Proper Decompositions of Finitely Presented Groups

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Introduction

This is an expanded set of slides of the talk the second author gave at the Chiswell Symposium at Queen Mary on November 9th 2009. In this set of slides proofs are included of the main result.

An action of a group G on a tree without involutions and with one orbit of edges corresponds to a decomposition (or splitting) of the group G either as a free product with amalgamation or an HNN-group. Michah Sageev (1995) described a cubing $\tilde{C}$ associated with a finite number of such decompositions. The space $\tilde{C}$ is a simply connected $CAT(0)$ cube complex with a G-action.

A cube complex is similar to a simplicial complex except that the building blocks are n-cubes rather than simplices.

For any group G a subgroup H of G is *G-unsplittable* if for any action of G on a tree the induced action of H is trivial. i.e. it fixes a vertex.

A trivial decomposition or splitting of G is a decomposition as a free product with amalgamation $G = G *_H H$ for some subgroup H of G.

Main Theorem

A finitely presented group G has a finite list of n splittings for which the associated G-cubing $\tilde{C}$ has edge and vertex groups which are G-unsplittable. Every G-unsplittable subgroup of G fixes a vertex of $\tilde{C}$.

The group G has a non-trivial action on a tree if and only if at least one splitting in the list is non-trivial. The list of splittings is computable.

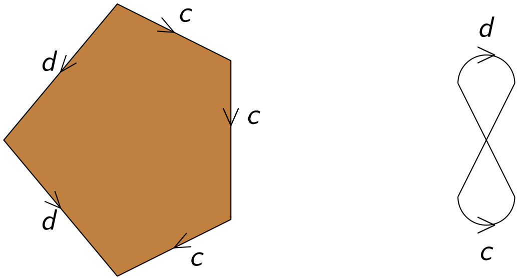
The list of splittings is computable in the sense that each splitting corresponds to a decomposition of G either as a free product with amalgamation or as an HNN-group. For each splitting in the list, sets of generators can be given for the vertex groups and edge groups of the decomposition.

It may be the case that this information will be insufficient to determine whether of not the splitting is trivial. Thus there is a presentation of a group G for which there is no programme which will determine if the group G is trivial. Clearly there is then a presentation for $G * G$ for which there is no programme which will determine if the group has a non-trivial splitting. If one takes the obvious presentation for $G * G$ in which one takes a repeat set of generators and relations, then one can easily specify a splitting which is non-trivial if the group has such a splitting, and so this accords with our theorem.

The cubing $\tilde{C}$ is simply connected and $CAT(0)$. A complex of groups $(\mathcal{C}, C)$ can be obtained from the G action on $\tilde{C}$ by taking $C = G\backslash\tilde{C}$ and assigning a subgroup of G to each edge or vertex of C that is the stabilizer of an appropriate lift of an edge or vertex in $\tilde{C}$. This complex of groups and its fundamental group can be regarded as a universal JSJ-decomposition for G.

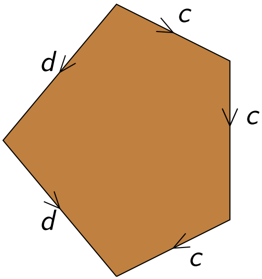
Group presentations and tracks

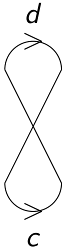
The cell complex for the trefoil group $G = \langle c, d | c^3 = d^2 \rangle$



Attach the 5-sided disc to the figure eight as specified by the letters and arrows. The space X has $\pi_1(X) = G$.

A presentation can be changed so that every relation has length at most three.





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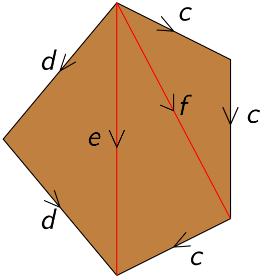

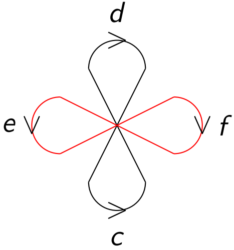

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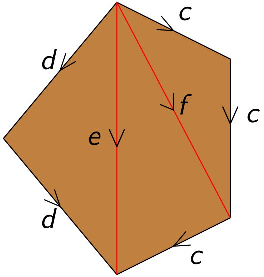
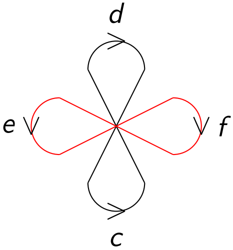




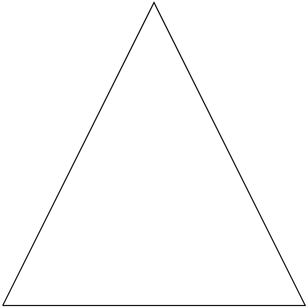
Thus $G = \langle c, d, e, f | d^2 = e, e = fc, f = c^2 \rangle$.

The cell complex X consists of three 3-sided 2-cells attached to a 4-leaved rose.

In fact any finitely presented group has a presentation in which each relator has length one or three. Thus one can replace the relation $a^2 = 1$ by the two relations $a^2 = b, b = 1$.

Let X be a cell complex in which each 2-cell is 3-sided, or 1-sided.

A pattern is a subset of X which intersects each 2-cell in a finite number of disjoint lines each of which intersects the boundary of the 2-cell in its two end points which lie in distinct edges. A pattern has empty intersection with any 1-sided 2-cell.



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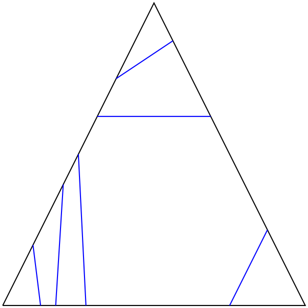
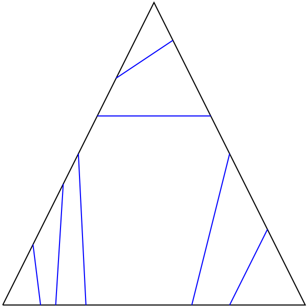


If X has m 2-cells then a pattern is specified (up to an obvious equivalence) by a $3m$-vector in which there are three coefficients for each 2-cell which record the number of lines joining the two edges at each corner. Thus for previous 2-cell



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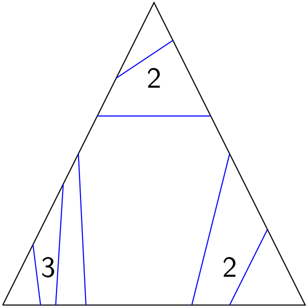
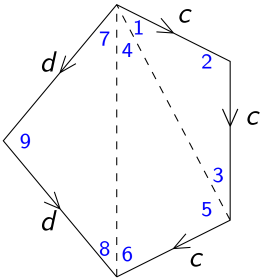


the coefficients $2, 2, 3$ record the intersection of the pattern with that particular 2-cell.

