A SHORT PROOF OF THE POINCARÉ CONJECTURE

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Abstract. A short, fairly self-contained proof is given of the Poincaré Conjecture.

1. Introduction

In 2002 I attempted a proof of the Poincaré Conjecture and put it on my home page. A number of errors were pointed out. At the time I was unable to resolve all of them and came to the conclusion that the approach could not work. Recently I wrote an account of my research [2], particularly relating to Stallings' Theorem and the accessibility of finitely generated groups and that made me think again about my aborted proof. It now seems to me that the approach was a good one and I have come up with a proof that I think resolves the earlier problems. I think the approach could provide the proof of the Sphere Theorem that I was looking for on the last page of [1].

The proof, then and now, was inspired by the beautiful algorithm of Hyam Rubinstein [5] for recognising the 3-sphere and the proof of this by Abigail Thompson [6]. In fact we show that the argument of [6] can be generalised to apply to a sequence of homotopies rather than a sequence of isotopies.

Perelman gave a proof of the Poincaré Conjecture in 2002.

My understanding of simple closed curves on tetrahedrons has benefitted from correspondence with Sam Shepherd and with Andrew Bartholomew.

I am very grateful to Peter Kropholler for his interest and encouragement.

2. Patterns in a Tetrahedron

Recall from [1] the definition of a pattern. Let K be a finite 2-complex with polyhedron $|K|$. A pattern is a subset P of $|K|$ satisfying the following conditions:-

(i) For each 2-simplex σ of K, $P \cap |\sigma|$ is a union of finitely many disjoint straight lines joining distinct faces of σ.

(ii) For each 1-simplex γ of K, $P \cap |\gamma|$ consists of finitely many points in the interior of $|\gamma|$.

A track is a connected pattern. If two patterns P and Q intersect each 1-simplex in the same number of points then the patterns are said to be *equivalent*. Two equivalent disjoint tracks in the same 2-complex are said to be *parallel.* We investigate tracks and patterns in a tetrahedron T, which we regard as the 2-skeleton $|\rho^2|$ of a 3-simplex ρ. We call a track in T an n-track if it has n intersections with edges.

If a pattern is as in Figure 1 then the tracks are all 3-tracks or 4-tracks. A pattern in a 3-manifold is called a normal pattern if the intersection with the boundary of every 3-simplex is like this.

An 8-track is shown in Figure 2. The only tracks one can have in a tetrahedron are n-tracks where $n = 3$ or $n = 4m$ for $m = 1, 2, \ldots$.

Figure 3 shows a 12-track. A pattern in T can only have two types of track. There can be 3-tracks, each of which separates one of the corner vertices from the other three vertices. There can be one other parallel set of tracks each of which is a $4n$-track for the same positive integer n, and for which there is a fixed sum $n = a + b$, where a, b are coprime integers. If n is even, then a, b are both odd integers, and the track is as in Figure 2 with a lines going from uv to wz and each of the other 4 lines replaced by b parallel lines. If n is odd, then exactly one of a, b is even. If a is even, so that

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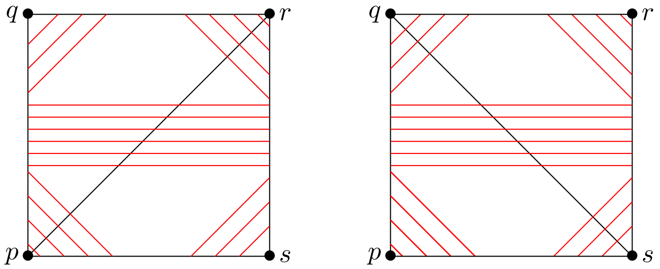


FIGURE 1.

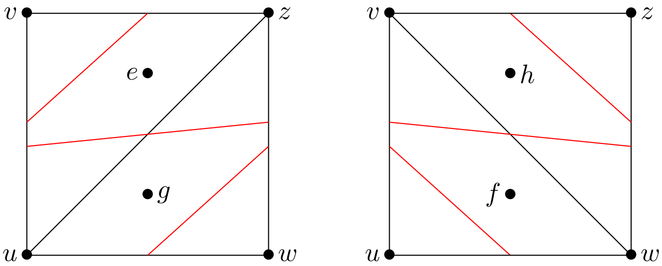


FIGURE 2.

