ALMOST INVARIANT SETS

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Abstract. A short proof of a conjecture of Kropholler is given. This gives a relative version of Stallings’ Theorem on the structure of groups with more than one end. A generalisation of the Almost Stability Theorem is also obtained, that gives information about the structure of the Sageev cubing.

1. Introduction

Let $G$ be a group. A subset $A$ of $G$ is said to be almost invariant if the symmetric difference $A + Ag$ is finite for every $g \in G$. In addition $A$ is said to be proper if both $A$ and $A^* = G - A$ are infinite. The group $G$ is said to have more than one end if it has a proper almost invariant subset.

Theorem 1.1. A group $G$ contains a proper almost invariant subset (i.e. it has more than one end) if and only if it has a non-trivial action on a tree with finite edge stabilizers.

This result was proved by Stallings [13] for finitely generated groups and was generalized to all groups by Dicks and Dunwoody [3]. The action of a group $G$ on a tree is trivial if there is a vertex that is fixed by all of $G$. Every group has a trivial action on a tree.

Let $T$ be a tree with directed edge set $ET$. If $e$ is a directed edge, then let $\bar{e}$ denote $e$ with the reverse orientation. If $e, f$ are distinct directed edges then write $e > f$ if the smallest subtree of $T$ containing $e$ and $f$ is as below.

Suppose the group $G$ acts on $T$. We say that $g$ shifts $e$ if either $e > ge$ or $ge > e$. If for some $e \in ET$ and some $g \in G$, $g$ shifts $e$, then $G$ acts non-trivially on a tree $T_e$ obtained by contracting all edges of $T$ not in the orbit of $e$ or $\bar{e}$. In this action there is just one orbit of edge pairs. Bass-Serre theory tells us that either $G = G_u *_{G_v} G_v$ where $u, v$ are the vertices of $e$ and they are in different orbits in the contracted tree $T_e$, or $G$ is the HNN-group $G = G_u *_{G_e} G_v$ if $u, v$ are in the same $G$-orbit. If either case occurs we say that $G$ splits over $G_e$.

If there is no edge $e$ that is shifted by any $g \in G$, (and $G$ acts without involutions, i.e. there is no $g \in G$ such that $ge = \bar{e}$) then $G$ must fix a vertex or an end of $T$. If the action is non-trivial, it fixes an end of $T$, i.e. $G$ is a union of an ascending sequence of vertex stabilizers, $G = \bigcup G_{v_n}$, where $v_1, v_2, \ldots$ is a sequence of adjacent vertices and $G_{v_1} \leq G_{v_2} \leq \ldots$ and $G \neq G_{v_n}$ for any $n$.

Thus Theorem 1.1 could be restated as

Theorem 1.2 ([13], [3]). A group $G$ contains a proper almost invariant subset (i.e. it has more than one end) if and only if it splits over a finite subgroup or it is countably infinite and locally finite.

The if part of the theorem is fairly easy to prove. We now prove a stronger version of the if part, following [2].
Let $H$ be a subgroup of $G$. A subset $A$ is $H$-finite if $A$ is contained in finitely many right $H$-cosets, i.e. for some finite set $F$, $A \subseteq HF$. A subgroup $K$ is $H$-finite if and only if $H \cap K$ has finite index in $K$. Let $T$ be a $G$-tree and suppose there is an edge $e$ and vertex $v$.

We say that $e$ points at $v$ if there is a subtree of $T$ as below. We write $e \rightarrow v$.

\[
\begin{array}{c}
e \\
v
\end{array}
\]

Let $G[e,v] = \{ g \in G| e \rightarrow gv \}$.

If $h \in G$, then $G[e,v]h = G[e,h^{-1}v]$, since if $e \rightarrow gv, e \rightarrow gh(h^{-1}v)$.

It follows from this that if $K = G_e$, then $G[e,v]K = G[e,v]$. Also if $H = G_e$, then $HG[e,v] = G[e,v]$.

If $e = \cdot v$, then $G_e = H \leq K = G_v$ and if $A = G[e,\cdot e]$, then $A = HAK$.

\[
\begin{array}{c}
v \\
\cdot \\
e \\
\cdot
\end{array}
\]

Consider the set $Ax, x \in G$. If $g \in A, gx \notin A$, then $e \rightarrow gv, e \rightleftharpoons gxv$. This means that $e$ is on the directed path joining $gxv$ and $gv$. This happens if and only if $g^{-1}e$ is on the path joining $xv$ and $v$. There are only finitely many directed edges in the $G$-orbit of $e$ in this path. Hence $g^{-1} \in FH$, where $F$ is finite, and $H = G_e$, and $g \in HF^{-1}$. Thus $A - Ax^{-1} = HF^{-1}$, i.e. $A - Ax^{-1}$ is $H$-finite.

It follows that both $Ax - A$ and $A - Ax$ are $H$-finite and so $A + Ax$ is $H$-finite for every $x \in G$, i.e. $A$ is an $H$-almost invariant set.

If the action on $T$ is non-trivial, then neither $A$ nor $A^*$ is $H$-finite. We say that $A$ is proper.

Peter Kropholler has conjectured that the following generalization of Theorem 1.1 is true for finitely generated groups.

**Conjecture 1.3.** Let $G$ be a group and let $H$ be a subgroup. If there is a proper $H$-almost invariant subset $A$ such that $A = AH$, then $G$ has a non-trivial action on a tree in which $H$ fixes a vertex $v$ and every edge incident with $v$ has an $H$-finite stabilizer.

We have seen that the conjecture is true if $H$ has one element. The conjecture has been proved for $H$ and $G$ satisfying extra conditions by Kropholler [8], Dunwoody and Roller [6], Niblo [10] and Kar and Niblo [7].

If $G$ is the triangle group $G = \langle a, b| a^2 = b^3 = (ab)^7 = 1 \rangle$, then $G$ has an infinite cyclic subgroup $H$ for which there is a proper $H$-almost invariant set. Note that in this case $G$ has no non-trivial action on a tree, so the condition $A = AH$ is necessary in Conjecture 1.3.
A discussion of the Kropholler Conjecture is given in [11]. I first learned of this conjecture in a letter Peter wrote to me in January 1988, a page of which is shown here.