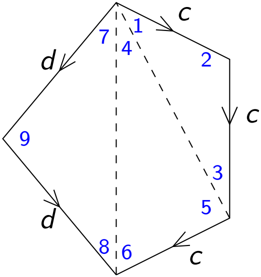
For the complex X for the trefoil group G a pattern is specified by a 9-vector. The i-th coefficient corresponds to the number of lines crossing the i-th corner as indicated below



An example is as follows. The 9-vector

$t_1 = (1, 1, 1, 0, 2, 0, 0, 0, 0)$

corresponds to the pattern



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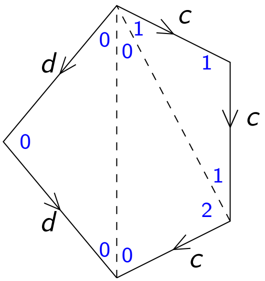

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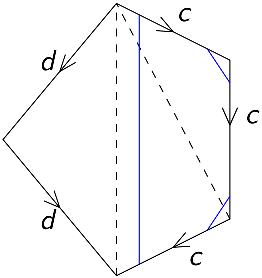


This pattern is connected. A connected pattern is called a track.

A $3m$-vector corresponds to a pattern, if and only if

- (i) Each entry is a non-negative integer.
- (ii) It is a solution vector to a finite set of linear equations called the matching equations.

For the trefoil complex X a vector of non-negative integers $x = (x_1, x_2, \ldots, x_9)$ is a pattern if it satisfies the matching equations

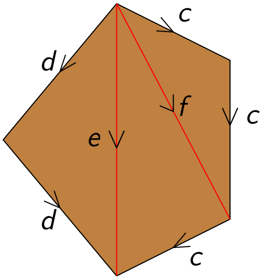
$x_1 + x_2 = x_2 + x_3 = x_5 + x_6$, (number of intersection points with edge c)

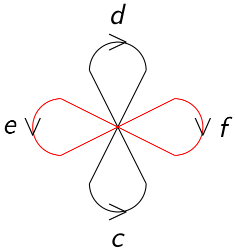
$x_1 + x_3 = x_4 + x_5$ (number of intersection points with edge f)

$x_4 + x_6 = x_7 + x_8$ (number of intersection points with edge e)

$x_7 + x_9 = x_8 + x_9$ (number of intersection points with edge d)

The universal cover $\tilde{X}$ is a 2-dimensional cell complex. Its 1-skeleton is the Cayley graph of G with respect to the generating set corresponding to the 1-cells of X.





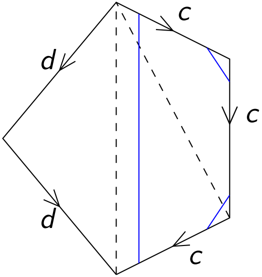
In this case $\tilde{X}$ has vertex set which can be identified with G, four orbits of 1-cells and three orbits of two cells. There is a free action of G on $\tilde{X}$ (on the left) and $X = G\backslash\tilde{X}$

A pattern P in X lifts to a pattern $\tilde{P}$ in $\tilde{X}$. Each component track t of $\tilde{P}$ separates $\tilde{X}$, i.e. $\tilde{X} - t$ has two components. This means that the dual graph to $\tilde{P}$ is a tree $T = T(P)$. The action of G on $\tilde{X}$ induces an action of G on T. By Bass-Serre theory we obtain a decomposition of G as the fundamental group of a graph of groups. This graph of groups has underlying graph $G \backslash T$, the edges and vertices of which are labelled by appropriate stabilizers of T.

Thus a pattern in X determines a decomposition of G. If the pattern consists of a single track then the decomposition is as a free product with amalgamation if the track in X is separating and it is as an HNN-group if the track is untwisted and non-separating.

The track shown in blue is separating and gives a decomposition of G as a free product with amalgamation

$G = \langle d \rangle *_{\langle d^2 = c^3 \rangle} \langle c \rangle.$



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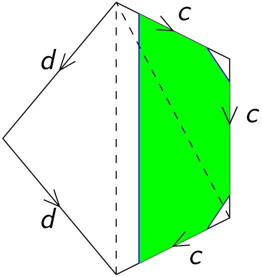
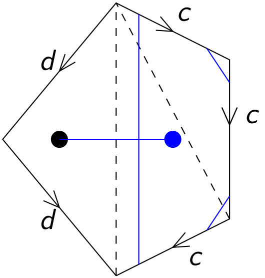
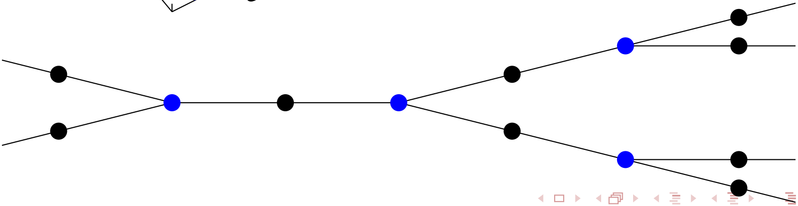
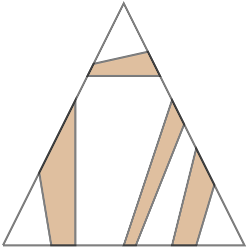


The decomposition of the group $G = \langle d \rangle *_{\langle d^2 = c^3 \rangle} \langle c \rangle$ corresponding to this track corresponds to an action of G on a tree T for which $G \backslash T$ has one orbit of edges and two orbits of vertices.

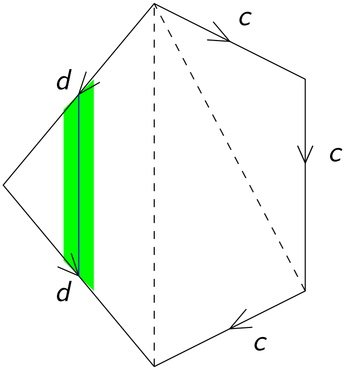




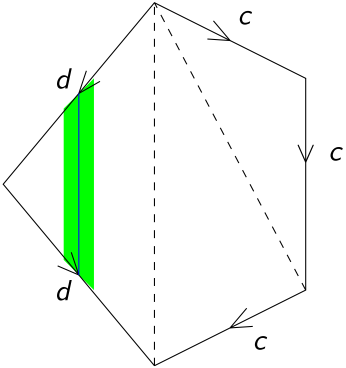
A track is separating if and only if it intersects each 1-cell an even number of times. More generally a pattern is separating, i.e each component track is separating, if it intersects each 1-cell an even number of times. This is because if a pattern intersects each 1-cell an even number of times then X can be 2-coloured so that adjacent regions have different colours.



