A Proof of the Poincaré Conjecture ?

by

M.J.Dunwoody

Colin Rourke has pointed out there is a problem in the statement

"This 2-sphere will have the property that for any equivalence class $\{p, q\}$ and any 2-simplex σ containing it, the arcs of C pp', qq' containing p, q are uncrossed, i.e. it uncrosses every configuration."

It is certainly true for any configuration as in Fig 2 but I do not see how to prove it for the Fig 3 ones. There may be an argument using thin position (I hope!).

We give a prospective proof of the Poincaré Conjecture. The proof was inspired by the beautiful algorithm of Hyam Rubinstein [2] for recognizing the 3-sphere and the proof of this by Abigail Thompson [3]. The philosophy is that of the final chapter of Dicks and Dunwoody [1].

I am grateful to Graham Niblo for his very perceptive observations and pointing out errors in earlier versions.

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In the Recognition Algorithm one determines a maximal set of disjoint normal surfaces in a triangulated 3-manifold M that are 2-spheres. Each such surface separates M and so the set of surfaces correspond to the edges of a finite tree. It is proved that M is a 3-sphere if each region corresponding to a vertex of this tree of valency one either contains a single vertex of the triangulation or contains no vertices but does contain an almost normal surface, i.e. one for which the intersections of the surface with 3-simplices are all 3 or 4-sided except for one exceptional 8-sided disc.

In our case we will be dealing with a fake 3-ball M. The algorithm now says that after finding a maximal set of disjoint normal 2-spheres one only has to examine those regions corresponding to vertices of valency one which do not contain the boundary region or a single vertex.

Let M be a fake 3-ball with boundary 2-sphere δM. We assume that M has a fixed trangulation T. We also assume that T is such that every vertex is in δM. Let $f' : S^2 \to \delta M$ be a homeomorphism and let $f = \iota f' : S^2 \to M$ where $\iota : \delta M \to M$ is the inclusion map. Since M is a fake 3-ball the map f is null homotopic. Let $F : S^2 \times I \to M$ be a homotopy between f and a constant map. For $t \in I$ let $f_t : S^2 \to M, f_t(s) = F(s,t)$. We can assume that for all but finitely many values t, f_t meets the 1-skeleton T^1 of the triangulation transversely and for each t for which the map f_t does not meet T^1 transversely there is precisely one point where f_t is tangential to T^1. Note that $f_0 = f$. Let $t'_0 = 0, t'_1, t'_2, \ldots, t'_n$ be the values of t for which f_t does not meet T^1 transversely and put $t'_{n+1} = 1$. For $i = 0, 1, \ldots, n$ choose $t_i \in (t'_i, t'_{i+1})$ and put $f_i = f_{t_i}$.

Let W_i be the set of intersections of $f_i(S^2)$ with the 1-skeleton of T and let $w_i = |W_i|$. Rearrange the weights w_i into a finite non-increasing sequence. Order these sequences lexicographically. The width of T is the minimum sequence of weights as F ranges over all possible homotopies.

If F realizes the width of T then F is said to be in *thin position.* Now revert to the original ordering of the w_i's Clearly for each i either $w_{i+1} = w_i + 2$ or $w_{i+1} = w_i - 2$. Note that w_0 which is the number of times a boundary of a regular neighbourhood of δM meets the 1-skeleton is twice the number of internal edges, since each edge has both vertices in δM. We are particularly interested in the values i for which $w_{i-1} = w_{i+1} = w_i - 2$. For such an $i, f_i(S^2)$ is called a *thick* sphere, while if $w_{i-1} = w_{i+1} = w_i + 2$, then $f_i(S^2)$ is called a *thin* sphere .

Given two finite subsets F_1, F_2 of the open interval $(0, 1)$ and a bijection $\beta_I : F_1 \to F_2$, there is a continuous map $\phi_I : I \to I$ which restricts to β on F_1 and to the identity on $\{0, 1\}$. Building up from such maps on the edges of T, it can be seen that given a bijection $\beta : G_1 \to G_2$ between two finite subsets of points in the interior of edges taking a point in a given edge to a point in the same edge, there is a continuous map $\nu : M \to M$ preserving simplices and restricting to β on G_1. Such a map will be homotopic to the identity map on M.