We give a proof of the conjecture when $G$ is finitely generated over $H$, i.e. it is generated by $H$ together with a finite subset.

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2. Infinite Networks

Let $X$ be an arbitrary connected simple graph. It is not even assumed that $X$ is locally finite. Let $BX$ be the set of all edge cuts in $X$. Thus if $A \subset VX$, then $A \in BX$ if $\delta A$ is finite. Here $\delta A$ is the set of edges which have one vertex in $A$ and one in $A^*$.

A ray $R$ in $X$ is an infinite sequence $x_1, x_2, \ldots$ of distinct vertices such that $x_i, x_{i+1}$ are adjacent for every $i$. If $A$ is an edge cut, and $R$ is a ray, then there exists an integer $N$ such that for $n > N$ either $x_n \in A$ or $x_n \in A^*$. We say that $A$ separates rays $R = (x_n), R' = (x'_n)$ if for $n$ large enough either $x_n \in A, x'_n \in A^*$ or $x_n \in A^*, x'_n \in A$. We define $R \sim R'$ if they are not separated by any edge cut. It is easy to show that $\sim$ is an equivalence relation on the set $\Phi X$ of rays in $X$. The set $\Omega X = \Phi X/\sim$ is the set of edge ends of $X$. An edge cut $A$ separates ends $\omega, \omega'$ if it separates rays
representing \( \omega, \omega' \). A cut \( A \) separates an end \( \omega \) and a vertex \( v \in VX \) if for any ray representing \( \omega \), \( R \) is eventually in \( A \) and \( v \in A^* \) or vice versa.

We define a network \( N \) to be a simple, connected graph \( X \) and a map \( c : EX \to \{1, 2, \ldots \} \). If \( X \) is a network in which each edge has capacity 1, then \( BX \) is the set of edge cuts, and if \( A \in BX \), then \( c(A) = |\delta A| \).

The following result is proved in [5].

**Theorem 2.1.** Let \( N(X) \) be a network in which \( X \) is an arbitrary connected graph. For each \( n > 0 \), there is a network \( N(X) \) on a tree \( T_n \) and a map \( \nu : VX \cup \Omega X \to VT \cup \Omega T \), such that \( \nu(VX) \subset VT \) and \( \nu x = \nu y \) for any \( x, y \in VX \cup \Omega X \) if and only if \( x, y \) are not separated by a cut \( A \) with \( c(A) \leq n \).

The network \( N(T_n) \) is uniquely determined and is invariant under the automorphism group of \( N(X) \).

**Theorem 2.2.** There is a uniquely defined nested set \( E_n \) of generators of \( B_n X \), with the following properties:

(i) If \( G \) is the automorphism group of \( N(X) \), then \( E_n \) is invariant under \( G \).

(ii) For each \( i < j \), \( E_i \subseteq E_j \).

We will only really be using Theorem 2.2 for networks in which every edge has capacity one.

**Theorem 2.3.** Let \( X \) be a connected graph. There is a uniquely determined sequence of structure trees \( T_n \) and a map \( \nu : VX \cup \Omega X \to VT \cup \Omega T \), such that \( \nu(VX) \subset VT \) and \( \nu x = \nu y \) for any \( x, y \in VX \cup \Omega X \) if and only if \( x, y \) are not separated by a cut \( A \) with \( |\delta A| \leq n \). Each tree \( T_n \) admits an action of the automorphism group of \( X \).

In this case \( ET_n = E_n \).

In any tree \( T \) if \( p \) is a vertex and \( Q \) is a set of unoriented edges, then there is a unique set of vertices \( P \) such that \( v \in P \) then the geodesic \([v, p]\) contains an odd number of edges from \( Q \). We then have \( \delta P = Q \). Note that \( BT = B_1 T \) and every element of \( BT \) is uniquely determined by the set \( Q \) together with the information for a fixed \( p \in VT \) whether \( p \in A \) or \( p \in A^* \). The vertex \( p \) induces an orientation \( O_p \) on the set of pairs \( \{e, \bar{e}\} \) of oriented edges by requiring that \( e \in O \) if \( e \) points at \( p \). For \( A \in BT \), \( A \) is uniquely determined by \( \delta A \) together with the orientation \( O_p \cap \delta A \) of the edges of \( \delta A \).

In \( X \) it is the case that a cut \( A \) is uniquely determined by \( \delta A \) together with the information for a fixed \( p \in VX \) whether \( p \in A \) or \( p \in A^* \).

Since \( B_n X \) is generated by \( E_n = ET_n \), the cut \( A \) can be expressed in terms of a finite set of oriented edges of \( T_n \). This set is not usually uniquely determined. Thus if \( \nu \) is not surjective, and \( v \) is not in the image of \( \nu \), and the set of edges incident with \( v \) is finite, then \( VX \) is the union of these elements in \( BX \). The empty set is the intersection of the complements of these sets. Orienting the edges incident with \( v \) towards \( v \) gives the empty set and orienting them away from \( v \) gives all of \( VX \). However there is a canonical way of expressing an element of \( B_n X \) in terms of the generating set \( E_n \).

To see this let \( A \in B_n X - B_{n-1} X \). There are only finitely many \( C \in E_n \) with which \( C \) is not nested. This number is \( \mu(A, E_n) = \mu(A) \). We use induction on \( \mu(A) \). Our induction hypothesis is that there is a canonically defined way of expressing \( A \) in terms of the \( E_n \). Any two ways of expressing \( A \) in terms of \( E_n \) differ by an expression which gives the empty set in terms of \( E_n \). Such an expression will correspond to a finite set of vertices each of which has finite degree in \( T_n \) and none of which is in the image of \( \nu \). The canonical expression is obtained if there is a unique way of saying whether or not each such vertex is in the expression for \( A \). Thus the canonical expression for \( A \) is determined by a set of vertices of \( VT \) which consists of the vertices of \( \nu(A) \) together with a recipe for deciding for each vertex which is not in the image of \( \nu \) whether it is in the expression for \( A \).