A tracks t is twisted if the pattern $2t$ is also a track. A track is untwisted if and only if a small neighbourhood of it is homeomorphic to $t \times I$

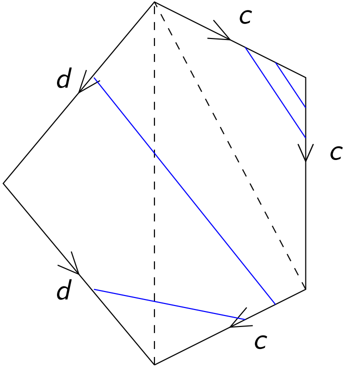


A track t is twisted if the pattern $2t$ is also a track. A track is untwisted if and only if a small neighbourhood of it is homeomorphic to $t \times I$. For this track a small neighbourhood is a Möbius band.

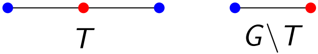


A separating track is always untwisted. It t is twisted, then $2t$ is separating and hence untwisted.

The track t is twisted so the pattern $2t$ is also a track. The separating track $2t$ gives the trivial decomposition $G = G *_H H$ where H has index two in G

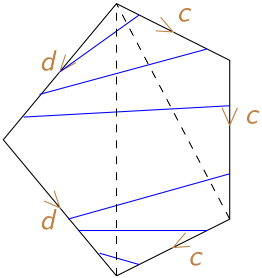


The trivial decomposition $G = G *_H H$ where H has index two in G corresponds to an action of G on the tree T.



In this case G fixes the centre vertex and H is the stabilizer of each of the other vertices. If a track corresponds to a trivial decomposition then the corresponding tree has diameter two. It may not be possible to decide if a decomposition is trivial or not. One has to be able to decide if a given a set of elements of a group whether it generates a proper subgroup.

The track shown in blue is non-separating and untwisted, and gives a decomposition of G as an HNN-group.



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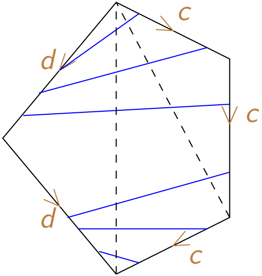
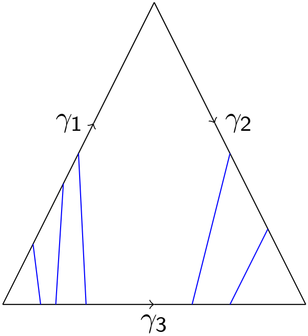


Such a track is always associated with a homomorphism $G \to \mathbb{Z}$.
In this case $c \mapsto 2, d \mapsto 3$.

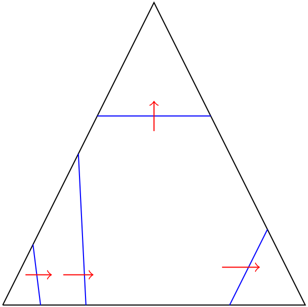
Suppose we are given a homomorphism $\theta : G \to \mathbb{Z}$. There is a natural way to construct a pattern $P(\theta)$ corresponding to θ. The oriented 1-cells of X correspond to generators of G. Choose the orientation so that $\theta(\gamma) \geq 0$ for each oriented 1-cell γ. Let $\gamma_1, \gamma_2, \gamma_3$ be the oriented faces of a 2-cell σ. By relabeling we can assume that $0 \leq \theta(\gamma_1) \leq \theta(\gamma_2) \leq \theta(\gamma_3)$. The 2-cell corresponds to the relation $\gamma_3 = \gamma_1\gamma_2$ and hence

$\theta(\gamma_3) = \theta(\gamma_1) + \theta(\gamma_2)$.

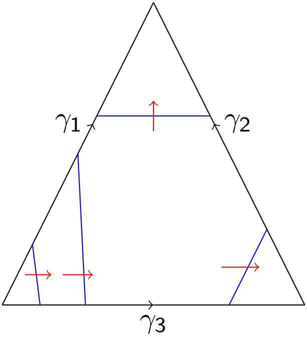
We obtain a pattern in which the corners of σ are assigned the integers $0, \theta(\gamma_1), \theta(\gamma_2)$ as below, where $\theta(\gamma_1) = 3, \theta(\gamma_2) = 2$.



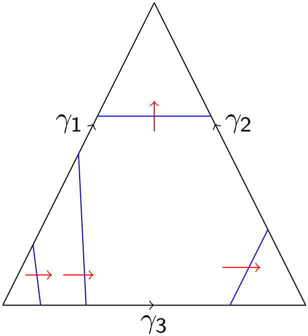
Conversely a pattern p in which every component track is untwisted determines a homomorphism $\theta_p : G \to \mathbb{Z}$, and this homomorphism is surjective if the pattern is a non-separating track. To see this note that since the components of p are untwisted we can put transverse arrows on the components of p as below.



One can do this so that the arrows on a track t indicate an orientation of I in a small neighbourhood of t homeomorphic to $t \times I$. One can change this orientation for a particular track reversing the arrows crossing that track. Now choose an orientation for each 1-cell.

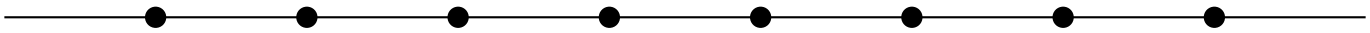


Thus we get the relation $\gamma_1 = \gamma_3\gamma_2$. The value $\theta_p(\gamma)$ is found by taking the number of arrows on the pattern that are in the same direction as the arrow on γ minus the number of arrows on the pattern that are in the opposite direction. In the diagram below $\theta_p(\gamma_1) = \theta_p(\gamma_3) = 3, \theta_p(\gamma_2) = 0$.



We have $\theta_{P(\theta)} = \theta$. But not usually $P(\theta_p) = p$. Note $\theta_p = 0$ if and only if p is a separating pattern. And $p = P(\theta_p) + q$ where $q = 0$ or q intersects each 1-cell an even number of times and so is a separating pattern.