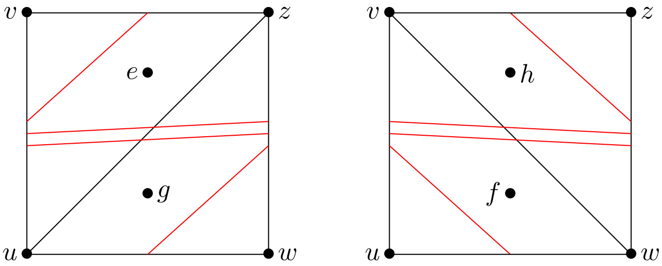


FIGURE 3.

b is odd, then the $4n$-track is as in Figure 3 again with a lines going from uv to wz and b parallel lines replacing each of the other 4 lines. These are patterns because the numbers of intersection points on edges match up. Each is a track because any proper subpattern must be of the same form with smaller integers a, b. Each such pattern separates the four vertices and the four centre points into two pairs. If a is odd as in Figure 2, the $4n$-pattern separates u, v from w, z and e, f from g, h. While if a is even as in Figure 3, the $4n$-track separates u, z from v, w and e, g from f, h. Thus each such subpattern will separate vertices and centre points in the same way and the space between two component tracks of the pattern could contain no vertex or centre point, so that in the terminology of [1] it is an untwisted band, making the two tracks parallel. But a, b will not be coprime if all the tracks in the pattern are parallel and there is more than one. Thus each pattern is a $4n$-track.

In fact for our proof here, we only need to know that there is an 8-track as in Figure 2 that intersects each of two opposite edges of T in two points and each of the other four edges just once.

A track in T is a simple closed curve, which will bound a disc in $|\rho|$. If M is a 3-manifold and M is triangulated so that $M = |K|$ where K is a finite 3-complex, then a pattern P in $|K^2|$ determines a *patterned surface* S such that for each 3-simplex ρ, $S \cap |\rho|$ consists of disjoint properly embedded discs and $S \cap |K^2| = P$. A patterned surface is determined, up to isotopy, by the intersection $P \cap |K^1|$. If the pattern in $|K^2|$ is normal, then the patterned surface is a normal surface.

Orient a track in T by choosing a positive direction as one goes along the track. At adjacent points of intersection of a track with an edge the directions of the track will be opposite to each other. This gives what we call a + - pair, i.e. a pair of points - not necessarily adjacent - of intersection points of an edge with a track where the track has opposite directions. This will be of more significance for singular "tracks " or *stracks* as shown in Figures 5 and 6, as if a pair of points of intersection on an edge can be removed by a homotopy, then the pair is a + - pair. A track in T has a + - pair if and only if it is not a 3-track or a 4-track, since adjacent points of intersection of a track with an edge will be a + - pair.

A *spattern* sP in K is defined to be a subset of $|K|$ satisfying

(i) For each 2-simplex σ of K, $sP \cap |\sigma|$ is a union of finitely many straight lines joining distinct faces of σ.

(ii) For each 1-simplex γ of K, $sP \cap |\gamma|$ consists of finitely many points in the interior of $|\gamma|$. Each such point belongs to exactly one straight line in each of the 2-simplexes containing γ.

A *strack* is a spattern that has no proper subspatterns. Every spattern is a union of finitely many stracks. A strack in T is the image of a circle. If M is a 3-manifold and M is triangulated so that $M = |K|$ where K is a 3-complex, then a spattern sP in $|K^2|$ determines a *spatterned surface* S such that for each 3-simplex ρ, $S \cap |\rho|$ consists of singular discs and $S \cap |K^2| = sP$.

Let $f : S^2 \to M$ be a general position map (see Hempel [4], Chapter 1), in which f is in general position with respect to a triangulation K of M. An i-piece of f is defined to be a component of $f^{-1}(\sigma)$ where σ is an $(i+1)$-simplex of K. Thus a 0-piece is a point of S^2. A 1-piece is either an scc (simple closed curve) or an arc joining two 0-pieces. If there are no 1-pieces that are scc's, then each 2-piece has boundary that is is a union of 1-pieces. One can use surgery along simple closed curves to change f to a map in which there are no 1-pieces that are scc's, and in which every 2-piece is a disc. The 2-pieces will then give a cell decomposition (tessellation) of the 2-sphere.

