

FINITELY PRESENTED GROUPS ACTING ON TREES

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ABSTRACT. It is shown that for any action of a finitely presented group G on an \mathbb{R} -tree, there is a decomposition of G as the fundamental group of a graph of groups related to this action. If the action of G on T is non-trivial, i.e. there is no global fixed point, then G has a non-trivial action on a simplicial \mathbb{R} -tree.

1. INTRODUCTION

A group G is said to split over a subgroup C if either $G = A *_C B$, where $A \neq C$ and $B \neq C$ or G is an HNN-group $G = \langle A *_C \langle A, t | t^{-1}at = \theta(a) \rangle$ where $\theta : C \rightarrow A$ is an injective homomorphism. It is one of the basic results of Bass-Serre theory (see [5] or [[15]]), that a finitely generated group G splits over some subgroup C if and only if there is an action of G on a tree T , without inversions, such that for no vertex $v \in VT$ is v fixed by all of G . Here the tree is a combinatorial tree, i.e. a connected graph with no cycles, and an action without inversions is one in which no element $g \in G$ transposes the vertices of an edge. Tits [19] introduced the idea of an \mathbb{R} -tree, which is a non-empty metric space in which any two points are joined by a unique arc, and in which every arc is isometric to a closed interval in the real line \mathbb{R} . Alternatively an \mathbb{R} -tree is a 0-hyperbolic space. A tree in the combinatorial sense can be regarded as a 1-dimensional simplicial complex. The polyhedron of this complex will be an \mathbb{R} -tree - called a simplicial \mathbb{R} -tree. However not every \mathbb{R} -tree is like this. A point p of an \mathbb{R} -tree T is called *regular* if $T - p$ has two components. An \mathbb{R} -tree is simplicial if the points of T which are not regular form a discrete subspace of T . It is fairly easy to construct examples of \mathbb{R} -trees where the set of non-regular points is not discrete. There are good introductory accounts of groups acting on \mathbb{R} -trees in [1] and [16]. We assume that all our actions are by isometries. It is a classical result that a group is free if and only if it has a free action on a simplicial tree. As the real line \mathbb{R} is an \mathbb{R} -tree and \mathbb{R} acts on itself freely by translations, any free abelian group has a free action on an \mathbb{R} -tree. Morgan and Shalen [12] showed that the fundamental group of any compact surface other than the projective plane and the Klein bottle has a free action on an \mathbb{R} -tree. Rips showed that the only finitely generated groups that act freely on an \mathbb{R} -tree are free products of free abelian groups and surface groups. Rips never published his proof, but there are proofs of more general results by Bestvina-Feighn [3] and by Gaboriau-Levitt-Paulin (see [14] or [4]). Bestvina and Feighn classify the *stable* actions of finitely generated groups on \mathbb{R} -trees. Recall, that an action of a group G on an \mathbb{R} -tree is said to be *stable* if there is no sequence of arcs l_i such that l_{i+1} is properly contained in l_i for every i , and for which the stabilizer G_I of l_i is properly contained in $G_{l_{i+1}}$ for every i . In particular [3] Bestvina and Feighn proved that if

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a finitely presented group has a non-trivial minimal stable action on an \mathbb{R} -tree then it has a non-trivial action on some simplicial tree.

A group is said to be (FA) if it has no non-trivial action on a simplicial \mathbb{R} -tree and it is said to be $(F\mathbb{R})$ if it has no non-trivial action on any \mathbb{R} -tree. A *trivial* action is one in which there is a point of the tree that is a global fixed point. In contrast A.Minasyan and I [9] have recently given examples of finitely generated groups that are (FA) but not $(F\mathbb{R})$. This provides a negative answer to Shalen's Question A of [16]. In this paper it is shown that there is a positive answer to this question for finitely presented groups.

A *morphism* from a segment I to an \mathbb{R} -tree T is a continuous map $\phi : I \rightarrow T$ such that I may be subdivided into finitely many subsegments that ϕ maps isometrically into T . Let T, T' be \mathbb{R} -trees with actions of groups G, G' respectively. Let $\rho : G \rightarrow G'$ be a homomorphism. A *morphism* from T to T' is a map ϕ equivariant with respect to ρ which induces a morphism on every segment $I \subset T$.

In this paper the following theorem is proved.

Theorem 1.1. *Let G be a finitely presented group and let T be a G -tree, i.e. an \mathbb{R} -tree on which G acts by isometries.*

Then G is the fundamental group of a finite graph (\mathcal{Y}, Y) of groups, in which every edge group is finitely generated and fixes a point of T . If $v \in VY$, then either $\mathcal{Y}(v)$ fixes a vertex of T or there is a homomorphism from $\mathcal{Y}(v)$ to a target group $Z(v)$ (an augmented parallelepiped group), which is the fundamental group of a cube complex of groups based on a single n -cube $c(v)$.

Every hyperplane of $c(v)$ is associated with a non-trivial splitting of G .

There is a marking of the cube $c(v)$ so that the corresponding \mathbb{R} -tree with its $Z(v)$ -action is the image of a morphism from a $\mathcal{Y}(v)$ -tree T_v and this tree is the minimal $\mathcal{Y}(v)$ -subtree of T .

The action of a target group on an \mathbb{R} -tree is usually unstable, but a parallelepiped group of rank n contains a free abelian group of rank n and this acts freely on \mathbb{R} by translation.

In another paper A.N.Bartholomew and I hope to show that there is a decomposition of a finitely presented group G as in Theorem 1.1 that can be used for any action of G on a tree. It will also be shown that if G has a solvable word problem, then this decomposition can be computed.

I thank Ashot Minasyan for very helpful conversations.

2. TARGET GROUPS

In [8] *rectangle groups* were constructed.

The rectangle group $R = R(m.n.p.q)$, $m, n, p, q \in \{2, 3, \dots\} \cup \{\infty\}$ is the group with presentation

$$R = \langle a, b, c, d \mid a^m = b^n = c^p = d^q = 1, ab^{-1} = cd^{-1}, ac^{-1} = bd^{-1} \rangle.$$

Think of the relations as saying that opposite edge vectors are equal and that the corners are assigned orders, a corner can have infinite order.

In the group R above, let $x = ab^{-1} = cd^{-1}$, $y = ac^{-1} = bd^{-1}$, then $xy = ab^{-1}bd^{-1} = ad^{-1} = ac^{-1}cd^{-1} = yx$, and x, y generate a free abelian rank 2 group.

Also R has *incompatible* decompositions as a free product with amalgamation

$$R = \langle a, b \rangle *_{ab^{-1}=cd^{-1}, b^{-1}a=d^{-1}c} \langle c, d \rangle,$$

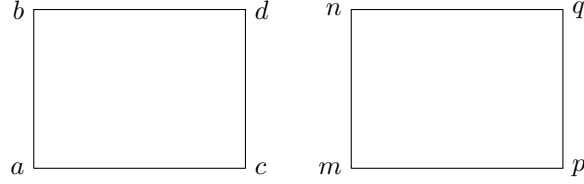


FIGURE 1. rectangle group

and

$$R = \langle a, c \rangle *_{ac^{-1}=bd^{-1}, c^{-1}a=d^{-1}b} \langle b, d \rangle,$$

the amalgamated subgroup in each case is free of rank two.

The decompositions are incompatible because the lines dividing the rectangle intersect.

