Universal JSJ-Decompositions for Finitely Presented Groups

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Introduction

A group $G$ is said to split over a subgroup $C$ if either $G = A \ast C B$, where $A \neq C$ and $B \neq C$ or $G$ is an HNN-group $G = \langle A \ast C = < A, t^-1 at = \theta(a) > \rangle$ where $\theta : C \to A$ is an injective homomorphism. It is one of the basic results of Bass-Serre theory (see [DD] or [Se]), that a finitely generated group $G$ splits over some subgroup $C$ if and only if there is an action of $G$ on a tree $T$, without inversions, such that for no vertex $v \in VT$ is $v$ fixed by all of $G$. Here the tree is a combinatorial tree, i.e. a connected graph with no cycles, and an action without inversions is one in which no element $g \in G$ transposes the vertices of an edge. Tits [T] introduced the idea of an $R$-tree, which is a non-empty metric space in which any two points are joined by a unique arc, and in which every arc is isometric to a closed interval in the real line $R$. Alternatively an $R$-tree is a 0-hyperbolic space. A tree in the combinatorial sense can be regarded as a 1-dimensional simplicial complex. The polyhedron of this complex will be an $R$-tree - called a simplicial $R$-tree. However not every $R$-tree is like this. A point $p$ of an $R$-tree $T$ is called regular if $T - p$ has two components. An $R$-tree is simplicial if the points of $T$ which are not regular form a discrete subspace of $T$. It is fairly easy to construct examples of $R$-trees where the set of non-regular points is not discrete. There are good introductory accounts of groups acting on $R$-trees in [Be] and [Sh]. We assume that all our actions are by isometries. It is a classical result that a group is free if and only if it has a free action on a simplicial tree. As the real line $R$ is an $R$-tree and $R$ acts on itself freely by translations, any free abelian group has a free action on an $R$-tree. Morgan and Shalen [MS] showed that the fundamental group of any compact surface other than the projective plane and the Klein bottle has a free action on an $R$-tree. Rips showed that the only finitely generated groups that act freely on an $R$-tree are free products of free abelian groups and surface groups. Rips never published his proof, but there are proofs of more general results by Bestvina-Feighn [BF] and by Gaboriau-Levitt-Paulin (see [P] or [C]). Shalen in [Sh, Question D] considers groups which act on an $R$-tree with small arc stabilizers. He asks if such a group must admit an action on a simplicial tree with small arc stabilizers. In [D] I proved that this is the case if $G$ is finitely presented and the action on the $R$-tree has infinite cyclic arc stabilizers. In [D2] an example is given of a finitely generated group which acts on an $R$-tree with finite cyclic arc stabilizers but which does not split over a small subgroup.

In his seminal work [St] Stallings showed that a finitely generated group with more than one end splits over a finite subgroup. In [D1] I showed that a finitely presented group is accessible. This means that a finitely presented group $G$ has a decomposition as the fundamental group of a graph of groups in which vertex groups are one ended and edge groups are finite. This decomposition provides information about every action of $G$ on a simplicial tree with finite edge groups. Thus, let $S$ be the Bass-Serre $G$-tree associated with
the decomposition described and let \( T \) be an arbitrary \( G \)-tree with finite edge stabilizers, then there is a \( G \)-morphism \( \theta : S \to T \). We say that any action is resolved by the action on \( S \). In [D3, D4] I gave examples of inaccessible groups. These are finitely generated groups - but not finitely presented - for which there is no such \( G \)-tree \( S \). These groups do have actions on a special sort of \( \mathbb{R} \)-tree (a realization of a protree) but there appears to be no such action which resolves all the other actions.

