

Patterns, Tracks and the Poincaré Conjecture

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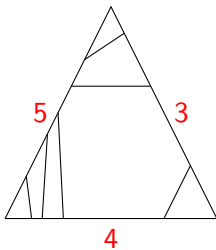
In the 90's Hyam Rubinstein put forward an algorithm for recognising if a 3-manifold is a 3-sphere and proved that it did. What follows here is a reworking of the proof of Abigail Thompson, together with a short explanation of how it can be adapted to give a proof of the Poincaré Conjecture. The Poincaré Conjecture was proved by Perelman in 2002.

Patterns and Tracks

Let K be a finite 2-complex with polyhedron $|K|$. A pattern is a subset P of $|K|$ satisfying the following conditions:-

- (i) For each 2-simplex σ of K , $P \cap |\sigma|$ is a union of finitely many disjoint straight lines joining distinct faces of σ .
- (ii) For each 1-simplex γ of K , $P \cap |\gamma|$ consists of finitely many points in the interior of $|\gamma|$.

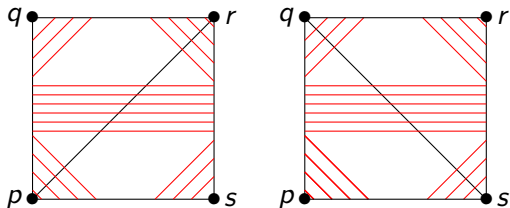
A track is a connected pattern. A pattern is uniquely determined by its intersection with the 1-skeleton.



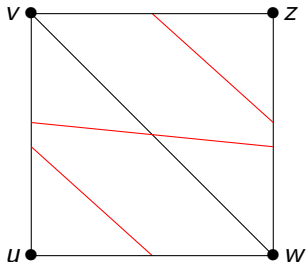
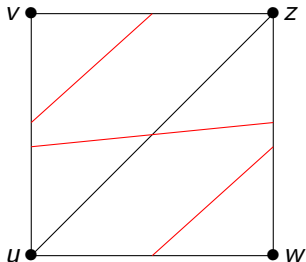
If two patterns P and Q intersect each 1-simplex in the same number of points then the patterns are said to be *equivalent*. Two equivalent disjoint tracks in the same 2-complex are said to be *parallel*. We call a track in T an n -track if it has n intersections with edges (1-simplexes).

Normal Pattern

If a pattern in a tetrahedron T is as in Figure 1 then the tracks are all 3-tracks or 4-tracks. A pattern in a 3-manifold is called a normal pattern if the intersection with the boundary of every 3-simplex ρ is like this.



There are other patterns in a tetrahedron.

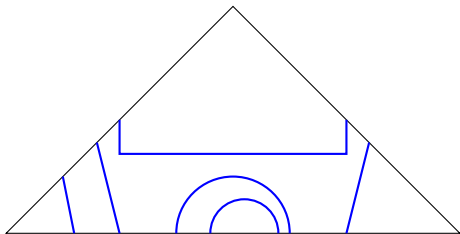


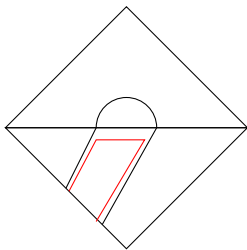
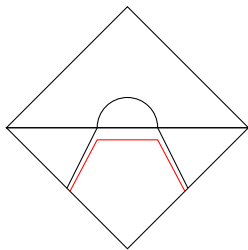
This is an 8-track. It is important in this theory.

Two tracks are parallel unless the space between them contains a vertex or the centre point of a face. This is used in the accessibility of finitely presented groups,

A tetrahedron is a triangulation of a 2-sphere. There are many triangulations of a 2-sphere. Another one is the icosahedron. Let K be such a 2-complex. Every edge (1-simplex) belongs to two faces and the neighbourhood of each vertex is a disc. A pattern P in K is a set of tracks that are simple closed curves and each track separates so the tracks are the edges of tree D_P in which the vertices are the connected components after removing the tracks.

Suppose T is a simple closed curve in K that is in general position with respect to the triangulation. Then T is a track (or rather it can be straightened to be a track) if and only if in the intersection with each 2-simplex there are no returning arcs, i.e. one as in diagram for which the end points are in the same edge of the triangulation.