By composing F with such a map it can be assumed that both $W_{i-1} \subset W_i$ and $W_{i+1} \subset W_i$. Put $W = W_i$, and let C be the collection of (possibly intersecting) arcs in $f_i(S^2) \cap \delta\sigma$ as σ ranges over the 2-simplexes of T. In fact by altering F we can assume $W_j \subseteq W$ for every j and pairs of vertices which are joined by arcs of C in W_j are also joined in W.

Let $\{u, v\} = W_i - W_{i-1}$ and let $\{x, y\} = W_i - W_{i+1}$.

Consider the inverse image (under f_i) of the 2-skeleton of T. This will give a tessellation of the 2-sphere. In the case we will be mainly considering we will show that it can be assumed that the regions are simply connected. In this case we get a cell decomposition Σ of the 2-sphere. A 2-cells is called an n-gon if its boundary meets the inverse image of the 1-skeleton exactly n times.

The 1-skeleton of Σ is a graph Γ of which the vertices are the inverse images of the 1-skeleton *vertices*, and the edges are the arcs of C.

Orient the sphere by selection a direction for a normal at every point and put an orientation on every edge of the triangulation. Call an intersection of an edge with the sphere a $+$ intersection if the orientations agree and a $-$ intersection if they do not.

First note that if two vertices of a region lie in the same 1-simplex and one is $+$ and one is $-$ we can join them be an arc in the region (or by an edge if the vertices are adjacent) and find a map of S^2 homotopic to f_i and equal to f_i outside a neighbourhood of this arc, so that the new map has 2 less intersections with the 1-skeleton. There is a (singular) disc in the relevant 3-simplex which is bounded by the image of the arc and the interval of the 1 simplex between the end points of the image of the arc and the homotopy takes place in a neighbourhood of this disc.

It can be assumed that f_{i-1} and f_{i+1} are obtained from f_i by such homotopies as described above. If the regions involved in the two homotopies are different then the order of the homotopies can be changed and the weight of the total homotopy reduced - one peak is replaced by two smaller peaks. Also if x, y, u, v are in the same region and x, y are still in the same region for the map f_{i-1} then we can also swap the homotopies round and reduce weight. Similarly if u, v are in the same region for the map f_{i+1} then we can swap the homotopies. It follows that the exceptional region has two $+, -$ pairs of vertices and the two arcs between the pairs involved in the two homotopies cross.

The simplest such region is as in Fig 1 (a). Note that x, y will be two points on one edge and u, v will be the vertices on the opposite edge. Removing u, v creates two 3-gons each containing one of x, y.

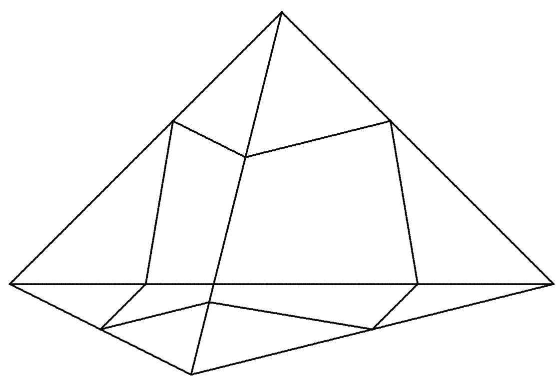
We will show that all the other regions have no $+, -$ pairs of vertices which map to the same 1-simplex. This is similar to the proof in [3].

Suppose p, q are such a pair of vertices in a region. By the above they can be removed by a homotopy. Suppose first that p, q are both different from any of u, v, x, y. Starting from the $(i-1)$-th stage, we can carry out a homotopy that removes p, q. We then add in the pair u, v and remove x, y and finally replace p, q so that we are at the $(i+1)$-th stage. This sequence of homotopies has a lower sequence of weights than the original as the weight of the highest peak has been reduced by two. If, say $p = u$ but $q \neq v$ then we can remove p, q this will not separate x, y and so we can remove x, y and then add in u, v this has replaced the points of W_{i+1} except that we have replaced q by v and this can be achieved by a homotopy which only affects the points on one 1-simplex. Again this sequence of homotopies has a lower weight sequence.