My result -and its proof - on the accessibility of finitely presented groups can be seen as a generalization of a result by Kneser (see [He]) - and its proof - that a compact 3-manifold (without boundary) has a prime decomposition, i.e. it can be expressed as a connected sum of a finite number of prime factors. A compact 3-manifold \( M \) is prime if for every decomposition \( M = M_1 \# M_2 \) as a connected sum, either \( M_1 \) or \( M_2 \) is a 3-sphere. Expressed as a result about fundamental groups, it says that the fundamental group of a compact 3-manifold is a free product of finitely many factors, which, of course, is true for any finitely generated group by Grushko’s Theorem. The JSJ-decomposition of a compact 3-manifold \( M \), due to Jaco-Shalen [JS] and Johannson [J] concerns the embeddings of tori in compact 3-manifolds. They show that there exists a finite collection of embedded 2-sided incompressible tori, such that the pieces obtained by cutting \( M \) along these tori are either Seifert fibered spaces or simple manifolds (acylindrical and atoroidal). The JSJ-decomposition provides information about the splittings of \( \pi_1(M) \) over rank 2 free abelian subgroups. In a group theoretic setting JSJ-decompositions were discussed first by Kropholler [Kr] and subsequently by many authors. The first result for all finitely presented groups (over cyclic subgroups) was by Rips-Sela [RS]. In their result 2-orbifold groups appear as special vertex groups for the first time.

In this paper it is shown that a finitely presented group \( G \) has a decomposition as a fundamental group of a graph of groups which provides information about every action of \( G \) on an \( \mathbb{R} \)-tree.

We use the definition of a complex of groups as in [D] which is a slightly different notion to that of [Ha]. An attaching map of a 2-cell is said to be quadratic (or a quadratic word) if each 1-cell occurs exactly twice or not at all in the attaching word. A standard complex \( X \) of groups is one in which each 2-cell is attached to the one skeleton by a quadratic word \( w \cup w' \), in which both \( w \) and \( w' \) have at least two distinct letters. The group attached to each 2-cell is trivial and all the other groups attached to the cells are finitely generated.

The decomposition of the quadratic words into sub-words \( w \cup w' \) must be such that the complex admits a marking. A marking is an assignment of a positive length to each of the 1-cells so that the total length of the 1-cells in each \( w \) is the same as that in the corresponding \( w' \). If each \( w \) and \( w' \) contains each 1-cell at most once then we can assign arbitrary positive lengths to each 1-cell. Another example would be when \( w = a\bar{a}b \) and \( w' = \bar{c}\bar{c}b \). This admits a marking in which \( a \) and \( c \) are assigned the same length and \( b \) is assigned an arbitrary length. In some ways it would be better to write \( w' \) as the reverse path, as then \( w, w' \) would represent paths starting and ending at the same vertex. However I have decided to keep with the notation of [D] so that in this example the attaching word is \( a\bar{a}b\bar{c}c\bar{b} \). It can be seen that a marking is a solution to a set of homogenous linear equations with integer coefficients. For both an \( A \)-complex and a \( B \)-complex we require that the
solution space should have dimension at least two. For an $A$-complex this is equivalent to both $w$ and $w'$ involving at least two distinct letters.

The attaching maps of the 2-cells induce an equivalence relation $\sim$ on the 1-cells of $X$ generated by $e \sim f$ if $e, f$ lie in the same 2-cell. If $G$ is the fundamental group of the complex of groups $X$, then $G$ has a natural decomposition as the fundamental group of a graph of groups in which the edges are those which do not belong to any 2-cell. A vertex group corresponds to a subcomplex of $X$. The edges of this subcomplex consist of a single equivalence class under $\sim$.

If the complex consists of the 1-cells in an equivalence class with a single attached 2-cell then the complex is called an $A$-complex. A complex consisting of an equivalence class of 1-cell and there is more than one attached 2-cell, then the complex is called a $B$-complex. The structure of $A$-complexes and their fundamental groups is not complicated. In particular it is possible to say when the complex is developable.

We further require of a standard complex that distinct $A$ or $B$-complexes are disjoint. This means that they do not share any edges or vertices. From our proof the fact that they cannot share a vertex is because if they did then the group would have a non-trivial decomposition obtained by expanding that vertex to an edge.

Particular examples of $A$-complexes are discussed in [D]. The vertices and faces of an $n$-cube give an example of a $B$-complex if parallel faces are identified. This gives a space with one vertex (0-cell) whose fundamental group is free abelian of rank $n$. Other examples are provided by the rectangle groups and parallelepiped groups of [D5]. For a rectangle group, the 1-skeleton of $X$ consists of a rectangle with 4 vertices $A, B, C, D$ and 4 1-cells, $e = AB, f = AC, g = BD, h = CD$ and there are 2 2-cells $\sigma_1, \sigma_2$ where $\sigma_1$ is attached via $egh(che)(cd^{-1}g)(ae)$. This word can be read off from Fig 1 in [D5] by looking at the rectangle with vertices $aB, B, D, cD$ consisting of two smaller rectangles. Note that since $ab^{-1} = cd^{-1}$, the edge $cd^{-1}g$ joins $cd^{-1}D = cD$ and $ab^{-1}B = aB$. Thus $\sigma_1$ corresponds to the relation $ab^{-1} = cd^{-1}$. The 2-cell $\sigma_2$ is attached via the word $fhgbdh(beat^{-1})h(ae)$ which can be read off from the rectangle with vertices $C, D, bD, aC$ in Fig 1 of [D2]. This corresponds to the relation $bd^{-1} = ace^{-1}$. A marking is obtained in which $e, h$ are assigned the same length, and $f, g$ are assigned the same length.