If R is a 1-piece with end points u, v whose images under f are in the same 1-simplex, then the restriction of f to R is called a returning arc.

If $f : S^2 \to M$ has no 1-pieces that are returning arcs or scc's, then the intersection of $f(S^2)$ with the 2-skeleton of M is a spattern sP. In the case in which we are interested, there is a homotopy from f to $f' : S^2 \to M$ in which the image is the spatterned surface determined by sP.

Let γ be a 1-simplex of M. Two points $p, q \in \gamma \cap f(S^2)$ are said to be removable if there is a homotopy from f to a map $f' : S^2 \to M$ such that $f(x) = f'(x)$ for every x that is not in the interior of a simplex with γ as a face and $\gamma \cap f'(S^2)$ is the same as $\gamma \cap f(S^2)$ but with p, q removed.

The pair of end points of a returning arc R are removable by the following homotopy. Let σ be the 2-simplex of K such that $f(R) \subset \sigma$. Let V be a regular neighbourhood of R in S^2. Let V° be the interior of V regarded as a subspace of V. Let βV be the boundary of V regarded as a subspace of S^2, so that $\beta V = V - V^\circ$. Let γ be the 1-simplex containing the end points of R. The regular neighbourhood V is a disc and $\beta V = \delta V$ is a simple closed curve in S^2. The union of all the 3-simplexes that contain γ is a closed ball B and $f(\beta V) \subset B^\circ - \sigma$, which is contractible. Define $f' : S^2 \to M$ so that f' is continuous, f' and f are the same when restricted to $S^2 - V^\circ$, and $f'(V) \subset B^\circ - \sigma$. Note that removing p may create more 1-pieces that are returning arcs or sccs, but the size of the intersection with the 1-skeleton goes down by two. In the case in which we are interested intersections, in which sccs can be removed as above, the maps f and f' are homotopic. This is illustrated in Figure 4, where it is shown how the removal of the ends of a returning arc will create returning arcs in two other simplexes that have the same 1-face.

There are also removable pairs of points if a strack intersects a 1-simplex in a + - pair. In [1] there is a mistake on page 253 in the section on simplifying surface maps. It is incorrectly asserted there that any pair of points in the intersection of the boundary of a 2-piece with a 1-simplex can be removed by a homotopy.

Suppose Q is a 2-piece and that γ is a 1-simplex for which $\delta Q \cap \gamma$ contains at least one + - pair. Let $s : S^1 \to \delta Q$ be as in the definition of a strack. There will be at least one + - pair in γ for which there is an arc $I = [p, q] \subset S^1$ such that $s(p) = u$ and $s(q) = v$ and $s(I)$ intersects γ only in its end points u, v. Such a pair is removable. Thus if γ contains a + - pair, then it contains a removable pair. Let p, q be a removable pair as above. There will be a map $s' : [p, q] \to Q$ which is close to s, for which $s'(p) = u, s'(q) = v$ and the image of the open interval (p, q) is contained in the interior of the 3-simplex ρ containing Q. Any two maps of I into the interior of ρ are homotopic. In particular there will be such a map that is close to the interval J in γ joining p, q. We can adjust $f : S^2 \to M$ by a homotopy so that the piece Q contains the image of this map. Thus $f' : S^2 \to M$ is the same outside Q and on the boundary of Q, but in the interior of Q takes the arc close to s to the arc J. Now f and f' are homotopic. Another homotopy in a neighbourhood of I, will give a new map f'' in which the pair p, q has been removed. This homotopy is similar to the one for removing a returning arc.

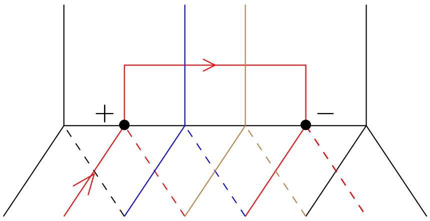


Figure 4.

In both cases, a removable pair can be removed without disturbing any other points of intersection with the 1-skeleton.

