. Let the group of each edge be infinite cyclic generated by $z$. For each edge $e$ the map $G(e) \to G(\tau e)$ is the map $z \mapsto z^2$. There is also a 4-sided 2-cell attached by the path corresponding to the word $uba\bar{b}\bar{a}$. The complex $X$ is a torus with one cone point order 2. We get a presentation for the fundamental group $G$ of this complex of groups to be

$$G = \langle a, b | a^{-1}za = b^{-1}zb = z^2, u = aba^{-1}b^{-1}, uz = zu, u^2 = 1 \rangle.$$

Let $N$ be the normal closure of $z$ in $G$. So $G$ is the splitting extension of $N$ by $F$, the subgroup generated by $\{a, b\}$ which is the Fuchsian group of signature $(1, 2)$.

Let $D = \{n/2^m | n, m \in \mathbf{Z}\}$ be the dyadic rationals. There is an action of $F$ on $D$, in which each of $a, b$ maps to the automorphism which is multiplication by 2. Let $G_1$ be the group which is the splitting extension of $D$ by $H$, with this action of $H$. Clearly there is a surjective homomorphism from $G$ to $G_1$ which restricts to the identity map on $H$, and takes $z$ to $1 \in D$. We will show that this map is not an isomorphism and so $N$ is not locally cyclic.

Consider a tessellation of the hyperbolic plane by regular 4-gons with angle $\pi/4$. Let $\Gamma$ be the graph determining this tessellation. Thus $\Gamma$ is an 8-regular plane graph, whose dual graph $\Gamma'$ is 4-regular. There is an action of $H$ on the hyperbolic plane by isometries which induces an action on $\Gamma$ in which there is one orbit of vertices and 2 orbits of edges. Diametrically opposite edges of an octagonal face lie in the same edge orbit. Let $A$ be one of the faces of $\Gamma$. Orient the edges of $A$ as shown in Fig 4.

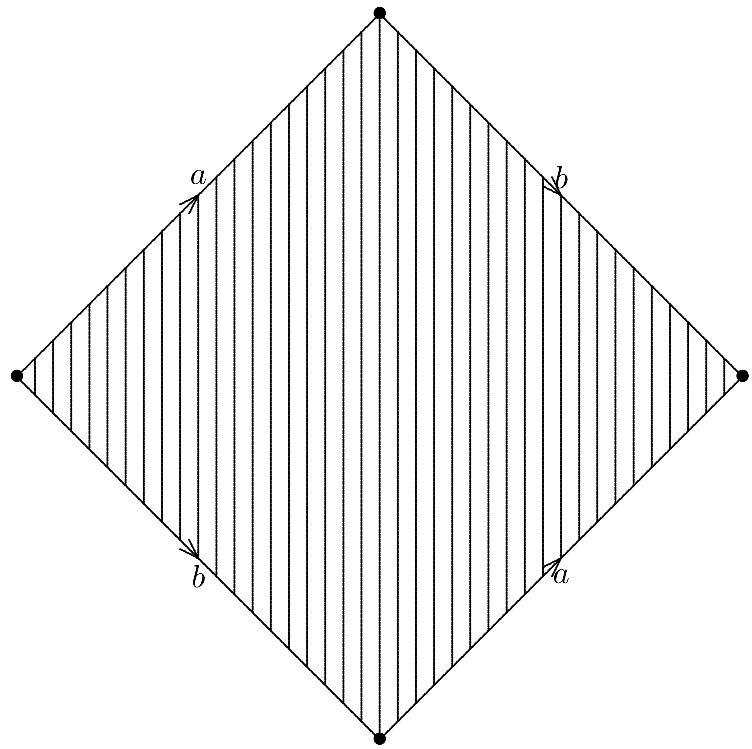



Fig 4

Extend this orientation to an orientation $\mathcal{O}$ of $E\Gamma$ in such a way that the orientation is preserved under the action of $H$. Define $\phi : V\Gamma \to \mathbf{Z}$ in the following way. For the left hand vertex of $A$ put $\phi(v) = 0$. For any other vertex $u$, choose an oriented edge path $e_1, e_2, \ldots, e_n$ from $v$ to $u$. let $\phi(u) = \epsilon_1 + \epsilon_2 + \ldots \epsilon_n$ where $\epsilon_i = 1$ if $e_i$ is in $\mathcal{O}$ and $\epsilon_i = -1$ if $\bar{e}_i$ is in $\mathcal{O}$. The fact that $\phi$ is well defined follows from the fact that it is the unique function such that $\phi(v) = 0$ and such that $\delta\phi(e) = 1$ for each edge in $\mathcal{O}$.

Let $p, q \in V\Gamma$. Consider a vertex path $\alpha = (v_1, v_2, \ldots, v_n)$ from $p$ to $q$, so that $p = v_1$ and $q = v_n$ and $v_i, v_{i+1}$ are adjacent for $i = 1, 2, \ldots, n-1$. Define $\nu(\alpha) = min\{\phi(v_1), \phi(v_2), \ldots, \phi(v_n)\}$, and put $c(p,q)$ to be the maximum of $\nu(\alpha)$ as $\alpha$ ranges over all vertex paths from $p$ to $q$.

This maximum exists since $\nu(\alpha) \le min(\phi(p), \phi(q))$. Note that if $p, q, r \in V\Gamma$, we have

$$c(p,q) \ge min\{c(p,r), c(r,q)\}.$$

This means that $\{c(p,q), c(p,r), c(q,r)\}$ is an isosceles triple (see [C]), i.e. at least two of the values are equal and not greater than the third. If we put $d(p,q) = \phi(p) + \phi(q) - 2c(p,q)$, then $d$ is a pseudometric.Thus $d(p,q) \ge 0$, and for example, if $c(p,q) = c(r,p) \le c(r,q)$, then $\phi(r) \ge c(r,q)$ and so $\phi(r) - c(r,p) \ge c(r,q) - c(r,p)$ and $d(p,r) + d(r,q) = \phi(p) + \phi(q) + 2\phi(r) - 2c(p,r) - 2c(r,q) \ge \phi(p) + \phi(q) + c(r,q) - c(r,p) - c(p,r) - c(r,q) \ge \phi(p) + \phi(q) - 2c(p,q) = d(p,q)$. The other cases are similar.

If we identify all pairs of points $p, q$ for which $d(p,q) = 0$ we get a metric space $M$ (with integer valued distance function) which in fact is a tree. To see this, note that if

$d(p, q) = 0$ then $\phi(p) = \phi(q) = c(p, q)$. Thus $\phi$ takes a single value on each equivalence class of $M$ and so defines an integer valued function - also denoted $\phi$ - on $M$. Let $p, q \in M$. A path in $M$ is a set of points $v_1, v_2, \ldots, v_n$, such that $d(v_i, v_{i+1}) = 1$ for $i = 1, 2, \ldots, n-1$. It is a geodesic if $n = d(v_1, v_n)$. Let $p', q'$ be representives in $V\Gamma$ of $p, q$. Given a path joining $p, q$ in $M$ there is a path in $\Gamma$ joining $p', q'$ for which the minimum value of $\phi$ attained on the path joining $p', q'$ in $\Gamma$ is the same as that joining $p, q$ in $M$. It follows that we obtain a well defined value $c(p, q)$ which is $c(p', q')$ for any representatives $p', q'$ of $p, q$ respectively. Also for any path joining $p, q$ the minimum value of $\phi(v_i)$ is at most $c(p, q)$. A geodesic joining $p, q$ has length $\phi(p) + \phi(q) - 2c(p, q)$. If the minimum value of $\phi(v_i)$ on the geodesic is $m$ then the length of the geodesic is at least $\phi(p) - m + \phi(q) - m$. But $m \leq c(p, q)$. Hence $m = c(p, q)$, and there is exactly one point in the geodesic where this value is attained. If $q, q'$ are adjacent to $p$ in $M$ and $\phi(q) = \phi(q') = \phi(p) - 1$, then $c(p, q) = c(p, q') = \phi(q)$. It follows that $d(q, q') = 0$ and so $q = q'$. The above analysis shows that for any geodesic there is a unique minimal value for $\phi(v_i)$ and the values of $\phi$ decrease monotonically from either end towards this minimal value. However this means that the path traversed is unique. Thus there is exactly one geodesic joining $p, q$. Let $p, q, r \in M$. We know that $c(p, q), c(p, r), c(p, r)$ is an isosceles triple. If $c(p, q) = c(q, r) \leq c(p, r)$ then the unique point of $[p, q]$ where $\phi$ takes the value $c(p, q)$ must be the same as the unique point of $[q, r]$ where $\phi$ takes this value. Either this point is $q$ or the geodesics intersect in another point. Suppose this point is $q$, then it is not hard to see that the point on $[q, r]$ where $\phi$ takes the value $c(p, r)$ is the same as the point on $[r, p]$ where $\phi$ takes the value $c(p, r)$. Thus either this point is $q$ and the union of $[p, q]$ and $[q, r]$ is $[p, r]$ or the geodesics $[p, q], [q, r]$ intersect in points other than $q$. It follows that $M$ is a tree. We can direct the edges of $M$ so that for any edge $e, \phi(\iota e) < \phi(\tau e)$. If we do this every vertex $p$ has at most one edge directed into it, i.e. there is at most one edge for which $\tau e = p$. Construct a tree of groups on $M$ as follows. For each point $p \in M$ we have an infinite cyclic group generated by $z_p$. For each edge $e$ there is an infinite cyclic group generated by $z_e$. We identify $z_e = z_{\tau e}^2, z_e = z_{\iota e}$. Let $N_1$ be the fundamental group of this tree of groups.

Consider a vertex $u \in V\Gamma$. We can extend $\phi$ linearly to $E\Gamma$ and then to the hyperbolic plane, and we regard $\phi$ as defining a height function. The contours in $A$ will be as in Fig 4 and in the neighbourhood of a vertex as in Fig 5. If we start at the centre of $A$ there is a path (proceeding upwards) to the boundary intersecting only translates of edge $a$ and another path intersecting only translates of edge $b$. The value of $\phi$ on these paths is monotone - decreasing for the path intersecting translates of $a$ and increasing for the path intersecting translates of $b$.