An innermost

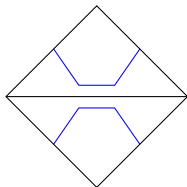
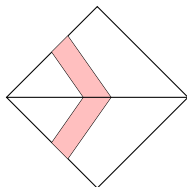
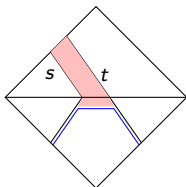
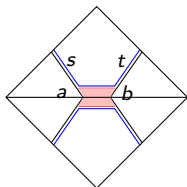
returning arc can be removed as shown by an isotopy of the simple closed curve, but a returning arc may be created in the other 2-simplex containing the edge.

Notice that the number of intersections with the 1-skeleton is reduced by two. The process of removing returning arcs must end with a track or with a loop inside a 2-simplex. The set of vertices inside the curve does not change and so it will end with a track if initially there were vertices inside the curve.

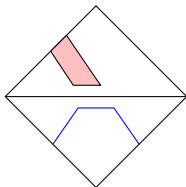
Theorem

Let P be a pattern in K consisting of a maximal set of tracks no two of which are parallel. Every vertex in the tree D_P has degree (valency) one or three. A component of degree one contains one vertex. A component of degree three contains no vertex and is a disc (2-ball) with two punctures.

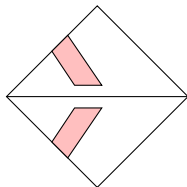
Proof Let γ be a 1-simplex. Let a, b be adjacent points of $\gamma \cap P$. Removing small open neighbourhoods of a, b and replacing them with lines parallel to γ joining the end points of the open neighbourhoods will create either one simple closed curve (scc) or two simple closed curves. If a, b lie in distinct tracks s, t then just one scc U is created. This scc will be a track if there are no returning arcs.



(i)

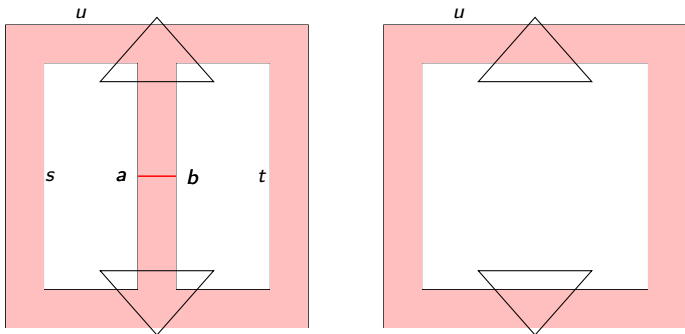


(ii)



(iii)

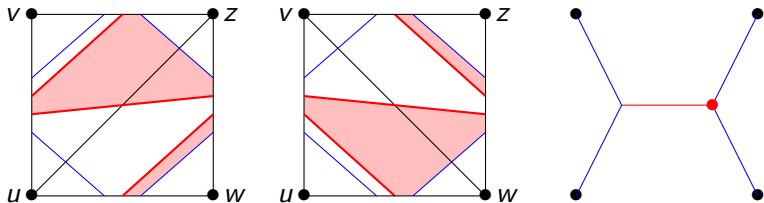
Returning arcs can be removed as shown, but a new returning arc may be created. Removing a returning arc in the 2-simplex σ joining two points a, b will create a returning arc in the other 2-simplex containing a, b if a, b are joined to points in the same edge. Thus it will be a track if and only the situation in the two 2-simplexes containing γ is as in (i). If the situation is as (ii) then one returning arc is created, and if it is as in (iii) then two returning arcs are created.



Removing returning arcs in a simple closed curve in K will eventually stop when either a track is reached or the whole simple closed curve disappears. This last possibility will only happen if s and t are parallel. If s and t are not parallel then removing returning arcs must end when a new track is obtained as in the diagram above.

The track created must be parallel to another track u in the pattern P . Since a, b are adjacent points of intersection of P with a 1-simplex the process of removing returning arcs must stay within the connected component corresponding to a vertex of D_P . This vertex will have s, t, u as its incident edges. The vertex will contain exactly two triangular regions.

In the case when K is the faces of a tetrahedron



The diagram shows a pattern P with a maximal set of non-parallel tracks in a tetrahedron. There are five tracks. Since tracks separate, they are the five edges of a finite tree D_P with six vertices, where the vertices are the regions between edges.

A. tetrahedron has 4 faces. Each vertex of degree 3 will contain 2 triangular regions each belonging to exactly one face. This gives the two vertices of D_P of degree 3. An icosahedron has 20 faces. So in this case there will be 10 vertices of D_P of degree 3.

There are infinitely many possibilities for the track in the tetrahedron that is not a 3-track, but the 4-track and the 8-track are the ones of interest to us.