Suppose \( \mu(A) = 0 \), so that \( A \) is nested with every \( C \in E_n \), and neither \( A \) nor \( A^* \) is empty. If \( A \in E_n \), then this gives an obvious way of expressing \( A \) in terms of the \( E_n \). If \( A \) is not in \( E_n \), then it
corresponds to a unique vertex \( z \in VT_n \). Thus because \( \mu(A) = 0 \), \( A \) induces an orientation of the edges of \( \mathcal{E}_n \). To see this, let \( C \in \mathcal{E}_n \), then just one of \( C \subseteq A, C^* \subseteq A, C \subseteq A^*, C^* \subseteq A^* \) holds. From each pair \( C, C^* \) we can choose \( C \) if \( C \subseteq A \) or \( C \subseteq A^* \) and we choose \( C^* \) if \( C^* \subseteq A \) or \( C^* \subseteq A^* \). Let \( \mathcal{O} \) be this subset of \( \mathcal{E} \). Then if \( C \in \mathcal{O} \) and \( D \in \mathcal{E} \) and \( D \subseteq C \), then \( D \in \mathcal{O} \). This means that the orientation \( \mathcal{O} \) determines a vertex \( z \) in \( VT_n \). Intuitively the edges of \( \mathcal{O} \) point at the vertex \( z \). It can be seen that \( A \) or \( A^* \) will be the union of finitely many edges \( E \) of \( \mathcal{E}_n = ET_n \), all of which have \( \tau E = z \). If \( A \) is such a union, then we use this to express \( A = C_1 \cup C_2 \cup \cdots \cup C_k \). If \( A \) is not such a union, but \( A^* = C_1 \cup C_2 \cup \cdots \cup C_k \), then we write \( A = (C_1 \cup C_2 \cup \cdots \cup C_k)^* = C_1^* \cap C_2^* \cap \cdots \cap C_k^* \). Note that this gives a unique way of expressing cuts corresponding to a vertex \( z \) of finite degree not in the image of \( \nu \). The vertex \( z \) is included in the expression for \( A^* \) if and only if only finitely many cuts in \( \mathcal{E}_n \) incident with \( z \) and pointing at \( z \) are subsets of \( A \). Suppose then that the hypothesis is true for elements \( B \in B_nX \) for which \( \mu(B) < \mu(A) \). Let \( C \in \mathcal{E}_n \) be not nested with \( A \). Then \( \mu(A \cap C) + \mu(A \cap C^*) \leq \mu(A) \). Thus each of \( A \cap C \) and \( A \cap C^* \) can be expressed in a unique way in terms of the \( \mathcal{E}_n \). If at most one of these expressions involves \( C \) then we take the expression for \( A \) to be the union of the two expressions for \( A \cap C \) and \( A \cap C^* \). If both of the expressions involve \( C \), then we take the expression for \( A \) to be the union of the two expression with \( C \) deleted. The expression obtained for \( A \) is independent of the choice of \( C \). In fact the decomposition will involve precisely those \( C \) for which \( C \) occurs in just one of the decompositions for \( A \cap C \) and \( A \cap C^* \). We therefore have a canonical decomposition for \( A \). To further clarify this proof observe the following.

The edges \( C \) which are not nested with \( A \) form the edge set of a finite subtree \( F \) of \( T_n \). If \( EF \neq \emptyset \) we can choose \( C \) so that it is a twig of \( F \), i.e. so that one vertex \( z \) of \( F \) is only incident with a single edge \( C \) of \( F \). By relabelling \( C \) as \( C^* \) if necessary we can assume that \( \mu(A \cap C) = 0 \). The vertex determined by \( A \cap C \) as above is \( z \), and we have spelled out the recipe for if this vertex is to be included in the expression for \( A \). The induction hypothesis gives us a canonical expression for \( A \cap C^* \), which together with the expression for \( A \cap C \) gives the expression for \( A \).

3. Relative Structure Trees

We prove Conjecture 1.3 in the case when \( G \) is finitely generated over \( H \), i.e. \( G \) is generated by \( H \cup S \) where \( S \) is finite.

First, we explain the strategy of the proof. Suppose that we have a non-trivial \( G \)-tree \( T \) in which every edge orbit contains an edge which has an \( H \)-finite stabiliser, and suppose there is a vertex \( \bar{o} \) fixed by \( H \). Let \( T_H \) be an \( H \)-subtree of \( T \) containing \( \bar{o} \) and every edge with \( H \)-finite stabiliser. The action of \( H \) on \( T_H \) is a trivial action, since it has a vertex fixed by \( H \), and so the orbit space \( H \setminus T_H \) is a tree, which might well be finite, but must have at least one edge. Our strategy is to show that if \( G \) is finitely generated over \( H \) and there is an \( H \)-almost invariant set \( A \) satisfying \( AH = A \), then we can find a \( G \)-tree \( T \) with the required properties by first deciding what \( H \setminus T_H \) must be and then lifting to get \( T_H \) and then \( T \).

We show that if \( G \) is finitely generated over \( H \), then there is a \( G \)-graph \( X \) if which there is a vertex with stabiliser \( H \) and in which a proper \( H \)-almost invariant set \( A \) satisfying \( AH = A \) corresponds to a proper set of vertices with \( H \)-finite coboundary. It then follows from the theory of [5], described in the previous section, that there is a sequence of structure trees for \( H \setminus X \). We choose one of these to be \( H \setminus T_H \), and show that we can lift this to obtain \( T_H \) and then \( T \) itself.

For example if \( G = H \ast K \) then there is a \( G \)-tree \( Y \) with one orbit of edges and a vertex \( \bar{o} \) fixed by \( H \), and every edge incident with \( \bar{o} \) has \( H \)-finite stabiliser. Suppose that \( K, L \) are such that these are the only edges with \( H \)-finite stabilisers. Then \( H \setminus T_H \) has two vertices and one edge. When we lift to \( T_H \) we obtain an \( H \)-tree of diameter two in which the middle vertex \( \bar{o} \) has stabiliser \( H \). The tree \( T \) is covered by the translates of \( T_H \).