Now construct a group $N_2$ as follows. For each $v \in V\Gamma$ there is a copy of $\mathbf{Z}$ generated (multiplicatively) by $z_v$. For each oriented edge $e$ with $\iota e = p$ and $\tau e = q$ we have the relation $z_p^2 = z_q$. In fact $N_1$ and $N_2$ are isomorphic. For suppose $p, q \in V\Gamma$ and $d(p, q) = 0$, which means that $\phi(p) = \phi(q)$ and there is a vertex path joining $p$ and $q$ for which every intermediate vertex $v$ has $\phi(v) \geq \phi(p)$. Using induction on the number of local maxima of $\phi$ as one goes along the path, one proves that $z_p = z_q$.

As remarked above there is an action of $H$ on the hyperbolic plane by isometries, which induces an action on $\Gamma$. If in Fig 5 the edges incident with $u$ are labelled as shown then each of $a, b$ takes $u$ to an adjacent vertex by a hyperbolic isometry. This action now gives an action of $H$ on $N_2$ in which $h^{-1} z_u h = z_{h(u)}$. Thus $a(u) = v, b(u) = w, a^{-1}(u) = x, b^{-1}(u) = y$. It is not hard to see that $N_2$ is isomorphic to $N$. If we put $\phi(u) = 0$, then $\phi(v) = \phi(y) = \phi(v') = \phi(y') = 1$, and $\phi(x) = \phi(w) = \phi(x') = \phi(w') = -1$. Now $\phi(v) = \phi(v') = 1$ but $v, v'$ represent different points of $M$ since $c(v, v') = 0$. To see this note that the shortest path from $v$ to $v'$ passes through $u$ and $\phi(u) = 0$. There is no path from $v$ to $v'$ for which $\phi$ only takes positive values. This is because on either side of the saddle point $u$ between $v$ and $v'$ there are paths to the boundary on which $\phi$ is monotone decreasing. On the other hand, $c(v, y) = 1$ as there is a path from $v$ to $y$ passing round 2 sides of a 4-gon octagon for which $\phi$ takes positive values.

Note this means that $z_v, z_{v'}$ are distinct elements of $N_2$ whose squares are equal to $z_u$. In particular this means that $N$ is not locally cyclic.

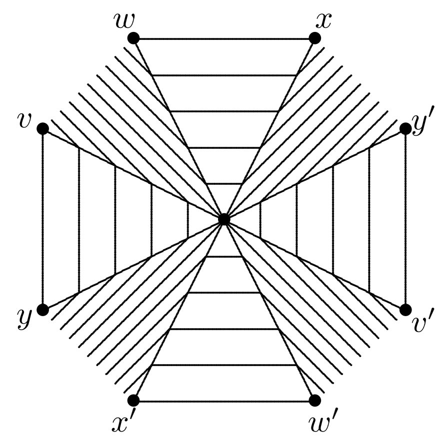



**Fig 5**

Consider now a *track* in the hyperbolic plane. In this context, this means a connected 1-dimensional subset whose intersection with an octagonal region is either empty or it is a segment joining two points in the interior of distinct edges. In particular a track has empty intersection with $V\Gamma$. We will show that a track determines a decomposition of $N$ as a free product with amalgamation.

The complex $X$ of groups is developable, i.e. there is a cell complex $\tilde{X}$ on which $G$ acts so that $X = G \backslash \tilde{X}$ and the complex of groups on $C(X)$ is the one described above. This follows from [H] Theorem 4.1 if we can show that $z$ generates an infinite cyclic subgroup of $G$. But there is a homomorphism of $G$ to $G_1$ in which $z$ is mapped to an element of infinite order and so this is clear.

The group $F$ acts on the hyperbolic plane $\mathbf{H}$ with fundamental region the 4-gon already considered. The quotient $Q = F \backslash \mathbf{H}$ is a torus (with one exceptional point $p$)

An essential simple closed curve in $Q$ avoiding $p$ determines a splitting of the Fuchsian group $F$.

An essential simple closed curve in $Q$, i.e. one which does not bound a disc, lifts to collection of curves in $\mathbf{H}$ invariant under $F$. Since each such curve separates, the dual graph is an $F$-tree. By Bass-Serre theory this determines the splitting of $F$. This is described in detail for a general finitely presented group in [DD], Chapter VI. It is also shown there that any action of a finitely presented group $G$ on a simplicial tree is resolved by such a geometric splitting, i.e. if $S$ is a simplicial $G$-tree then there is a geometric simplicial $G$-tree $T$ for which there is a $G$-morphism $\theta : T \to S$. If $p \in T$ then the stabilizer $G_p$ is a subgroup of $G_{\theta(p)}$. Thus if there is a splitting of $G$ over a slender (small) subgroup then there must be a geometric splitting over a slender (small) subgroup. In our case instead of the Fuchsian group $F$ we have the fundamental group $G$ of a complex of groups corresponding to a complex which is a cell decomposition of a torus. The same theory will work however in this case, if we can show that in any action of $G$ on a simplicial tree each cell group, i.e. a group corresponding to a vertex of $C(X)$, fixes a vertex of the simplicial tree. Thus we need to show that in any action of $G$ on a simplicial tree $z$ does not act hyperbolically.

Suppose $z$ acted hyperbolically with axis $A_z$. Now any two elements of $N$ have a power in common. It follows from [C] Chapter 3, Lemma 1.7, that all elements of $N$ have a common axis $A_z$. Also $A_{gzg^{-1}} = gA_z$ and so $A_z = gA_z$ for every $g \in G$ and we have an abelian action. But $z$ is in the derived subgroup of $G$ and so we have a contradiction.

The cell complex $C(X)$ lifts to a cell complex $C(Y)$ based on the tessellation of the hyperbolic plane described above. We know that $G$ is the fundamental group of the complex of groups specified. Assigning infinite cyclic groups $N(v), N(e)$ to each 0-cell $v$ and 1-cell $e$ of $C(Y)$ so that the maps $N(e) \to N(\iota e)$ commute with the projection to $X$, we obtain a complex of groups whose fundamental group is $N$. Suppose we have a track $t$ in $Y$. Now $t$ separates $Y$ into two pieces $L, R$. We assume that $L, R$ are closed regions so that $L \cap R = t$. We see that $L, R$ and $t$ inherit cell stuctures $Y_R, Y_L, Y_t$ from $Y$. For $t$ the 0-cells are the intersections of $t$ with the 1-cells of $Y$ and the 1-cells are the intersections with the 2-cells of $Y$. The tracks we are interested in are non-compact and intersect each 2-cell of $Y$ at most once.

Assume that $t$ is a track. The cell structures on $R, L$ will include the cell structure on $t$ and also a cell for each cell of $Y$ which has non-trivial intersection with $R$ or $L$ respectively. We get a complex of groups for each of $R, L, t$ by assigning to each cell of $Y_R, Y_L, Y_t$ (apart from the 1-cells of $Y_t$) the group of the corresponding cell of $Y$. Each 1-cell of $Y_t$ is assigned the intersection of its incident 0-cells. Let $N_R, N_L, N_t$ be the fundamental groups of the respective complexes of groups. Then $N = N_R *_{N_t} N_L$. To see this, note that we have shown that $N$ is the fundamental group of a tree of cyclic groups, the vertices of which are equivalence classes of vertices of $\Gamma$. Adjacent vertices in $\Gamma$ which are not in the same equivalence class are adjacent in this tree. The group $N_R$ will be generated by the groups corresponding to the vertices of $\Gamma$ which lie in $R$, which is connected. Thus $N_R$ is the subgroup of $N$ which is the fundamental group of a connected subgraph, i.e. a subtree. The same must be true for $L$ and, hence, also for $t$, since the intersection of two subtrees is a subtree.

Let us consider some examples. The simple closed curve joining the midpoints of the $a$ edges lifts to a track which has a small stabilizer. The stabilizer in $N$ is locally cyclic. It induces a splitting of $G$ as an HNN-group over a small subgroup which is a locally cyclic group extended by a cyclic group. The simple closed curve shown in Fig 6 lifts to an infinite countour in $Y$. It corresponds to a splitting (as an HNN group) over a free abelian rank two group.

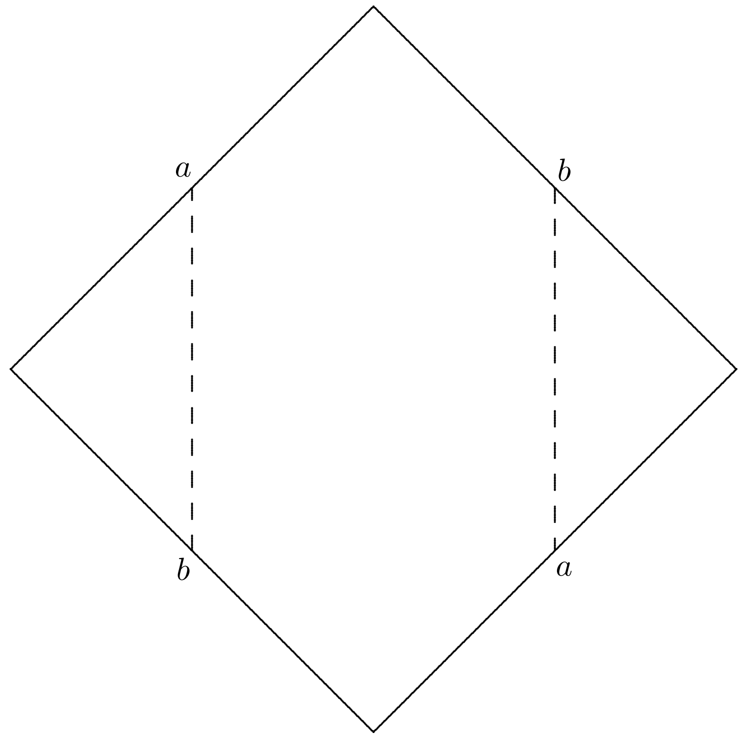



**Fig 6**

Any track, as described above gives a two way infinite sequence of $\phi$ values corresponding to the $\phi$ value of successive 1-cells that it intersects. The sequence for the first example goes $\dots, 0, 1, 2, 3, \dots$ and satisfies the property that the $n+1$-st term is the $n$-th term plus one.

In the second example, shown in Fig 6 the sequence is constant.

We show that $G$ admits actions on $\mathbf{R}$-trees in which point stabilizers are locally cyclic and arc stabilizers are cyclic.

