A SHORT PROOF OF THE POINCARÉ CONJECTURE

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ABSTRACT. A short, fairly self-contained proof is given of the Poincaré Conjecture.

1. Introduction

In 2002 I attempted a proof of the Poincaré Conjecture and put it on my home page. A number of errors were pointed out. At the time I was unable to resolve all of them and came to the conclusion that the approach could not work. Recently I wrote an account of my research [2], particularly relating to Stallings' Theorem and the accessibility of finitely generated groups and that made me think again about my aborted proof. It now seems to me that the approach was a good one and I have come up with a proof that I think resolves the earlier problems. I think the approach could provide the proof of the Sphere Theorem that I was looking for on the last page of [1].

The proof, then and now, was inspired by the beautiful algorithm of Hyam Rubinstein [4] for recognising the 3-sphere and the proof of this by Abigail Thompson [5].

Perelman gave a proof of the Poincaré Conjecture in 2002.

My understanding of simple closed curves on tetrahedrons has benefitted from correspondence with Sam Shepherd.

2. Patterns in a Tetrahedron

Recall from [1] the definition of a pattern. Let K be a finite 2-complex with polyhedron $|K|$. A pattern is a subset P of $|K|$ satisfying the following conditions:-

(i) For each 2-simplex σ of K, $P \cap |\sigma|$ is a union of finitely many disjoint straight lines joining distinct faces of σ.

(ii) For each 1-simplex γ of K, $P \cap |\gamma|$ consists of finitely many points in the interior of $|\gamma|$.

A track is a connected pattern. If two patterns P and Q intersect each 1-simplex in the same number of points then the patterns are said to be *equivalent*. Two equivalent disjoint tracks in the same 2-complex are said to be *parallel*. We investigate tracks and patterns in a tetrahedron T, which we regard as the 2-skeleton $|\rho^2|$ of a 3-simplex ρ. We call a track in T an n-track if it intersects n edges.

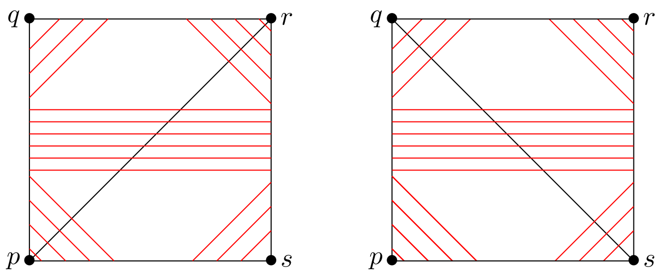

FIGURE 1.

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If a pattern is as in Figure 1 then the tracks are all 3-tracks or 4-tracks. A pattern in a 3-manifold is called a normal pattern if the intersection with the boundary of every 3-simplex is like this.

An 8-track is shown in Figure 2. The only tracks one can have in a tetrahedron are n-tracks where $n = 3$ or $n = 4m$ for $m = 1, 2, \ldots$.

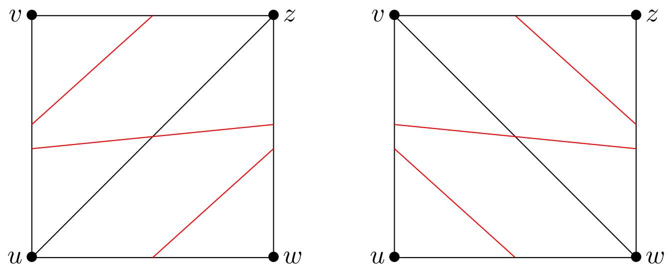

FIGURE 2.

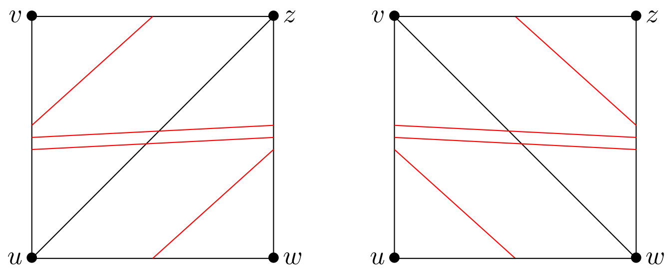

FIGURE 3.

