A SHORT PROOF OF THE POINCARÉ CONJECTURE

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Abstract. A short, fairly self-contained proof is given of the Poincaré Conjecture.

1. Introduction

In 2002 I attempted a proof of the Poincaré Conjecture and put it on my home page. A number of errors were pointed out. At the time I was unable to resolve all of them and came to the conclusion that the approach could not work. Recently I wrote an account of my research [2], particularly relating to Stallings' Theorem and the accessibility of finitely generated groups and that made me think again about my aborted proof. It now seems to me that the approach was a good one and I have come up with a proof that I think resolves the earlier problems. I think the approach could provide the proof of the Sphere Theorem that I was looking for on the last page of [1].

The proof, then and now, was inspired by the beautiful algorithm of Hyam Rubinstein [4] for recognising the 3-sphere and the proof of this by Abigail Thompson [5]. The proof uses techniques from my proof of the Equivariant Sphere Theorem (see [1]) and a labelling of points associated with a null homotopy. Perelman gave a proof of the Poincaré Conjecture in 2002.

My understanding of simple closed curves on tetrahedrons has benefitted from correspondence with Sam Shepherd.

2. Patterns in a Tetrahedron

Recall from [1] the definition of a pattern. Let K be a 2-complex with polyhedron $|K|$. A pattern is a subset P of $|K|$ satisfying the following conditions:-

(i) For each 2-simplex σ of K, $P \cap |\sigma|$ is a union of finitely many disjoint straight lines joining distinct faces of σ.

(ii) For each 1-simplex γ of K, $P \cap |\gamma|$ consists of finitely many points in the interior of $|\gamma|$.

A track is a connected pattern. If two patterns P and Q intersect each 1-simplex in the same number of points then the patterns are said to be *equivalen*t. Two equivalent disjoint tracks in the same 2-complex are said to be *parallel*. We investigate tracks and patterns in a tetrahedron $|K|$, which we regard as the set of faces of a 3-simplex ρ. We call such a track an n-track if it intersects n edges.

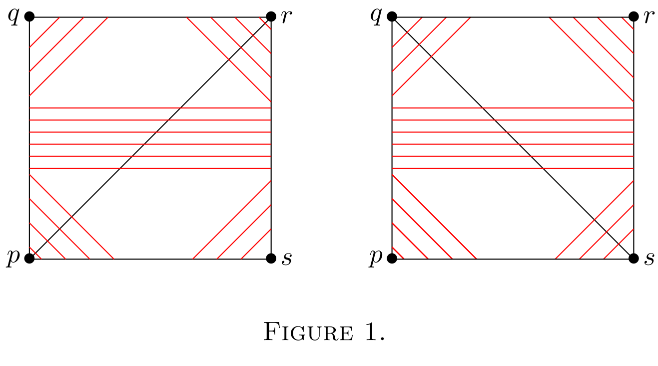

Figure 1.

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If a pattern is as in Figure 1 then the tracks are all 3-tracks or 4-tracks. A pattern in a 3-manifold is called a normal pattern if the intersection with the boundary of every 3-simplex is like this. Note the symmetry in Figure 1, in which the right hand pattern is the same as the left hand pattern, except that we have swapped diagonals. There is symmetry like this in the faces of a tetrahedron if and only if it is a normal pattern in which all the tracks are 3-tracks or 4-tracks.

An 8-track is shown in Figure 2. The only tracks one can have in a tetrahedron are n-tracks where $n = 3$ or $n = 4m$ for $m = 1, 2, \ldots$.

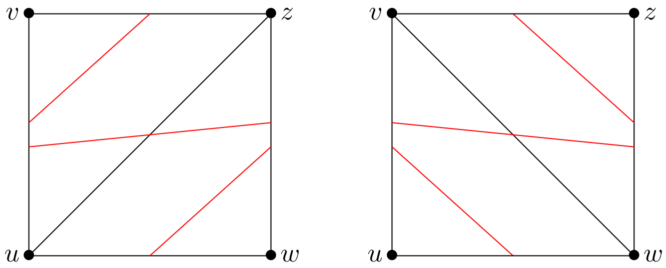

FIGURE 2.