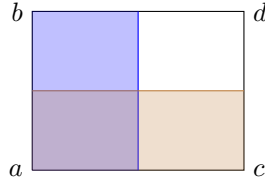


FIGURE 2. Decompositions of a rectangle group

A *cube complex* is similar to a simplicial complex except that the building blocks are n -cubes rather than n -simplexes. A rectangle group R acts on a simply connected 2-dimensional cube complex \tilde{C} with orbit space C . This is illustrated in Fig 3. Apart from some exceptional cases, when two or more of m, n, p, q are 2, there are three orbits of 2-cells, each with trivial stabilizer. In all cases there are four orbits of edges also with trivial stabilizers, and four orbits of vertices labelled A, B, C, D with stabilizers which are cyclic of orders m, n, p, q respectively. In the group $R(2, 2, 2, 2)$ the subgroup $\langle x, y \rangle$ has index 2 and there is one orbit of 2-cells. In $R(2, 2, 2, q)$ or $R(2, 2, p, q)$ for $p, q \geq 3$ there are two orbits of 2-cells and both C and \tilde{C} are 2-orbifolds.

In [8] it is shown that for any action of $J = \langle x, y \rangle$ on an \mathbb{R} -tree there is an action of the rectangle group on an \mathbb{R} tree T which restricts to the given action on the minimal J -subtree of T . This action is unstable in all cases when there are 3 orbits of 2-cells.

There is a Euclidean 2-dimensional subspace E of \tilde{C} acted on by $\langle x, y \rangle$. For the action of $\langle x, y \rangle$ on E there is one orbit of 2-cells, each of which is made up of 4 smaller rectangles of \tilde{C} .

Note that the blue rectangle A, aB, aC, aD is in the same R -orbit as the red rectangle A, B, C, D , and $bD = bd^{-1}D = yD, cD = cd^{-1}D = xD$.

We now define an *augmented rectangle group*. Let S be 1-complex that is a 4-cycle, i.e. the edges and vertices of a rectangle. Let v_1, v_2, v_3, v_4 be the vertices of S where v_1, v_4 are not adjacent. Let (\mathcal{S}, S) be a graph of groups with underlying graph S . Let $\mathcal{S}(v_1) = A, \mathcal{S}(v_2) = B, \mathcal{S}(v_3) = C, \mathcal{S}(v_4) = D$. Let $a \in A, b \in B, c \in C, d \in D$ and we require that no conjugate of a, b, c or d is in an edge group. We require that $m, n, p, q \in \{2, 3, \dots\} \cup \{\infty\}$ are such that $a^m = b^n = c^p = d^q = z$, so

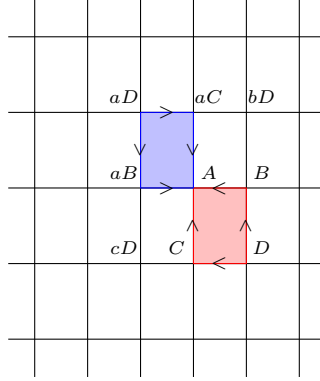


FIGURE 3. The Euclidean space for a rectangle group

that z is in every edge group. We allow $z = 1$. The augmented rectangle group R corresponding to this data is the fundamental group $\pi(\mathcal{S}, S)$ of this graph of groups together with the extra relations $ab^{-1} = cd^{-1}$, $ac^{-1} = bd^{-1}$. It can be seen that R is the fundamental group of a cube complex of groups, where the complex is the same as for a rectangle group. However the vertex groups have been enlarged. We show that this complex of groups is still developable, so that there is a simply connected cube complex \tilde{C} with an action of R and $R \backslash \tilde{C} = C$ is S together with one, two or three attached 2-cells. We prove that, like an ordinary rectangle group, the augmented rectangle group R has incompatible splittings.

$$R = \langle A, B \rangle *_{A \cap C, B \cap D, ab^{-1}=cd^{-1}, b^{-1}a=d^{-1}c} \langle C, D \rangle,$$

and

$$R = \langle A, C \rangle *_{A \cap B, C \cap D, ac^{-1}=bd^{-1}, c^{-1}a=d^{-1}b} \langle B, D \rangle,$$

In the first splitting, the subgroup $\langle A, B \rangle$ is a free product with amalgamation of A and B and $\langle C, D \rangle$ is a free product with amalgamation of C and D . In order to see that there are indeed such splittings, one has to show that - for the first splitting - the subgroup $\langle A \cap C, B \cap D, ab^{-1}, b^{-1}a \rangle$ of $\langle A, B \rangle$ is isomorphic to the subgroup $\langle A \cap C, B \cap D, cd^{-1}, d^{-1}c \rangle$ of $\langle A, B \rangle$. However if one considers the action of $\langle A \cap C, B \cap D, ab^{-1}, b^{-1}a \rangle$ on the Bass-Serre tree for $\langle A, B \rangle$ which is a tree with one orbit of edges and two orbits of vertices, then the minimal subtree for this subgroup has two orbits of vertices and either one two or three orbits of edges depending on whether $ab^{-1} \in A \cap B$ or $b^{-1}a \in A \cap B$. It can be seen that there is a graph isomorphism between this subtree and the corresponding subtree for $\langle A \cap C, B \cap D, cd^{-1}, d^{-1}c \rangle$ on the Bass-Serre tree for $\langle C, D \rangle$ which is equivariant with respect to the required isomorphism between the two subgroups. Thus the target group has incompatible decompositions as free products with amalgamation as indicated. There is an action of R on the Sageev cubing \tilde{C}' associated with the two incompatible splittings. The fact that our cube complex is developable now follows as for the ordinary rectangle group. In fact there is an embedding of \tilde{C} in \tilde{C}' that is equivariant with respect to R .

A *parallelepiped group* of dimension n has 2^n generators corresponding to the vertices of an n -cube. The generators corresponding to a 2-dimensional face satisfy the relations of a rectangle group. Such a group has an action on an n -dimensional

cube complex C_n for which the orbit space is an n -cube. There is a subgroup J_n that is free abelian of rank n , which acts on a subcomplex E_n of C_n so that the orbit space $J_n \backslash E_n$ consists of 2^n smaller cubes.

A *target group* is an augmented parallelepiped group. Thus a target group is the fundamental group of an n -dimensional cube complex in which the quotient complex is the same as that of an ordinary parallelepiped group, but in which vertex groups have been expanded so that every 2-dimensional face the labelled complex is that of an augmented rectangle group.

3. THE HIGMAN GROUP

In this section we give an example of a target group that is the homomorphic image of the well known Higman group.

Example 3.1. Let $H = \langle a, b, c, d \mid aba^{-1} = b^2, bcb^{-1} = c^2, cdc^{-1} = d^2, dad^{-1} = a^2 \rangle$.