Choose positive real numbers $\alpha, \beta$. Draw our 4-gon $A$ so that the projections on the $x$-axis of the sides have lengths $\alpha, \beta$. Foliate $A$ by vertical lines. This foliation lifts to a foliation both of $Y$ and of $\tilde{X}$. The leaves of the foliation on $Y$ can be regarded as the points of an $\mathbf{R}$-tree. The action of $F$ on this tree is free if the values $\alpha, \beta$ are rationally independent. We show that the set $S$ of leaves of $\tilde{X}$ can also be regarded as the points of an $\mathbf{R}$-tree. There is a transverse measure on $\tilde{X}$, obtained by lifting the $dx$ measure on $X$ which also lifts the measure on $Y$. We get a pseudometric on $S$ by taking the infimum of integrals of the measure along smooth curves joining points in the leaves. We need to show that the distance between distinct leaves is never zero. If two leaves in $\tilde{X}$ project to different leaves in $Y$ then the distance between them is at least as big as the distance in $Y$ which is non zero. Suppose then that we have two leaves which project to the same leaf in $Y$. If this leaf does not contain a vertex then it is a track $t$ and we have seen that a track determines a decomposition of $N$ as a free product with amalgamation and the lifts of $t$ in

$\tilde{X}$ can be regarded as the edges of a simplicial $N$-tree with one orbit of edges. Clearly it suffices to show that tracks which are adjacent edges in this simplicial tree are a non-zero distance apart. Two such tracks are $\tilde{t}$ and $n\tilde{t}$ where $n \in N_R$ or $n \in N_L$. If say $n \in N_R$, then there there is a finite set $J$ of vertices of $Y_R$ such that $n$ is in the subgroups generated by $z_f, f \in J$. We can find a track $t_1$ in the foliation of $Y$ which separates $J$ from $t$. The lifts of $t$ and $t_1$ to $\tilde{X}$ will give a simplicial $N$-tree in which $\tilde{t}$ and $n\tilde{t}$ are no longer adjacent - they will be separated by a lift of $t_1$. But we have seen that a lift of $t$ is always non-zero distance from a lift of $t_1$, and so the argument is complete in this case. If the leaf contains a vertex, it projects to a leaf in $Y$ which contains a vertex. If the values $\alpha, \beta$ are rationally independent then the leaf in $Y$ will contain just one vertex and it will divide the plane up into finitely many regions. There will be a decompositon of $N$ into a tree of groups with one vertex group for the leaf and a vertex for each region. An argument similar to the one above then shows that any two lifts of the leaf to $\tilde{X}$ are a non-zero distance apart.

It follows then as in [LP] that $S$ is an $\mathbf{R}$-tree. If we choose a leaf of the foliation which does not contain a vertex and examine the values of $\phi$ on the edges that it intersects then we obtain a doubly infinite sequence, i.e. a map of $\mathbf{Z}$ into the integers. This sequence will be monotone, whatever the values of $\alpha$ and $\beta$. This means that the stabilizer in $S$ of the point corresponding to such a leaf is locally cyclic. Arc stabilizers will be cyclic since it will contain the intersection of the stabilizers of more than one such point and distinct leaves will have different stabilizers as the corresponding tracks will eventually intersect distinct sets of 1-simplices.

Let us consider the folding sequence corresponding to particular values of $\alpha$ and $\beta$. Let us take $\alpha = 1, \beta = \sqrt{2}$. After the first fold we have new generators $a, ab^{-1} = c$, where the translation length of $c$ is $\gamma = \sqrt{2} - 1$. With these generators $G$ has presentation

$$G =< a, c, z | a^{-1}za = z^2, c^{-1}zc = z > .$$

A further fold gives generators $c, d = ca^{-1}$ where $d$ has length $\delta = 2 - \sqrt{2} = \sqrt{2}\gamma$ and a new presentation

$G =< c, d, z | c^{-1}zc = z, d^{-1}zd = z^2 >$.

Further folding gives a recurring pattern of presentations.

In the next example, which is similar to Example 1, we allow the cone point to have infinite index.

**Example 2**

Let the complex $X$ have one vertex $U$ and two oriented edges $a, b$ which necessarily are loops at $U$ . Let the group of $U$ be $< z, u | uz = zu >$. Let the group of each edge be infinite cyclic generated by $z$. For each edge $e$ the map $G(e) \to G(\tau e)$ is the map $z \mapsto z^2$. There is also 4-sided 2-cell attached by the path corresponding to the word $uba\bar{b}\bar{a}$. The complex $X$ is a torus with one cone point of infinite order. We get a presentation for the fundamental group $G$ of this complex of groups to be

$$G =< a, b | a^{-1}za = b^{-1}zb = z^2, u = aba^{-1}b^{-1}, uz = zu > .$$

Corresponding to this complex we get a tessellation of the hyperbolic plane in which all the vertices are ideal points on the boundary and each region is four sided. Let $\Gamma$ be the

1-skeleton of this tessellation. Let $F =< a, b >$ the free group on 2 generators, which is a subgroup of $G$. The dual graph is a 4-regular tree. Let $\phi : F \to \mathbf{Z}$ be the homomorphism such that $\phi(a) = 1, \phi(b) = 1$.

We can regard $\phi$ as a height function on $\Gamma$ by choosing a particular vertex $p$ to have height 0 and any other vertex $gp$ to have height $\phi(g)$. Let $N$ be the normal closure of $z$ in $G$. As in Example 1 the structure of $N$ is determined by $\phi$ and is a tree product of cyclic groups. If we assign length 1 to $a$ and length $\sqrt{2}$ to $b$, we get a foliation of $X$ which lifts to a foliation of $Y$ the hyperbolic plane and $\tilde{X}$. As in Example 1, we get an unstable action of $G$ on an $\mathbf{R}$-tree with cyclic arc stabilizers, in which the points of the $\mathbf{R}$-tree are leaves of the foliation in $\tilde{X}$. This is because a leaf of the foliation in $Y$ which is a line has a monotone increasing height function.

Note also that restricted to $F$ we have a non-simplicial action of $F$ on an $\mathbf{R}$-tree in which arc stabilizers are trivial, but the commutator $u = aba^{-1}b^{-1}$ fixes a point.

We will see in Section 5 that there are fairly severe constraints on the T2 vertex groups which give rise to unstable actions with cyclic arc stabilizers. In particular the 2-orbifold involved in such an action must be orientable. To see why this is likely to be case we present another example.

**Example 3**

Let

$$G =< a, b, c, d, z | a^{-1}za = z^2, b^{-1}zb = z^2, c^{-1}zc = z^2, d^{-1}zd = z^2, a^2b^2 = c^2d^2 > .$$

As in Example 1 the structure of $N$, the closure of $z$, is determined by a tessellation of the plane by 8-gons and a height function on the vertices. If however we attempt to foliate the plane so that the height function is monotone on leaves and the foliation is invariant under the $G$ action we run into difficulties. Thus in the tessellation the regions can be two-coloured (black and white) so that an element of $g$ takes a region of one colour to a region with the same colour if and only if it is orientation preserving. If we assign lengths $\alpha, \beta, \gamma, \delta$ to $a, b, c, d$ respectively so that $\alpha + \beta = \gamma + \delta$, there will be an induced a foliation of each region and hence of the entire plane. Taking $\alpha = \delta = 1$ and $\beta = \gamma = \sqrt{2}$, for example, will give a folding sequence which does not terminate as in Example 1. In a particular region the leaves going downwards will all be monotone with regard to the height function $\phi$ however if the function is decreasing in a white region then it will be increasing in a black region. The foliation does correspond to a non-simplicial action on an $\mathbf{R}$-tree, but arc stabilizers will not be small, let alone cyclic.

Consider the general situation for a T2 vertex group. We restrict attention to the case when arc stabilizers are cyclic. It may be possible to generalize the argument to the case when arc stabilizers are 2-ended or even polycyclic-by-finite. Thus we take $G$ to be the fundamental group of a complex of groups in which the underlying complex is a 2-orbifold. We assume that there is one 2-cell and that the complex resolves an action of $G$ on an $\mathbf{R}$-tree $T$ with cyclic arc stabilizers. Thus all edge groups in the complex of groups are cyclic. Let $N$ be the normal closure in $G$ of the edge groups. Then $G/N$ is a 2-orbifold group.

Let $z$ be an element which lies in one of the edge groups. Now $G$ is the commensurizer of $z$ and as in [K] there is a homomorphism $\rho$ from $G$ to the multiplicative group of the rationals in which $\rho(x) = m/n$ if $x^{-1}z^m x = z^n$. We consider first the case when the image of $\rho$ lies in the cyclic subgroup of the rationals generated by a particular prime $p$.

Thus $\rho$ takes values $p^n, n \in \mathbf{Z}$ and there is a homomorphism $\phi : G \to \mathbf{Z}, \phi(x) = n$

As in the example the structure of $N$ is a tree product. Thus $N$ has a set of generators $z_u, z_e, u \in V\Gamma, e \in E\Gamma$ and for each edge $e$, with vertices $u, v$ there are relations $z_e = z_u^{p^a} = z_v^{p^b}$ where $a, b$ are integers depending on the $G$-orbits of $e, u, v$. For each vertex $u$ all the incident edges have edge groups which are totally ordered under inclusion, if we assume that $z_u$ generates the same group as the largest of the incident edge groups. Thus if $e, f$ are edges incident with $u$, then for some integer $k = k(e, f)$ we have $x_f^{p^k} = z_e$.

Let $e$ be a fixed edge of $\Gamma$. Let $e_1, e_2, \ldots, e_n$ be an edge path with $e = e_1$. Let $\phi(e) = 0$ and let $\phi(e_n) = k(e_1, e_2) + k(e_2, e_3) + \ldots k(e_{n-1}, e_n)$. This is well defined, since the relation corresponding to a path going round the fundamental region ensures that the sum of the $k(e_i, e_{i+1}$'s for any loop is zero. We extend $\phi$ to $V\Gamma$ by defining $\phi(u)$ to be the maximum value of $\phi(e)$ as $e$ ranges over the edges incident with $e$. It is easy to check that this is consistent with the way we defined $\phi$ as a homomorphism, in that $\phi(gu) = \phi(g)$. As in our example we can extend $\phi$ to get a height function on the whole hyperbolic plane. Specifically we do this by triangulating the fundamental region and then extending $\phi$ linearly first to edges and then triangular faces. Now the argument given for the example works in this situation to show that $N$ is a tree product of cyclic groups.