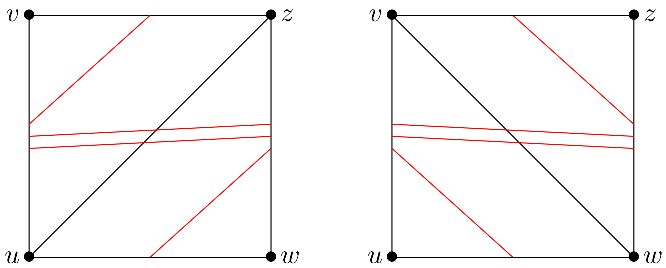

FIGURE 3.

Figure 3 shows a 12-track. A pattern in K can only have two types of track. There can be 3-tracks, each of which separates one of the corner vertices from the other three vertices. There can be one other parallel set of tracks each of which is a $4n$-track for the same positive integer n. Each such track separates the four vertices into two pairs. If $n > 0$ is even, it separates u, v from w, z and if $n > 1$ is odd it separates u, z from v, w.

A track in K is a simple closed curve, which will bound a disc in $|\rho|$. If M is a 3-manifold and M is triangulated so that $M = |K|$ where K is a finite 3-complex, then a pattern P in $|K|$ determines a *patterned surface* S such that for each 3-simplex ρ, $S \cap |\rho|$ consists of disjoint properly embedded discs and $S \cap |K^2| = P$. A patterned surface is determined, up to isotopy by the intersection $P \cap |K^1|$. If the pattern in $|K^2|$ is normal, then the patterned surface is a normal surface.

Orient a track in K by choosing a positive direction as one goes along the track. At adjacent points of intersection of a track with an edge the directions of the track will be opposite to each other. This gives what we call a + - pair, i.e. a pair of points - not necessarily adjacent - of intersection points of an edge with a track where the track has opposite directions. This will be of more significance for singular "tracks " or *stracks* as shown in Figures 4 and 5, as a pair of points of intersection on an edge can be removed by a homotopy if and only if the pair is a + - pair. A track in K has a + - pair if and only if it is not a 3-track or a 4-track, since any track which intersects an edge more than once will have a + - pair.

A *spattern* sP in K is defined to be a subset of $|K|$ satisfying

(i) For each 2-simplex σ of K, $sP \cap |\sigma|$ is a union of finitely many straight lines joining distinct faces of σ.

(ii) For each 1-simplex γ of K, $sP \cap |\gamma|$ consists of finitely many points in the interior of $|\gamma|$. Each such point belongs to exactly one straight line in each of the 2-simplexes containing γ.

A *strack* is a spattern that is the image of a circle. Every spattern is a union of finitely many stracks. If M is a 3-manifold and M is triangulated so that $M = |K|$ where K is a 3-complex, then a spattern sP in $|K|$ determines a *spatterned surface* S such that for each 3-simplex ρ, $S \cap |\rho|$ consists of singular discs and $S \cap |K^2| = sP$.

Let $f : S^2 \to M$ be a general position map (see Hempel [3], Chapter 1), in which f is in general position with respect to a triangulation K of M. An i-piece of f is defined to be a component of $f^{-1}(\sigma)$ where σ is an $(i+1)$-simplex of K. Thus a 0-piece is a point of S^2. A 1-piece is either an scc (simple closed curve) or an arc joining two 0-pieces, and 2-pieces are surfaces with boundary, whose boundary is a union of 1-pieces.

The map f can be simplified via a homotopy as described in Section VI. 9 in [1], so that the image is a spatterned surface and each 2-piece is a disc. Looking at all the 2-pieces will give a cell decomposition (tessellation) of the 2-sphere. In fact there is a mistake on page 253 of that section. It is incorrectly asserted there that any pair of points in the intersection of the boundary of a 2-piece with a 1-simplex can be removed by a homotopy. In fact the pair needs to be a + - pair. The proof there does work for a + - pair and it shows the pair can be removed without disturbing any other points of intersection with the 1-skeleton. The proof given there is for homotopies and not just isotopies. As shown there one can use surgery along a simple closed curve to make any 2-piece that is not a disc into a disc.