This group was investigated by Higman [11]. He showed that it was infinite and had no non-trivial finite homomorphic images. His proof that it was non-trivial involved showing that it had a decomposition as a free product with amalgamation

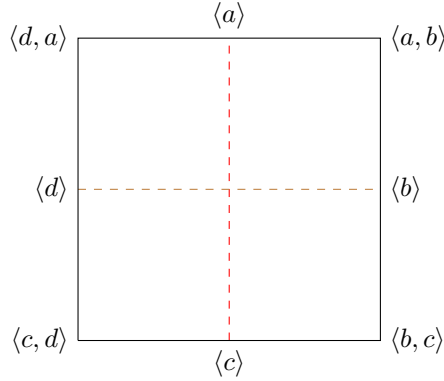
$$H = \langle a, b, c \rangle *_{\langle a, c \rangle} \langle a, d, c \rangle.$$

Also $\langle a, b, c \rangle$ is the free product with amalgamation

$$\langle a, b, c \rangle = \langle a, b \rangle *_{\langle b \rangle} \langle b, c \rangle,$$

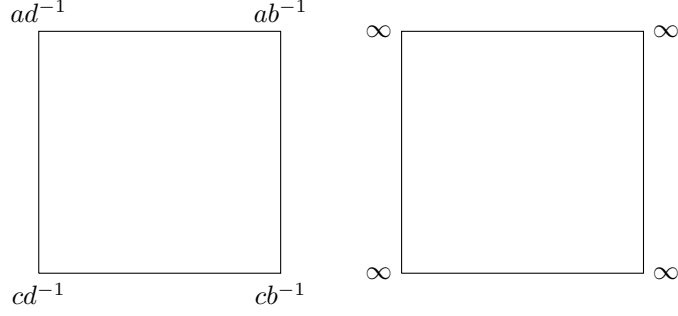
where both $\langle a, b \rangle$ and $\langle b, c \rangle$ are isomorphic to the Baumslag-Solitar group $BS(1, 2)$.

The group H acts on a cube complex for which the labelled orbit space has one-skeleton as indicated. There will be two orbits of 2-cells, as can be seen from the following argument.

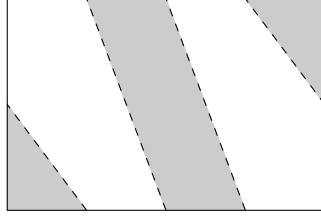


Let $K = \langle da^{-1}, ab^{-1}, bc^{-1}, cd^{-1} \rangle$. Put $x = ad^{-1}, y = ab^{-1}, u = cb^{-1}, v = cd^{-1}$. Then $yu^{-1}vx^{-1} = 1$ and K is free rank 3. There is a homomorphism from K to $R(\infty, \infty, \infty, \infty)$, which has the extra relation $yx^{-1}vu^{-1} = 1$. This group has an action on a cube complex with three orbits of 2-cells. Without the extra relation there will be two orbits of 2-cells. The group K acts on the planar orbifold cover of a 2-sphere with four cone points.

In fact K and H have infinitely many decompositions, each given by a pair of integers, and K and H have non-simplicial actions on an \mathbb{R} -tree each given by a pair

FIGURE 4. Target group for H

of real numbers, which are independent over \mathbb{Q} . The decomposition corresponding to the *marking* $(2, 3)$ is illustrated in Figure 5 .

FIGURE 5. Splitting associated with $(2, 3)$

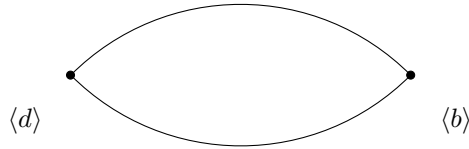
Let L be the group

$$L = \langle a, b, c, d \mid aba^{-1} = b^2, bcb^{-1} = c^2, cdc^{-1} = d^2, dad^{-1} = a^2, ab^{-1}da^{-1} = cb^{-1}dc^{-1} \rangle.$$

This is a non-trivial proper homomorphic image of H , as noted by Schupp [17]. The fact that L is non-trivial can be seen from the splitting

$$L = \langle b, a, d \rangle *_{\langle b, d, ab^{-1}da^{-1} = cb^{-1}dc^{-1} \rangle} \langle b, c, d \rangle.$$

Here the two factors are again the free product with amalgamation of two $BS(1, 2)$ groups with cyclic amalgamated subgroup. In the splitting of L , the amalgamated subgroup $C = \langle b, d, ab^{-1}da^{-1} = cb^{-1}dc^{-1} \rangle$ is free of rank 3. This is because in the action of C on the Bass-Serre tree for $A = \langle b, d, a \rangle$ corresponding to the splitting $A = \langle d, a \rangle *_{\langle a \rangle} \langle a, b \rangle$, the element $ab^{-1}da^{-1}$ is hyperbolic with hyperbolic length 2 and the minimal C -subtree is the Bass-Serre tree for the graph of groups in which the edge groups are trivial. If we now put $x = ad^{-1}, y = ab^{-1}, u =$



$cb^{-1}, v = cd^{-1}$, then $yu^{-1}vx^{-1} = 1$ and $yx^{-1}vu^{-1} = 1$ and so $\langle x, y, u, v \rangle$ is isomorphic to $R(\infty, \infty, \infty, \infty)$. Clearly L is a target group. Since L is an epimorphic image of H , every action of L on an \mathbb{R} -tree gives an action of H on an \mathbb{R} -tree. We will see that the converse is also true, i.e. every action of H on an \mathbb{R} -tree induces an action of L on an \mathbb{R} -tree. Thus we can call L the target group for H . Since H has no non-trivial finite homomorphic images the same is true for L . As remarked above, the group L appears in the paper [17] of Schupp. It also follows from the techniques of Schupp in that paper that L is SQ-universal.

4. FINITELY PRESENTED GROUPS

Let X be a finite CW 2-complex. We introduce the idea of a complex of groups $G(X)$ based on X . This is a slightly different notion to a special case of the complex of groups described by Haefliger [10]. Haefliger restricts X to be a simplicial cell complex. One can get from our situation to that of Haefliger by triangulating each 2 cell. We are only concerned with the situation when each group assigned to a 2-cell is trivial.

Thus the 1-skeleton X^1 of X is a graph. We take the edges to be oriented, and use Serre's notation, so that each edge e has an initial vertex ιe and a terminal vertex τe and \bar{e} is e with the opposite orientation. Let $G(X^1)$ be a graph of groups based on X^1 . The attaching map of each 2-cell σ is given by a closed path in X^1 . Let S be a spanning tree in X^1 . The fundamental group $\pi(G(X), S)$ of the complex of groups $G(X)$ is the fundamental group of the graph of groups $G(X^1)$ together with extra relations corresponding to the attaching maps of the 2-cells. Thus $\pi(G(X), S)$ is generated by the groups $G(v), v \in V(X^1)$ and the elements $e \in E(X^1)$. For each $e \in E(X^1)$, $G(e)$ is a distinguished subgroup of $G(\iota e)$ and there are injective homomorphisms $t_e : G(e) \rightarrow G(\tau e), g \mapsto g^{\tau e}$. The relations of $\pi(G(X), S)$ are as follows:-

- the relations for $G(v)$, for each $v \in V(X^1)$
- $e^{-1}ge = g^{\tau e}$ for all $e \in E(X^1), g \in G(e) \leq G(\iota e)$,
- $e = 1$ if $e \in E(S)$.