In the contour map, each infinite contour determines a decomposition of $G$ over a free abelian rank 2 subgroup. There will be such contours unless the height function is bounded - in which case the action on $T$ is stable. For if not, i.e. the height function is not bounded and all contours are compact, then there will be contours containing arbitrarily large numbers of vertices. Under the action of $G/N$ this contour will be mapped onto itself or disjoint from itself. Two vertices inside this contour which are in the same $G/N$ orbit must have the same height, since the isometry taking one point to the other must make the contours intersect. This means that the height function is bounded on the plane, which is a contradiction.

If $C(X)$ is the complex of groups in which the underlying complex is the 2-orbifold, then this lifts to a complex of groups $C(Y)$ in which the underlying complex is the tessellation of the hyperbolic plane. The resolution of $T$ gives a foliation of $Y$. The intersection of each leaf of this foliation with the 1-skeleton of $Y$ determines a sequence of cyclic subgroups of $N$ which are totally ordered by inclusion. For otherwise by considering a small neighbourhood of a relevant part of the leaf one can show that there is an arc of $T$ whose stabilizer is not cyclic.

Consider now another example.

**Example 4**

Let

$$J = \langle a, b, c, d, z | (abcd = dcba, a^{-1}za = z^6, b^{-1}zb = z^{18}, c^{-1}zc = z^6, d^{-1}zd = z^{18} \rangle.$$

Again this group has a normal subgroup $N$ which is the normal closure of $z$ and $J/N$ is an orientable surface of genus two. The structure of $N$ is determined by two height functions on the hyperbolic plane that is tessellated by 8-gons. If $\Gamma$ is the 1-skeleton of this tessellation, then $N$ is generated by elements $z_v$ corresponding to the vertices of the tessellation. For any two vertices $u, v$ there are integers $a, b$ such that $z_u^a = z_v^b$. In our case $a/b$ is in the subgroup of multiplicative group of positive rationals generated by 2 and 3. Let $u$ be a fixed vertex, then we can define the two height functions $\phi_2, \phi_3$ by taking $\phi_2(v) = \alpha, \phi_3(v) = \beta$, where $a/b = 2^\alpha 3^\beta$. For $\phi_2$ there is an increase of 1 along each edge of the 8-gon going from left to right and the contours are as illustrated in Fig 7.

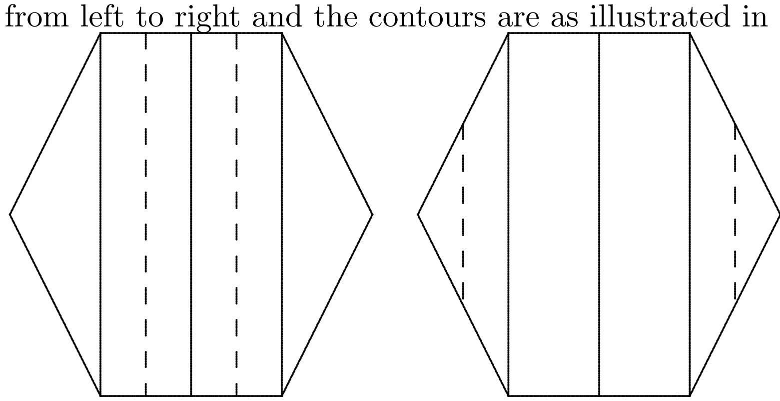


Fig 7

In the height function $\phi_3$, if the left hand vertex $u$ has height 0 then the vertices along the top - starting at $u$ - have heights $0, 1, 3, 4, 6$ and along the bottom $0, 2, 3, 5, 6$ and there is a contour as illustrated in Fig 8

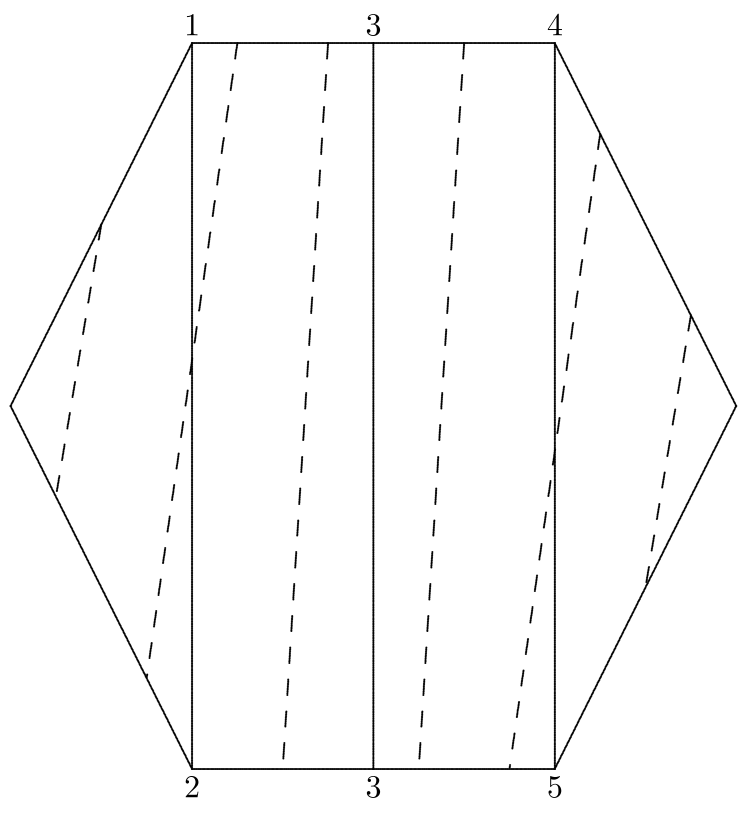



Fig 8

There is no track which is a contour for both height functions and so this group does not split over $\mathbf{Z} \oplus \mathbf{Z}$.

This group does have an action on an $\mathbf{R}$-tree with cyclic arc stabilizers. As in our earlier example, choose positive real numbers $\alpha, \beta, \gamma, \delta$. Draw our octagon $A$ so that the projections on the $x$-axis of the sides have lengths $\alpha, \beta$ etc. Foliate $A$ by vertical lines. This foliation lifts to a foliation both of $Y$ and of $\tilde{X}$ - here we use the notation of the earlier example. The leaves of the foliation on $Y$ can be regarded as the points of an $\mathbf{R}$-tree. The action of $J/N$ on this tree is free if the values $\alpha, \beta, \gamma, \delta$ are rationally independent. As in the earlier example the set $S$ of leaves of $\tilde{X}$ can also be regarded as the points of an $\mathbf{R}$-tree. There is a transverse measure on $\tilde{X}$, obtained by lifting the $dx$ measure on $X$ which also lifts the measure on $Y$. We get a pseudometric on $S$ by taking the infimum of integrals of the measure along smooth curves joining points in the leaves. Exactly as in the earlier example one can show that the distance between distinct leaves is never zero.

We now show that we can choose the values $\alpha, \beta, \gamma, \delta$ so that in the action of $J$ on $S$ point stabilizers are locally cyclic and arc stabilizers are cyclic. What we want to arrange is that the sequence of values of both $\phi_2$ and $\phi_3$ on each track of the foliation of $Y$ is monotone and that the direction of increase is the same for both functions as one goes along the track. This is the case if each of the three vertices in the top half of the polygon $A$ is to right of the corresponding vertex in the bottom half of the polygon, i.e. if $\alpha > \delta, \alpha + \beta > \delta + \gamma$. It is clear that $\alpha, \beta, \gamma, \delta$ can be chosen to be rationally independent and to satisfy these inequalities. An arc stabilizer must be contained in the intersection of stabilizers of two distinct points and it is not hard to see that this must be cyclic, as the corresponding tracks will eventually intersect distinct sets of 1-simplices.

Note that although $J$ does not split over a slender subgroup, it does split over a small subgroup and in fact there is a sequence of splittings over small subgroups which approximate the action described above.

**Example 5**

Let the complex $X$ have two vertices $U, V$ and three oriented edges $a, b, c$ where $a$ is a loop at $U$ and $\iota b = \tau c = U$ and $\tau b = \iota c = V$. Let the group of $U$ be $< z, u | uz = zu, u^2 = 1 >$, and the group of $V$ be $< z, v | vz = zv, v^3 = 1 >$. Let the group of each edge is generated by $z$. For each edge $e$ the map $G(e) \to G(\tau e)$ is the map $z \mapsto z^2$. There is also 6-sided 2-cell attached by the path corresponding to the word

$$abc\bar{a}\bar{c}v\bar{b}u.$$

The complex $X$ is a torus with two cone points. If we take a spanning tree for the 1-skeleton of this complex to consist of the edge $c$ and its vertices, we get a presentation for the fundamental group of this complex of groups to be

$$K =< a, b, d, e, f, u, v, z | aba^{-1}vb^{-1}u = u^3 = v^2 = 1, a^{-1}za = z^2, b^{-1}zb = z^4,$$

$$uz = zu, vz = zv > .$$

This example is similar to Example 1, except that we have put an extra vertex on the $b$ edge of Example 1. In the corresponding tessellation of the hyperbolic plane the fundamental region is a 6-gon. There are two orbits of vertices. Each vertex in the orbit corresponding to $U$ is incident with 8 edges and each vertex in the orbit corresponding to $V$ is incident with 6 edges. There is a single height functions $\phi_2$. We fix a particular vertex $p$ of $V\Gamma$ to be 0 and for any other vertex $q$ the height is determined by considering a path from $p$ to $q$ in which the height increases by 1 for an edge which is a lift of $a, b, c$ with the positive orientation and $-1$ for an edge with the reverse orientation.