On the other hand, if \( G = L \ast K \) where \( K \) is finite, and \( T \) is as above, then every edge of \( T \) is \( H \)-finite and so \( T_H \) is regarded as an \( H \)-tree. The fact that our construction gives a canonical construction for \( H \setminus T_H \) means that when we lift to \( T_H \) and \( T \) we will get the unique tree that admits the action of \( G \).

We proceed with our proof.
Lemma 3.1. The group $G$ is finitely generated over $H$ if and only if there is a connected $G$-graph $X$ with one orbit of vertices, and finitely many orbits of edges, and there is a vertex $o$ with stabiliser $H$.

Proof. Suppose $G$ is generated by $H \cup S$, where $S$ is finite. Let $X$ be the graph with $VX = \{ gH \mid g \in G \}$ and in which $EX$ is the set of unordered pairs $\{ gH, gsH \}$, $g \in G, s \in S$. We then have that $X$ is vertex transitive, there is a vertex $o = H$ with stabilizer $H$ and $G \setminus X$ is finite. We have to show that $X$ is connected. Let $C$ be the component of $X$ containing $o$. Let $G'$ be the set of those $g \in G$ for which $gH \in C$. Clearly $G'H = G'$ and $G's = G'$ for every $s \in S$. Hence $G' = G$ and $C = X$. Thus $X$ is connected.

Conversely let $X$ be a connected $G$-graph and $VX = Go$ where $G_o = H$. Suppose $EX$ has finitely many $G$-orbits, $G_{e_1}, G_{e_2}, \ldots, G_{e_r}$, where $e_i$ has vertices $o$ and $g_i o$. It is not hard to show that $G$ is generated by $H \cup \{ g_1, g_2, \ldots, g_r \}$.

Let $A \subset G$ be a proper $H$-almost invariant set satisfying $AH = A$. Let $G$ be finitely generated over $H$, and let $X$ be a $G$-graph as in the last lemma. There is a subset of $VX$ corresponding to $A$, which is also denoted $A$. For any $x \in G$, $A + Ax$ is $H$-finite. In particular this is true if $s \in S$. This means that $\delta A$ is $H$-finite. Note that neither $A$ nor $A^* = VX - A$ is $H$-finite. Thus a proper $H$-almost invariant set corresponds to a proper subset of $VX$ such that $\delta A$ is $H$-finite.

From the previous section (Lemma 2.2) we know that $B(H \setminus X)$ has a uniquely determined nested set of generators $E = E(H(X))$. For $E \in E$, let $H \setminus E$ be the set of all $v \in VX$ such that $hv \in E$. Let $C$ be a component of $E$.

Lemma 3.2. For $h \in H$, $hc = C$ or $hc \cap C = \emptyset$. Also $HC = \hat{E}, h\delta C \cap \delta C = \delta C$ or $h\delta C \cap \delta C = \emptyset$ and $H \setminus \delta C = \delta E$.

Proof. Let $h \in H$. Then $hc$ is also a component of $\hat{E}$, since $HC \subseteq E$. Thus either $hc = C$ or $hc \cap C = \emptyset$. Let $K$ be the stabilizer of $C$ in $H$. if $h \in C$ then $hv \in C$ if and only if $h \in K$. Thus $K \setminus C$ injects into $H \setminus C = E$ and $K \setminus \delta C$ injects into $\delta E$. But $E$ is connected, and so the image $HC$ is $E$. It follows that there is a single $H$-orbit of components.

It follows from the lemma that it is also the case that $C^*$ is connected, since any component of $C^*$ must have coboundary that includes an edge from each orbit of $\delta C$. Let $E(H, X)$ be the set of all such $C$, and let $E_n(H, X)$ be the subset of $E(H, X)$ corresponding to those $C$ for which $\delta C$ lies in at most $n$ $H$-orbits.

Lemma 3.3. The set $\hat{E}(H, X)$ is a nested set. The set $\hat{E}_n(H, X)$ is the edge set of an $H$-tree.

Proof. Let $C, D \in \hat{E}_n(H, X)$. Then $HC, HD$ are in the nested set $E$. Suppose $HC \subset HD$, then $C \subset D$ or $C \cap D = \emptyset$. It follows easily that $\hat{E}(H, X)$ is nested. It was shown in [1] that a nested set $E$ is the directed edge set of a tree if and only if it satisfies the finite interval condition, i.e. if $C, D \in E$ and $C \subset D$, then there are only finitely many $E \in E$ such that $C \subset C \subset D$. Thus we have to show that $\hat{E}_n(H, X)$ satisfies the finite interval condition. If $C \subset D$ and $C \subseteq E \subseteq D$ where $C, E, D \in \hat{E}_n(H, X)$, then $HC \subseteq HE \subseteq HD$. But $\hat{E}_n(H, X)$ does satisfy the finite interval condition and $HC = HE$ implies $E = C$. Now let $C \cap D = \emptyset$ and suppose that $o = H \in C^* \cap D^*$. There are only finitely many $E \in \hat{E}_n$ such that $C \subset E$ and $o \in E^*$ or such that $D \subset E^*$ and $o \in E$. Each $E \in \hat{E}_n$ such that $C \subset C \subset D^*$ has one of these two properties.

Let $\hat{T} = \hat{T}(H)$ be the tree constructed in the last Lemma. Let $T = H \setminus \hat{T}$. Note that in the above $\hat{T}(H)$ is the Bass-Serre $H$-tree associated with the quotient graph $T(H) = H \setminus \hat{T}(H)$ and the graph of groups obtained by associating appropriate labels to the edges and vertices of this quotient graph (which is a tree). Clearly the action of $H$ on $T(H)$ is a trivial action in that $H$ fixes the vertex $\hat{o} \equiv o_0$. The stabilisers of edges or vertices on a path or ray beginning at $\hat{o}$ will form a non-increasing sequence of subgroups of $H$. 