For each attaching closed path e_1, e_2, \dots, e_n in X of a 2-cell, there is a relation

$$g_0 e_1 g_1 e_2 x_2 \dots g_{n-1} e_n = 1,$$

where $g_i \in G_{\tau e_i} = G_{\iota e_{i+1}}$, called the *attaching word*. The elements g_i are called *joining* elements. Such a word represents both a path p , called the *attaching path* in the Bass-Serre tree T corresponding to the graph of groups $G(X^1)$, for which initial point ιp and end point τp are in the same $\pi(G(X^1), S)$ -orbit and an element $g \in \pi(G(X^1), S)$ for which $g\iota p = \tau p$. Adding the relation identifies the points ιp and τp and puts $g = 1$. If we carry out all these identifications, we obtain a G -graph Γ in which the attaching paths are all closed paths. We describe specifically how this path arises (as in [5], p15). We lift S to an isomorphic subtree of S_1 of Γ . Thus the vertex set of S_1 is a transversal for the action of G on Γ . For each edge e in $X - S$ we can choose an edge $\tilde{e} \in T$ such that \tilde{e} maps to e in the natural projection, and $\iota \tilde{e}$ is a vertex of S_1 . Let \tilde{S} be the union of S_1 with these extra edges. Note that it will not normally be the case that $\tau \tilde{e} \in \tilde{S}$ and so \tilde{S} is not usually a subtree of T , but there will be an element $c(e) \in G$ such that $c(e)^{-1}(\tau \tilde{e}) \in \tilde{S}$. These elements (called the *connecting* elements) together with the stabilizers of elements of VS_1 generate G . Clearly \tilde{S} consists of a transversal for the action of G on both the

edges and vertices of Γ . Let $vp = v_0$ be the vertex of \tilde{S} lying above ιe_1 , and put $x_0 = g_0$. Suppose we have constructed v_i and $x_i \in G$ so that v_i is the terminal vertex of the path corresponding to $g_0 e_1 g_1 e_2 g_2 \dots g_{i-2} e_{i-1}$, and so that if \tilde{v}_i is the element of \tilde{S} in the orbit of v_i , then $v_i = x_i \tilde{v}_i$. This is certainly true when $i = 0$. To construct v_{i+1} and x_{i+1} , put $x_{i+1} = x_i c(e_{i+1}) g_{i+1}$ where we put $c(e) = 1$ if $e \in S$. Then $x_{i+1} \tilde{v}_{i+1} = x_i c(e_{i+1}) \tilde{v}_{i+1}$ is the terminal vertex of the edge $x_i \tilde{e}_{i+1}$ with initial vertex v_i . Note that x_i is obtained from $g_0 e_1 g_1 e_2 g_2 \dots g_{i-2} e_{i-1}$ by replacing each e_i by $c(e_i)$.

We now foliate each 2-cell of X in a particular way. Thus let $D = \{(x, y) | x, y \in \mathbf{R}, x^2 + y^2 \leq 1\}$ be the unit disc.

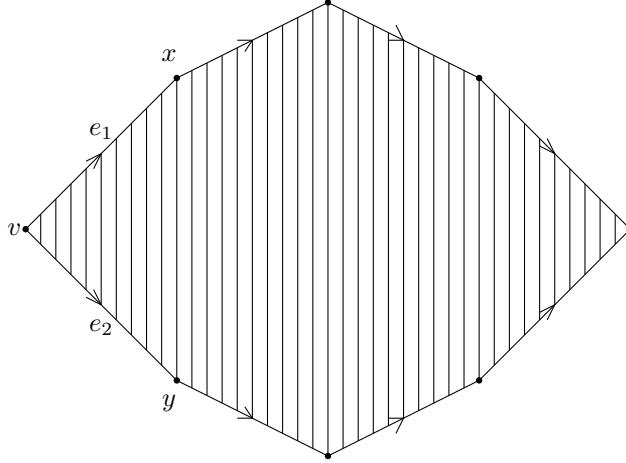


FIGURE 6. Foliated 2-cell

Give this the foliation in which leaves are the intersection of D with the vertical lines $x = c$ where c is a constant in the interval $[-1, 1]$. Let σ be a 2-cell of X which is attached via the closed path e_1, e_2, \dots, e_n . We map D to σ so that for some $j = 2, \dots, n-1$ the upper semi-circle joining $(-1, 0)$ and $(1, 0)$ is mapped to the path e_1, \dots, e_j . Thus there are points $z_0 = (-1, 0), z_1, \dots, z_j = (1, 0)$ on the upper semi-circle so that $z_i \mapsto \iota e_i, i = 1, 2, \dots, e_{j+1}$ and the map is continuous and injective on each segment $[z_i, z_{i+1}]$, except if $\iota e_i = \iota e_{i+1}$ in which case the map is injective on the interior points of this segment. In a similar way the lower semi-circle is mapped to the path $\bar{e}_n, \dots, \bar{e}_{j+1}$.

Let, then, X be a 2-complex of groups in which each 2-cell is foliated as described above and let T be a G -tree, i.e. T is an \mathbf{R} -tree on which G acts by isometries. We say that the X resolves T if there is an isomorphism $\theta : \pi(X, S) \rightarrow G$ which is injective on vertex groups (and hence on all groups G_σ for all cells σ of X). In this situation (see [10]), the complex of groups is developable, i.e. there is a cell complex \tilde{X} on which G acts and $G(X)$ is the complex of groups associated with this action. We also require that there be a G -map $\alpha : \tilde{X} \rightarrow T$ such that for each 1-cell γ the restriction of α to γ is injective and for each 2-cell σ and each $t \in T$, the intersection of σ with $\alpha^{-1}(t)$ is either empty or a leaf of the foliation described above.

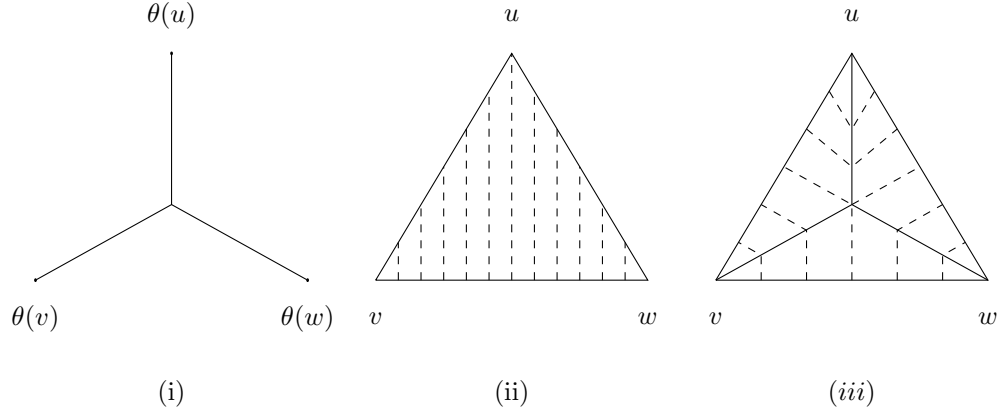


FIGURE 7. Foliating a simplex

We show that if G is finitely presented then any G -tree has a resolution, i.e. there is a cell complex X as above that resolves T . Our approach is similar to that of [13].