A homomorphism $G \to \mathbb{Z}$ gives an action on the tree

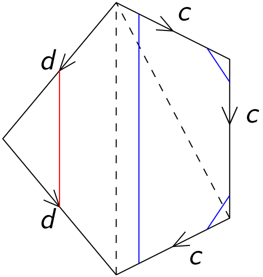


Not every action of a group on a tree corresponds to a pattern. An action corresponding to a pattern with untwisted component tracks is called geometric. However every action on a tree (without involutions) is resolved by a geometric action. What this means is that if S is a tree with a G-action then there is a G-morphism $T \to S$ where T is a tree with a geometric G-action.

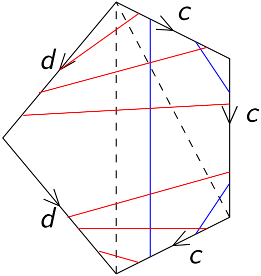
In particular this means that if G has a non-trivial action on a tree then it has a non-trivial geometric action on a tree. Thus there is an untwisted track for which the corresponding action is non-trivial.

A pattern is specified by a vector p. If p, q are patterns then so is $p + q$. Two tracks s, t are compatible if the pattern $s + t$ has component tracks s, t.

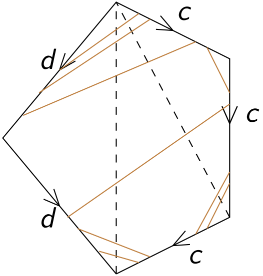
The tracks shown below are compatible



The tracks t_1. t_2 shown below are incompatible



The sum $t_1 + t_2$ (a twisted track) is shown below



It has the same intersection with the 1-skeleton as $t_1 \cup t_2$.

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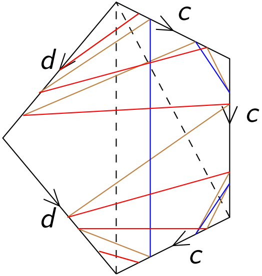


It has the same intersection with the 1-skeleton as $t_1 \cup t_2$.

A pattern is a vector solution with non-negative integer coefficients to a set of homogenous linear equations. The set of solutions with non-negative rational coefficients form a cone which in projective space is the convex closure of finitely many extreme solutions. Multiplying by a rational we obtain a track, called an extreme track.

A track t is extreme if and only if for a positive integer n the only patterns p, q for which $nt = p + q$ are $p = kt, q = (n-k)t$ where $k = 1, 2, \ldots, n-1$. The trefoil presentation has 5 extreme tracks. The Higman presentation $\langle a, b, c, d | a^2 = bab^{-1}, b^2 = cbc^{-1}, c^2 = dcd^{-1}, d^2 = dad^{-1} \rangle$ has 8769 extreme tracks.

The first author in his 1987 thesis wrote programmes which gave the extreme tracks for a presentation, decomposed patterns into tracks and gave the decomposition corresponding to a track. His programmes are available on his homepage http://www.layer8.co.uk/maths/index.htm

We were hoping at that time to show that if a group has a non-trivial action on a tree, then at least one extreme track gives a non-trivial decomposition.

We can now show that there is a longer (but still finite) list of tracks that does have this property.

A track t is minimal if t cannot be written as a sum of two patterns. An extreme track is minimal, but not every minimal track is extreme.

For the trefoil there are 5 extreme tracks and 8 minimal tracks. Thus $t_3 = (0, 2, 0, 0, 0, 2, 1, 1, 0)$, $t_4 = (2, 0, 2, 2, 2, 0, 1, 1, 0)$ are extreme tracks. $t_6 = (1, 1, 1, 1, 1, 1, 1, 1, 0)$ is a minimal track. It is not extreme since $2t_6 = t_3 + t_4$.

There are only finitely many minimal tracks.

There is a nice account of why there are only finitely many extreme and minimal tracks in Chapter 8 of G.Hemion's book.

For vectors p, q, write $p \leq q$ if $p_i \leq q_i$ for each i, and $p < q$ if $p \leq q$ and $p \neq q$. Thus a track t is minimal if there is no track t' with $t' < t$.

Every pattern p can be written as a sum $p = \alpha_1 t_1 + \alpha_2 t_2 + \cdots + \alpha_r t_r$ where $t_1, t_2, \ldots, t_r$ are extreme and $\alpha_1, \alpha_2, \ldots, \alpha_r$ are positive rationals.

But with minimal tracks, p can be written $p = \beta_1 m_1 + \beta_2 m_2 + \cdots + \beta_s m_s$ where $m_1, m_2, \ldots, m_s$ are minimal and the β_i's are positive integers.

An untwisted track is called minimal if it cannot be written as a non-trivial integer sum of untwisted tracks. Every untwisted minimal track is a minimal untwisted track, but there are minimal untwisted tracks that are not minimal tracks.

Every untwisted track can be written as an integer sum of minimal ones. The trefoil complex discussed at length has 13 minimal untwisted tracks.

Theorem

There are only finitely many minimal untwisted tracks.

Proof

We first show that a non-separating minimal untwisted track t is a minimal track. As there are only finitely many minimal tracks, it then follows that there are only finitely many non-separating minimal untwisted tracks.

We know that $\theta_t : G \to \mathbb{Z}$ is a surjective homomorphism. Since $t = P(\theta_t) + q$ where $q = 0$ or q is a separating pattern, and t is minimal, we must have $q = 0$. Here we also use the fact that the tracks of $P(\theta_t)$ are untwisted. This is because the orientation of the 1-cells given by the homomorphism also gives a transverse orientation to a small neighbourhood of a component track of $P(\theta_t)$. Thus $t = P(\theta_t)$. This means that in each 2-cell there is at least one corner for which the corresponding coefficient of t is zero. If $p < t$ then p will also have this property. But this will mean that $p = P(\theta_p)$ and each component of p is untwisted and non-separating. Since t is a minimal untwisted track, this cannot happen. Hence t is a minimal track.

We now show that there are only finitely many separating minimal untwisted tracks. Let t be such a track. There are no separating patterns p for which $p < t$, since, if both t and p are separating then so is $t - p$. Here we use the condition that a pattern is separating if and only if it intersects each 1-cell an even number of times.