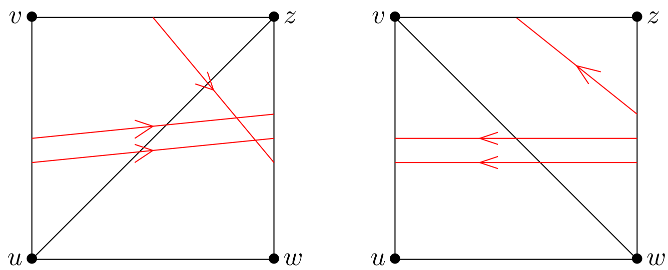

FIGURE 4.

Figure 4 shows a 12-strack that has no + - pairs of points. I conjecture that a strack must intersect all six edges to have a + - pair.

If S is a spattern in a 2-complex K, then there is a uniquely determined underlying pattern P that has the same intersection with the 1-skeleton of K. Put $W = S \cap |K^1| = P \cap |K^1|$.

Figure 5 shows a spattern that is a union of a red 12-track and a blue 3-track, and its underlying pattern, which is also a 3-track and a 12-track.

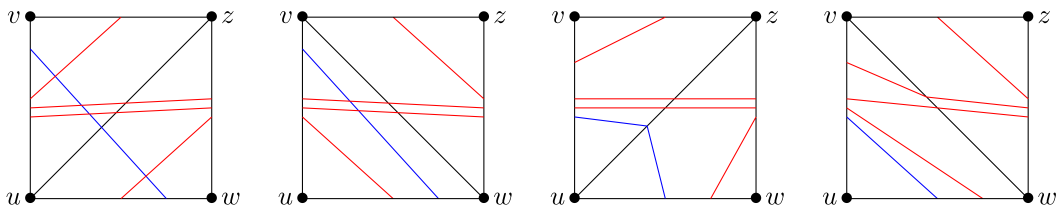

FIGURE 5.

Given two finite subsets F_1, F_2 of the closed interval $[0,1]$ and a bijection $\beta_I : F_1 \to F_2$, there is a continuous map $\phi_I : I \to I$ which restricts to β on F_1 and to the identity on $\{0,1\}$. There is a homotopy between ϕ_I and the identity map.

Building up from such maps, if $\nu : W \to W$ is a permutation that restricts to a permutation on $W \cap |\gamma|$ for each 1-simplex γ, then ν extends to a map of the 1-skeleton into itself which restricts to the identity on the 0-skeleton and which is homotopic to the identity map on $K^{(1)}$. This map can be further extended linearly to a map of the 2-skeleton and then, in the case of a triangulated 3-manifold, to a continuous map $\nu : M \to M$ which is homotopic to the identity map on M. It will have the property that if two points on the boundary of a 2-simplex σ are joined by a line in $S \cap \sigma$, then they are joined by a line in $\nu S \cap \sigma$. A spattern S in K is mapped to another spattern. Lines that were uncrossed may become crossed, and lines that were crossed may become uncrossed. The underlying pattern P is not changed. The tessellation of S together with the pieces of the tessellation are unchanged in such a map.

Our proof of the Poincaré Conjecture is to show that a certain spattern must occur in a homotopy from the boundary of a fake ball to a constant map, and this spattern is homotopic to its underlying pattern by such a homotopy.

3. The Proof

Let $M = |K|$ be a 3-manifold, where K is a finite 3-complex. It follows from a result of Kneser (see [1] or [3]) that there is a bound on the number of disjoint non parallel normal surfaces in a compact triangulated 3-manifold. I was able to prove that finitely presented groups are accessible by generalising this result to patterns in a finite 2-complex. In the Recognition Algorithm one determines a maximal set of disjoint normal surfaces in a triangulated 3-manifold M that are 2-spheres. If M is simply connected then each such surface separates M and so the set of surfaces correspond to the edges of a finite tree. It is proved that M is a 3-sphere if each region corresponding to a vertex of valency one in this tree either contains a single vertex of the triangulation or contains no vertices but does contain an almost normal surface, i.e. one for which the intersections of the surface with 3-simplexes are all 3 or 4-sided except for one exceptional 8-sided disc.