Since G is finitely presented, there is simplicial 2-complex X such that $\pi(X, S) \cong G$. Here S is a spanning tree in the 1-skeleton of X . Let \tilde{X} be the universal cover of X . Clearly there is a G -map $\theta_0 : V\tilde{X} \rightarrow T$, which can be obtained by first mapping a representative of each G -orbit of vertices into T and then extending so as to make the map commute with the G -action. Now extend this map to the 1-skeleton so that each 1-simplex γ with vertices u, v of \tilde{X} is mapped injectively to the geodesic joining $\theta_0(u)$ and $\theta_0(v)$. It may be necessary to subdivide X and choose the map θ_0 to ensure that $\theta_0(u) \neq \theta_0(v)$ for every 1-simplex γ . We can extend the map to every 1-simplex so that it commutes with the G -action giving a G -map $\theta_1 : \tilde{X}^1 \rightarrow T$. Now we extend the map to the 2-simplices. Let σ be a 2-simplex with vertices u, v, w . If $\theta_0(u)$ lies on the geodesic joining $\theta_0(v)$ and $\theta_0(w)$ then we can map σ as indicated in Fig 7 (ii). Each vertical line is mapped to a point. If $\theta_0(u), \theta_0(v)$ and $\theta_0(w)$ are situated as in Fig 7(i) so that no point is on the geodesic joining the other two, then we subdivide σ as in Fig 7 (iii). The new vertex is mapped to the point p of (i) and the three new simplexes now have the middle vertex mapped into the geodesic joining the images of the other two sides and are mapped as shown in (iii).

Again this map can be extended to every subdivided 2-simplex so that it commutes with the G -action. We change X to be this subdivided complex. Regard X as a 2-complex in which each cell is attached via a loop of length three. We can make a complex of groups in which each G_σ is the trivial group. Since G is the fundamental group of X it is the fundamental group of this complex of groups. We have described a way of foliating the 2-cells which shows that this complex of groups resolves T .

We now describe some moves on a resolving 2-complex which can be made on a resolving complex which change a resolving 2-complex to another resolving 2-complex.

Move 1. Subdividing a 1-cell.

Let γ be a 1-cell, with vertices u, v , which may be the same. This can be replaced by two 1-cells γ_1, γ_2 and a new vertex w , so that γ_1 has vertices u, w and γ_2 has vertices v, w . The groups associated with w, γ_1, γ_2 in the new complex of groups are all $G(\gamma)$. The attaching maps of 2-cells are adjusted in the obvious way.

Move 2. Folding the corner of a 2-cell.

Suppose that one end of a foliated 2-cell is as in Fig 6. Thus v is the end vertex of the 2-cell and adjacent vertices are x, y and x, y are mapped to the same point of T , so that they lie on the same vertical line. Let the adjacent 1-cells to v be e_1 and e_2 , which conflicts with our earlier notation but is in line with that of [7] and [2]. Let the groups associated with the cells (in the complex of groups) be denoted by the corresponding capital letters.

Folding the corner results in a fold of the graph of groups associated with the 1-skeleton of X . Such a fold is one of three types which are listed in [2] (as Type A folds) or in [7]. They are shown in Fig 8 for the reader's convenience. As the group acting is always G it is not necessary to carry out vertex morphisms (see [7]) which are necessary when carrying out morphisms of trees rather than graphs.

The attaching word of the 2-cell, whose corner has been folded is changed in a way which we will describe in an example. One can arrange that the joining element at the pivot vertex is trivial, by changing the lift of the spanning tree. In this case, any other attaching word of a 2-cell that involves e_1 or e_2 , \bar{e}_2 is replaced by the folded edge element $\langle e_1, e_2 \rangle$ and e_1 is replaced by $\langle e_1, e_2 \rangle$. Let the new complex of groups be X'

Clearly there is a surjective homomorphism $\phi : \pi(X, S) \rightarrow \pi(X', S')$ in which $g_v e_1 g_v^{-1}$ and e_2 are both mapped to $\langle e_1, e_2 \rangle$. In fact this homomorphism is an isomorphism since the resolving isomorphism $\alpha : \pi(X, S) \rightarrow G$ factors through ϕ . We conclude that X' also resolves the G -tree T .

If both the upper semi-circle and the lower semi-circle consist of a single 1-cell, then folding results in the elimination of a 2-cell, and a reduction in the number of 1-cells.

Move 3 Contracting a leaf.

Consider a foliated 2-cell. Let ℓ be a particular vertical line of the foliation. This will contain points u, v of the upper semi-circle and lower semi-circle respectively. After subdividing the relevant 1-cells, it can be assumed that these points are vertices. Contracting the leaf ℓ results in the 2-cell σ being replaced by two 2-cells σ_1 and σ_2 . The vertices u, v become a single vertex w and its group G_w is the subgroup of G generated by G_u and G_v in G , except if u, v belong to the same G -orbit, in which case G_w is generated by G_u and an element $g \in G$ such that $gv = u$. Let g_u, g_v be the respective elements of G_u and G_v in the attaching word for σ . Let the edge after reaching u in the attaching word end up in σ_2 . This means that the edge after reaching v ends up in σ_1 . Suppose first that u, v are in different orbits, then after the move the element for w in σ_2 is g_u and the element for w in σ_1 is g_v . Note that an edge has to be removed from the spanning tree S . If u, v are in the same orbit then the element for w in σ_2 is $g_u g$ and the element for w in σ_1 is $g_v g^{-1}$.

A similar argument to that for Move 2 shows that the complex we have created also resolves the G -tree T .

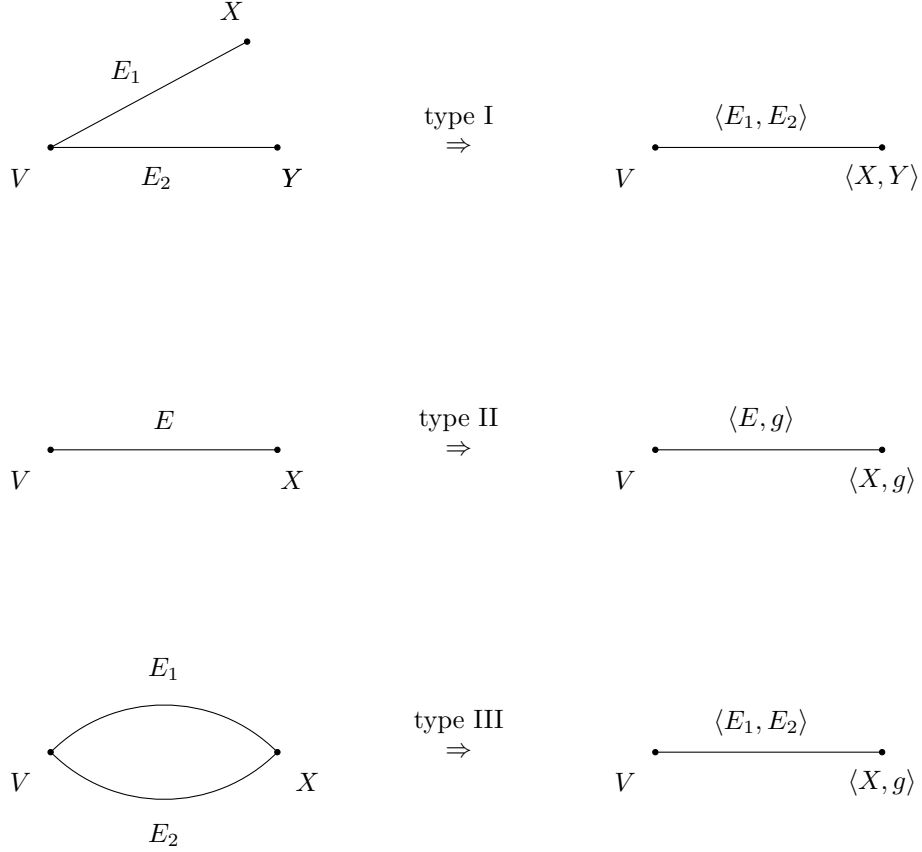


FIGURE 8. Folding operations

Let σ be a 2-cell of X . We now examine what can happen as we repeatedly fold corners of σ , at each stage replacing σ by the new 2-cell created. Since each 1-cell of \tilde{X} injects into T we can assign each 1-cell γ of X a length, namely the distance in T between $\theta(u)$ and $\theta(v)$ where u, v are the vertices of a lift of γ in \tilde{X} .