We can write $t = \beta_1 t_1 + \beta_2 t_2 + \cdots + \beta_r t_r$ where $t_1, t_2, \ldots, t_r$ are minimal and $\beta_1, \beta_2, \ldots, \beta_r$ are positive integers. There can be no separating tracks in this sum, unless it is the whole sum. Also for each $t_i, 2t_i$ is always a separating pattern. Thus the only possible sums with more than one summand (which will be t_i or $2t_i$) are of the form $t = n_1 + n_2 \cdots + n_r$ where the n_i's are distinct minimal tracks that are non-separating. Clearly there are only finitely many possibilities.

The trefoil has 5 extreme tracks and 8 minimal tracks. These are

$t_1 = (1, 1, 1, 0, 2, 0, 0, 0, 0)$

$t_2 = (0, 0, 0, 0, 0, 0, 0, 0, 1)$

$t_3 = (0, 2, 0, 0, 0, 2, 1, 1, 0)$

$t_4 = (2, 0, 2, 2, 2, 0, 1, 1, 0)$

$t_5 = (2, 0, 2, 4, 0, 2, 3, 3, 0).$

$t_6 = (1, 1, 1, 1, 1, 1, 1, 1, 0)$

$t_7 = (2, 0, 2, 3, 1, 1, 2, 2, 0)$

$t_8 = (1, 1, 1, 2, 0, 2, 2, 2, 0).$

In the above the first five tracks are extreme, the red tracks are separating, the blue track is untwisted and non-separating and the remaining tracks are twisted.

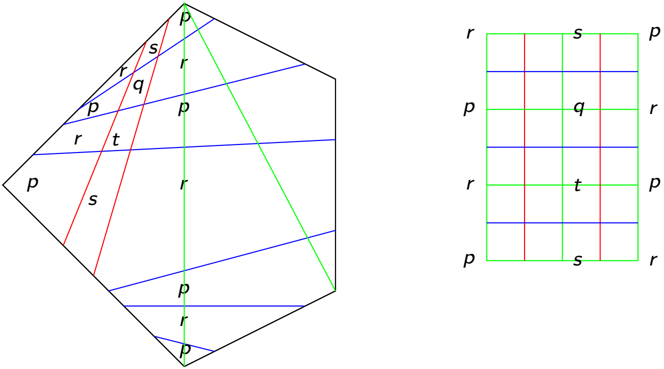
The minimal untwisted tracks are the 11 vectors $t_1, 2t_2, 2t_3, 2t_4, t_5, t_7, t_8, t_2 + t_3, t_2 + t_4, t_2 + t_6, t_3 + t_6$.

One arrives at this list by looking at $2t_i$ where t_i is twisted and sums of distinct non-separating minimal tracks and checking that there is no smaller untwisted track. Note $2t_6 = t_1 + t_8$ and $t_4 + t_6 = t_1 + t_7$ and so neither $2t_6$ not $t_4 + t_6$ is a minimal untwisted track.

Of these 11 tracks only $t_1, 2t_2$ and t_5 give non-trivial decompositions. The tracks t_1 and $2t_2$ are compatible and give the same decomposition as a free product with amalgamation

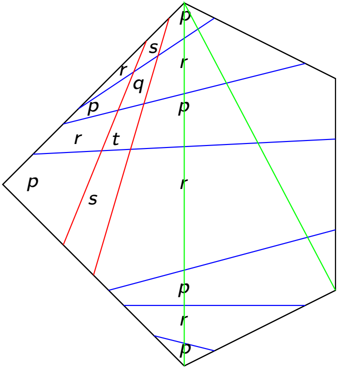
An untwisted track corresponds to a decomposition of the group G either as a free product with amalgamation or an HNN-group. Michah Sageev (1995) described a cubing $\tilde{C}$ associated with a finite number of such decompositions. The space $\tilde{C}$ is a $CAT(0)$ cube complex with a G-action.

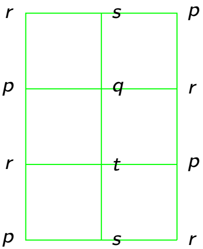
A cube complex is similar to a simplicial complex except that the building blocks are n-cubes rather than simplices.



The diagram show the cube complex $C = G\backslash\tilde{C}$ for the two decompositions given by the red and blue tracks t_1, t_2. The vertices of C correspond to the components of $X - (t_1 \cup t_2)$. Two vertices are joined by an edge if the components are adjacent in X.

Now note that the track $t_1 + t_2$ also corresponds to a “track ” in C. A pattern $\beta t_1 + \beta t_2$ where β_1, β_2 are non-negative integers will correspond to a pattern in C





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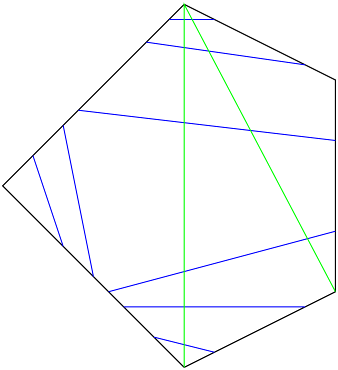
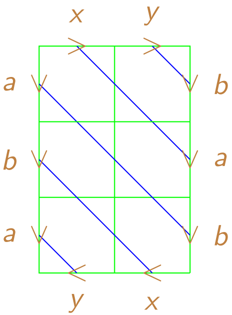
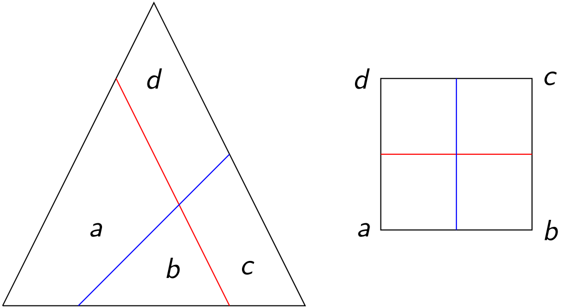
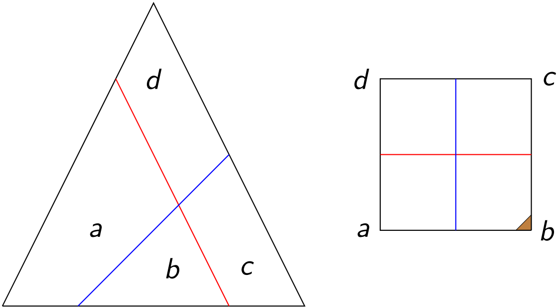




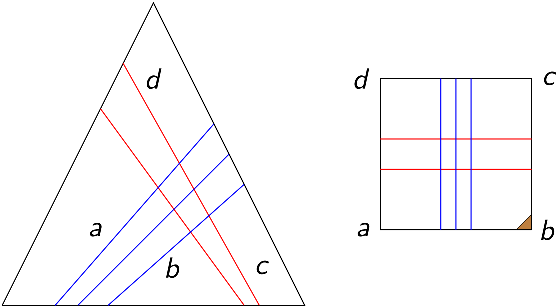
Each crossing point of tracks t_1, t_2 in X corresponds to a 2-cell of C. Let s_1, s_2 be the line segments of the two tracks in the three-sided two cell in X. One of the three sides contains 2 points of $s_1 \cup s_2$. In this case the bottom side.