Figure 3 shows a 12-track. A pattern in T can only have two types of track. There can be 3-tracks, each of which separates one of the corner vertices from the other three vertices. There can be one other parallel set of tracks each of which is a $4n$-track for the same positive integer n. Each such track separates the four vertices into two pairs. If $n > 0$ is even, the $4n$-track separating u, v from w, z is as in Figure 2 but with $n-1$ lines crossing from uv to wz. While if $n > 1$ is odd the $4n$-track separating u, z from v, w is as in Figure 3 but with $n-1$ lines crossing from uv to wz.

A track in T is a simple closed curve, which will bound a disc in $|\rho|$. If M is a 3-manifold and M is triangulated so that $M = |K|$ where K is a finite 3-complex, then a pattern P in $|K|$ determines a *patterned surface* S such that for each 3-simplex ρ, $S \cap |\rho|$ consists of disjoint properly embedded discs and $S \cap |K^2| = P$. A patterned surface is determined, up to isotopy, by the intersection $P \cap |K^1|$. If the pattern in $|K^2|$ is normal, then the patterned surface is a normal surface.

Orient a track in T by choosing a positive direction as one goes along the track. At adjacent points of intersection of a track with an edge the directions of the track will be opposite to each other. This gives what we call a + - pair, i.e. a pair of points - not necessarily adjacent - of intersection points of an edge with a track where the track has opposite directions. This will be of more significance for singular “tracks ” or *stracks* as shown in Figures 4 and 5, as if a pair of points of intersection on an edge can be removed by a homotopy iand then the pair is a + - pair. A track in T has a + - pair if and only if it is not a 3-track or a 4-track, since any track which intersects an edge more than once will have a + - pair.

A *spattern* sP in K is defined to be a subset of $|K|$ satisfying

(i) For each 2-simplex σ of K, $sP \cap |\sigma|$ is a union of finitely many straight lines joining distinct faces of σ.
(ii) For each 1-simplex γ of K, $sP \cap |\gamma|$ consists of finitely many points in the interior of $|\gamma|$. Each such point belongs to exactly one straight line in each of the 2-simplexes containing γ.

A *strack* is a spattern that has no proper subspatterns. Every spattern is a union of finitely many stracks. A strack in T is the image of a circle. If M is a 3-manifold and M is triangulated so that $M = |K|$ where K is a 3-complex, then a spattern sP in $|K^2|$ determines a *spatterned surface* S such that for each 3-simplex ρ, $S \cap |\rho|$ consists of singular discs and $S \cap |K^2| = sP$.

Let $f : S^2 \to M$ be a general position map (see Hempel [3], Chapter 1), in which f is in general position with respect to a triangulation K of M. An i-piece of f is defined to be a component of $f^{-1}(\sigma)$ where σ is an $(i+1)$-simplex of K. Thus a 0-piece is a point of S^2. A 1-piece is either an scc (simple closed curve) or an arc joining two 0-pieces. If there are no 1-pieces that are scc's, then each 2-piece has boundary that is is a union of 1-pieces. One can use surgery along simple closed curves to change f to a map in which there are no 1-pieces that are scc's, and in which every 2-piece is a disc. The 2-pieces will then give a cell decomposition (tessellation) of the 2-sphere.

If R is a 1-piece with end points u, v whose images under f are in the same 1-simplex, then the restriction of f to R is called a returning arc.

If $f : S^2 \to M$ is a general position map with no 1-pieces that are returning arcs, then the intersection of $f(S^2)$ with the 2-skeleton of M is a spattern sP and there is a homotopy from f to $f' : S^2 \to M$ in which the image is the spatterned surface determined by sP.

Let γ be a 1-simplex of M. Two points $p, q \in \gamma \cap f(S^2)$ are said to be removable if there is a homotopy from f to a map $f' : S^2 \to M$ such that $f(x) = f'(x)$ for every x that is not in the interior of a simplex with γ as a face and $\gamma \cap f'(S^2)$ is the same as $\gamma \cap f(S^2)$ but with p, q removed.