**Example 6**

Let the complex $X$ have four vertices $U, V, W, Y$ and three oriented edges $a, b, c$. Let $\iota a = U, \tau a = V, \iota b = V, \tau b = W, \iota c = U, \tau c = Y$. Let the groups of $U, V, W, Y$ be finite cyclic of order 3 and generated by $u, v, w, y$ respectively. Let the 6-sided 2-cell be attached via the word

$$abw\bar{b}v\bar{a}ucy\bar{c}.$$

In this case $X$ is a 2-sphere with 4 cone points. Consider the folding sequence corresponding to the values $\alpha = 1, \beta = \gamma = \sqrt{2}/2$, which are assigned to the edges $a, b, c$ respectively. Initially we have the 2-cell attached along $abw\bar{b}v\bar{a}ucy\bar{c}$, where the left hand vertex is $U$ and the right hand vertex is $V$. The total length along the top or bottom is $1+\sqrt{2}$. After the first subdivision and fold we have a new edge $d$ with length $1-\sqrt{2}/2$ and the attaching word has become $dbw\bar{b}v\bar{d}(uyu^{-1})\bar{c}uc$. After the next subdivision and fold we have a new edge $e$ with length $\sqrt{2}/2-(1-\sqrt{2}/2) = \sqrt{2}-1$ and the attaching word has become $bw\bar{b}v\bar{d}(uyu^{-1})deu\bar{e}$ and after the next subdivision and fold we have a new edge $g$ with length $1-\sqrt{2}/2$ and the attaching word is $gw\bar{g}\bar{e}v\bar{d}(uyu^{-1})deu = gw\bar{g}(w^{-1}uw)\bar{e}v\bar{d}((uyu^{-1})de$. But note that $(1-\sqrt{2})\sqrt{2}/2 = 1-\sqrt{2}/2$ and the situation we have reached is the initial situation scaled by $\sqrt{2}-1$. In the corresponding tessellation and foliation of the hyperbolic plane $\mathcal{H}$ there are 4 orbits of vertices, Each vertex in the orbits corresponding to $U$ and $V$ is incident with 12 edges. Each vertex in the orbits corresponding to $W$ and $Y$ is incident with 6 edges. Since we have an infinite folding sequence, in the corresponding interval exchange each orbit is dense and so there are no compact regular leaves. Also it can be seen from the folding sequence that the lift of an upper (or lower) semi-circle to the Cayley graph gets subdivided in the folding sequence but does not get folded, and so no leaf in the plane intersects this lift more than once. Clearly a leaf in the plane must separate. The leaf space made Hausdorff is an **R**-tree, for example by [LP]. In fact the leaf space is already Hausdorff. This is because distinct leaves are a non-zero distance apart. This is clear for leaves which intersect a fundamental region, where the distance is the difference in the projections on the $x$-axis. It is also true for leaves $\ell_1, \ell_2$ which do not intersect the same fundamental region, as one can find a leaf $\ell$ such that $\ell_1$ and $\ell_2$ lie in the distinct components of $\mathcal{H}-\ell$ and $\ell_1$ and $\ell$ do intersect the same fundamental region. The distance of $\ell_1$ and $\ell_2$ will be greater than the distance between $\ell$ and $\ell_1$. The orbifold group $G$ has presentation - obtained by taking a spanning tree of $X$ with edges $a, b, c$ -

$$G =< u, v, w, y | u^3 = v^3 = w^3 = y^3 = uwvy = 1 > .$$

We have described a non-simplicial action of $G$ on an **R**-tree with trivial arc stabilizers. We will see that there is no unstable action in which $X$ is a 2-sphere with 4 cone points.

We will see that for there to be such an action there has to be a non-zero vector field on the underlying surface. There is no such vector field on the 2-sphere and so there is no unstable action.

## 4. Actions with cyclic arc stabilizers

Let $G$ be a finitely presented group which has an action on an $\mathbf{R}$-tree $T$ with infinite cyclic arc stabilizers. Suppose we are in the T2 situation arrived at in the proof of Theorem 1. Thus $G$ is the fundamental group of a complex of groups in which the underlying complex $X$ is a hyperbolic 2-orbifold and edge groups are infinite cyclic. Let $H$ be the subgroup of $G$, which is the fundamental group of the 2-orbifold. There is a singular foliation of $X$ as in Fig 2. The upper semi-circle corresponds to a word $w$ and the lower semi-circle to a word $w'$ and every edge in the 1-skeleton of $X$ occurs exactly twice in $w \cup w'$ Suppose the action on $T$ is unstable. We aim to show that the orbifold is orientable and each edge in the defining word of the orbifold occurs once in both the upper semi-circle and the lower semi-circle. In all the examples given this was the case, but as yet we have not ruled out the possibility that an edge occurs twice in $w$ and not at all in $w'$. Note that the total length of edges which occur twice in $w$ must be equal to the total length of edges occurring twice in $w'$.

Consider first the case when $\rho$ takes values in the cyclic subgroup of the rationals generated by the prime $p$. In this case the structure of $N$ is determined by the height function $\phi$ as in Example 1. Let $u, v$ be two vertices of $Y$ on which $\phi$ takes the same value. Then $G_u = G_v$ if and only if there is a path from $u$ to $v$ in the 1-skeleton of $Y$ on which the value of the height function $\phi$ never falls below $\phi(u) = \phi(v)$. A track in $Y$ determines a decomposition of $N$ as a free product with amalgamation.

First we show that the singular foliation of the surface associated with the action is oriented, i.e. it is the foliation associated with a vector field which is non-zero except at the finitely many singuarities. First we do this for the foliation of the hyperbolic plane which is the lift of the foliation of $X$ .

Each leaf $\ell$ which does not include a vertex is an infinite track. Such leaves will certainly exist. The other leaves will each contain a single vertex. For each edge $e$ that $\ell$ intersects, the group of $e$ will lie in the stabilizer of an arc of the tree which contains the point corresponding to $\ell$, the size of the arc will depend on $e$. As we are dealing with the case when indices are powers of a fixed prime $p$ this means that the stabilizers of edges intersected by $\ell$ are totally ordered. Choose an arbitrary point $x \in \ell$. Removing $x$ from $\ell$ separates $\ell$ into two components $L$ and $R$. We claim that the value of $\phi$ on edges intersected by exactly one of $L$ and $R$ is bounded by a fixed amount more than the value at $x$. The fixed amount being the maximum difference between the values of $\phi$ on the edges of a fundamental region. For any vertex $u$ of $\Gamma$ proceeding away from $u$ along at least half of the incident edges results in no increase of the height function $\phi$. It is not hard to see that starting from any vertex near $x$ there is an infinite path not intersecting $\ell$ for which the value of $\phi$ is non-increasing. If there are edges intersected by $L$ and $R$ for which the values of $\phi$ are higher than the value at $x$ then there would have to be a path in the 1-skeleton of $Y$ connecting these two points and maintaining a high value of $\phi$. However such a path would have to intersect a path leading away from $x$ on which the height function is non-decreasing, which is a contradiction. Put an arrow on $\ell$ in the direction of $L$ if and only if $\phi$ takes unbounded values on $L$ and in the direction of $R$ if it takes unbounded values on $R$. On leaves on which the values of $\phi$ are bounded there are no arrows. Suppose that the leaf $\ell$ does get oriented in the above process. Consider the image of $\ell$ in $X$. If this image is compact, then the stabilizer of $\ell$ will be a soluble Baumslag-Solitar group and $G$ will split over this group. If the image is not compact and its closure in $X$ is $C$ then $\ell$ will be dense in $C$, and the boundary of $C$ - if it is non-empty - will consist of compact singular leaves ([LP, Proposition 1.5]. These boundary leaves can be contracted using Move 3 a finite number of times. Thus we can assume that $\ell$ is dense in $X$. We want to show that the arrows on the image of $\ell$ all point up or down. To see that this must be the case consider a point $x$ in $X$ which is not a vertex . We show that there is a neighbourhood $U$ of $x$ in $X$ in which each component of $U \cap \ell$ also has its arrow pointing up or all pointing down. Let $\bar{x}$ be a lift of $x$. Consider translates of $\ell$ that contain points close to $\bar{x}$. Since the image of $\ell$ is dense in $X$ there are translates of $\ell$ containing points arbitrarily close to $\bar{x}$. The leaf of the foliation of $Y$ containing $\bar{x}$ contains at most one vertex. There will be segments containing $\bar{x}$ of arbitrary length in this leaf which do not contain a vertex. Let $s$ be such a segment. A leaf containing a point sufficiently close to $\bar{x}$ will contain a segment parallel to $s$, i.e. it will intersect the same set of edges of $Y$. Thus the values of $\phi$ on this segment will be the same as the values on $s$ and the same as any other segment close to $s$. in particular a translate of $g\ell$ containing points sufficiently close to $\bar{x}$ will contain such a segment. By choosing $s$ long enough, there will be a segment in a translate $\ell$ parallel to $s$ on which $\phi$ varies by more than the maximum variation in $\phi$ on the edges of a fundamental region. On such a segment the direction of the arrow is determined. Thus all the arrows in a translate of $\ell$ close to $\bar{x}$ are oriented coherently. It follows then that in $X$ there is a neighbourhood of $x$ in which the arrows on the image of $\ell$ all point up or all point down. Suppose they all point up. This means that all the arrows on the image of $\ell$ must be up. If the same edge occurred twice in $w$ (or in $w'$), then the arrows could not all point in the same direction. It follows that every edge occurs exactly once in $w$ and exactly once in $w'$.

We can also see that $X$ must be orientable. For suppose that the action of $g$ on $Y$ is orientation reversing. Then $g$ must preserve the orientation of leaves. However it changes the height function from $\phi$ to $-\phi$ which means that $g$ must reverse the orientation of a leaf.