Much of the above theory still works when we consider patterns in a 3-manifold, or rather the 2-skeleton of a 3-manifold. Let K be a triangulation of a 3-manifold, so that $M = |K|$. If P is a pattern in the 2-skeleton, then P determines a **patterned surface** S in which for each 3-simplex ρ , $S \cap |\rho|$ consists of disjoint properly embedded discs and $S \cap |K^2| = P$. A patterned surface is determined, up to isotopy, by the intersection $P \cap |K^1|$. If the pattern in $|K^2|$ is normal, then the patterned surface is a normal surface. Thus a normal surface intersects each 3-simplex in 3-sided or 4-sided discs.

Let $H^1(M, \mathbb{Z}_2) = 0$, so that tracks separate. For example if M is simply connected.

Theorem

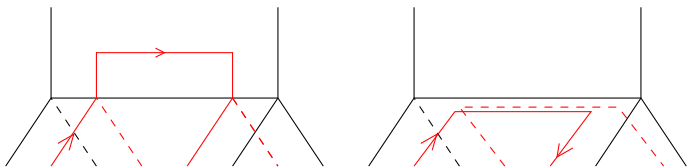
Let P be a pattern in K consisting of a maximal set of tracks such that for each track the corresponding patterned surface is a 2-sphere and no two of the tracks are parallel. Every vertex in the tree D_P has degree (valency) one, two or three. Each vertex of degree two or three is a component of $K^2 - P$ that is a punctured 3-ball. The same is true if the patterns are restricted to normal patterns.

Proof

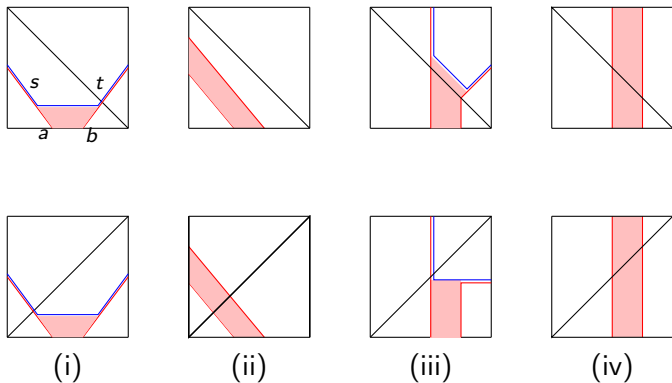
Part of the previous proof works with a few adjustments.

Let V be the closure of a component of $K^2 - P$ corresponding to a vertex of D_P of degree at least two. An argument from Thompson [94] shows that there will be a 1-simplex of K , intersecting two distinct tracks s, t of P in the boundary of V at adjacent points a, b on the 1-simplex γ where $a \in s \cap \gamma, b \in t \cap \gamma$.

There will now be a 2-sphere S which is the union of the patterned surface corresponding to s with a small neighbourhood of a removed and the patterned surface corresponding to t with small neighbourhoods of b removed together with a tube joining the boundary circles. The 2-sphere S will usually not be a patterned 2-sphere, but it will become one by removing 'returning' arcs.



It will now be the case that removing a returning arc may create several returning arcs in the other 2-simplexes which have γ as an edge. After all returning arcs have been removed one ends up with a track that determines a 2-sphere patterned surface U if the vertex region has degree more than 2, and it has degree 2 if all of s and t have been removed. This can happen without s and t being parallel, unlike the earlier theorem..



These diagrams show what can happen in the process of removing returning arcs in a 3-simplex if s and t are normal tracks. In the new track U the intersection with the 3-simplex will be a 4-track in (i) and a 3-track in (iii), shown in blue.

In (ii) and (iv) there will be no intersection with U . The intersection with s and t will be removed with the returning arcs. Thus U will also be a normal track.

The Recognition Algorithm

Let M be a compact triangulated simply connected 3-manifold with no boundary. Let P be a pattern in which the tracks are a maximal collection of tracks which correspond to 2-spheres and no two are parallel. These tracks are the edges of a finite tree D_P . Then M will be a 3-sphere if and only if the regions corresponding to the vertices of D_P of degree one are all 3-balls. This is because all the other vertex regions are punctured 3-balls and they will fit together to form one big punctured 3-ball.

A vertex region is a 3-ball if it contains a vertex of the triangulation, since, by the maximality of P , it will contain just one vertex. Unlike the 2-dimensional case, there can be vertex regions of degree one that do not contain a vertex of the triangulation.