The pair of end points of a returning arc R are removable by the following homotopy. Let σ be the 2-simplex of K such that $f(R) \subset \sigma$. Let V be a regular neighbourhood of R in S^2. Let V° be the interior of V regarded as a subspace of V. Let βV be the boundary of V regarded as a subspace of S^2, so that $\beta V = V - V^\circ$. Let γ be the 1-simplex containing the end points of R. The regular neighbourhood V is a disc and $\beta V = \delta V$ is a simple closed curve in S^2. The union of all the 3-simplexes that contain γ is a closed ball and $f(\beta V) \subset B^\circ - \sigma$, which is contractible. Define $f' : S^2 \to M$ so that f' is continuous, f' and f are the same when restricted to $S^2 - V^\circ$, and $f'(V) \subset B^\circ - \sigma$. Note that removing p may create more 1-pieces that are returning arcs or sccs, but the size of the intersection with the 1-skeleton goes down by two. In the case in which we are interested intersections which are sccs can be removed as above. The maps f and f' are homotopic. This is illustrated in Figure 4, where it is shown how the removal of the ends of a returning arc will create returning arcs in two other simplexes that have the same 1-face.

There are also removable pairs of points if a strack intersects a 1-simplex in a + - pair. In [1] there is a mistake on page 253 of that section on simplifying surface maps. It is incorrectly asserted there that any pair of points in the intersection of the boundary of a 2-piece with a 1-simplex can be removed by a homotopy. Suppose Q is a 2-piece and that γ is a 1-simplex for which $\delta Q \cap \gamma$ contains at least one + - pair. Let $s : S^1 \to \delta Q$ be as in the definition of a strack. There will be at least one + - pair in γ for which there is an arc $I = [p, q] \subset S^1$ such that $s(p) = u$ and $s(q) = v$ and $s(I)$ intersects γ only in its end points u, v. Such a pair is removable. Thus if γ contains a + - pair, then it contains a removable pair. Let p, q be a removable pair as above. There will be a map $s' : [p, q] \to Q$ which is close to s, for which $s'(p) = u, s'(q) = v$ and the image of the open interval (p, q) is contained in the interior of the 3-simplex ρ containing Q. Any two maps of I into the interior of ρ are homotopic. In particular there will be such a map that is close to the interval J in γ joining p, q. We can adjust $f : S^2 \to M$ by a homotopy so that the piece Q contains the image of this map. Thus $f' : S^2 \to M$ is the same outside Q and on the boundary of Q, but in the interior of Q takes the arc close to s to the arc J. Now f and f' are homotopic. Another homotopy in a neighbourhood of I, will give a new map f'' in which the pair p, q has been removed. This homotopy is similar to the one for removing a returning arc.

In both cases, a removable pair can be removed without disturbing any other points of intersection with the 1-skeleton.

Figure 5 shows an 11-strack in T that has no + - pairs of points. I conjecture that a strack in T must intersect all six edges to have a + - pair.

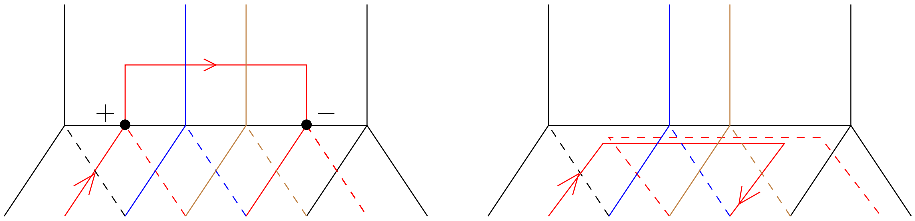

FIGURE 4.

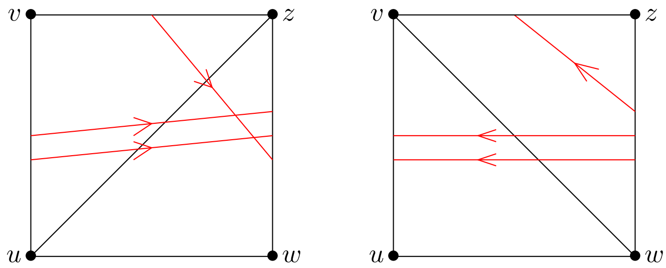

FIGURE 5.

If S is a spattern in a 2-complex K, then there is a uniquely determined underlying pattern P that has the same intersection with the 1-skeleton of K. Put $W = S \cap |K^1| = P \cap |K^1|$.

Figure 6 shows a spattern in T that is a union of a red 12-track and a blue 3-track, and its underlying pattern, which is also a 3-track and a 12-track. The underlying pattern for the strack of Figure 5 has two 4-tracks. and one 3-track.