By a similar argument we can show that the only $+, -$ pairs in the exceptional region are u, v and x, y.

Note that if two points in a region map to the same 1-simplex, and they are adjacent in the boundary then they must be a $+, -$ pair and so it follows that an arc in C must join points in distinct faces of a 2-simplex.

Note that the only simple closed curves on the boundary of a 3-simplex which intersect each 1-simplex so that the intersections are all $+$ or all $-$ intersect each 1-simplex only once and so they bound a 3-gon and a 4-gon. Also the only embedded curve with the intersection properties of the exceptional curve is the 8-gon as illustrated in Fig 1. We will show that we can alter a thick disc by a homotopy so that it is embedded. This means that it must originally have had only 3-gons and 4-gons plus one exceptional 8-gon.



$Fig1$

Consider first the case when $f_i(S^2)$ is the only thick sphere.

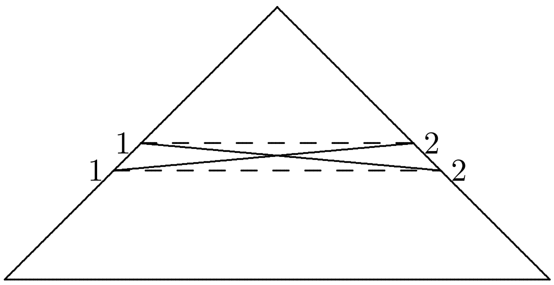
We can alter F so that no component of the intersection $f_i(S^2)$ with a 2-simplex is a closed curve (possibly self-intersecting) and so that the regions described above are simply connected. If a region is not simply connected then it contains an essentiol simple closed curve. If we carry out surgery along the image of this curve the surface S obtained is a union of two (possibly intersecting) 2-spheres. Only one of these can contain the exceptional disc and as such it will be the only 2-sphere containing a returning arc. It follows that the homotopy as t increases cannot affect the other 2-sphere. Hence it will not reach the constant map. It follows that the surgery only cuts off singular 2-spheres which do not intersect the 1-skeleton. It is possible to alter F to avoid these occurring. This means that we can assume the regions for f_i are all n-gons as we wanted.

As we only have one thick sphere $f_i(S^2)$ the sequence of weights satisfies $w_0 < w_1 < \ldots < w_i > w_{i+1} > \ldots > w_n = 0$ and $w_j = w_{j+1} - 2$ if $j < i$ while $w_{j+1} = w_j - 2$ if $j > i$.

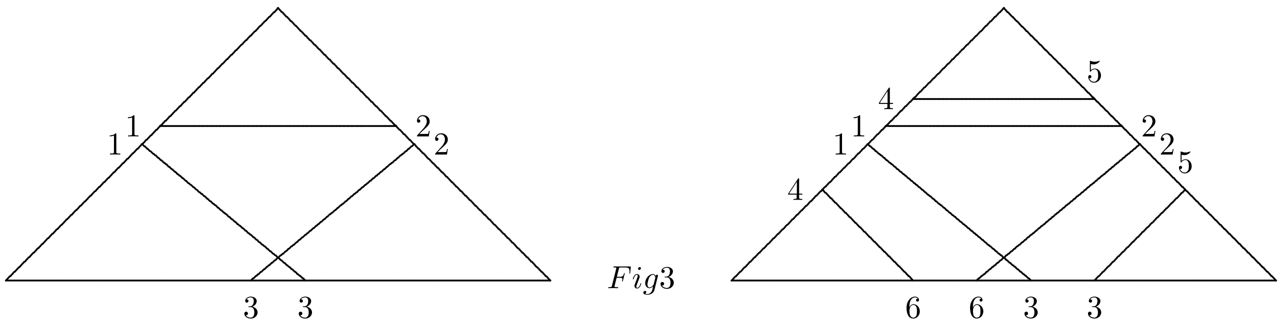
Define an equivalence relation on W as follows:- $p \sim q$ if $p \in W_j$ if and only if $q \in W_j$ for every $j \geq i$.