Almost Normal Pattern

A pattern P is said to be almost normal if every track is a 3-track or a 4-track apart from exactly one track which is an 8-track. A patterned surface is almost normal if it corresponds to an almost normal pattern.

Theorem

The manifold M is a 3-sphere if and only if every vertex region corresponding to a vertex of D_P of degree one contains a vertex of K or an almost normal 2-sphere.

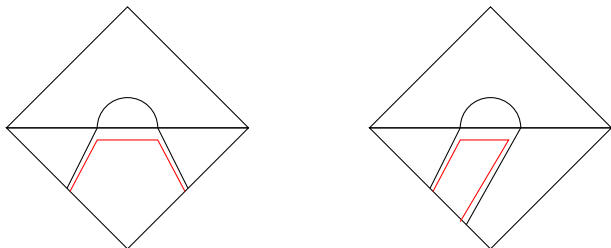
This result is due to Hyam Rubinstein [97]. We are reworking the proof by Abigail Thompson [94].

Let M be a triangulated 3-sphere.

Let M_0 be a component, obtained by cutting along the maximal collection of normal 2-spheres, which has one boundary component and which does not contain a vertex. By Van Kampen, M_0 is a 3-ball. There is isotopy between the boundary and the constant map.

We can break the isotopy up into finitely many steps in which the intersection with the 1-skeleton changes by a pair of points.

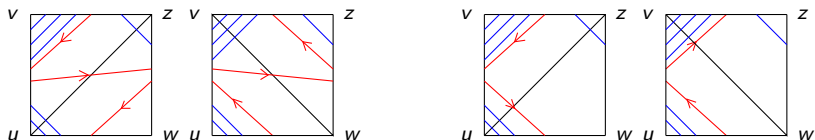
Isotopy moves



This is a returning arc in a 2-manifold. In a 3-manifold there will be more than two 2-simplexes with the same edge. However one of these two pictures will occur in each pair of 2-simplexes one of which is the 2-simplex containing the returning edge and the other 2-simplex is any one of the other 2-simplexes containing that edge. It will be an isotopy of the surface.

In fact every isotopy move is the removal or addition of a **removable pair**. A removable pair will be the ends of a returning arc or two points belonging to the same edge in an n -track in a tetrahedron, which will only exist for $n \geq 8$. A pattern is a normal pattern if and only if it has no removable pairs.

In the diagram see how the removal of the two right hand points of an 8-track gives two 3-tracks in which the opposite pair of points are no longer in the same track. Returning arcs will be created in at least one of the other 2-simplexes containing the edge wz .



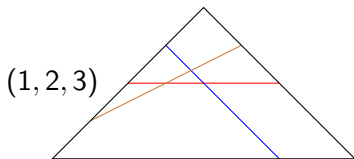
The pair of points in the opposite edge uv is also removable. Returning arcs will be created in the some or all of the other 2-simplexes containing uv . Now successively removing returning arcs (which can be done by isotopies) we will either end up with a normal pattern corresponding to a 2-sphere S_0 parallel to the boundary of M_0 or the whole track is removed. Each case will occur with the different choices of the pair of points removed initially.

In Thompson's proof of the Recognition Algorithm, she chooses the sequence of isotopies going from S_0 , the boundary of M_0 , to a constant map, that reaches the first peak at the earliest time possible. Suppose this happens at the k th step, so that S_k has two more intersections with edges than either S_{k-1} or S_{k+1} . She then shows that S_k must be almost normal by showing that there are exactly two removable pairs of points and they are in the one 8-track.

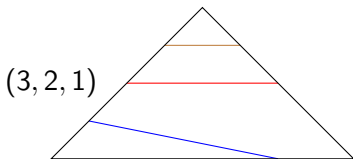
In our proof of the Poincaré Conjecture we instead consider a chain of homotopies rather than isotopies, so that we now allow lines of intersection with the 2-skeleton to cross. As for isotopies we choose the sequence that has the lowest first peak. Much of the Thompson argument still works, and so S_k intersects the 2-skeleton in a [spattern](#),

which is like a pattern but the lines are allowed to cross.