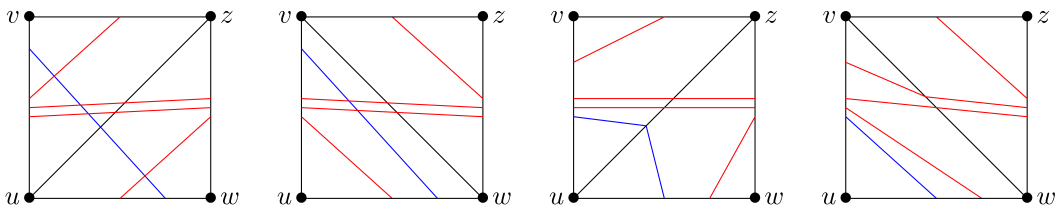

FIGURE 6.

Given two finite subsets F_1, F_2 of the closed interval $[0,1]$ and a bijection $\beta_I : F_1 \to F_2$, there is a continuous map $\phi_I : I \to I$ which restricts to β on F_1 and to the identity on $\{0,1\}$. There is a homotopy between ϕ_I and the identity map.

Building up from such maps, if $\nu : W \to W$ is a permutation that restricts to a permutation on $W \cap |\gamma|$ for each 1-simplex γ, then ν extends to a map of the 1-skeleton into itself which restricts to the identity on the 0-skeleton and which is homotopic to the identity map on $K^{(1)}$. This map can be further extended linearly to a map of the 2-skeleton and then, in the case of a triangulated 3-manifold, to a continuous map $\nu : M \to M$ which is homotopic to the identity map on M. It will have the property that if two points on the boundary of a 2-simplex σ are joined by a line in $S \cap \sigma$, then they are joined by a line in $\nu S \cap \sigma$. A spattern S in K is mapped to another spattern. Lines that were uncrossed may become crossed, and lines that were crossed may become uncrossed. The underlying pattern P is not changed. The tessellation of S together with the pieces of the tessellation are unchanged in such a map.

Our proof of the Poincaré Conjecture is to show that a certain spattern must occur in a homotopy from the boundary of a fake ball to a constant map, and this spattern is homotopic to its underlying pattern by such a homotopy.

3. The Proof

Let $M = |K|$ be a 3-manifold, where K is a finite 3-complex. It follows from a result of Kneser (see [1] or [3]) that there is a bound on the number of disjoint non parallel normal surfaces in a compact triangulated 3-manifold. I was able to prove that finitely presented groups are accessible by generalising this result to patterns in a finite 2-complex. In the Recognition Algorithm one determines a maximal set of disjoint normal surfaces in a triangulated 3-manifold M that are 2-spheres. If M is simply connected, then each such surface separates M and so the set of surfaces correspond to the edges of a finite tree. It is proved that M is a 3-sphere if each region corresponding to a vertex of valency one in this tree either contains a single vertex of the triangulation or contains no vertices but does contain an almost normal surface, i.e. one for which the intersections of the surface with 3-simplexes are all 3 or 4-sided except for one exceptional 8-sided disc.

Let M be a fake 3-ball. Let M_0 be a component, obtained by cutting along the maximal collection of normal 2-spheres, which has one boundary component and which does not contain a vertex. By Van Kampen, M_0 is simply connected, and so it is either a ball or a fake ball. In either case there is a homotopy between the boundary and the constant map. Let $f : S^2 \to M$ be an injective map whose image is the normal 2-sphere δM_0. Let $F : S^2 \times I \to M_0$ be a homotopy between f and a constant map. For $t \in I$ let $f_t : S^2 \to M_0, f_t(s) = F(s,t)$. We can assume that for all but finitely many values t, f_t meets the 1-skeleton T^1 of the triangulation transversely and for each t for which the map f_t does not meet T^1 transversely, there is precisely one point where f_t is tangential to T^1. Let $t'_1, t'_2, \ldots, t'_n$ be the values of t for which f_t does not meet T^1 transversely and put $t_0 = 0, t_{n+1} = 1$. For $i = 1, ..., n$ choose $t_i \in (t'_i, t'_{i+1})$ and put $f_i = f_{t_i}$.