Rectangle groups and parallelepiped groups play a key role in understanding $B$-complexes.

Associated with a marking is a measured foliation of the complex in which each 2-cell is foliated as in Fig 1 with $w$ being the cells along the upper semi-circle and $w'$ the cells...

![Fig 1](image_url)
along the lower semi-circle, each 1-cell has the transverse length assigned in the marking.

Let $G$ be the fundamental group of a standard complex $X$ of groups, and let $T$ be a $G$-tree, i.e. an $\mathbb{R}$-tree on which $G$ acts by isometries. We say that $X$ resolves the action on $T$ if the standard complex $X$ is developable, i.e. there is a simply connected complex of groups $\tilde{X}$ on which $G$ acts so that $X$ is the complex of groups associated with the action. It is also required that there is $G$-map from $\tilde{X}$ to $T$ so that the upper and lower sides of each 2-cell are mapped isometrically to segments of $T$: this means that 1-cells are mapped isometrically to segments of $T$. The induced marking on $X$ determines a foliation which lifts to a foliation of $\tilde{X}$ and each leaf of the foliation in $\tilde{X}$ is mapped to a point of $T$.

We prove the following theorem.

**Theorem 1.** Let $G$ be a finitely presented group. Then $G$ admits a decomposition as the fundamental group of a standard complex, which resolves any action on a $G$-tree. Each marking of the standard complex resolves an action on a tree.

It is also shown that for any finite presentation of a group a standard complex can be computed. This provides a way of deciding if a group has a non-trivial action on a tree or if it has more than one end.

I am very grateful to Andrew Bartholomew for his tenacity in trying to understand previous drafts of this paper. His questions have resulted in a considerable improvement in the exposition. He hopes to implement the algorithm described here.

**The Proof of Theorem 1**

Let $G$ be a finitely presented group and let $T$ be an $\mathbb{R}$-tree on which $G$ acts by isometries. In [D] it was shown how to construct a complex of groups with fundamental group $G$ that resolved the action. In [D] it is assumed that arc stabilizers are slender, but this was not necessary for the construction of the complex.

The first step is to take a simplicial complex $X$ with fundamental group $G$ and universal cover $\tilde{X}$. A $G$-map was then constructed by first defining a $G$-map $\theta_0 : V\tilde{X} \to T$ and then extending to the 1-skeleton and 2-skeleton. Each 1-cell of $\tilde{X}$ is mapped to a segment of $T$. So each 1-simplex $\gamma$ of $X$ can be assigned a length, namely the length of the segment of $T$ onto which a lift of $\gamma$ is mapped. If one studies the proof of Theorem 1 in [D] it can be seen that the whole of the folding process is determined by the lengths assigned to the edges. We see that each action of $G$ on an $\mathbb{R}$-tree is associated with a particular set of lengths associated with the 1-cells of a presentation complex $X$.

It is natural to ask if one assigns lengths to the edges of $X$, then can one construct an $\mathbb{R}$-tree with an action of $G$ for which there is a resolving map $\theta_0 : V\tilde{X} \to T$ giving rise to that set of lengths. Here one has the minimal requirement:

(L1) For any 2-simplex the length assigned to one edge should not exceed the sum of the lengths assigned to the other two sides.