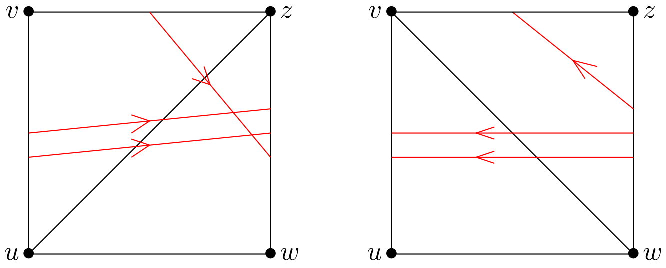


Figure 5.

Figure 5 shows an 11-strack in T that has no + - pairs of points.

If S is a spattern in a 2-complex K, then there is a uniquely determined underlying pattern P that has the same intersection with the 1-skeleton of K. Put $W = S \cap |K^1| = P \cap |K^1|$.

Figure 6 shows a spattern in T that is a union of a red 12-track and a blue 3-track, and its underlying pattern, which is also a 3-track and a 12-track. The underlying pattern for the strack of Figure 5 has two 4-tracks. and one 3-track.

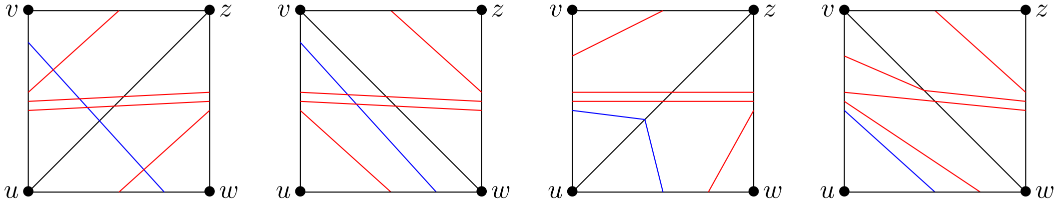


Figure 6.

Given two finite subsets F_1, F_2 of the closed interval $I = [0, 1]$ and a bijection $\beta_I : F_1 \to F_2$, there is a continuous map $\phi_I : I \to I$ which restricts to β on F_1 and to the identity on $\{0, 1\}$. There is a homotopy between ϕ_I and the identity map.

Building up from such maps, if $\nu : W \to W$ is a permutation that restricts to a permutation on $W \cap |\gamma|$ for each 1-simplex γ, then ν extends to a map of the 1-skeleton into itself which restricts to the identity on the 0-skeleton and which is homotopic to the identity map on $K^{(1)}$. This map can be further extended linearly to a map of the 2-skeleton and then, in the case of a triangulated 3-manifold, to a continuous map $\nu : M \to M$ which is homotopic to the identity map on M. It will have the property that if two points on the boundary of a 2-simplex σ are joined by a line in $S \cap \sigma$, then they are joined by a line in $\nu S \cap \sigma$. A spattern S in K is mapped to another spattern. Lines that were uncrossed may become crossed, and lines that were crossed may become uncrossed. The underlying pattern P is not changed. The tessellation of S together with the pieces of the tessellation are unchanged in such a map.

Our proof of the Poincaré Conjecture is to show that a certain spattern must occur in a homotopy from the boundary of a fake ball to a constant map, and this spattern is homotopic to its underlying pattern by such a homotopy.

3. The Proof

Let $M = |K|$ be a 3-manifold, where K is a finite 3-complex. It follows from a result of Kneser (see [1] or [4]) that there is a bound on the number of disjoint non parallel normal surfaces in a compact triangulated 3-manifold. I was able to prove that finitely presented groups are accessible by generalising this result to patterns in a finite 2-complex. In the Recognition Algorithm one determines a maximal set of disjoint normal surfaces in a triangulated 3-manifold M that are 2-spheres. If M is simply connected, then each such surface separates M and so the set of surfaces correspond to the edges of a finite tree. It is proved in [6] that every region corresponding to a vertex of valency (degree) at least two is a punctured 3-ball. A new proof of this is given in [3]. It follows from this that it suffices to consider regions corresponding to vertices of degree one in the Recognition Algorithm or the Poincaré Conjecture. It is proved that M is a 3-sphere if each region corresponding to a vertex of valency one in this tree either contains a single vertex of the triangulation or contains no vertices but does contain an almost normal surface, i.e. one for which the intersections of the surface with 3-simplexes are all 3 or 4-sided except for one exceptional 8-sided disc.