Let W_i be the set of intersections of $f_i(S^2)$ with the 1-skeleton of T and let $w_i = |W_i|$. Rearrange the weights w_i into a finite non-increasing sequence. Order these sequences lexicographically. The width of T is the minimum sequence of weights as F ranges over all possible homotopies. If F realises the width of T, then F is said to be in thin position. Now revert to the original ordering of the w_i's. Clearly for each i either $w_{i+1} = w_i + 2$ or $w_{i+1} = w_i - 2$. Note that w_0 is the number of intersections of δM_0 with the 1-skeleton. For F in thin position, we are particularly interested in the values i for which $w_{i-1} = w_{i+1} = w_i - 2$. For such an $i, f_i(S^2)$ is called a thick sphere, while if $w_{i-1} = w_{i+1} = w_i + 2$, then $f_i(S^2)$ is called a thin sphere. Our interest will be in the first thick sphere. Since there are no removable pairs in a normal surface, $w_1 = w_0 + 2$. Let $S = f_k(S^2)$ be the first thick sphere. Each of the preceding $f_j(S^2) = S_j$, $1 \leq j \leq k$, satisfies $w_{j-1} = w_j - 2$, so that $W_j = W_{j-1} \cup \{u_j, v_j\}$ where u_j, v_j are a pair of points, labelled j, j, from a particular 1-simplex γ_j.

Now consider $f_k : S^2 \to M_0$ with $S = f_k(S^2)$. We follow the argument of [5]. We know that the homotopy going from f_k to f_{k+1} results in the deletion a removable pair in a 1-simplex γ and the homotopy going from f_k to f_{k-1} results in the removal of another pair. Both pairs must belong to the same 2-piece, for if there is no 2-piece that contains both pairs, so that one pair is in one 2-piece and the other removable pair is in another 2-piece, then the order of the homotopies can be changed and the weight of the total homotopy reduced - one peak is replaced by two smaller peaks. Also if both pairs are in the same 2-piece and one pair is not separated by the removal of the other pair, then we can also swap the homotopies round and reduce weight. It follows that the exceptional 2-piece has two removable pairs and that removing one pair disconnects the 2-piece. The simplest such 2-piece has boundary an 8-track as in Figure 2. Removing one pair (labelled k, k) creates two 3-tracks, with the pair on the opposite edge separated as shown in Figure 7. In an isotopy this is the only possibility for the exceptional 2-piece. For a homotopy there are other possibilities such as the 12-strack in Figure 8. In this case removing the pair labelled k, k gives a blue 3-track and a red 7-strack. In both cases, note that the lines joining the pair labelled k, k connect to points on the other two edges of the 2–simplex. They are neither returning arcs, nor do they connect to points on the same 2-simplex.. If they did, then removal of the pair labelled k, k would not disconnect the strack.

We will show that all the 2-pieces apart from the exceptional one have no removable pairs. This is similar to the proof in [5]. Suppose p, q are a removable pair of 0-pieces in a 2-piece different from the exceptional 2-piece. Starting from the $(k-1)$-th stage, we can carry out a homotopy that

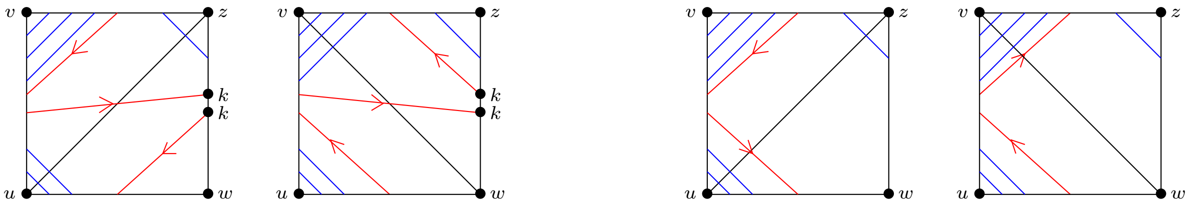

FIGURE 7.

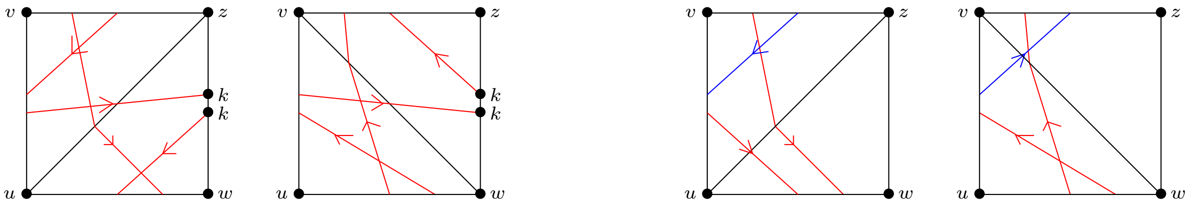

FIGURE 8.

removes p, q. We then carry out the homotopies f_k and f_{k+1} before replacing p, q. This sequence of homotopies has a lower sequence of weights than the original, as the weight of the highest peak has been reduced by two lower ones. If p, q are in the exceptional 2-piece but are also in W_{k-1} or W_{k+1}, then we can also construct a sequence of homotopies with a lower weight sequence.