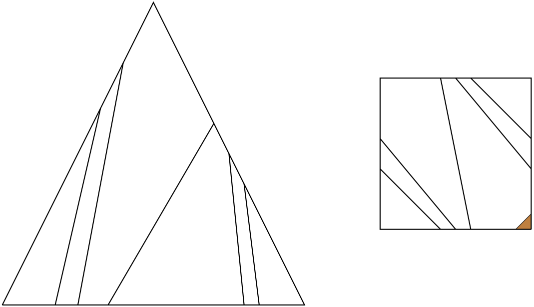
Now mark the corner (b) in the square in C corresponding to that side.



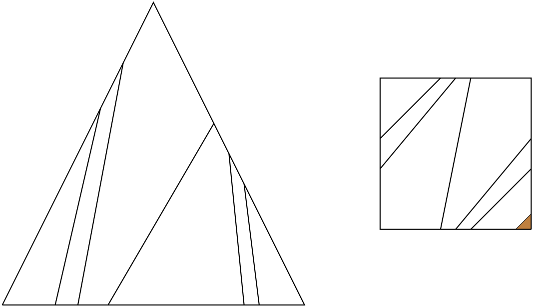
There are two ways to join the intersection points of $2t_1 \cup 3t_2$ with the boundary of the 2-cell by disjoint lines in the interior of the 2-cell. In the track corresponding to $2t_1 + 3t_2$ avoid lines crossing the marked corner.



In the track corresponding to $2t_1 + 3t_2$ avoid lines crossing the marked corner. Join this way!



In the track corresponding to $2t_1 + 3t_2$ avoid lines crossing the marked corner. Not this way!



An untwisted track in C will determine a decomposition of G, i.e. an action on a tree. This is because $\tilde{C}$ is simply connected and so a track in $\tilde{C}$ separates and again the dual graph will be a tree. Let $\tilde{C}$ be the cubing corresponding to all minimal untwisted tracks. The advantage of using tracks in C rather than tracks in X is that the action of G on $\tilde{C}$ is not usually free. Thus if all the minimal untwisted tracks give trivial decompositions then G fixes a vertex of $\tilde{C}$ which means that any track in C will give a trivial action. Any untwisted track in X will correspond to an untwisted track in $C = G \backslash \tilde{C}$.

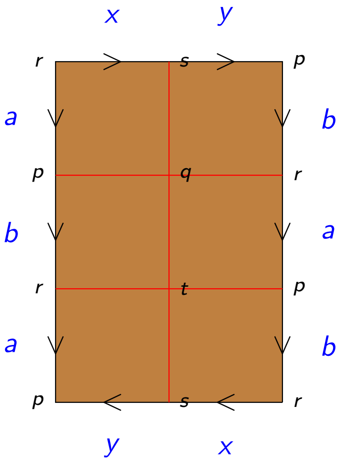
We have the following theorem.

[For any group G a subgroup H of G is *G-unsplittable* if for any action of G on a tree the induced action of H is trivial. i.e. it fixes a vertex.]

Main Theorem

A finitely presented group G has a finite list of n splittings for which the associated G-cubing $\tilde{C}$ has edge and vertex groups which are G-unsplittable, and every G-unsplittable subgroup of G fixes a vertex of $\tilde{C}$.

The group G has a non-trivial action on a tree if and only if at least one splitting in the list is non-trivial. The list of splittings is computable.

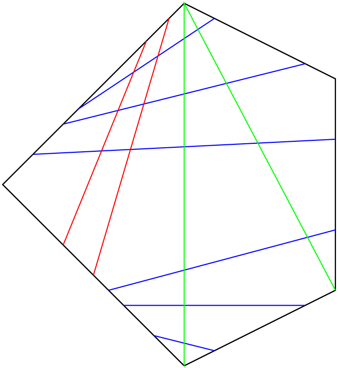


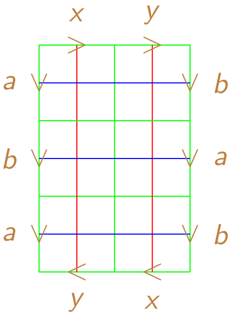
The above is the complex $C = G\backslash\tilde{C}$ for the trefoil group.

Proof

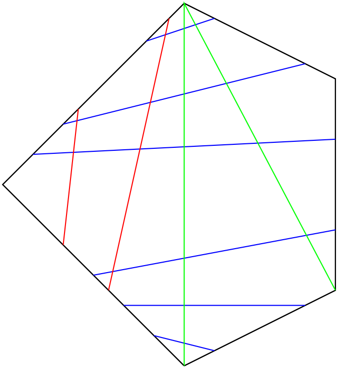
Let $t_1, t_2, \ldots, t_n$ be the minimal unwisted tracks. Consider their union in X. There are different ways of embedding the tracks. We assume that they intersect transversely with intersection points in the interior of the 3-sided 2-cells. For a particular embedding there is a G-map $\tilde{\theta} : \tilde{X} \to \tilde{C}$ (inducing $\theta : X \to C$) so that the lift $\tilde{t}_i$ of a track t_i is the inverse image of an orbit of hyperplanes in $\tilde{C}$.

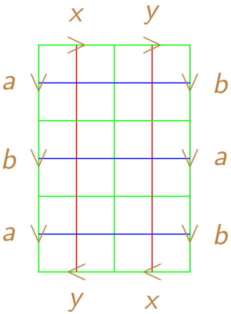
Here is one embedding of two tracks





Here is another embedding of the two tracks





If t is an untwisted track then $t = \beta_1 t_1 + \beta_2 t_2 + \cdots + \beta_n t_n$ for non-negative integers β_i and we can form a track in $C = G \backslash \tilde{C}$ corresponding to t as described above. Thus each 2-cell of C corresponds to a pair i, j and we replace the β_i line segments in the i-direction and the β_j segments in the j-direction with $i + j$ segments joining the same boundary points as above. Then t will be the inverse image of this track t' in C.