Let M be a compact, triangulated, simply connected 3-manifold . Let M_0 be a component, obtained by cutting along the maximal collection of normal 2-spheres, which has one boundary component and which does not contain a vertex. By Van Kampen, M_0 is simply connected, and so it is either a ball or a fake ball. In either case there is a homotopy between the boundary and the constant map. Let $f : S^2 \to M$ be an injective map whose image is the normal 2-sphere δM_0. Let $F : S^2 \times I \to M_0$ be a homotopy between f and a constant map. For $t \in I$ let $f_t : S^2 \to M_0, f_t(s) = F(s,t)$. We can assume that for all but finitely many values t, f_t meets the 1-skeleton T^1 of the triangulation transversely and for each t for which the map f_t does not meet T^1 transversely, there is precisely one point where f_t is tangential to T^1. Let $t'_1, t'_2, \ldots, t'_n$ be the values of t for which f_t does not meet T^1 transversely and put $t_0 = 0, t_{n+1} = 1$. For $i = 1, ..., n$ choose $t_i \in (t'_i, t'_{i+1})$ and put $f_i = f_{t_i}$.

Let W_i be the set of intersections of $f_i(S^2)$ with the 1-skeleton of T and let $w_i = |W_i|$.

For each i either $w_{i+1} = w_i + 2$ or $w_{i+1} = w_i - 2$. Note that w_0 is the number of intersections of δM_0 with the 1-skeleton. Since there are no removable pairs in a normal surface, $w_1 = w_0 + 2$. Let k be the smallest integer for which $w_k = w_{k-1} + 2. = w_{k+1} + 2$. There must be such a k as eventually every point of intersection is removed. Choose the homotopy F for which this value k is the smallest among all possible homotopies. Let $S = f_k(S^2)$. It will be shown that S is a spatterned sphere, so that the underlying patterned sphere is an embedded sphere. Each of the preceding $f_j(S^2) = S_j$, $1 \leq j \leq k$, satisfies $w_{j-1} = w_j - 2$, so that $W_j = W_{j-1} \cup \{u_j, v_j\}$ where u_j, v_j are a pair of points, labelled j, j, from a particular 1-simplex γ_j.

Now consider $f_k : S^2 \to M_0$ with $S = f_k(S^2)$. We follow the argument of [6]. We know that the homotopy going from f_k to f_{k+1} results in the deletion of a removable pair in a 1-simplex γ and the homotopy going from f_k to f_{k-1} results in the removal of another pair. Both pairs must belong to the same 2-piece, for if there is no 2-piece that contains both pairs, so that one pair is in one 2-piece and the other removable pair is in another 2-piece, then the order of the homotopies can be changed and in the weight sequence of the total homotopy one peak is replaced by two smaller peaks, and we have chosen F so that the first peak is the lowest possible. Also if both pairs are in the same 2-piece and one pair is not separated by the removal of the other pair, then we can also swap the homotopies round and get a lower first peak. It follows that the exceptional 2-piece has two removable pairs and that removing one pair disconnects the 2-piece. The simplest such 2-piece has boundary an 8-track as in Figure 2. Removing one pair (labelled k, k) creates two 3-tracks, with the pair on the opposite edge separated as shown in Figure 7. In an isotopy this is the only possibility for the exceptional 2-piece. For a homotopy there are other possibilities such as the 12-strack in Figure 8. In this case removing the pair labelled k, k gives a blue 3-track and a red 7-strack. In both cases, note that the lines joining the pair labelled k, k connect to points on the other two edges of the 2–simplex. They are not returning arcs, and they do not connect to points on the same 1-simplex. If they did, then removal of the pair labelled k, k would not disconnect the strack.

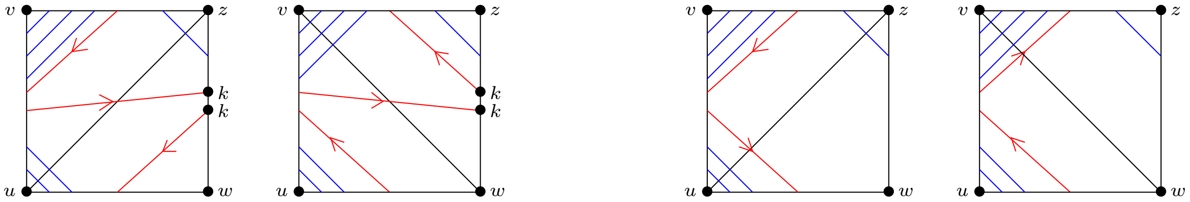


FIGURE 7.