As above let x be the corner vertex and let e_1, e_2 be the incident edges.

If e_1, e_2 have the same length, then we can fold the corner of σ . If e_1 is shorter than e_2 then subdivide e_2 so that the initial part has the same length as e_1 and then fold the corner. If e_2 is shorter than e_1 then we subdivide e_1 and then fold the corner. Now repeat the process. This process may terminate when all the 2-cell is folded away.

However it may happen that the folding sequence is infinite i.e. it never terminates.

First we give an example making it easier to understand the following general explanation. This example is a corrected version of Example 6 of [6].

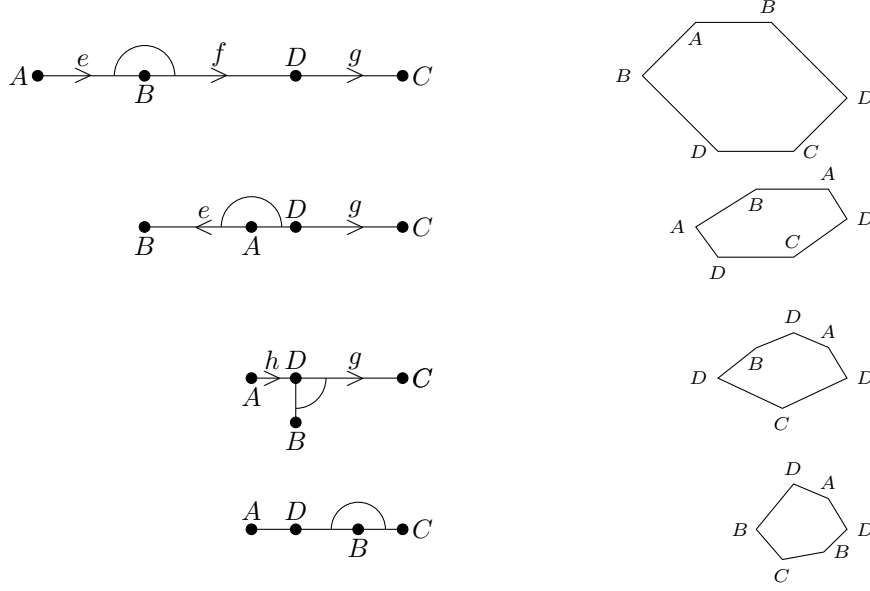


FIGURE 9. Folding sequence

Example 4.1. Let the complex X_1 have four vertices A, B, C, D and three oriented edges e, f, g . Let $\iota e = A, \tau e = B, \iota f = B, \tau f = D, \iota g = D, \tau g = C$. Let the groups of A, B, C, D be finite cyclic of order 3 and generated by u, v, y, z respectively. Let the 6-sided 2-cell be attached via the word

$$w \cup w' = \bar{e}a^{-1}ebfgc^{-1}\bar{g}d\bar{f}$$

Here $w = \bar{e}a^{-1}ebf$ and $w' = fd^{-1}gcb$. In this case X_1 is a 2-sphere with 4 cone points. Let G_1 be the group of this complex of groups.

The attaching word is describing a loop in \tilde{X}_1 the universal cover of the complex of groups. This loop maps to the loop, starting at B $\bar{e}efg\bar{g}\bar{f}$ in X_1 . This loop is obtained by omitting the joining elements, which are elements of the vertex group that has been reached at that point. How a path in \tilde{X}_1 corresponds to such a word was described earlier. We now discuss how the joining elements occur in $w \cup w'$.

In this case the 1-skeleton of X_1 is a tree. We choose a particular lift of this tree in X_1 to the universal orbifold cover \tilde{X}_1 , which is the (hyperbolic) plane tessellated by 6-gons. Here \tilde{X}_1 is the universal cover of the complex of groups described above. The attaching word traces out a loop in \tilde{X}_1 , which is the boundary of a fundamental region. Note that although the image of the path backtracks in X_1 , it is not allowed to backtrack in \tilde{X}_1 . This means that there must be non-trivial joining elements where the image backtracks. In \tilde{X} each fundamental region has 6 vertices, including one point (incident with 3 edges in \tilde{X}_1) from the orbits corresponding to A and C and two vertices (incident with 6 edges in \tilde{X}_1) from each of B and D . At one of the visits of the attaching word to vertices corresponding to B (or D) we have to use a non-trivial joining element. We can choose where this is. We get a presentation for G_1 in which the generators are a, b, c, d and a relation obtained by deleting the edges in the attaching word. This is because the 1-skeleton of X_1 is a tree. Thus there is a relation $a^{-1}bd^{-1}c = 1$. There are also relations $a^3 = b^3 = c^3 = d^3 = 1$.

Clearly there is a surjective homomorphism ϕ from G_1 to the rectangle group

$$G = \langle a, b, c, d | a^3 = b^3 = c^3 = d^3, x = a^{-1}b = c^{-1}d, y = a^{-1}c = b^{-1}d \rangle.$$

Consider the folding sequence corresponding to the “marking” in which lengths are assigned to the edges with $|e| = |g| = 1, |f| = \sqrt{2}$. Initially we have the 2-cell attached along $w \cup w' = \bar{e}a^{-1}ebfgc^{-1}\bar{g}d\bar{f}$. The attaching word is quadratic - its image in X_1 is $\bar{e}efg\bar{g}\bar{f}$ - and we will see that there is an infinite folding sequence in which $w = \bar{e}a^{-1}ebf$ is folded against $\bar{w}' = fd^{-1}gc\bar{g}$. The total length along top or bottom is $1 + \sqrt{2}$. After the first subdivision and fold we have a new complex X_2 with the same vertices A, B, C, D and with edges b, c and a new edge h with length $\sqrt{2} - 1$ with $\iota h = A, \tau h = D$ and the attaching word has become $eb\bar{e}b^{-1}a^{-1}bhgc^{-1}\bar{g}d\bar{h}$. Note that the joining element a^{-1} has changed to a conjugate $b^{-1}a^{-1}b$ as its position has changed.