The stabilizers of different edges of $Y$ are commensurable. Let $S =< z >$ be such a stabilizer. For each $g \in G$, there are integers $m, n$ such that $g^{-1}z^m g = z^n$. The map $g \mapsto m/n$ is a homomorphism from $G$ to the multiplicative group of the non-zero rationals

Suppose now that $\rho$ takes values is the multiplicative subgroup $< P >$ of the rationals $\mathbf{Q}$ generated by the finite set of primes $P$, let $p \in P$ and let $Q = \{q \in P | q \neq p\}$. Let $\mathbf{Z}_P, \mathbf{Z}_Q$ be the additive group consisting of the rationals $m/n$ where $m$ is an integer and $n$ is an integer whose only prime divisors are in $P, Q$ respectively. There is a homomorphism $\phi_p : G \to \mathbf{Z}$, such that $\phi_p(g) = \nu$ if $\rho(g) = p^\nu m/n$ where $m, n$ are integers coprime to $p$. We will now justify defining $\phi_p$ in this way as it has already been used to denote a height function on $Y$. Choose a spanning tree $Q$ in the 1-skeleton of $X$ and lift this to a subtree $\bar{Q}$ in $Y$ which projects bijectively onto $Q$. Thus $V\bar{Q}$ is a transversal for the action of $H$ on $Y$. Define the height function $\phi_p : VY \to \mathbf{Z}$ so that it is 0 on $\bar{Q}$ and for any other vertex

$v \in VY$ such that $v = hv', v' \in V\bar{Q}, , \phi_p(v) = \phi_p(hv') = \phi_p(h)$. We can then extend $\phi_p$ to $EY$ by defining $\phi_p(e)$ to be the minimum of $\phi_p(\iota e)$ and $\phi_p(\tau e)$. Let $G_1$ be the splitting extension of $\mathbf{Z}_P$ by $< P >$ in which the automorphism of $\mathbf{Z}_P$ induced by an element of $< P >$ is multiplication by that element. We can regard $G_1$ as the group of $2 \times 2$ matrices

$$\begin{pmatrix} x & m \\ 0 & 1 \end{pmatrix}$$

where $x \in < P >, m \in \mathbf{Z}_P$ There is a homomorphism from $G$ to $G_1$ in which

$$z \mapsto \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad h \mapsto \begin{pmatrix} \rho(h) & 0 \\ 0 & 1 \end{pmatrix} \quad h \in H$$

Consider the complex of groups in which the underlying complex $X$ is the same as that of $G$. However instead of all the edge and vertex groups being cyclic, as in $C(X)$, in the new complex $C_p(X)$ they are copies of $\mathbf{Z}_Q$. We also require that there are inclusion maps from the vertex groups and edge groups of $C(X)$ to those of $C_p(X)$ such that for each oriented edge $e \in EX$ the diagram

$$\begin{array}{ccc} G(e) & \longrightarrow & G(\iota e) \\ \downarrow & & \downarrow \\ G_p(e) & \longrightarrow & G_p(\iota e) \end{array}$$

commutes.

In Example 4, $P = \{2.3\}$ and $\rho : J \to \mathbf{Q}, \rho(a) = 6, \rho(b) = 18, \rho(c) = 6, \rho(d) = 18$. If we take $p = 2$, then $Q = \{3\}$ and $C_p(X)$ is the complex of groups in which every edge and vertex group is a copy of $A = \mathbf{Z}_{(3)} = \{m/3^n | m, n \in \mathbf{Z}\}$.

The inclusion maps for edge groups in vertex groups in both $C(X)$ and $C_p(X)$ are as subgroups of finite index The fundamental group $G_p$ of $C_p(X)$ commensurizes each vertex group and there is a homomorphism $\rho_p : G_p \to \mathbf{Q}$ in which conjugation by $x$ induces multiplication by $\rho_p(x)$ on some finite index subgroup of each vertex group. The homomorphism $\rho : G \to \mathbf{Q}$ factors through $\rho_p$. Let $N'$ be the subgroup of $G_p$ which is the normal closure of the vertex groups, so that $G_p/N'$ is the fundamental group of the underlying hyperbolic orbifold. The structure of $N'$ is a tree product of copies of $\mathbf{Z}_Q$ and is determined by a height function $\phi_p$ on the hyperbolic plane just as is the case for when $P$ consists of a single prime. Thus $N'$ acts on the hyperbolic plane and if $v \in V\Gamma$, then the stabilizer $N'_v$ of $v$ is isomorphic to $\mathbf{Z}_Q$ and if $u$ is an adjacent vertex, then $N'_u = p^\alpha N'_v$, where $\alpha = \phi_p(u) - \phi_p(v)$. In Example 4 if $u, v$ are the vertices of a lift of $b$ for example, then $\phi_2(v) - \phi_2(u) = 1$ while $\phi_3(v) - \phi_3(u) = 2$, and for $p = 2, N'_u = 18N'_v = 2N'_v$ since $N' = \mathbf{Z}_{(3)}$ and so $9N' = N'$.

We assign a positive real number to each edge. This determines a foliation on the disc $B$ as indicated in Fig . We now associate a number $\Omega_p$ with this foliation - the sign of $\Omega_p$ will indicate in which direction the arrows point. Each edge $e$ has been assigned a length $\alpha(e)$. Also $e$ occurs once in the upper semi-circle and once in the lower semi-circle. Let $\bar{e}, \underline{e}$ be the lifts of $e$ in the top and bottom of a fundamental region in $Y$ Let $n(e) = \phi_p(\bar{e}) - \phi_p(\underline{e})$, and let $\Omega_p = \Sigma_e n(e)\alpha(e)$. It can be seen that $\Omega_p$ measures the difference in height of the upper semi-circle and the lower semi-circle.

We will show that our Moves 1 and 2 leave $\Omega_p$ unchanged. While Move 3 replaces the disc by two discs for which the sum of the two $\Omega_p$'s is the same as that of the original disc. It is easy to see that the only non-trivial case is Move 2 and even this case is straightforward. Let $e_1$ be the first edge in $w_1$ and let the value of $\phi$ on this occurrence of $e_1$ be $h_1$ and let $e_2$ be the first edge in $w_2$ with corresponding height $h_2$. After subdivision we may assume that they have the same length $\alpha(e_1) = \alpha(e_2) = \alpha$ assigned. Note that $e_1$ also occurs once in $w_2$ and $e_2$ also occurs once in $w_1$. Let $h_1', h_2'$ be the corresponding heights, so that $n(e_1) = h_1 - h_1'$ and $n(e_2) = h_2' - h_1$ The contribution to $\Omega_p$ involving $e_1$ and $e_2$ is $\alpha(n(e_1) - n(e_2))$. Suppose $h_1 \geq h_2$. After folding $e_1$ and $e_2$ become the same edge $e$ and its associated group in $C(p)$ will be the larger of the two subgroups $G_p(e_1), G_p(e_2)$, i.e. $G_p(e_1)$. To see how $\Omega_p$ is changed by folding, we have to work out $n(e)$. Now in the top semi-circle $e$ is the old $e_2$ and its height $h_2'$ has increased by $h_1 - h_2$. The edge $e$ in the bottom semi-circle is the old $e_1$ and its height is not changed. Thus $n(e) = h_2' + (h_1 - h_2) - h_1' = n(e_1) - n(e_2)$ and so $\Omega_p$ is unchanged. The argument when $h_2 > h_1$ is similar.

Let $e_1, e_2, \ldots, e_k$ be the edges in $w_1$. They are also the edges of $w_2$, since each edge involved occurs exactly once in $w_1$ and $w_2$. For each $p \in P$ we obtain a vector $\mathbf{n}_p = (n(e_1), n(e_2), \ldots, n(e_k))$ of integers. Let $\mathbf{a} = (\alpha_1, \alpha_2, \ldots, \alpha_k)$ be a $k$-vector of positive real numbers. As we have seen above this provides a foliation of $X$ and the arrows given by the height function given by $\mathbf{n}_p$ point upwards if $\Omega_p = \mathbf{n}_p.\mathbf{a} > 0$.

We will show that the vector $\mathbf{a}$ is associated with an action of $G$ on an $\mathbf{R}$-tree if and only each of the vectors $\mathbf{n}_p$ lies in the half-space $H_{\mathbf{a}} = \{\mathbf{x} = (x_1, x_2, \ldots, x_k) | \ \mathbf{x}.\mathbf{a} > 0\}$ or they all lie in half-space $H_{-\mathbf{a}} = \{\mathbf{x} = (x_1, x_2, \ldots, x_k) | \ \mathbf{x}.\mathbf{a} < 0\}$.

In Example 4 $\mathbf{n}_2 = (3, 1, -1, -3)$ and $\mathbf{n}_3 = (5, 2, -1, -4)$ and $\mathbf{a} = (\alpha, \beta, \gamma, \delta)$ needs to satisfy $3\alpha + \beta - \gamma - 3\delta > 0$ and $5\alpha + 2\beta - \gamma - 4\delta > 0$ (or both terms negative). This is certainly the case if $\alpha > \delta$ and $\alpha + \beta > \gamma + \delta$ since $3\alpha + \beta - \gamma - 3\delta = 2(\alpha - \delta) + (\alpha + \beta - \gamma - \delta > 0$ and $5\alpha + 2\beta - \gamma - 4\delta = 3(\alpha - \delta) + 2(\alpha + \beta - \gamma - \delta) + \gamma + \delta > 0$.

In general if all the vectors $\mathbf{n}_p$ lie in $H_{\mathbf{a}}$, then in $Y$ the stabilizers of the 1-cells that $\ell$ intersects will be totally ordered, since $\phi_p$ will be increasing in the same direction for each $p$. This means that the stabilizer of $\ell$ is locally cyclic and we have an action on an $\mathbf{R}$-tree with cyclic arc stabilizers.

Conversely if exactly one of $\mathbf{n}_p, \mathbf{n}_q$ lies in $H_{\mathbf{a}}$ then the height functions $\phi_p, \phi_q$ will increase in opposite directions along a leaf $\ell$ in Y. We will show that it intersects 1-cells whose stabilizers do not commute. As we have seen in Example 1, we can regard $N$ as the fundamental group of a graph of groups with $Y$ as the underlying complex. This graph of groups is developable and in fact $\tilde{Y} = \tilde{X}$, so that $N$ acts on $\tilde{X}$ and $N \backslash \tilde{X} = Y$ and the groups labeling the cells of $Y$ are the stabilizers of appropriately chosen lifts of the cells in $\tilde{X}$. A track $t$ in $Y$ lifts to a pattern of tracks in $\tilde{X}$. Since $\tilde{X}$ is simply connected, a track in $\tilde{X}$ separates, and the pattern of tracks in $\tilde{X}$ determines an $N$-tree (see [DD, Chapter VI]). In fact $N$ will decompose as a free product with amalgamation $N = N_L *_{N_t} N_R$ where $N_L$ is generated by the groups labelling vertices to the left of $t$, $N_R$ is generated by groups labelling vertices to the right of $t$ and $N_t$ is generated by groups labelling the 1-cells intersected by $t$. It is reasonably clear from the hyperbolic geometry of $Y$ that one can find a contour $t_p$ for $\phi_p$ and a contour $t_q$ of $\phi_q$ which are disjoint but each of which intersects $\ell$. Let $s$ be a segment of $\ell$ containing these intersections. We can choose $s$ so that its end points $P, Q$ are on 1-cells, with $P$ closer to the intersection with $t_p$ than the intersection with $t_q$. We can also choose the contours and $s$ so that the value of $\phi_p$ at $P$ is higher than its value on $t_p$, and the value of $\phi_q$ at $Q$ is higher than its value on $t_q$. Now each of $t_p, t_q$ will determine a decomposition of $N$ as a free product with amalgamation. Suppose there is a cyclic subgroup $C =< c >$ of $N$ which contains both the subgroups labelling the 1-cells containing $P$ and $Q$. On the Bass-Serre tree corresponding to any decomposition of $N$ determined by a track $t$ the element $c$ must fix a vertex, since it has a power which fixes a vertex. In particular in the decomposition given by $t_p$ it must lie in the subgroup generated by the labels on the same side of $t_p$ as $P$. Similarly it must lie in the subgroup generated by the labels on the same side of $t_q$ as $Q$. But these regions are disjoint, and the intersection of the corresponding subgroups lies in both $N_{t_p}$ and $N_{t_q}$. Also $\phi_p$ takes a fixed value $\gamma$ on $t_p$, and we have chosen $c$ so that $\phi_p(c) > \gamma$. We have a contradiction and so we have an action on an $\mathbf{R}$-tree with cyclic arc stabilizers if and only if all the vectors $\mathbf{n}_p$ lie in $H_{\mathbf{a}}$ or they all lie in $H_{-\mathbf{a}}$.