Let $W = W_k$. It is clear that $S = f_k(S^2)$ has no returning arcs and so it is a spattern. Let P be the underlying pattern. It will be shown that there is a permutation of W, restricting to a permutation on each intersection with a 1-simplex for which the corresponding homotopy changes S to P.

We see that W_0 is the set of vertices of a normal 2-sphere $S_0 = \delta M_0$. The pair u_1, v_1 will be the ends of a returning arc in at least one 2-simplex. In another 2-simplex u_1, v_1 will be joined by lines in both S_1 and S to the vertices of an edge in S_0. Having such a situation in a 2-simplex is the only way that removing u_1, v_1 will give a normal pattern.

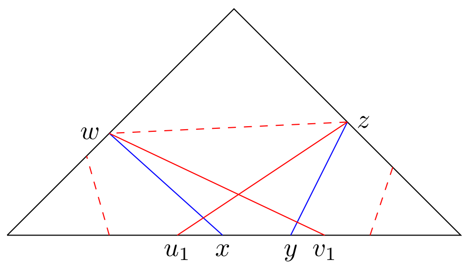

FIGURE 9.

Figure 9 shows the situation for u_1, v_1 which lie in the 1-simplex γ_1. In at least one 2-simplex containing γ_1 as a face, u_1, v_1 will be joined to the vertices w, z of an edge of S_0, shown dashed red. In P, w, z will be joined to points x, y by lines, shown blue in Figure 9. Let ν_1 be the permutation of W, which restricts to $(v_1, x)(u_1, y)$ for $W \cap \gamma_1$ and is the identity map on all the other 1-simplexes. Under the corresponding homotopy the two lines become uncrossed if they were crossed before. Note that the end points are adjacent and stay adjacent because the homotopy takes place in M_0 and so the new lines of the homotopy do not intersect the lines of S_0. The pattern for S_0 in each 3-simplex is as in Figure 1 and so the homotopy ν_1 must take place in one of the regions which is not bounded by two parallel lines. This is illustrated in Figure 10, which indicates what can happen in the faces of a 3-simplex. If a 2-simplex σ contains γ_1 as one of its faces, but has no intersection with S_0, then the points will be joined by a returning arc in σ. In σ, a 4-track has been replaced by two 3-tracks. If instead of joining the ordered pair u_1, v_1 to the vertices of the second edge, we joined v_1, u_1 to those vertices we would get a 6-strack. To get a subgraph of P one must join up the points to give two 3-tracks.

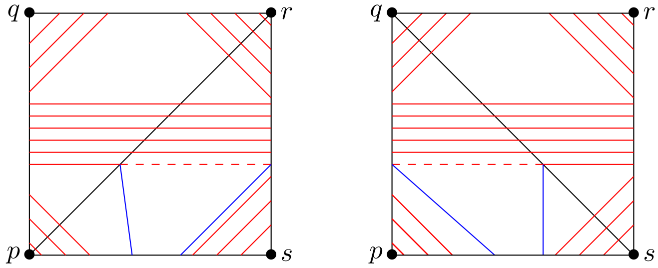

FIGURE 10.

We use an induction argument for defining ν_i for $2 \leq i \leq k$.

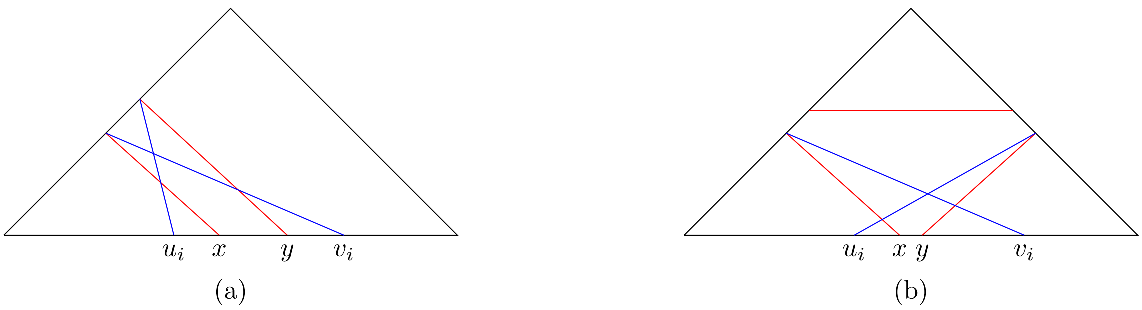

FIGURE 11.