We do the same for all embeddings of the union of the tracks t_i in X. We always get t as the inverse image of a track t' in C. Although the embedding changes the preimage does not. If all the t_i's give trivial decompositions, then G fixes a vertex of $\tilde{C}$. If this happens then the pattern in C corresponding to $\beta_1 t_1 + \beta_2 t_2 + \cdots + \beta_n t_n$ has $\beta_1 + \beta_2 + \cdots + \beta_n$ components and so the only tracks in C are those corresponding to the t_i's.

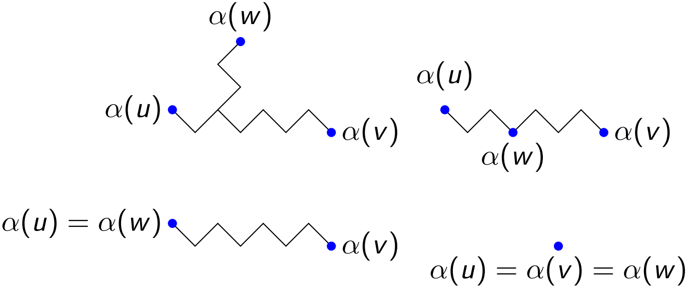
Thus if all the t_i's give trivial decompositions then every action of G on a tree is trivial.

More generally we know that every action on a tree (without involutions) is resolved by a geometric action. What this means is that if S is a tree with a G-action then there is a G-morphism $\theta : T \to S$ where T is a tree with a geometric G-action. We show now that θ factors through a G-tree T' arising from a pattern in C. In fact T and T' are isomorphic.

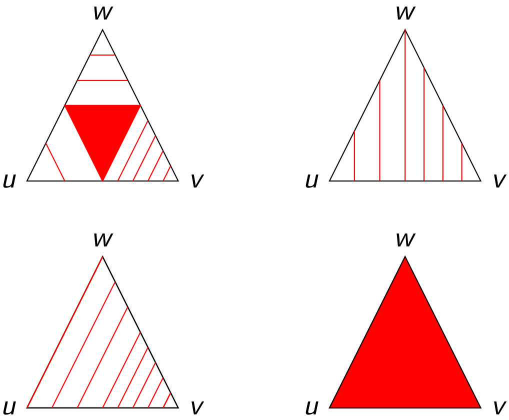
We first recall how θ is obtained. The vertex set $V\tilde{X}$ of $\tilde{X}$ consists of a single orbit of points. Since G acts freely on $\tilde{X}$, by choosing arbitrarily an image for one vertex v_0 of $V\tilde{X}$ in VS, we obtain a G-map $\alpha : V\tilde{X} \to VS$, where $\alpha(gv_0) = g\alpha(v_0)$.

Let u, v be the vertices of a 1-cell (edge) γ of $\tilde{X}$. There is a unique path p in S joining $\alpha(u)$ and $\alpha(v)$. Suppose this path has length d. It is possible that $d = 0$. If $d \neq 0$ then divide γ into d sub-segments of equal length and map γ to p so that the i-th segment is mapped to the i-th edge of p. If $d = 0$ then map the whole of γ to $\alpha(u) = \alpha(v)$. We have then extended α to the 1-skeleton of $\tilde{X}$ so that it still commutes with the G-action. We now want to extend α to all of $\tilde{X}$.

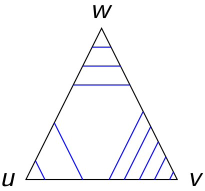
Let u, v, w be the vertices of a 2-cell of $\tilde{X}$. There are essentially four possibilities (after relabeling vertices) for the image of the three boundary edges in S as below.

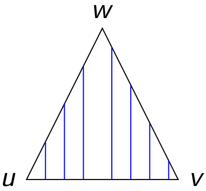


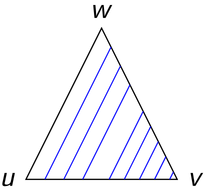
The corresponding maps of the 2-cell are illustrated below. The red regions are mapped to vertices and the blue patterns are mapped to the midpoints of edges.

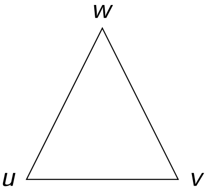


The corresponding maps of the 2-cell are illustrated below. The red regions are mapped to vertices and the blue patterns are mapped to the midpoints of edges.



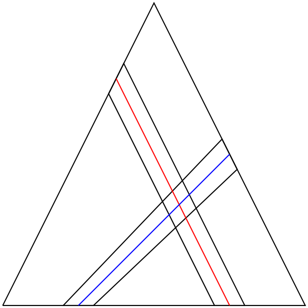




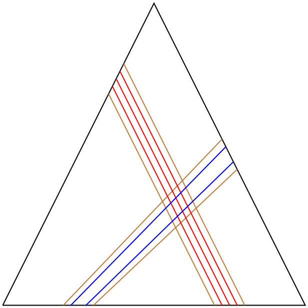


The G-pattern $\tilde{p}$ obtained in this way will be such that all the component tracks are untwisted. Its image p in X can be written $p = \beta_1 t_1 + \beta_2 t_2 + \cdots + \beta_n t_n$ where the β_i's are positive integers and the t_i's are minimal untwisted tracks. Choose an embedding of the tracks t_i in X so that the t_i's intersect transversely, specifically (for $i \neq j$) so that if σ is a 2-cell then a component of $t_i \cap \sigma$ and a component of $t_j \cap \sigma$ intersect transversely in at most one point in the interior of σ.

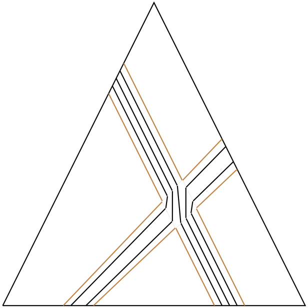
We can then choose a small closed neighbourhood b_i of each t_i so that each component of $b_i \cap b_j$ is a 4-sided disc containing exactly one point of intersection.



Now replace each t_i by β_i parallel copies lying within a b_i. Below $\beta_i = 2, \beta_j = 3$.

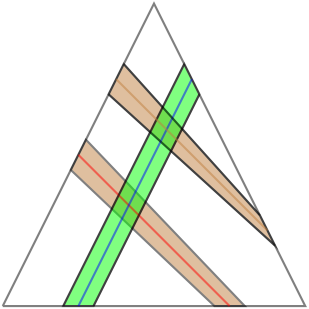


In each intersection of b_i with b_j replace by non-intersecting lines as described earlier. The pattern p is obtained. Note the number of lines between each pair of edges does not change. The number of intersections reduces.