Let $X$ be a compact orientable hyperbolic 2-orbifold with $k$ cone points of finite index and $m$ punctures (or cone points of infinite index), which is given as a cell complex. The 2-orbifolds that we exclude are the 2-spheres with at most three cone points and the torus with no cone points. Let $H$ be the fundamental group of $X$. Then $H$ has a presentation in the form

$$H = gp < a_1, a_2, \ldots, a_{2n}, b_1, b_2, \ldots b_k \mid a_1 a_2 \ldots a_{2n} a_1^{-1} a_2^{-1} \ldots a_{2n}^{-1} b_1 b_2 \ldots b_k, c_1, c_2, \ldots, c_m$$

$$= b_1^{r_1} = b_2^{r_2} = \ldots = b_k^{r_k} = 1 >,$$

where $r_i > 1$ and either $n = 0$ and $k \geq 4$ or $n = 1$ and $k \geq 1$ or $n \geq 2$ and $k \geq 0$.

Let $\rho$ be a homomorphism from $H$ to the multiplicative group of non-zero rationals and let $P$ be the smallest set of primes which generates a subgroup containing the image of $\rho$. For each prime $p \in P$, let $\phi_p : G \to \mathbf{Z}$ be the homomorphism such that $\phi_p(g) = \nu$ if $\rho(g) = p^\nu m/n$ where $m, n$ are integers coprime to $p$. For each $p \in P$ there is a $2n$-vector of integers

$$\mathbf{n}_p = (\phi_p(a_1), \phi_p(a_2), \ldots, \phi_p(a_{2n})),$$

and an $m$-vector

$$\mathbf{m}_p = (\phi_p(c_1), \phi_p(c_2), \ldots, \phi_p(c_m)),$$

such that the sum of the coefficients is zero, i.e $\mathbf{m}_p.(1, 1, \ldots, 1) = 0$, and these vectors determine $\rho$. When $X$ is the 2-sphere with any number of cone points, the map $\rho$ is trivial .

Let $G$ be the group with presentation

$$G = gp < a_1, a_2, \ldots, a_{2n}, b_1, b_2 \ldots, b_k, z \mid (a_1 a_2 \ldots a_{2n} a_1^{-1} a_2^{-1} \ldots a_{2n}^{-1} b_1 b_2 \ldots b_k$$

$$= b_1^{r_1} = b_2^{r_2} = \ldots = b_k^{r_k} = 1, a_i^{-i} z^s a_i = z^t \text{ if } \rho(a_i) = s/t, gcd(s,t) = 1,$$

$$c_i^{-i} z^s c_i = z^t \text{ if } \rho(a_i) = s/t, gcd(s,t) = 1 > .$$

Then $G$ has an unstable action on an $\mathbf{R}$-tree with cyclic arc stabilizers if and only if $n > 0$ and there is a $2n$-tuple of positive reals $\mathbf{a} = (\alpha_1, \alpha_2, \ldots, \alpha_{2n})$ such that $\mathbf{a}.\mathbf{n}_p > 0$ for each $p \in P$ or $\mathbf{a}.\mathbf{n}_p < 0$ for each $p \in P$. One gets such an action by assigning the lengths $\alpha_i$ to 1-cells $a_i$. An extra $m$ vertices are obtained by subdividing these cells. There is no constraint on their positions except that they should represent distinct points in the orientable surface obtained by omitting the points. The 2-cell is then attached along the word $a_1 a_2 \ldots a_{2n} \bar{a}_1 \bar{a}_2 \ldots \bar{a}_{2n}$ with vertex elements $c_1, c_2, \ldots, c_m$ inserted at the extra vertices. If $m \geq 2$ there will certainly be homomorphisms $\rho : G \to \mathbf{Q}$ for which $\rho(c_1) = 2/3$. In this case the stabilizer of the corresponding cone point will be the group of multiplicative rationals generated by $2, 3$ extended by an infinite cyclic group generated by an element $x$ inducing multiplication by $2/3$ on this subgroup. This group could therefore show up as an edge group in the decomposition of a group obtained in Theorem 1.

Conversely a group occurring as a T2 vertex group in Theorem 1 and giving rise to an unstable action must be such a group. If $\rho$ is trivial then we get a stable action, in which each arc stabilizer is normalized by $H$. We can get stable actions when $X$ is any hyperbolic 2-orbifold which is not necessarily orientable, except for when $X$ is a 2-sphere with 3 cone points. In this case it is well known that the orbifold group - a triangle group - only has trivial actions on an $\mathbf{R}$=tree. An action when $X$ is a 2-sphere with 4 cone points is given in Example 6, and it is easy to adapt this to give examples with $n$ branch points for any $n > 3$.

Finally we want to show that the action of a finitely presented group $G$ on an $\mathbf{R}$-tree $T$ with cyclic arc stabilizers is a limit of simplicial actions with small arc stabilizers. From Theorem 1 it is only necessary to do this in the T2 situation. Thus we assume that $G$ is as above, so that the action on $T$ is determined by the $k$-tuple $\mathbf{a}$ of positive reals and the $k$-tuples $\mathbf{n}_p, p \in P$. We assume, without loss of generality, that $\mathbf{a}.\mathbf{n}_p > 0$ for each $p \in P$. A $k$-tuple $\mathbf{a}'$ of rational numbers will satisfy $\mathbf{a}'.\mathbf{n}_p > 0$ for each $p \in P$ if $||\mathbf{a}' - \mathbf{a}||$ is sufficiently small. The leaves of the foliation corresponding to $\mathbf{a}'$ will all be compact, and there will be ones which are simple closed curves. A lift $\ell'$ of such a leaf to $Y$ will be an infinite track, and there will be an element $h \in H$ generating the stabilizer of $\ell$. By the choice of $\mathbf{a}'$ each of the values $\phi_p(h) > 0$ (possibly after replacing $h$ by $h^{-1}$). But this means $\rho(h)$ is a positive integer $m$ and the original leaf in $X$ determines a splitting of $G$ over a soluble Baumslag-Solitar group $gp < z, h | h^{-1} z h = z^m >$. For closer rational approximations to $\mathbf{a}$ the value of $m$ will increase. If we attach the values $\mathbf{a}'$ to the edges of $X$, it will determine a folding sequence, which will stop after finitely many steps, since the edges will always be multiples of the greatest common divisor of the entries of $\mathbf{a}'$, and so cannot tend to zero. We define the limit of a folding sequence of $G$-graphs, equipped with the path metric. Let $X_n$ be such a sequence, so that $X_{n+1}$ is obtained from $X_n$ by a folding operation as in Section 4, so there is a $G$-morphism $\rho_n : X_n \to X_{n+1}$. Let $\theta_n = \rho_n \rho_{n-1} \ldots \rho_1 : X_1 \to X_{n+1}$. We require that for any edge $e = (u, v)$ of $X_1, \theta_n$ restricts to an isometry on $e$ for every $n$. Thus $d(\theta_n(u), \theta_n(v))$ is constant. Let $d_n$ be the path metric on $X_n$. One can show that there is a pseudometric $d$ on $X_1$ by $d(x, y) = \lim_{n \to \infty}(d_{n+1}(\theta_n(x), \theta_n(y))$. We put $X = X_1 / \sim$, where $x \sim y$ if $d(x,y)=0$, and we have a metric space which we define to be the limit of the sequence. In the cases in which we are interested this limit is an **R**-tree $T$. This is because the limit space is the leaf space of a foliated complex (see [LP]). In fact in the T2 situation when we are resolving an action of $G$ - with small arc stabilizers - on an **R**-tree the limit is (isometric to) the tree being resolved. This follows from the proof of [Sk], who shows that every action of a hyperbolic surface group is geometric.

We now consider these actions in an apparently different way following Shalen [S]. For each space $X_n$ we define a translation length function $\ell_n : G \to \mathbf{R}$, where $\ell_n(g) = \inf\{d_n(x, gx) | x \in X_n\}$. Since $\ell_n$ is constant on conjugacy classes, it represents an element of $[0.\infty)^{\mathcal{C}} - \{0\}$, where 0 is the function identically zero on the set $\mathcal{C}$ of conjugacy classes of $G$. Clearly $\ell_n$ converges to $\ell$ the translation length function of $G$ on $T$. If we use $\mathbf{a}'$, a rational approximation to $\mathbf{a}$, instead of $\mathbf{a}$, then we get a finite folding sequence and the corresponding translation length functions eventually become constant as the length function $\ell_{\mathbf{a}'}$ for $G$ acting on a simplicial tree. It is not too hard - I hope - to show that as we choose a sequence of rational vectors tending to $\mathbf{a}$, then the corresponding translation length functions tend to $\ell$.

## 5. Finitely generated groups

We can also use the techniques developed here together with those of [D2] to obtain a result about stable actions of finitely generated groups on **R**-trees.

Let $G$ be a finitely generated group and let $T$ be a $G$-tree, i.e. an **R**-tree on which $G$ acts by isometries.