Let M be a fake 3-ball. Let M_0 be a component, obtained by cutting along the maximal collection of normal 2-spheres, which has one boundary component and which does not contain a vertex. Let $f : S^2 \to M$ be an injective map whose image is the normal 2-sphere δM_0. Since M_0 is a fake 3-ball, the map f is null homotopic. Let $F : S^2 \times I \to M_0$ be a homotopy between f and a constant map. For $t \in I$ let $f_t : S^2 \to M_0, f_t(s) = F(s,t)$. We can assume that for all but finitely many values t, f_t meets the 1-skeleton T^1 of the triangulation transversely and for each t for which the map f_t does not meet T^1 transversely, there is precisely one point where f_t is tangential to T^1. Let $t'_1, t'_2, \ldots, t'_n$ be the values of t for which f_t does not meet T^1 transversely and put $t'_0 = 0, t'_{n+1} = 1$. For $i = 1, ..., n$ choose $t_i \in (t'_i, t'_{i+1})$ and put $f_i = f_{t_i}$.

Let W_i be the set of intersections of $f_i(S^2)$ with the 1-skeleton of T and let $w_i = |W_i|$. Rearrange the weights w_i into a finite non-increasing sequence. Order these sequences lexicographically. The width of T is the minimum sequence of weights as F ranges over all possible homotopies. If F realizes the width of T, then F is said to be in thin position. Now revert to the original ordering of the w_i's. Clearly for each i either $w_{i+1} = w_i + 2$ or $w_{i+1} = w_i - 2$. Note that w_0 is the number of intersections of δM_0 with the 1-skeleton. For F in thin position, we are particularly interested in the values i for which $w_{i-1} = w_{i+1} = w_i - 2$. For such an $i, f_i(S^2)$ is called a thick sphere, while if $w_{i-1} = w_{i+1} = w_i + 2$, then $f_i(S^2)$ is called a thin sphere. Our interest will be in the last thick sphere. Let f_k be the last thick sphere and put $S = f_k(S^2)$. Each of the following $f_j(S^2)$, $j \geq k$, satisfies $w_{j+1} = w_j - 2$ and results in the removal of a pair of points, labelled j, j, from the intersection of $f_j(S^2)$ with a particular 1-simplex. When a pair is removed, the pair are points in a 1-simplex γ, and, for $j > k$, in one of the 2-simplexes σ having γ as a face, the pair are the ends of a returning arc, i.e. an arc in the intersection of $f_j(S^2)$ with σ with endpoints in γ. Such a pair is a + - pair.

Now consider $f_k : S^2 \to M_0$ with $S = f_k(S^2)$. We follow the argument of [5]. We know that the homotopy going from f_k to f_{k+1} results in the deletion of two points u, v in a 1-simplex γ and the homotopy going from f_k to f_{k-1} deletes points x, y. We also know that these pairs must be + - pairs. All four points must belong to the same 2-piece, for if there is no 2-piece that contains all

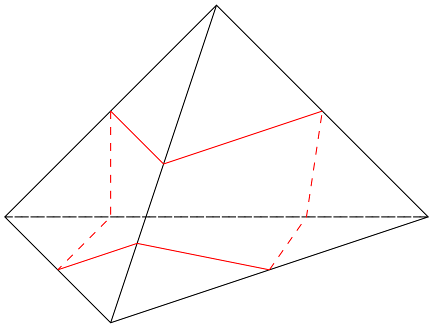

FIGURE 6.

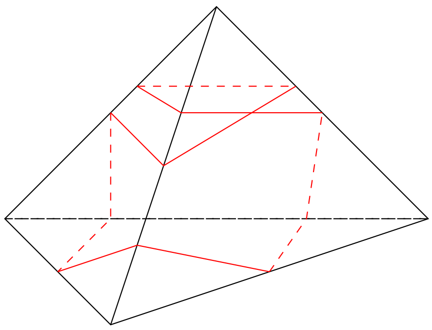

FIGURE 7.