We now adapt the argument of the previous section to show that if \( A \subseteq VX \) is such that \( \delta A \) lies in at most \( n \) \( H \)-orbits, then there is a canonical way of expressing \( A \) in terms of the set \( E_n(H, X) \). In this case we have to allow unions of infinitely many elements of the generating set. Our induction hypothesis is that if \( \delta A \) lies in at most \( n \) \( H \)-orbits, then \( A \) is canonically expressed in terms of \( E_n(H, X) \). First note that there are only finitely many \( H \)-orbits of elements of \( E_n(H, X) \) with which \( A \) is not nested. This is because if \( C \subseteq E_n \) is not nested with \( A \) and \( F \) is a finite connected subgraph of \( H \setminus X \) containing all the edges of \( H \delta A \), then \( H \delta C \) must contain an edge of \( F \) and there are only finitely many elements of \( E_n \) with this property. We now let \( \mu(A) \) be the number of \( H \)-orbits of elements of \( E_n \) with which \( A \) is not nested. If \( \mu(A) = 0 \), then \( A \) is nested with every \( C \subseteq E_n \). This then means that if neither \( A \) nor \( A^* \) is empty and it is not already in \( E_n \), then \( A \) determines a vertex \( z \) of \( T_n \) and either \( A \) or \( A^* \) is the union (possibly infinite) of edges of \( T_n \) that lie in finitely many \( H \)-orbits. If \( A \) is such a union, then we use this union for our canonical expression for \( A \). If \( A \) is not such a union, then \( A^* \) is a union of finitely many elements of \( E \). We write \( A = \bigcup \{C_\lambda | \lambda \in \Lambda \} \), where each \( C_\lambda \) has \( \tau C_\lambda = z \) and the edges lie in finitely many \( H \)-orbits. We show that the hypothesis is true for elements \( B \) for which \( \mu(B) < \mu(A) \). Let \( C \subseteq E_n \) be not nested with \( A \). Then \( \mu(A \cap HC) + \mu(A \cap HC^*) \leq \mu(A) \). Thus each of \( A \cap HC \) and \( A \cap HC^* \) can be expressed in a unique way in terms of the \( E_n \). We take the expression for \( A \) to be the union of the two expressions for \( A \cap HC \) and \( A \cap HC^* \) except that we include \( hC \) for \( h \in H \), only if just one of the two expressions involve \( hC \).

If \( g \in G \), then \( gT(H) \) is a \((gHg^{-1})\)-tree. It is the tree \( T(gHg^{-1}) \) obtained from the \( G \)-graph \( X \) by using the vertex \( go \) instead of \( o \). We now show that there is a \( G \)-tree \( T \) which contains all of the trees \( gT(H) \). We know that the action of the group \( G \) on \( X \) is vertex transitive and that \( X \) has a vertex \( o \) fixed by \( H \). Also \( G \) is generated by \( H \cup S \) where \( S \) is finite.

Clearly there is an isomorphism \( \alpha_g : T(H) \to T(gHg^{-1}) \) in which \( D \mapsto gD \).

Suppose now that \( \alpha o \neq \nu(g o) \). Let \( A, B \) be \( H \)-almost invariant sets satisfying \( AH = A, BH = B \) and let \( g \in G \). We regard \( A, B \) as subsets of \( VX \), so that \( \delta A \) and \( \delta B \) are \( H \)-finite.

Suppose that \( o \in gB^* \) and \( go \in A^* \). The following Lemma is due to Kropholler [8, 9]. We put \( K = gHg^{-1} \).

Lemma 3.4. In this situation \( \delta(A \cap gB) \) is \((H \cap K)\)-finite.

Proof. Let \( x \in G \). We show that the symmetric difference \((A \cap gB)x + (A \cap gB)\) is \((H \cap K)\)-finite. Since \( A, B \) are \( H \)-almost invariant, there are finite sets \( E, F \) such that \( A + Ax \subseteq HE \) and \( B + Bx \subseteq HF \). We then have

\[
(A \cap gB)x + (A \cap gB) = Ax \cap (gBx + gB) + (Ax + A) \cap gB = Ax \cap gHF + g^{-1}HE \cap B.
\]

Now \( Ax \cap gHF \) is \( K \)-finite, but it is also \( H \)-finite because \( gH \) is contained in \( A^* \), since \( go \in A^* \). A set which is both \( H \)-finite and \( K \)-finite is \((H \cap K)\)-finite. Thus \( Ax \cap gHF \) is \((H \cap K)\)-finite. Similarly using the fact that \( g^{-1}o \in B^* \), it follows that \( g^{-1}HE \cap B \) is \((H \cap g^{-1}Hg)\)-finite, and so \( g(g^{-1}HE \cap B) \) is \((H \cap K)\)-finite. Thus \( A \cap gB \) is \((H \cap K)\)-almost invariant. But this means that \( \delta(A \cap gB) \) is \((H \cap K)\)-finite. \( \square \)

What this Lemma says is that if \( A, gB \) are not nested then there is a special corner - sometimes called the Kropholler corner - which is \((H \cap K)\)-almost invariant.

Notice that in the above situation all of \( \delta A, \delta(A \cap gB^*) \) and \( \delta(A \cap gB) \) are \( H \)-finite. If we take the canonical decomposition for \( A \), then it can be obtained from the canonical decompositions for \( A \cap gB \) and \( A \cap gB^* \) by taking their union and deleting any edge that lies in both. Also \( \delta(gB) \) is \( K \)-finite and the decomposition for \( gB \) can be obtained from those for \( gB \cap A \) and \( gB \cap A^* \). But the edges in the decomposition for \( A \cap gB \) which is \((H \cap K)\)-almost invariant are the same in both decompositions.
We will now show that it follows from Lemma 3.4 that the set $G\bar{\mathcal{E}}_n$ is a nested $G$-set which satisfies the final interval condition, and so it is the edge set of a $G$-tree. We have seen that $\mathcal{E}_n$ is a nested $H$-set where $\mathcal{E}_n = H \setminus \bar{\mathcal{E}}_n$ is the uniquely determined nested subset of $\mathcal{B}_n(H \setminus X)$ that generates $\mathcal{B}_n(H \setminus X)$ as an abelian group. It is the edge set of a tree $T_n(H \setminus X)$.