Let $\Gamma$ be a $G$-graph. We suppose that $\Gamma$ is simple, i.e. there are no loops and at most one edge between any pair of vertices. Also suppose there are no edges that are inverted by any element of $G$. In this case the stabilizer of any edge is the intersection of the stabilizers of its vertices. Let $\alpha : V\Gamma \to T$ be a $G$-map. Such a map is specified by the images of a $G$-transversal of the vertices of $V\Gamma$, the only requirement on these images is that $Stab(v) \leq Stab(\alpha(v))$. We make $\Gamma$ into a topological space by giving each edge with vertices $u, v$ the topology of a closed interval whose length is that of the geodesic $[\alpha(u), \alpha(v)]$ in $T$. If $\alpha(u) = \alpha(v)$ then we change $\Gamma$ by identifying all pairs $gu, gv, g \in G$ and removing the orbit of edges containing $e$. We can extend $\alpha$ to a continuous map $\alpha : \Gamma \to T$ which restricts to an isometry on edges.

Note that the Cayley graph of $G$ with respect to finite generating set $\{g_1, g_2, \ldots, g_n\}$ is such a $\Gamma$ and there will be such a map $\alpha$ in this case since stabilizers are trivial. In this case $\Gamma$ is a tree if and only if $G$ is a free group, freely generated by $g_1, g_2, \ldots, g_n$. In the general case suppose $\Gamma$ contains a cycle subgraph $C$. The image of $C$ under $\alpha$ will be a finite simplicial subtree of $T$. There must be two adjacent edges of $\Gamma$ (or a subdivided $\Gamma$ that are folded together under $\alpha$. In [BF1] and [D2] folding operations are described on simplicial $G$-trees. We now want to generalize these operations to $G$-graphs. There is no difficulty in doing this. We again get the three types of fold listed in [BF1] and [D2] which are described in terms of the quotient graphs and stabilizers of edges. Note also that the Proposition of [BF1] remains true if we allow $T'$ to be a $G$-graph. Thus if $G$ is finitely generated and $\alpha : \Gamma \to T$ is a $G$-map of graphs and $T$ is a tree and all edge stabilizers are finitely generated, then $\alpha$ can be represented as a finite composition of folds.

We show that if the action of $G$ on $T$ is stable then a finite sequence of folds on $\Gamma$ will reduce it to standard form which will give information as to the structure of $G$.

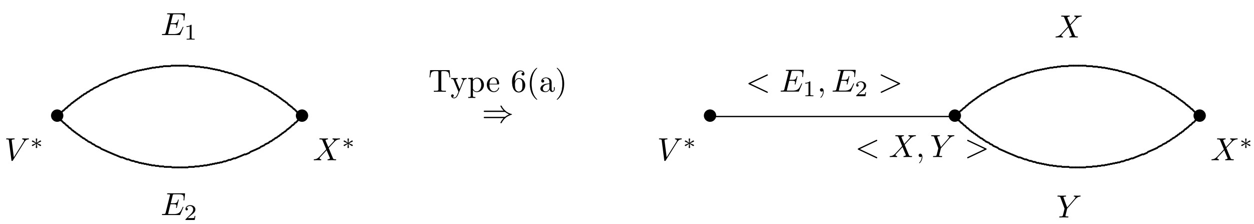
The approach is as follows. If $\Gamma$ is a tree then we are done, since the Bass-Serre theory gives the structure of $G$. If $\Gamma$ is not a tree then it contains a cycle $C$. The image of $C$ in $T$ is a finite simplicial subtree $D$, i.e. it will have finitely many branch points. We describe how to fold $\Gamma$ to produce a new $G$-graph in which $C$ is null homotopic if 2-cells are attached to quadratic words representing cycles in $\Gamma$. The map $C \to D$ is a composition of finitely many folds, in each of these folds segments $w, w'$ in $C$ are folded together, i.e. each of $w, w'$ are mapped isometrically to the same segment of $D$.

Such a folding may induce a simplicial folding in the whole graph $\Gamma$. If this does not happen, then folding together part of the segments must induce an infinite folding sequence of $\Gamma$.

Suppose first that all the induced foldings involved in the morphism $C \to D$ are simplicial. As in [D2] it is helpful to consider the effect of folds on reduced graphs, i.e. the graphs obtained by contracting a compressing forest $F$.

In the list of induced moves on the reduced trees given in [D2] I managed to omit a case. This is when

$e_1 \notin F, e_2 \notin F, x, y$ in the same component of $F, v, x$ in distinct components of $F$. The diagram change is shown in the diagram below and the explanation is similar to that of Types 6 and 7 in [D2].




Note that under the natural map $\Gamma \mapsto G\backslash\Gamma, C$ is not mapped into $F$, since the preimage of the forest $F$ is a forest in $\Gamma$, which will contain no cycles.

It follows that at least one of the folds induced by the folding of $C$ to $D$ is non-trivial for the reduced graph (after collapsing $F$). The last such fold must either be Type II or Type III on the reduced graph. This either results in an increase in an edge stabilizer or a reduction in the first Betti number of the quotient graph.

If we successively fold away any cycles in $\Gamma$ that we can by simplicial folding then this process will either stop, or, as in the proof of [D2, Theorem 3.3], $G$ will contains a subgroup $H$ that which does not fix an arc but every finitely generated subgroup of $H$ does fix an arc of $T$. This cannot happen if the action is stable.

If the folding induced by folding $w, w'$ together is not simplicial, then there will be an infinite folding sequence. This situation was analysed earlier in this paper. This can only happen if there are points $y, y'$ in $w, w'$ repectively which map to the same point in the quotient graph $G\backslash\Gamma$ and if $w_y, w'_{y''}$ are initial subsegments of $w.w'$ up to and including $y, y'$ respectively, then the image of $w_y \cup w'_{y'}$ will be a quadratic word in the edges of the quotient graph. Again it follows from Proposition 1 that if there is any edge of the quotient graph that is involved in more than one such folding, then either there is a subgroup $H$

of $G$ as above, ensuring the action is not stable. or both sets of edges map to a real line, and the structure of that part of the group consisting of the elements that map that line to itself is a normal subgroup fixing the real line, extended by a finitely generated group of isometries of the line. The number of times these infinite folding sequences can occur is bounded by the number of edges in the quotient graph.

It follows that it is possible to fold cycles in $\Gamma$ using simplicial folds until the only cycles that remain either correspond to a real line as above or they become null-homotopic after attaching $G$-orbits of 2-cells, corresponding to 2-orbifolds in the quotient space. Let $e, f$ be edges of $\Gamma$. Let $\sim$ be the equivalence relation on $E\Gamma$ generated by $e \sim f$ if $e, f$ lie in the same attached 2-cell. An equivalence class will consist of the edges in a single attached 2-cell if and only if there is no 2-cell in a different $G$-orbit containing any of these edges. (Each edge will be contained in one other 2-cell in the same $G$-orbit.) If this is not the case, i.e. there is an edge conained in 2-cells in diferent orbits, then the argument earlier shows that all the edges in the equivalence class are mapped to a single line in $T$. Construct a new graph $\Gamma'$ as follows: for each vertex $v \in V\Gamma$ let $B_v$ be the squivalence classes which contain an edge incident with $v$. Replace vertex $v$ by a subtree $S_v$ whose vertex set consists of a vertex also labelled $v$ and vertices labelled with the elements of $B_v$. The edges of $S_v$ join $v$ to each of the other vertices of $S_v$. In addition $E\Gamma'$ will have an edge corresponding to each edge of $\Gamma$. Suppose $e \in E\Gamma$ has vertices $u, v$. The equivalence class of $e$ lies both in $B_u$ and $B_v$, and so it labels vertices of $S_u$ and $S_v$. These are the vertices of $e$ in $\Gamma'$.

Note that contracting the edges of each $S_v$ will give $\Gamma$. There is a subgraph of $\Gamma'$ consisting of the edges of a particular equivalence class and their vertices in $\Gamma'$. Note that the subgraphs for distinct equivalence classes are disjoint. Since any cycle in $\Gamma$ has edges in a particular equivalence class, the only cycles in $\Gamma'$ lie in these subgraphs. Hence contracting all the edges of $\Gamma'$ which corresponded to edges of $\Gamma$ gives a $G$-tree $T'$. Each vertex of $T'$ corresponds to a vertex of $\Gamma$ or an equivalence class of edges. If $v \in V\Gamma$ then $Stab(v)$ fixes a point of $T$. If $v \in VT'$ corresponds to an equivalence class of edges in a single attached 2-cell, then $Stab(v)$ is the extension of a normal subgroup $N$ by a 2-orbifold group. For any other vertex $v \in VT', Stab(v)$ is a normal subgroup $N$, extended by a group of isometries of $\mathbf{R}$, in both the latter cases $N$ is the common stabilizer of all the edges in the equivalence class. A slightly different way of pulling vertex groups off the T1 and T2 vertices is given in Section 2. If the action is stable then in the infinite folding sequence associated with a T2 2-cell the groups attached to the 1-cells eventually become constant and this limit group will be normal in the group of the T2 complex. If this does not happen then there will be a group which does not fix an arc but all of whose finitely generated subgroups do.

This then completes the proof of Theorem 2.

## Concluding Remarks

1. I think it probable that the techniques of this paper can be used to classify finitely presented group that admit actions on $\mathbf{R}$-trees when arc stabilizers are allowed to lie in a larger class $\mathcal{C}$ or groups than just the class of cyclic groups. It is probable not too hard to extend to the case when $\mathcal{C}$ is the class of virtually cyclic groups. It might even be possible to extend to the case when $\mathcal{C}$ is the class of virtually polycyclic groups.

2. Mark Sapir asked me if it is possible to obtain similar results for an action of a group $G$ on an $\mathbf{R}$-tree $T$ when instead of acting by isometries the group $G$ acts by homeomorphisms with a non-nesting condition. I think this is the case. In obtaining the folding sequence in the proof of Theorem 1, a non-nesting condition suffices. One only needs to know that if an edge is mapped into itself by $g \in G$ then it is a Type II fold. Once one has an infinite folding sequence (in this case of graphs rather than trees) then one can apply Theorem 3.1 of [D2] to show that there is an $\mathbf{R}$-tree metric on $T$ for which $G$ acts by isometries. I think Theorem 3.1 of [D2] can be extended to folding sequences of graphs. This would then generalize the result of Levitt [L].

3. If an action is resolved by a T2 action then the resolution is an isometry. This follows from the result of Skora [S] that any action of a surface group on an bf R-tree with small arc stabilizers is geometric. This contrasts with actions of free groups on $\mathbf{R}$-trees. In this case Levitt type non-simplicial free actions are resolved by simplicial actions.

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