Each equivalence class contains two elements. Let σ be a 2-simplex of T.

Define a *configuration* in a 2-simplex σ to be a minimal collection of arcs of C in σ whose end points are a complete collection of equivalence classes. There are configurations with 2 pairs as in Fig 2. There are configurations with $3n$ pairs as in Fig 3 (this shows the cases $n = 1$ and 2). There may even be more exotic configurations. The set of configurations which are not as in Fig 2 is denoted A_3. A configuration is *uncrossed* if for every equivalent pair p, q that it contains the arcs containing p and q are uncrossed. Otherwise the configuration is said to be *crossed*. The the configuration in Fig 2 is crossed as is the first configuration in Fig 3. The more exotic configurations only exist in crossed form.



$Fig2$



$Fig3$

Eliminating the pairs $\{p_j, q_j\}$ in Σ for each $j = 1, 2, \ldots n$ creates a region in which p_{n+1}, q_{n+1} are a $+, -$ pair in the boundary.

For such a pair the normals to the surface either point inwards or outwards. Construct a product of transpositions on the points of W.

First p_1, q_1 are left fixed. Transpose p_n, q_n if and only if the normals to the surface point outwards.

The product of these transposition can be used to produce a continuous map $\nu : M \to M$ as above which after"straightening" composed with f_i produces a new 2-sphere with the same set W of intersections with the 1-skeleton.

This 2-sphere will have the property that for any equivalence class $\{p, q\}$ and any 2-simplex σ containing it, the arcs of $C\ pp', qq'$ containing p, q are uncrossed, i.e. it uncrosses every configuration.

Thus we see that the above process reduces the number of pairs of crossing arcs, unless all the ones involving a pair of points in an equivalence class were uncrossed to start with.

We now repeat the above process, except that we work down from $W = W_i$ to W_0 instead of from W_i to W_n. Note that W_0 determines a normal surface S_0 parallel to the boundary of M. As before define an equivalence relation on W. Put $x \sim y$ if $\{x, y\} \subset W_j - W_{j-1}$ for any $j = 1, 2, \ldots, i$. In this case W_0 is an equivalence class and all other equivalence classes have two elements. We repeat the unravelling using the new equivalence relation on W. This new unravelling will produce non-intersecting configurations going from i to 1. However the new map $f_i' : S^2 \to M$ may give rise to a new equivalence relation corresponding to going from i to n. If we keep swapping the processes we must eventually reach a stage when the configurations for both processes are non-intersecting. When this happens the patterned surface corresponding to f_i is an embedding and so are each of the intermediate homotoped maps. It follows easily that M is a 3-ball.

In the general case - in which there are more than one thick sphere - let $f_i : S^2 \to M$ be the last thick sphere and let $g : S^2 \to M$ be the last thin sphere. Now $g(S^2)$ may contain returning arcs, i.e. it may not intersect the 2-simplices in (possibly intersecting) arcs joining the same edge. However any such can be removed by homotopies and we end up with a surface in which the intersections with the 2-skeleton are (intersecting) arcs joining distinct edges.

The intersections with the 1-skeleton determine a patterned (embedded) surface in M (see [1]). We show that this surface is a normal 2-sphere N. Using the unravelling procedure described above we can alter F so that f_i is a 2 sphere which homotops via embeddings both to a point and also to N. Hence N is a 2-sphere. To complete the proof we use the Recognition Algorithm. The algorithm now says that after finding a maximal set of disjoint normal 2-spheres one only has to examine those regions corresponding to vertices of valency one which do not contain the boundary region or singleton vertices.

We can create a fake 3-ball M_1 by separating such a region from the rest of M by cutting along the normal 2-sphere. It is clear from the above that M_1 must contain an almost normal surface and so M is not fake at all.

References

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University of Southampton, Southampton, SO17 1BJ