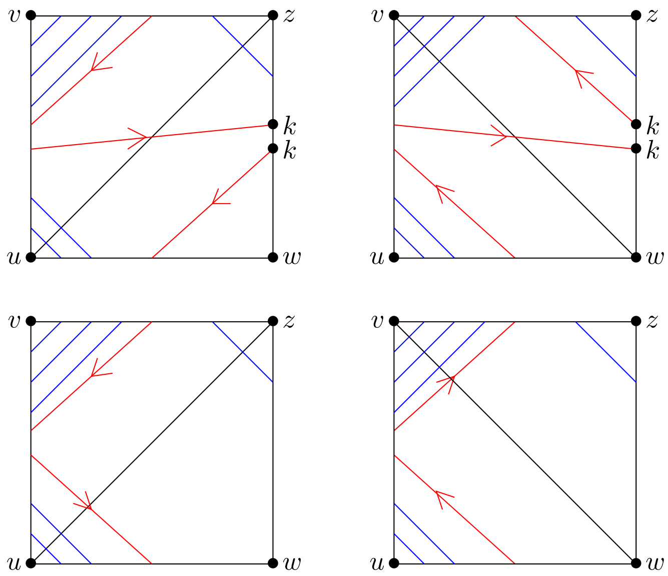

FIGURE 8.

four points, then the order of the homotopies can be changed and the weight of the total homotopy reduced - one peak is replaced by two smaller peaks. Also if x, y, u, v are in the same 2-piece and x, y are still in the same 2-piece for the map f_{k-1}, then we can also swap the homotopies round and reduce weight. Similarly if u, v are in the same 2-piece for the map f_{k+1}, then we can swap the homotopies. It follows that the exceptional 2-piece has two pairs of vertices that can be removed by a homotopy and that removing one pair disconnects the 2-piece. The simplest such 2-piece has boundary an 8-track as in Figure 2 or Figure 6. Note that x, y will be two points on one edge and u, v will be the vertices on the opposite edge. Removing u, v (labelled k, k) creates two 3-gons, with x, y separated as shown in Figure 8. In an isotopy this is the only possibility for the exceptional 2-piece. For a homotopy there are other possibilities such as in Figure 7.

We will show that all the 2-pieces apart from the exceptional one have no + - pairs of vertices which map to the same 1-simplex. This is similar to the proof in [5].

Suppose p, q are such a pair of vertices in a 2-piece. These can be removed by a homotopy. This is illustrated in Figure 9, where it is shown how the removal may create returning arcs in two other simplexes that have the same 1-face. Suppose first that p, q are both different from any of u, v, x, y. Starting from the $(k-1)$-th stage, we can carry out a homotopy that removes p, q. We then add in the pair u, v and remove x, y and finally replace p, q so that we are at the $(k+1)$-th stage. This sequence of homotopies has a lower sequence of weights than the original as the weight of the highest peak has been reduced by two. If, say $p = u$ but $q \neq v$, then we can remove p, q. This will not separate x, y and so we can remove x, y and then add in u, v. This has changed the points of W_{k+1} by replacing q by v and this can be achieved by a homotopy which only affects the points on one 1-simplex. Again this sequence of homotopies has a lower weight sequence.

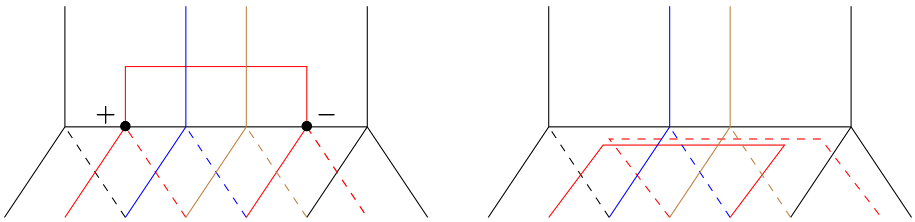

FIGURE 9.

We have both $W_{k-1} \subset W_k$ and $W_{k+1} \subset W_k$, and $W_{j+1} \subset W_j$ for $j > k$. We then have $W_{k+i} = W_{k+i+1} \cup \{u_i, v_i\}$, where u_i, v_i are the end points of the returning arc. Put $W = W_k$. Carrying on this process, we eventually get $w_n = 2$ and every point of W_k has received a label of an integer between k and n.

In the case of an isotopy this means that all the 2-pieces, apart from the exceptional one, intersect each 1-simplex at most once and so are 3-sided or 4-sided and so S is an almost normal 2-sphere. In the homotopy case, we see that S can have no returning arcs, i.e. 1-pieces that join two points on the same 1-simplex, and so S is a spattern. There are stracks that contain no + - pairs such as the one in Figure 4. As there are no + - pairs in S, the + - pairs that occur in the homotopies f_j for $j > k$ must have been created by the removal of an earlier + - pair in f_i for $k \leq i < j$.