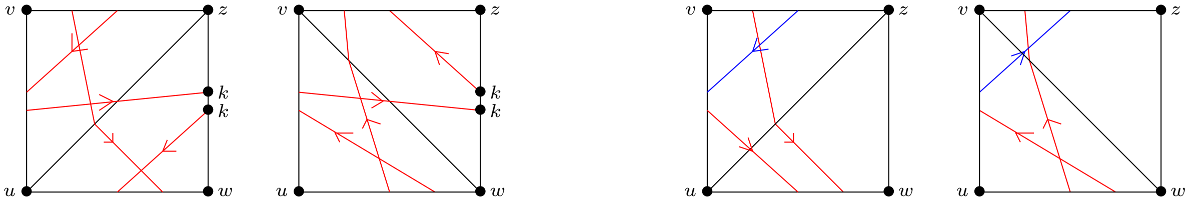


FIGURE 8.

We will show that all the 2-pieces apart from the exceptional one have no removable pairs. This is similar to the proof in [6]. Suppose p, q are a removable pair of 0-pieces in a 2-piece different from the exceptional 2-piece. Starting from the $(k-1)$-th stage, we can carry out a homotopy that removes p, q. We then carry out the homotopies f_k and f_{k+1} before replacing p, q. This sequence of homotopies has a lower first peak than the original, as the height of the first peak has been reduced by two. If p, q are in the exceptional 2-piece but are also in W_{k-1} or W_{k+1}, then we can also construct a sequence of homotopies with a lower weight sequence.

It is clear then that $S = f_k(S^2)$ has no returning arcs. Also it can be assumed that it has no 1-pieces that are scc's. For if there was a 1-piece that was an scc then surgery along this curve would produce two 2-spheres, one of which will contain the exceptional 2-piece. If this 2-sphere S' had fewer intersections with the 1-skeleton, i.e. $S' \cap K^1$ is a proper subset of $W = W_k$, then it would contradict the choice of F. Thus $S' \cap K^1 = W$ and the other 2-sphere will not intersect the 1-skeleton. All the 1-pieces for this 2-sphere must be scc's, and starting from an innermost one, these can be removed from f by a homotopy. Using a similar argument one can do surgery along scc's to change every 2-piece in S that is not a disc to one that is a disc.

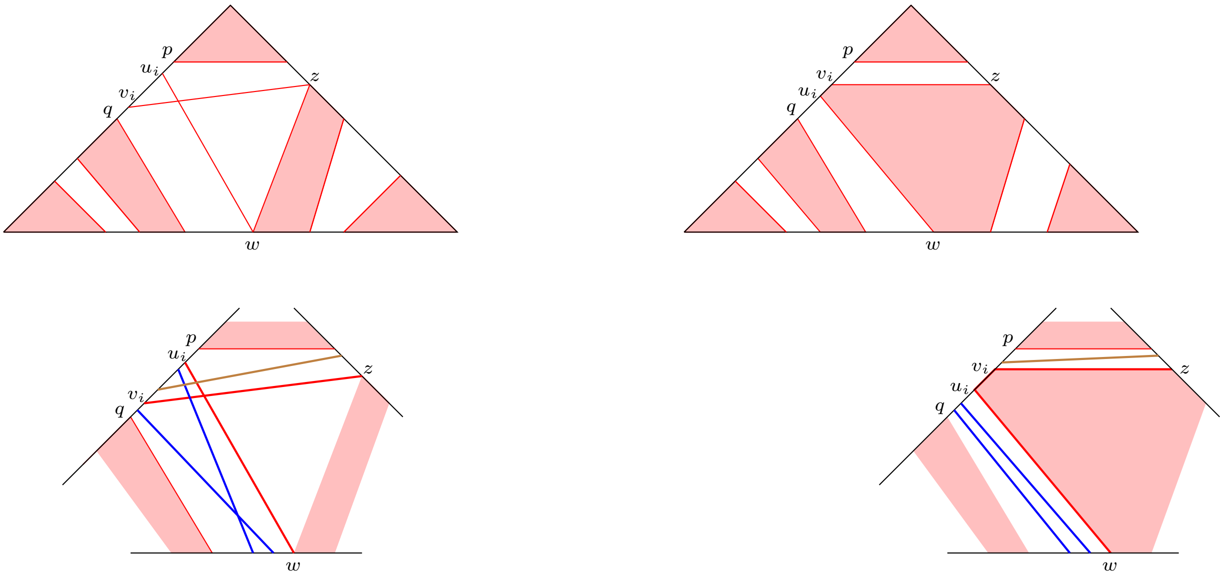
We can assume, then, that S is a spatterned surface. Let P be the underlying patterned surface. It will be shown that there is a permutation of W, restricting to a permutation on each intersection with a 1-simplex for which the corresponding homotopy changes S to P.