![Diagram](image)

**Figure 1.** Crossing cuts

If $A, B \in \mathcal{E}_n$ and $A, gB$ are not nested for some $g \in G$, then by Lemma 3.4 there is a corner -the Kropholler corner-, which we take to be $A \cap gB$, for which $\delta(A \cap gB)$ is $(H \cap K)$-finite. We then have canonical decompositions for $A \cap gB$ and $A \cap gB^*$ as above. This is illustrated in Fig 1. The labels $a, b, c, d, e, f$ are for sets of edges joining the indicated corners. In this case the letters do not represent edges of $X$ but elements of $\mathcal{E}_n$. Although each $E \in \mathcal{E}_n$ comes with a natural direction, in the diagram we only count the unoriented edges, i.e. we count the number of edge pairs $(E, E^*)$.

In the diagram, $A \cap gB$ is always taken to be the Kropholler corner. Thus we have that any pair contributing to $a, f$ or $e$ must be $(H \cap K)$-finite. Any pair contributing to $e$ or $b$ must be $H$-finite and any pair contributing to $e$ or $d$ must be $K$-finite.

We have that $a + e + f + b = 1$ and $e + e + f + d = 1$. Suppose that the Kropholler corner $A \cap B$ is not empty. It is the case that each of $a$ and $go$ lies in one of the other three corners. We know that $o \in gB^*$, $go \in A^*$. If $o \in A \cap gB^*$ and $go \in A^* \cap gB$, then $a = c = 1$ and $e = f = b = d = 0$ and $A^* \cap gB^* = \emptyset$. If $o \in A^* \cap gB$ and $go \in A^* \cap gB^*$, then $a = d = 1$ and $A \cap gB^* = \emptyset$, while if both $o$ and $go$ are in $A^* \cap gB^*$, then either $a = d = 1$ and $A \cap gB^* = \emptyset$ or $a = c = 1$ and $A^* \cap gB = \emptyset$ or $f = 1$ and both $A \cap gB^*$ and $A^* \cap gB$ are empty, so that $A = gB$. In all cases $A, gB$ are nested.

We need also to show that $G\bar{\mathcal{E}}_n$ satisfies the finite interval condition. Let $g \in G$ and let $K = gHg^{-1}$. Consider the union $\mathcal{E} \cup g\mathcal{E}$. This will be a nested set. In fact it will be the edge set of a tree that is the union of the trees $T(H)$ and $T(K)$. In the diagram the red edges are the edges that are just in $T(H)$. The blue edges are the ones that are in $T(K)$. The brown edges are in both $T(H)$ and $T(K)$. An edge is in the geodesic joining $o$ and $go$ if and only if it has stabiliser containing $H \cap K$, it will also lie in both $T(H)$ and $T(K)$ (i.e. it is coloured brown) if and only if it its stabiliser contains...
$H \cap K$ as a subgroup of finite index. It may be the case that $T(H)$ and $T(K)$ have no edges in common, i.e. there are no brown edges. An edge lies in both trees if and only if it has a stabiliser that is $(H \cap K)$-finite. If there are such edges then they will be the edge set of a subtree of both trees. They will correspond to the edge set $E(H \cap K)$.

It follows that $T(H)$ is always a subtree of a tree constructed from a subset of $G\bar{E}_n$ that contains $\bar{E}_n$. If $T(H)$ and $T(K)$ do have an edge in common, then $T(H) \cup T(K)$ will be a subtree of the tree we are constructing. If $e \in EX$ has vertices $go$ and $ko$ and there is some $C \in G\bar{E}_n$, then $e \in \delta C$, then $C \in gET(g^{-1}Hg) \cap kET(k^{-1}Hk)$. If there is no such $C$, i.e. there is no cut $C \in G\bar{E}_n$ that separates $o$ and $k^{-1}go$ then $T(H) = k^{-1}gT(H)$. As there is a finite path connecting any two vertices $u, v$ in $X$, it can be seen that there are only finitely many edges in $G\bar{E}_n$ separating $u$ and $v$ since any such edge must separate the vertices of one of the edges in the path. Thus $G\bar{E}_n$ is the edge set of a tree.

We say that a $G$-tree $T$ is reduced if for every $e \in ET$, with vertices $ie$ and $re$ we have that either $ie$ and $re$ are in the same orbit, or $G_e$ is a proper subgroup of both $G_{ie}$ and $G_{re}$.

**Theorem 3.5.** Let $G$ be a group that is finitely generated over a subgroup $H$. The following are equivalent:

(i) There is a proper $H$-almost invariant set $A = HAK$ with left stabiliser $H$ and right stabiliser $K$, such that $A$ and $gA$ are nested for every $g \in G$.

(ii) There is a reduced $G$-tree $T$ with vertex $v$ and incident edge $e$ such that $G_v = K$ and $G_e = H$.

**Proof.** It is shown that (ii) implies (i) in the Introduction.

Suppose that $A = HAK$ and let $g \in G$. We will show that there is a $G$-tree $T$ such that $T(K)$ is non-trivial. We have shown in the previous theorem that $A = HAK$ and $gA$ is a union of finitely many cosets $g^{-1}A, g^{-1}A, \ldots, g^{-1}A$. Then $\{g_i^{-1}A, g_i^{-1}A, \ldots, g_i^{-1}A\}$ is the edge set of a finite tree $F$. Since $A$ is a proper subset of $G$, we can show that $T(K)$ is non-trivial. But this must be the case as the edges separating vertices $A$ and $Ax$ will be the edges of $F$.

**Theorem 3.6.** Let $G$ be a group and let $H$ be a subgroup, and suppose $G$ is finitely generated over $H$. There is a proper $H$-almost invariant subset $A$ such that $A = AH$, if and only if there is a non-trivial reduced $G$-tree $T$ in which $H$ fixes a vertex and every edge orbit contains an edge with an $H$-finite edge stabilizer.

**Proof.** The only if part of the theorem is proved in Theorem 3.5. In fact it is shown there that if $G$ has an action on a tree with the specified properties, then there is a proper $H$ almost invariant set $A$ for which $HAH = A$.