Figure 10 shows what happens when one edge of a 3-simplex has a pair of points labelled j, j, which both lie on the boundaries of 4-sided discs. The intersection of these discs with the 1-skeleton will eventually receive labels m, p and l. Figure 10 shows what happens if $j < m < p < l$. The j removal takes red to blue, and the m removal takes blue to brown. Note that the removal process creates returning arcs between adjacent points.

Now that we have a homotopy, one might expect that the intersection with a 3-simplex to be quite complicated. In fact this does not happen as we will show that the homotopy can be changed to an isotopy by homotopies that restrict to homotopies on each 3-simplex.

Let u_i, v_i be the points labelled $k+i, k+i$ in the 1-simplex γ_i, so that $W_{k+i} = W_{k+i+1} \cup \{u_i, v_i\}$.

Our aim is to show in a finite induction argument that for each 1-simplex γ we can permute the finite set $\gamma \cap S$ in such a way that under the associated homotopies the spattterned 2-sphere S becomes the underlying patterned 2-sphere P. It will then be the case that F becomes an isotopy. Put $D_0 = \{u_0, v_0\}$. For $i = 1, \ldots, n-k$ let $D_i = D_{i-1} \cup \{u_i, v_i\}$. Let γ_i be the 1-simplex containing u_i and v_i.

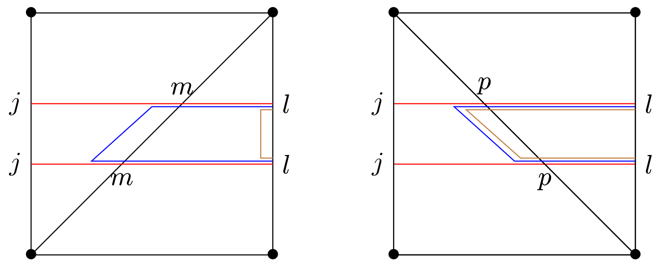

Figure 10.

One starts the induction process by permuting the points in $S \cap \gamma_0$. Let γ_0 have vertices a, b. We need to get u_0 and v_0 in the right position as all the subsequent permutations depend on this. Let σ_0 be one of the two 2-simplexes that contain γ_0 and which are faces of the 3-simplex containing the exceptional 2-piece. The points of $W \cap \gamma_0$ fall into two disjoint sets A, B where A is the set of points on lines in $\sigma_0 \cap S$ which are across the vertex a and B is the se of points on lines in $\sigma_0 \cap S$ which are across the vertex b. The permutation ν_0 moves the points of A so they are closest to a and the points of B so they are closest to b and so that u_0, v_0 are adjacent. This can be done since $u_0 \in A$ and $v_0 \in B$. It must be the case that we get the same position for u_0 and v_0 if we used the other 2-simplex, which is the other face containing γ_0 of the 3-simplex containing the exceptional 2-piece. This is because if u_0 and v_0 were joined by lines to two points in the same 1-simplex, then these points would be a + - pair that are not separated by the removal of u_0, v_0.

Having defined ν_j for $j < i$ to be a homotopy associated with a permutation of the points of $W \cap \gamma_j$, which moves the points of D_j to the positions they should have in P, we define the homotopy ν_i. Put $S_0 = \nu_0 S$ and $S_j = \nu_j S_{j-1}$. We know that for some $j < i$ either there are lines in S joining u_i to u_j and v_i to v_j as in Figure 11(a) or there are three labelled pairs joined by lines as in Figure 11 (b). This is because the homotopy f_{k+i} involves removing a returning arc joining u_i and v_i and this returning arc was created by one or more homotopy f_{k+j} for $j < i$.