As a spattern is determined by its intersections with 2-simplexes, we will consider what happens to a single 2-simplex in the homotopy sequence. Initially the 2-simplex will intersect S_0 as in Figure 9(a) or Figure 9(b). The unshaded parts will be the intersection with M_0. Note that since M_0 contains no vertices, each vertex is contained in a shaded region. The intersection with S will consist of the intersection with S_0 together with straight lines joining edges in the unshaded regions. The extra intersection points with an edge are paired - each pair lying in an unshaded component.

If our sequence of homotopies was a sequence of isotopies, then each isotopy would result in an increase in shaded area. Thus we could go from Figure 9(a) to Figure 9(b), since removing the pair u_i, v_i from 9(b) gives 9(a). For each 2-simplex, such a move can occur just once as the central region, which is initially unshaded becomes shaded. If this region is initially shaded then, obviously, no such move can occur. The other move that can occur is adding a shaded region to an unshaded region as shown in Figure 10.

It will be shown that our sequence of homotopies can be converted to a sequence of isotiopies by using the homotopies corresponding to permutations of the points of W on a 1-simplex.

Let W_0 ibe the set of vertices of the normal 2-sphere $S_0 = \delta M_0$. In S_1 the pair u_1, v_1 will be the ends of a returning arc in at least one 2-simplex. In another 2-simplex u_1, v_1 will be joined by lines in both S_1 and S to the vertices of an edge in S_0. Having such a situation in a 2-simplex is the only way that removing u_1, v_1 will give a normal pattern. Thus we are in the situation of Figure 9, with $i = 1$, rather than Figure 10. In the sequence of homotopies the lines joining u_1, v_1 to the ends w, z of an edge may cross. They can be uncrossed by transposing u_1 and v_1. We need ν_1 to do more than this. We permute the points of W that lie in the interval $[p, q]$ so that there are no points in the open interval (u_1, v_1) and there are no lines that cross the lines from either u_1 or v_1. This will happen if the points in $W \cap [p, q]$ are permuted so that the ones joined to a point on the bottom edge are in $[q, u_1]$ and the points in $[p, v_1]$ are joined to points in the right hand edge. The lines $u_1 w$ and $v_1 z$ will now be lines in P. This is illustrated in Figure 9(b).