Suppose then that $G$ has an $H$-almost invariant set $A$ such that $AH = A$. Since $G$ is finitely generated over $H$, we can construct the $G$-graph $X$ as above, in which $A$ can be regarded as a set of vertices for which $\delta A$ lies in finitely many $H$-orbits. Let the number of orbits be $n$. Then we have seen that there is a $G$-tree $T_n$ for which $H$ fixes a vertex $\bar{o}$ and every edge is in the same $G$-orbit as an edge in $T(H)$. The edges in this tree are $H$-finite. The set $A$ has an expression in terms if the edges of $T(H)$. Finally we need to show that the action on $T_n$ is non-trivial. If $G$ fixes $\bar{o}$, then $\nu(A)$ consists of the single vertex $\bar{o}$ and so $A$ is not proper. In fact the fact that $A$ is proper ensures that $T_n$ is fixed by $G$.

It can be seen from the above that $T(H) \cap \bar{T}(g^{-1}Hg) = \bar{T}(H \cap gHg^{-1})$ so that if $e \in ET(H)$, and $g \in G_e$, then $e \in \bar{T}(gHg^{-1})$ and so $G_e$ is $H$-finite.

The Kropholler Conjecture follows immediately from the last Theorem.

### 4. $H$-almost stability

Let $G$ be a group with subgroup $H$, and let $T$ be a $G$-tree.

Let $T \subset VT$ be such that $\delta A \subset ET$ consists of finitely many $H$-orbits of edges $e$ such that $G_e$ is $H$-finite. Also let $H$ fix a vertex of $T$. Note that $\delta A$ consists of whole $H$-orbits, so that $e \in \delta A$.
implies \( he \in \delta \tilde{A} \) for every \( h \in H \). The fact that \( G_e \) is \( H \)-finite for \( e \in \delta \tilde{A} \) follows from the fact that \( \delta \tilde{A} \) is \( H \)-finite. If \( H_e \) is the stabiliser of \( e \in \delta \tilde{A} \), then \([G_e : H_e] \) is finite.

Let \( v \in VT \), and let \( A = A(v) = \{ g \in G | gv \in A \} \). Note that \( A(xv) = A(v)x^{-1} \), so that the left action on \( T \) becomes a right action on the sets \( A(v) \). If \( x \in G \) and \( [v, xv] \) is the geodesic from \( v \) to \( xv \), then \( g \in A + Ax \) if and only the geodesic \([gv, gxv]\) contains an odd number of edges in \( \delta \tilde{A} \). If \([v, xv]\) consists of the edges \( e_1, e_2, \ldots, e_r \), then \( ge_i \in \delta \tilde{A} \) if and only if \( Hge_i \in \delta \tilde{A} \). It follows that \( H(A + Ax) = A + Ax \). It is also clear that for each \( e_i \) there are only finitely many cosets \( Hg \) such that \( Hge_i \in \delta \tilde{A} \). Thus \( A \) is \( H \)-almost invariant. We also have \( A(v)H = A(v) \) if \( H \) fixes \( v \).

For each \( v \in ET \), let \( d(v) \) be the number of cosets \( Hg \) such that \( Hge \in \delta \tilde{A} \). We see that \( d(v) = d(xv) \) for every \( x \in G \) and so we have a metric on \( VT \), that is invariant under the action of \( G \). We will show that if \( G \) has an \( H \)-almost invariant set such that \( HAH = A \) then there is a \( G \)-tree with a metric corresponding to this set.

From now on we are interested in the action of \( G \) on the set of \( H \)-almost invariant sets. But note that we are interested in the action by right multiplication. The Almost Stability Theorem [3], also used the action by right multiplication. Let \( C \subseteq G \) be \( H \)-almost invariant and let \( HA = A \) for the moment we do not assume that \( AH = A \).

Let \( M = \{ B | B =_{SA} A \} \) so that for \( B, C \in M \), \( B + C = HF \) where \( F \) is finite.

Note that for \( H = \{ 1 \} \) it follows from the Almost Stability Theorem that \( M \) is the vertex set of a \( G \)-tree.

We define a metric on \( M \). For \( B, C \in M \) define \( d(B, C) \) to be the number of \( H \)-cosets in \( B + C \).

This is a metric on \( M \), since \( (B + C) + (C + D) = (B + D) \), and so an element which is in \( B + D \) is in just one of \( B + C \) or \( C + D \). Thus \( d(B, D) \leq d(B, C) + d(C, D) \).

Also \( G \) acts on \( M \) by right multiplication and this action is by isometries, since \( (B + C)z = Bz + Cz \). Let \( \Gamma \) be the graph with vertex set \( M \) and two vertices are joined by an edge if they are distance one apart. Every edge in \( \Gamma \) corresponds to a particular \( H \)-coset. There are exactly \( n! \) geodesics joining \( B \) and \( C \) if \( d(B, C) = n \). Each geodesic will correspond to a permutation of the cosets in \( B + C \). The vertices of \( \Gamma \) on such a geodesic form the vertices of an \( n \)-cube.

The edges corresponding to a particular coset \( Hb \) disconnect \( \Gamma \), since removing this set of edges gives two sets of vertices, \( B \) and \( B^* \), where \( B \) is the set of those \( C \in M \) such that \( Hb \subseteq C \).

It has been pointed out to me by Graham Niblo that \( \Gamma \) is the \( 1 \)-skeleton of the Sageev cubing introduced in [12]. For completeness we describe this alternative characterization of \( \Gamma \).

Let \( G \) be a group with subgroup \( H \) and let \( A = HA \) be an \( H \)-almost invariant subset. Let

\[
\Sigma = \{ gA | g \in G \} \cup \{ gA^* | g \in G \}.
\]

We define a graph \( \Gamma' \). A vertex \( V \) of \( \Gamma' \) is a subset of \( \Sigma \) satisfying the following conditions:-

1. For all \( B \in \Gamma' \), exactly one of \( B, B^* \) is in \( V \).
2. If \( B \in V, C \in \Sigma \) and \( B \subseteq C \), then \( C \in V \).