Our aim is to show that for each 1-simplex γ we can permute the finite set $\gamma \cap S$ in such a way that under the associated homotopies the spattterned 2-sphere S becomes the underlying patterned 2-sphere P. It will then be the case that F becomes an isotopy.

Let γ_i be the 1-simplex containing u_i and v_i.

We have defined ν_1. Let $\mu_1 = \nu_1$. We now define ν_i and μ_i for $i = 2, \ldots, k$ to be permutations of W, where ν_i restricts to a permutation on the points of $W \cap \gamma_i$ and is the identity on the other points of W, and so that the homotopy associated with $\mu_i = \nu_i \mu_{i-1}$ moves the points of W_i to the positions they should have in P. In at least one of the 2-simplexes containing γ_i as a face there are lines in S joining u_j to u_i for some $j < i$ or there is a situation as in Figure 9, in which u_i, v_i are joined to the vertices of an edge of S_j for $j < i$.

If there are lines in $\mu_{i-1}S$ joining u_i to u_j and v_i to v_j it will be as in Figure 11(a) or there will be three labelled pairs joined by lines as in Figure 11(b). This is because S_i has a returning arc joining u_i and v_i and this returning arc was created by one or more homotopy f_j for $j < i$.

In Figure 11 (a) we see what happens in a 2-simplex σ_i containing γ_i, if the lines joining u_i and v_i come from a single pair u_j, v_j. In Figure 11(b) the lines to u_i, v_i come from two pairs. The lines in blue are those of $\mu_{i-1}S \cap \sigma_i$. The lines in red are the ones we move them to in $\mu_i S$. The bottom edge is the 1-simplex γ_i containing the pair u_i, v_i which have labels i, i in the labelling for S. In at least one of the 2-simplexes containing γ there will be a 1-face different from γ containing an adjacent pair with labels l, l for $l < i$, which are joined by non-intersecting lines in P to the pair x, y in γ_i, or the situation will be as in Figure 11(b) so that there are pairs with labels less than i in each of the other two faces. The induction step is to apply the permutation $\nu_i = (v_i, x)(u_i, y)$ to the set $\gamma_i \cap W$. Of course if $u_i = y$ we apply the transposition (v_i, x). Again with each transposition we also transpose all attaching lines in all 2-simplexes containing γ. This change will not disturb any of the vertices of W_{i-1} or the edges joining them. Note that x, y will have labels bigger than i as if, say, $x \in W_{i-1}$, then in S_{i-1} there would be a point in the intersection of a 1-simplex with $\mu_{i-1}S$ joined to both u_i and y and this does not happen in a spattern.. Put $\mu = \mu_k$.

For each labelled point of W we can choose a line in μS joining the point to a point of W_0 or a point with a smaller label. This will give us a connected subgraph of both P and μS that contains every point of W. This must mean that P is connected. Since S is a 2-sphere and μS is obtained from S by permuting the points of W and the lines joining them, μS is also a 2-sphere. But we now know P and μS have a common spanning subtree. Also P and μS have the same number of edges. This means that P is also a 2-sphere, as P and μS will have the same Euler characteristic. The addition of each edge not in a spanning tree results in a region being divided in two, and so there is one extra face for each such edge..

It now follows that at the end of the induction $P = \mu_k S$ so that P determines a patterned 2-sphere.

We now know that μS is a patterned surface. All the 2-pieces, apart from the exceptional one, intersect each 1-simplex at most once and so are 3-sided or 4-sided. The 8-track shown in Figure 6(a) is the only possibility for the exceptional 2-piece. Thus S has become an almost normal 2-sphere and we have a proof of the Poincaré Conjecture. In the homotopy F, the first thick sphere S will also be the last thick sphere. After applying μ, then for $j > k$ each step of the homotopy becomes an isotopy in which a returning arc joining adjacent points is removed. There is now a new labelling of W in which every point receives a label j, where $n + 1 \leq j \leq n$. The pair of points labelled $j + 1$ will be a removable pair in $f_j(S^2)$.

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