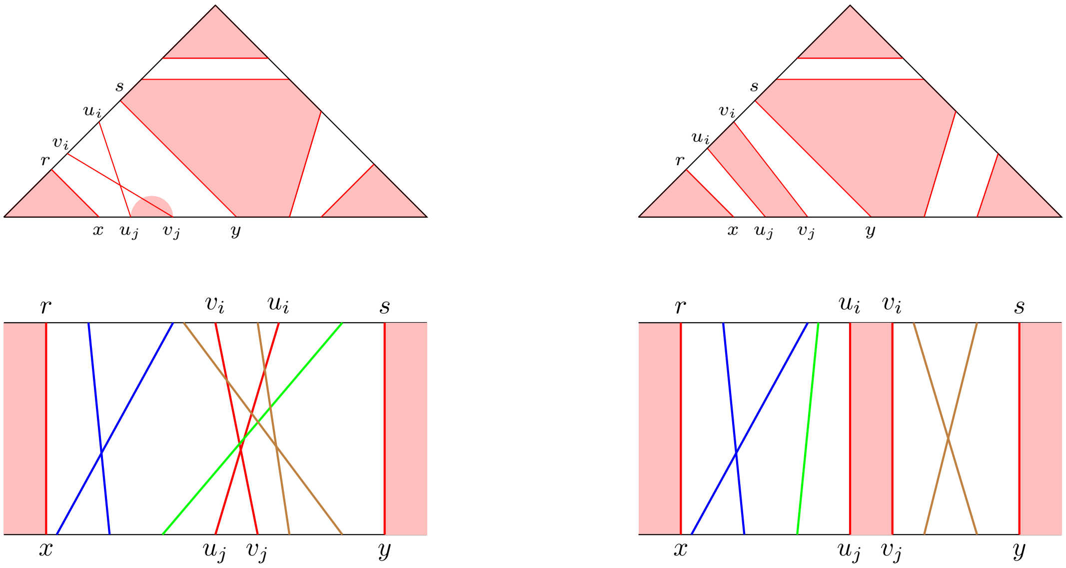


(a) (b)

Figure 9.

We use an induction argument for defining ν_i for $2 \leq i \leq k$.

Our aim is to show that for each 1-simplex γ we can permute the finite set $\gamma \cap S$ in such a way that under the associated homotopies the spattterned 2-sphere S becomes the underlying patterned 2-sphere P. It will then be the case that F becomes an isotopy.



(a) (b)

Figure 10.

Let γ_i be the 1-simplex containing u_i and v_i.

Our induction hypothesis is that there are permutations ν_i, μ_i of W for $i = 1, \dots, k$, such that ν_i restricts to a permutation on the points of $W \cap \gamma_i$ and is the identity on the other points of W, and so that the homotopy associated with $\mu_i = \nu_i \mu_{i-1}$ moves the points u_j and v_j for $j < i$ so that any line joining two such points is moved to a line in P, and so that the lines joining u_i to u_j and joining v_i to v_j are lines in P and they do not cross any other line from γ_j to γ_i.

We have defined ν_1. Let $\mu_1 = \nu_1$.

We now define ν_i and μ_i. In at least one of the 2-simplexes containing γ_i as a face there are lines in S joining u_j to u_i for some $j < i$ or there is a situation as in Figure 9, in which u_i, v_i are joined to the vertices of an edge of S_j for $j < i$. This is because there cannot be a returning arc joining u_i and v_i in every 2-simplex containing γ_i, as if there were, then S_i would not be connected. If u_i and v_i are joined to points u_j and v_j for $j < i$ then u_j and v_j are the ends of a returning arc in S_l for every $l, j \leq l < i$.

In Figure 10 we see what happens in a 2-simplex σ_i containing γ_i, if the lines joining u_i and v_i come from the pair u_j, v_j.

We permute the points of W that lie in the interval $[r, s]$ so that there are no points in the smaller interval $[u_i, v_i]$ and there are no lines that cross the lines from either u_i or v_i. This will happen if the points in $W \cap [r, s]$ are permuted so that the ones joined to a point in $[x, u_j]$ are in $[r, u_i]$ and the the points in $[s, v_i]$ are joined to points in $[v_j, y]$. The lines $u_i u_j$ and $v_i v_j$ will now be lines in P. This is also illustrated in Figure 10. Note that every point in the interval xy apart from u_j and v_j has label bigger than j and every label in the interval rs apart from u_i and v_i has label bigger than i.

A similar argument can be used if both pairs are in the central region.

At the end of the induction a connected subgraph, containing every vertex of the 1-skeleton of S, has been moved to the position it should have in P. In fact we could have chosen a line in any 2-simplex to be in this connected subgraph. Thus if a line connects u_i to u_j where $i > j$, we could have chosen the 2-simplex containing this line at the ith step in the induction. This means that $P = \mu_k S$ and so P determines a patterned 2-sphere.

We now know that $\mu_k S$ is a patterned surface. All the 2-pieces, apart from the exceptional one, intersect each 1-simplex at most once and so are 3-sided or 4-sided. The 8-track shown in Figure 7 is the only possibility for the exceptional 2-piece. Thus S has become an almost normal 2-sphere and we have a proof of the Poincaré Conjecture.

After applying μ, for $j > k$ each step of the homotopy becomes an isotopy in which a removable pair of adjacent points is removed. There is now a new labelling of W in which every point receives a label j, where $n \leq j \leq n$. The pair of points labelled j will be a removable pair in $f_j(S^2)$. In fact the pair of points will be joined by a returning arc for $j > k$.

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