Two vertices are joined by an edge in \( \Gamma' \) if they differ by one element of \( \Sigma \). For \( g \in G \), there is a vertex \( V_g \) consisting of all the elements of \( \Sigma \) that contain \( g \). Then Sageev shows that there is a component \( \Gamma_1 \) of \( \Gamma' \) that contains all the \( V_g \). In fact this graph \( \Gamma_1 \) is isomorphic to our \( \Gamma \).

By (1) for each \( V \in \Sigma \) either \( A \in V \) or \( A^* \in V \) but not both. Let \( \Sigma_A \) be the subset of \( \Sigma \) consisting of those \( V \in \Sigma \) for which \( A \subseteq V \). The edges joining \( \Sigma_A \) and \( \Sigma_A^* \) in \( \Gamma_1 \) form a hyperplane. Each edge in the hyperplane joins a pair of vertices that differ only on the set \( A \). For each \( xA \) there is a hyperplane joining vertices that differ only on the set \( xA \). Clearly \( G \) acts transitively on the set of hyperplanes.

With \( V \) as above, consider the subset \( A_V \) of \( G \)

\[
A_V = \{ x \in G | x^{-1}A \subseteq V \}.
\]

Then \( HA_V = A_V \) and \( A_{V_1} = A \). Also \( A_V + A \) is the union of those cosets \( Hx \) for which \( V \) and \( V_1 \) differ on \( x^{-1}A \), which is finite. Thus \( A_V \in VT \).

Thus there is a map \( VT^1 \to VT \) in which \( V \mapsto A_V \). This map is a \( G \)-map and an isomorphism of graphs.
If the set $A$ is such that $A$ and $gA$ are nested for every $g \in G$, then there is a $G$-subgraph of $\Gamma_1$ which is a $G$-tree. This will also be true of $\Gamma$.

In $\Gamma$ a hyperplane consists of edges joining those vertices that differ only by a particular coset $Hx$. Every edge of $\Gamma$ belongs to just one hyperplane. The group $G$ acts transitively on hyperplanes. The hyperplane corresponding to $Hx$ has stabilizer $x^{-1}Hx$.

Suppose now that $A$ is $H$-almost invariant with $HAK = A$. Here $H$ is the left stabiliser and $K$ is the right stabiliser of $A$, and we assume that $H \leq K$, so that in particular $HAK = A$. Note that it follows from the fact that $A$ is $H$-almost invariant that it is also $K$ almost invariant. Suppose that $G$ is finitely generated over $K$. We have seen, in the previous section, that there is a $G$-tree $T$ in which $A$ uniquely determines a set $A$ of vertices with $H$-finite coboundary $\delta \tilde{A}$. Here $T = T_0$ for $n$ sufficiently large that in the graph $X$ as defined in the previous section - the set $\delta \tilde{A}$ is contained in at most $n$ $H$-orbits of edges. Note that if $e$ is an edge of $\tilde{T}(H) = \tilde{E}(H,X)$, then $\delta e$ is $H_\infty$-finite, and will consist of finitely many $H_\infty$-orbits. It is then the case that $[G_e : H_]$ is finite, since $\delta e$ will consist of finitely many $G_e$-orbits each of which is a union of $[G_e : H_e]$ $H_\infty$-orbits of edges.

We also know that $K$ fixes a vertex $\tilde{o}$ of $T$, and that $H\delta \tilde{A} = \delta \tilde{A}$. Thus $\delta \tilde{A}$ consists of finitely many $H$-orbits of edges. We can contract any edge whose $G$-orbit does not intersect $\tilde{A}$. We will then have a tree that has the properties indicated at the beginning of this section. Thus $\tilde{A} \subset VT$ is such that $\delta \tilde{A} \subset ET$ consists of finitely many $H$-orbits of edges $e$ such that $G_e$ is $H$-finite. We see that the metric $d$ on $M$ is the same as the metric defined on $VT$. Explicitly we have proved the following theorem in the case when $G$ is finitely generated over $K$.

**Theorem 4.1.** Let $G$ be a group with subgroup $H$ and let $A = HAK$ where $H \leq K$ and $A$ is $H$-almost invariant. Let $M$ be the $G$-metric space defined above. Then there is a $G$-tree $T$ such that $VT$ is a $G$-subset of $M$ and the metric on $M$ restricts to a geodesic metric on $VT$. If $e \in ET$ then some edge in the $G$-orbit of $e$ has $H$-finite stabiliser.

This is illustrated in Fig 1 and Fig 2.

**Proof.** It remains to show that the theorem for arbitrary $G$ follows from the case when $G$ is finitely generated over $K$. Thus if $F$ is a finite subset of $G$, then there is a finite convex subgraph $C$ of $\Gamma$ containing $AF$. We can use the graph $X$ of the previous section for the subgroup $L$ of $G$ generated by $H \cup F$ to construct an $L$-tree which has a subtree $S(F)$ with vertex set contained in $VC$. These subtrees have the nice property that if $F_1 \subset F_2$ then $S(F_1)$ is a subtree of $S(F_2)$. They therefore fit together nicely to give the required $G$-tree. We give a more detailed argument for why this is the case. We follow the approach of [1].

Let $M'$ be the subspace of $M$ consisting of the single $G$-orbit $AG$. Define an inner product on $M'$ by $(B,C) = \frac{1}{4}(d(A,B) + d(A,C) - d(B,C))$.

This turns $M'$ into a 0-hyperbolic space, i.e. it satisfies the inequality

$$(B,C) \geq \min\{(B,D), (C,D)\}$$

for every $B, C, D \in M'$. This is because we know that if $L \leq G$ is finitely generated over $H$, then there is an $L$-tree which is a subspace of $M$. But $A, B, C, D$ are vertices of such a subtree which is 0-hyperbolic. It now follows from [1], Chapter 2, Theorem 4.4 that there is a unique $\mathbb{Z}$-tree $VT$ (up to isometry) containing $M'$. The subset of $VT$ consisting of vertices of degree larger than 2 will be the vertices of a $G$-tree and can be regarded as a $G$-subset of $M$ containing $M'$. 

□
Fig 1

Fig 2
References