A spattern always has an underlying pattern, which has the same intersection with the 1-skeleton.



spattern



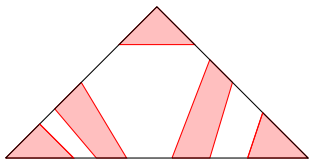
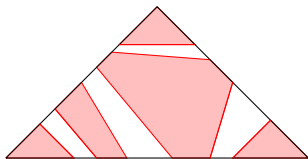
underlying pattern

Edge homotopy

Let I be the closed interval $[0, 1]$. Let F be a finite subset of the open interval $(0, 1)$ and a permutation β of F , there is a continuous map $\phi_I : I \rightarrow I$ which restricts to β on F and to the identity on $\{0, 1\}$. Think of an elastic string stretched between 0 and 1. There is a homotopy between ϕ_I and the identity map. In the diagram permuting the three points on the left hand edge and the attached edges will take the spattern to its underlying pattern

The intersection of M_0 with a 2-simplex will be as the unshaded regions in one of the two possible cases in the diagram, depending as to whether it includes the central region. Note that M_0 contains no vertex, and so the vertices are all in shaded regions.

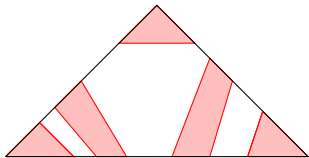
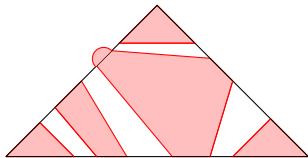
The peak pattern and its underlying pattern will intersect each 2-simplex in lines joining two edges in an unshaded region.

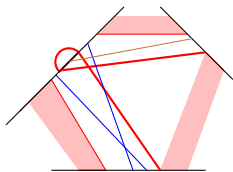
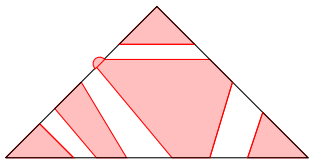
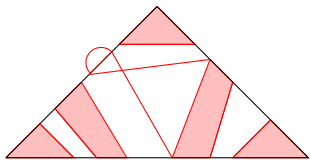


An edge homotopy will not change the underlying pattern as the number of intersections on each 1-simplex remains the same. Nor will it change the tessellation of the surface. Thus we will have a proof of the Poincaré Conjecture if we can show that the spattern reached at the first peak of the sequence of homotopies can be changed into its underlying pattern by edge homotopies. This is done by carrying out an edge homotopy on the peak spattern for each intermediary pair of points added in getting to the peak.

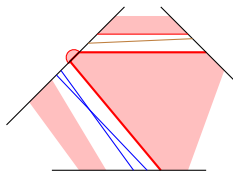
Let S_0 be the normal 2-sphere that is the boundary of M_0 . Now a normal pattern contains no removable pair and so S_1 has two extra vertices

Consider now the last removal of a returning arc before S_0 is reached. The only way one can end up with a normal surface with no returning arcs is for there to be a 2-simplex as in the left hand picture with pair of points in the top left hand edge joined by a returning arc in another 2-simplex containing that edge, which then becomes the right hand picture on removing the returning arc.





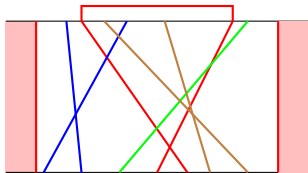
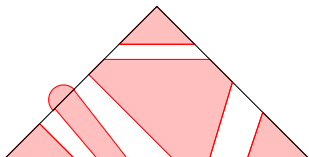
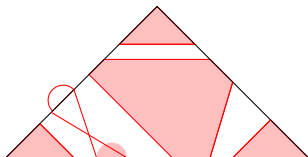
(1,2,3,4,5)



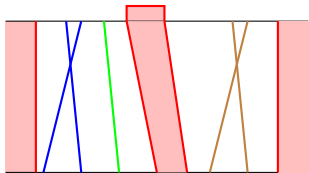
(1,4,5,3,2)

The edge homotopy that produces the required isotopy is now shown. The ends of the returning arc are moved to their correct position in the underlying pattern.

The next diagram shows what will happen in most of the removals of returning arcs which do not involve the centre region.



(1,2,3,4,5,6,7)



(1,5,6,2,7,4,3)

Notes and References

A fuller account of the results described here are available on my homepage in [D26] and [D26A]. In the final chapter of [D89] there is a full account of much of the theory of patterns and tracks. In the section on Simplifying Surface Maps there is a detailed account of how removing removable pairs (especially returning arcs) will usually result in a normal surface. However I failed to notice that one could end up with no intersections with the 1-skeleton or that in a homotopy sequence not every pair of points in the intersection of a 2-piece with a single 1-simplex are removable.

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Acknowledgement

Most of my efforts on this theory have attracted little interest but no mistakes have been brought to my attention. However I have had very enjoyable discussions of the theory and related matters with Peter Kropholler.