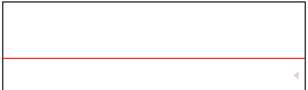


This process produces the pattern p, which is uniquely determined by its intersection with the one skeleton. Since the pattern p arises in this way we can define a contraction $\rho : X \to D \subset C$ which restricts to a contraction on p. This lifts to a contraction $\tilde{\rho}; \tilde{X} \to \tilde{D} \subset \tilde{C}$ and the pattern $\tilde{\rho}(\tilde{p})$ in $\tilde{D}$ has a dual graph which can be identified with the resolving tree corresponding to $\tilde{p}$.

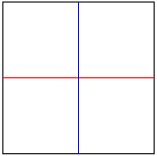
To define ρ consider the union of the bands b_i in X. Each point of X lies in either $0, 1$ or 2 bands.



Let x be a point in 0 bands. In $\tilde{X}$ a point $\tilde{x}$ lying above x determines a vertex of $\tilde{C}$ as for each component track t of the pattern $\tilde{t}_i$ it lies in one side of t. We define $\tilde{\rho}(\tilde{x})$ to be this vertex. Let now x be a point which lies in a single band, and let $\tilde{x}$ be a point lying above x. Then $\tilde{x}$ will lie in a region as below. The top (or bottom) side will border a region of points belonging to 0 bands. This side is mapped by $\tilde{\rho}$ to the vertex already assigned to that region. The vertices assigned to the two sides will be the vertices of an edge in $\tilde{C}$, since there is exactly one track that separates the points in $\tilde{X}$. The region is contracted by $\tilde{\rho}$ to that edge, so that any horizontal line is mapped to the same point of the edge.



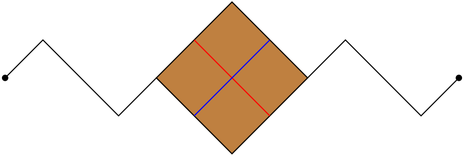
Finally the intersection of two bands has component regions as below. We have already defined $\tilde{\rho}$ on the boundary of this region, and the image of this boundary is the boundary of a unique 2-cube in $\tilde{C}$. We map the region in $\tilde{X}$ to this 2-cube in the obvious way.



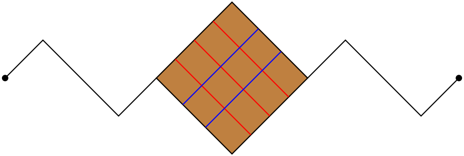
The map $\tilde{\rho}$ is a G-map and so it induces a map $\rho : X \to C$. Note that this map will not usually be surjective. A different embedding of the tracks in X will produce a different image in C.

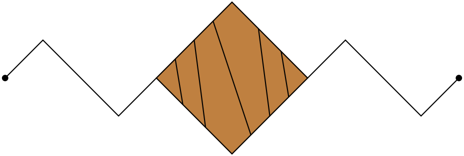
The image of a 1-cell under $\tilde{\rho}$ will be a geodesic in $\tilde{C}$.

In Sageev's paper, he shows that for any two geodesics between two points in a cubing one can get from one to the other by a finite number of moves that involve changing two consecutive edges for two edges that run on opposite sides of some 2-cube.

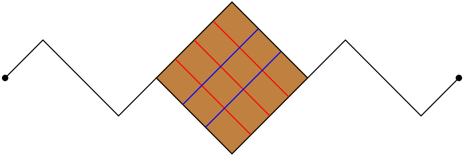


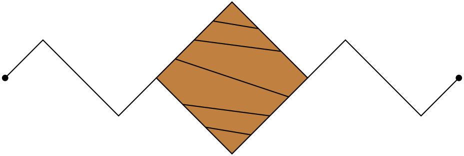
The intersection of the two geodesics and the 2-cube with $3t_i + 2t_j$ is shown below.





The above intersection with the 2-cube is chosen rather than the one below, to correspond to the right pattern in $\tilde{X}$.





Every 2-cell of $\tilde{C}$ will lie in at least one of the G-subcomplexes which are the images of the different contractions ρ corresponding to the different embeddings of the t_i's in X. The pattern $\tilde{p}$ defined on all of $\tilde{C}$ will restrict to the correct pattern in each of the different images of contractions of $\tilde{X}$ to a subcomplex of $\tilde{C}$ and the dual graph to $\tilde{p}$ in $\tilde{C}$ can be identified with the dual graph in each of these subcomplexes.

Thus the resolution $\theta : T \rightarrow S$ factors through the dual graph T' in $\tilde{C}$.

We have shown that for any G-tree S there is G-tree T' arising from a pattern in $\tilde{C}$ and a resolution $\theta' : T' \to S$. This means that if H fixes a vertex of $\tilde{C}$ then it will fix a vertex of S. Hence H is G-unsplittable.

Conversely if H is G-unsplittable then it fixes a vertex in each of the trees corresponding to a t_i. This means that it chooses a side for each track in the pattern $\tilde{t}_i$. From the theory of the Sageev cubing there is then a vertex of $\tilde{C}$ fixed by H.

This completes the proof of our main theorem.

Notes and References

Similar results have been obtained when X is the spine of a 3-manifold. In this case a track corresponds to a patterned surface in the 3-manifold (see Dicks and Dunwoody [1989]), which is a generalization of a normal surface.

Jaco-Oertel [1984] andJaco-Tollefson [1995] have shown that extreme normal surfaces carry important information. Thus Jaco and Oertel give an algorithm for deciding if a manifold is Haken, and Jaco and Tollefson show that the the extreme solutions contain a set of 2-spheres giving a complete factorization of a closed 3-manifold. Our algorithm is a generalization of part of Haken's algorithm for deciding the genus of a knot. (See Hemion [1992]).

Nicholas Touikan has recently given an algorithm for finding the Grusko decomposition of a finitely presented group using tracks.

It seems likely that if G has more than one end then at least one of the splittings corresponding to a minimal untwisted track will be a non-trivial decomposition over a finite subgroup. In fact we would conjecture that the list contains a set of compatible splittings over finite subgroups giving a graph of group decomposition of G with one-ended vertex groups.

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W.Jaco and U.Oertel, An algorithm to decide if a 3-manifold is a Haken manifold, Topology 23 (1984) 195-201.

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