In Figure 11 (a) we see what happens in a 2-simplex σ_i containing γ_i, if the lines joining u_i and v_i come from a single pair u_j, v_j. In Figure 11 (b) the lines to u_i, v_i come from two pairs. The lines in blue are those of $S_{i-1} \cap \sigma_i$. The lines in red are the ones we move them to in S_i. The bottom edge is the 1-simplex γ_i containing the pair $x = u_i, y = v_i$ which have labels $k+i, k+i$ in the labelling for S. In at least one of the 2-simplexes containing γ there will be a 1-face different from γ containing a pair of adjacent vertices with labels l, l for $l < k+i$, which are joined by non-intersecting lines in P to the pair u, v in γ_i, or there are pairs with labels less than i in each of the other two faces. The induction step is to apply the permutation $\nu_i = (u, x)(v, y)$ to the set $\gamma_i \cap S_i$. Of course if $u = x$ we apply the transposition (v, y). Again with each transposition we also transpose all attaching lines in all 2-simplexes containing γ. This change will not disturb any of the vertices of D_{i-1} or the edges joining them. Note that u, v will have labels bigger than $k+i-1$ as if, say, $u \in D_{i-1}$, then in S_{i-1} there would be a point in the intersection of a 1-simplex with S_{i-1} joined to both u and x and this does not happen in a spattern..

For each i we can identify a subgraph of both P and S_i consisting of either a pair of parallel lines or three lines shown in red in Figure 11 that join u_i and v_i to points which have already been moved to the right position in the induction.

There may be more than one such j, but we choose one of them to provide the lines joining u_i to u_j and v_i to v_j. Let $\nu = \nu_{n-k}\nu_{n-k-1} \ldots \nu_0$. Then ν moves all the points of W to the positions they should have in P.

The induction process is such that the points on every 1-simplex γ, end up with paired adjacent labels.

In defining ν_i, we have identified a subgraph of both P and νS that connects u_i and v_i to earlier pairs of points. The union of all these subgraphs will contain every point of W and have at most

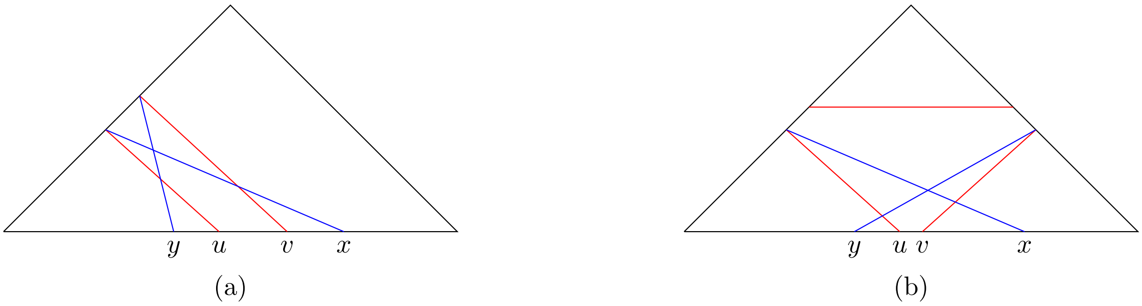

Figure 11.

two components. There will be only one if u_0 and v_0 are joined be a path in the union. This must mean that P is connected. Since S is a 2-sphere and νS is obtained from S by permuting the points of W and the lines joining them, νS is also a 2-sphere. But we now know P and νS have a common spanning subtree, or possibly they have spanning subtrees that differ by one edge. Also P and νS have the same number of edges. This means that P is also a 2-sphere, as P and νS will have the same Euler characteristic. The addition of each edge not in a spanning tree results in a face being divided in two, and so there is one extra face for each such edge.

As a spanning tree must contain all but one of the edges around each face, we know that P and νS can only differ on at most on edge around each face. But this means $P = \nu S$ as there is only one way the extra edges can be added to get the same number of faces.

It follows that we can replace the homotopy f by an isotopy, and so we have a proof of the Poincaré Conjecture. Note that because the homotopy can be changed to an isotopy by a homotopy that restricts to homotopies on 3-simplexes, there can be no 4-sided discs in the exceptional 3-simplex or crossing 4-sided discs in any 3-simplex. In the induction one is moving lines in S to their positions in P, and eventually all lines are uncrossed.

Note that the fact that F is a null homotopy is equivalent to getting a labelling of all the points of W. One can get immersed almost normal 2-spheres occurring in homotopies in 3-manifolds that have non-trivial fundamental group but the homotopy will not produce a labelling of all the points of W.

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