The 2-cell has $w = eb\bar{e}b^{-1}a^{-1}bh, \bar{w}' = hd^{-1}\bar{g}cg$. After the next subdivision and fold we have a new complex X_3 with the same vertex set but with edges h, g, j where $\iota j = D, \tau j = B$ and j has length $1 - (\sqrt{2} - 1) = 2 - \sqrt{2}$ and the attaching word has become $jb\bar{j}(b^{-1}db)\bar{h}(b^{-1}a^{-1}bhgc\bar{g})$. Note that in this graph there is a vertex D of valency 3 whereas previously no vertex had valency more than 2. The attaching word visits the vertex D three times. As before we move the non-trivial joining element so that it is not at the start or end point of $w = jb\bar{j}(b^{-1}db)\bar{h}(b^{-1}a^{-1}b)h$. As noted above this change of position of the joining element corresponds to a change of the lift of a spanning tree - in this case the whole of the 1-skeleton S_1 of X_1 . Having chosen a lift \tilde{D} of D there are 27 different lifts of S_1 to \tilde{X}_1 . These are acted on by the stabilizer of \tilde{D} and there are nine different orbits under this action. The attaching map must have at least one non-trivial joining element on a visit to \tilde{D} , since otherwise one could have used the trivial group as the group at D . We can choose the lift of S_1 so that the joining element is non-trivial at exactly one visit. In this case we do it so that the non-trivial joining element is at a visit which is not the start or end point of w . We now have $w = jb\bar{j}(b^{-1}db)\bar{h}(b^{-1}a^{-1}b)h$ and $\bar{w}' = gc^{-1}\bar{g}$. We can translate the whole lift by b giving an attaching word $jb\bar{j}d\bar{h}a^{-1}hg(bcb^{-1})\bar{g}$. All we have done here is conjugate all the elements by b to make the elements shorter. The next subdivision and fold starts at D and folds j the shorter edge against g , so that we then have a new edge k with length $1 - (2 - \sqrt{2}) = \sqrt{2} - 1$ replacing g . Here $\iota k = B, \tau k = C$ and the attaching word is $\bar{j}d\bar{h}a^{-1}hj(c^{-1}bc)k(bcb^{-1})\bar{k}$. Now note that the situation we have reached is similar to the initial situation scaled by $\sqrt{2} - 1$. In fact the positions of A and C have been transposed from the original position. To get an exact scaling carry out the next 3 folds to get the initial position scaled by $(\sqrt{2} - 1)^2$.

The foliation of X_1 corresponding to our marking, lifts to a foliation of \tilde{X} and there is an \mathbb{R} -tree T_1 in which the points are leaves of this foliation. Clearly T_1 is a G_1 -tree. Let C be the cube complex for the rectangle group G . As described in [8] there is a cube complex \tilde{C} on which G acts. Thus

$$G = \langle a, b, c, d | a^3 = b^3 = c^3 = d^3, x = a^{-1}b = c^{-1}d, y = a^{-1}c = b^{-1}d \rangle.$$

There is a Euclidean subspace E of \tilde{C} acted on by $\langle x, y \rangle$. Assume that x acts on E by translation 2 in the x -direction, and y by translation of $2\sqrt{2}$. There is a foliation on C which induces a foliation on \tilde{C} and E given by the lines $x + y = c$. The leaves of the foliation on \tilde{C} give an \mathbb{R} -tree T . There is a map $\theta : X_1 \rightarrow C$, which induces a homomorphism, denoted $\theta' : G_1 \rightarrow G$. The map θ induces a map $\tilde{\theta} : \tilde{X}_1 \rightarrow \tilde{C}$ in

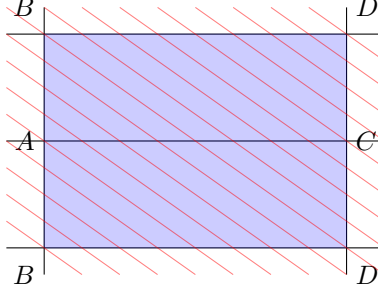


FIGURE 10. The image of a 2-cell

which a 2-cell of \tilde{X}_1 maps into E as indicated in Fig 10. It can be seen that the foliation of E lifts to the foliation on \tilde{X}_1 which is the one induced on the 2-cell of X_1 corresponding to the marking with $|e| = |f| = 1, |g| = \sqrt{2}$. Thus there is a map $\bar{\theta} : T_1 \rightarrow T$ which commutes with the actions of G_1 and G via θ' .

In G_1 the three elements $y = b^{-1}d, y' = a^{-1}c$ and $x = a^{-1}b = c^{-1}d$ freely generate a subgroup F and the minimal F -subtree of T_1 is non-simplicial. The elements y, y' are hyperbolic elements with the same hyperbolic length $2\sqrt{2}$. Their axes intersect in a segment of length $2\sqrt{2}$. In G these elements become equal and so have the same axis. There may be a G_1 -tree T' for which there are morphisms $T_1 \rightarrow T' \rightarrow T$, in which the two axes have a larger intersection than in T_1 .

Proof of Theorem 1.1. Choose a resolving complex X that has fewest 1-cells. Each attachment of a 2-cell must induce an infinite folding sequence, since Type I and Type III folds result in a reduction in the number of edges, so any Type I fold must be preceded by a subdivision, and there are no Type III folds. We define an equivalence relation on the 1-cells of X . We require that $e \sim f$ if there is a 2-cell that includes both e, f in its attaching map. We take \sim to be the smallest equivalence relation for which this is the case. For any 2-cell σ of X all the 1-cells to which it is attached lie in a single equivalence class. Thus for each equivalence class there is a subcomplex consisting of those 1-cells and its vertices together with the 2-cells attached to that class. Any two such complexes intersect in a set of vertices, but no edges. If two 2-cells σ, σ' share an edge e and vertex u in their boundaries, then we will see that we can choose the same joining elements in G_u for the two 2-cells. This is because, as in the example above, if a vertex requires a joining element, then at some stage in the folding sequence the attaching words for both σ and σ' will contain a subword of the form $fj_u\bar{f}$. In the tree T , f will map to an arc, $\tau f = u$ will map to a point v and $j_u f$ will map to an arc intersecting f in the single point v . Thus f and $j_u f$ determine different *directions* d_1, d_2 at v , such that $d_2 = j_v d_1$. This will be true for both the attached two cells, so that we can choose the same j_u for both attaching words.

For the moment let us assume that there is a single equivalence class, and so there is a single subcomplex X itself. We consider first the case when the one skeleton of X is a tree, as in Example 4.1 above.

In the resolution of the action of G on T each 1-cell e is effectively assigned a length $|e| \in \mathbb{R}$, which is the length of the arc in T joining the images in T of the end points of a lift of e to the universal cover \tilde{X} . Let A be the subgroup of \mathbb{R} ,

regarded as an additive group, generated by the set $\{|e||e \in X^1\}$. The group A is isomorphic to \mathbb{Z}^n for some n . Let P_n be a parallelepiped group corresponding to an n -cube, in which we will assign orders to the vertex elements in a certain way.

As described in [8] the group P_n acts on a 1-connected, n -dimensional cubing C_n that contains an n -dimensional Euclidean space E_n and P_n contains a free abelian rank n subgroup J_n that acts on E_n by translations of 2 units in each of the coordinate directions. The space $P_n \backslash C_n$ is obtained from $J_n \backslash E_n$ by identifying a single antipodal pair of n -cells in $J_n \backslash E_n$. We show that there is a subgroup G' of G generated by cyclic subgroups of distinct vertex groups of X and a map $\theta : \tilde{X} \rightarrow C_n$ which is equivariant with respect to a homomorphism $G' \rightarrow P_n$. For each 2-cell σ in X there is a lift $\tilde{\sigma}$ such that $\theta(\tilde{\sigma}) \subset E_n$, and the map θ is defined by specifying how θ acts on these 2-cells.

Suppose A is generated by the real numbers $\alpha_1, \alpha_2, \dots, \alpha_n$. We assume now that J_n acts on E_n by translations of $2\alpha_i$ in each of the coordinate directions. We give E_n the structure of a cell complex in the obvious way so that J_n acts cellularly and there are 2^n orbits of n -cells. For the left hand vertex v in a particular 2-cell $\tilde{\sigma}$ let $\theta(v)$ be the origin in E_n . Each 1-cell in \tilde{X} has a particular length in A , and this length will determine a vertex of E_n . Proceeding around the boundary of $\tilde{\sigma}$ will determine a loop in the positive quadrant of E_n . The distance from the origin will increase as one passes along the top or bottom of $\tilde{\sigma}$ away from v and one will reach the same point which is the image of the right hand vertex of the 2-cell. Since the 1-skeleton of X is a tree X^1 , in the loop in X^1 corresponding to the boundary of σ the path corresponding to successive visits to a particular vertex will pass over each edge an even number of times. If the edge is oriented then it must pass over the edge the same number of times in each direction. This means that in E_n if two vertices of $\tilde{\sigma}$ are in the same G -orbit, then their images in E_n are in the same J_n -orbit. Consider the subgroup G_σ of G generated by the joining elements of σ . The group G_σ is the fundamental group of the complex X_σ of groups corresponding to σ . All the vertex groups and edge groups are cyclic. Each one is generated by a power of a joining element. An element that fixes an edge of \tilde{X}_σ must fix every edge for the reason explained above. Thus G_σ has a cyclic normal subgroup N_σ such that G_σ/N_σ acts on \tilde{X}_σ with trivial edge stabilizers. We now show that there is a map $\theta_\sigma : \tilde{X}_\sigma \rightarrow \tilde{C}_n$ which is equivariant with respect to a homomorphism from G_σ to P_n with kernel N_σ .

We map G_σ into P_n by mapping N_σ to the identity element and giving each joining element to a vertex element in which its order is the order of that element modulo N_σ . The defining relations between the joining elements of G_σ are given by the attaching maps of σ as described above. If we map $\tilde{\sigma}$ into E_n then the relation is a consequence of the relations of J_n . In particular if we add in extra relations that say that if x, y, u, v are joining elements, then xy^{-1} commutes with uv^{-1} and add in the relation that the generator of N_σ is trivial, then we have a presentation for a subgroup of P_n .

If a different 2-cell σ' of X shares an edge e with σ , then there will be a lift $\tilde{\sigma}'$ that shares an edge with $\tilde{\sigma}$ and the boundary map of σ' will determine a closed path in E_n . Thus the maps θ_σ and $\theta_{\sigma'}$ match up nicely and carrying out the extension to every 2-cell we see that there will be a map $\theta : \tilde{X} \rightarrow C_n$ which restricts to θ_σ on each 2-cell $\tilde{\sigma}$. This map will be equivariant with respect to G' , the subgroup of G generated by all the G_σ for every 2-cell σ .

We now want to extend θ so that it is a homomorphism on G rather than G' . In order to do this we have to define it on the vertex groups of X . Let $v \in VX$ have vertex group $G(v)$. Then $G(v)$ contains a non-trivial joining element g_v that determines a vertex $\theta(v)$ of E_n . Choose a particular n -cube c in E_n : the 1-skeleton of this cube will be a transversal for the action of J_n on the 1-skeleton of E_n . For some $g \in J_n$ (which is a product of elements of the form xy^{-1} , where x, y are joining elements) we have that gv is a vertex of c . We define a complex of groups on the 1-skeleton of C by assigning the group $gG(v)g^{-1}$ to the vertex gv . The complex of groups has also to contain the complex of groups associated with the action of P_n on C_n , so that usually there will be three 2-cells with the same boundary 4-cycle, for each face of the parallelepiped. For any edge of c the group assigned to that edge will be the intersection of its vertex groups. I assumed in an earlier version of this paper that this intersection would be trivial and so one could “pull off” vertex groups, creating a graph of groups in which there is a vertex group with G' as its fundamental group. It can be seen from the Higman group considered in Example 3.1 that this is not always possible. If we enlarge edge and vertex groups of C in this way we end up with the target group for G , which is the fundamental group of the complex of groups based on c .

Note that any folding sequence results in a sequence of complexes that resolve the action on T and it can never be the case that the joining element becomes trivial in the folding. This is because at some stage in a folding sequence the joining element will lie between an edge e and \bar{e} and if the joining element is trivial, then the action on T will not be resolved. Two vertices in different G orbits in \tilde{X} may end up in the same J_n -orbit (I don't know if this can happen - it may be that if two vertices are mapped to the same J_n -orbit, then some folding sequence will result in a Type III fold and the images of the vertices lying in the same G -orbit). If it can happen, then a way of dealing with this is to give the vertex element in P_n as its order the lowest common multiple of the finite orders of any joining elements mapped to it (modulo the smallest power of that element that fixes an edge) and map each joining element to an appropriate power of the vertex element in P_n .

If the one skeleton X^1 of X is not a tree, then let Y be a spanning tree for X^1 . In this case a generating set of G will consist of vertex groups and the joining elements corresponding to a lift of Y to \tilde{X} together with a connecting element generator for each edge e of X that is not in Y . If u, u' are the vertices of e then the lift of Y to \tilde{X} will contain unique lifts \tilde{u}, \tilde{u}' of u, u' . There will not usually be an edge of \tilde{X} joining \tilde{u}, \tilde{u}' but there is a lift \tilde{e} of e with $\iota\tilde{e} = \tilde{u}$. The generator corresponding to e is an element $c(e)$ of G such that $c(e)^{-1}\tau\tilde{e} = \tilde{u}'$. The edge e is given a length $\frac{1}{2|e|}$ in our action on T . We want the corresponding generator to be mapped to a translation by $|e|$ in J_n . In this case let A be the subgroup of \mathbb{R} generated by $\{|e||e \in EY\} \cup \{\frac{1}{2}|e||e \in EX^1 \setminus EY\}$. Taking a generator $\frac{1}{2}|e|$ for $e \in EX^1 \setminus EY$ means that there is a translation of $|e|$ in J_n since it is through an even number of units. Let G' be the subgroup of G generated by all the joining elements and connecting elements. We can now define a homomorphism from G' to P_n . We map each joining element as before, as it will correspond to a vertex of the spanning tree. For each edge e that is not in the spanning tree, we introduce a vertex that is the midpoint of the subdivided edge. In P_n we give the corresponding vertex element $v(e)$ the order two. If the vertex element in P_n corresponding to the initial vertex of e in Y is u , then we map the connecting element $c(e)$ to the element $v(e)u^{-1}$,

which will then correspond to a translation of length $|e|$. We now expand the vertex groups to get an augmented parallelepiped group that is a target group for G as before.

If there is more than one equivalence class for the relation \sim then corresponding to each equivalence class is a subcomplex of X and any two such subcomplexes intersect in finitely many vertices but no edges. Let Y be the graph in which VY is the set of equivalence classes EX/\sim and there is an edge joining two vertices for each vertex of X that the equivalence classes share. Each vertex $v \in VY$ is associated with a subcomplex X_v of X and a subgroup $\mathcal{Y}(v)$ of G which is the fundamental group of a complex of groups based on X_v . Then G is the fundamental group $\pi(\mathcal{Y}, Y)$ of the graph of groups so that if $v \in VY$ then $\mathcal{Y}(v)$ is as above and if $e \in EY$ is associated with $v \in VY$ that lies in two equivalence classes then $\mathcal{Y}(e)$ is the group of that vertex.

The graph of groups we have constructed has the properties listed in the statement of the theorem, and so the proof is complete. \square

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