It turns out that this condition is sufficient for such an action to exist. We are able to do this by constructing a complex of groups as above with fundamental group $G$ which can be regarded as a universal JSJ-decomposition of $G$, in that it contains information about every splitting of $G$, not just splittings over a particular class of subgroup.
Suppose then that we assign lengths to the 1-simplexes of $X$ so that $L_1$ is satisfied. Let $\sigma$ be a 2-simplex with vertices $u, v, w$ with lengths of edges $|uv|, |vw|, |wu|$ where $\alpha = \frac{1}{2}(|uv| + |vw| + |wu|)$. We subdivide $\sigma$ into three triangular 2-cells. Thus there is an extra vertex $p$ and three extra edges $up, vp, wp$. We assign $up, vp, wp$ the lengths $\alpha - |vw|, \alpha - |uw|, \alpha - |uv|$ respectively. Then $|uv| = |up| + |vp|$ and we now have a 2-complex $X_1$ in which each 2-simplex has lengths assigned to edges so that one edge has a length assigned which is the sum of the lengths assigned to the other two sides. Think of each 2-simplex as a 2-cell foliated in the obvious way. In $X_1$ each simplex of $X$ has been replaced by 3 simplexes. Each simplex of $X_1$ has a foliation which relates to an attaching map in which $w$ is a long edge (uv say, which was an edge in $X$ and $w'$ consists of 2 short edges, $up, vp$. The matching equation says $|uv| = |up| + |vp|$. In $X_1$ one gets such an equation $|uv| = |up| + |vp|$, for each simplex which is regarded as a 2-cell. These are the matching equations for $X_1$. Carrying out a subdivision and fold in a marked 2-cell in which the edges at one end have lengths $x, y$ with $x < y$ results in the creation of a new edge with length $y - x$ and retaining the edge with length $x$. Thus the pair $(x, y)$ of lengths becomes $(x, y - x)$. Once one has operated on one 2-cell one has to revise the attaching maps of all the other 2-cells so they are expressed in terms of the new set of edges.

If $X$ has $n$ 2-simplexes and $m$ 1-simplexes (edges) then $X_1$ has $3n$ 2-cells and $3n + m$ 1-cells. A marking of $X_1$ is a solution to the matching equations. A marking will be any point of a compact, convex linear cell in $\mathbb{R}^{3n+m}$ called the projective solution space $\mathcal{P}$. This theory is a generalization of the theory of normal surfaces or patterned surfaces in 3-manifolds (see [JO], [JT] and [DD, Chapter VI]). The vertex solutions are the ones corresponding to vertices of the projective solution space. Jaco-Oertel [JO] and Jaco-Tollefson [JT] have shown that vertex solutions carry important information about normal surfaces in a 3-manifold. Thus in [JT] it is shown that there is a face of $\mathcal{P}$ for which the vertex solutions give a set of 2-spheres giving a complete factorization of a closed 3-manifold. A solution is a vertex solution $v$ if it has integer coefficients and integer multiples of $v$ are the only solutions to $nv = v_1 + v_2$, where $n$ is a positive integer and $v_1, v_2$ are non-zero vectors in $\mathcal{P}$ with non-negative integer coefficients. Two of my students, Andrew Bartholomew [B] and Tom Barker [Br] have investigated this solution space for a group presentation on a computer. We were hoping to show that at least one vertex solution gives a non-trivial decomposition if the group has such a decomposition. It is a source of satisfaction to eventually show here that this is the case.

Choose a basis $u_1, u_2, \ldots, u_n$ for the solution space to the marking equations. Choose the vectors $u_i$ so that they are markings, i.e. so that all the coefficients are non-negative. Let $p = \alpha_1 u_1 + \alpha_2 u_2 + \ldots + \alpha_n u_n$ where $\alpha_1 = 1, \alpha_i = \sqrt{p_{i-1}}$, where $p_1, p_2, \ldots, p_{n-1}$ are distinct primes. Then $p$ is a marking for $X_1$.

Each 2-cell $\sigma$ of $X_1$ has a foliation as in Fig 1, for the marking given by $p$. We see that $X_1$ satisfies all the conditions of a standard complex except that the attaching maps are not necessarily quadratic. We describe a process that reduces $X_1$ to a standard complex. Note that $X_1$ does have a marking, i.e. there is always at least one non-zero solution to the matching equations. Thus we can assign the integer 2 to each edge of the original complex $X$ and the integer 1 to each of the edges created by subdividing each 2-simplex. If the solution space is one dimensional then $G$ has no non-trivial action on an $\mathbb{R}$-tree.
If for any 2-cell the same vertex occurs in the upper and lower semi-circles in the same position, i.e. in the same leaf of the foliation, then we replace the 2-cell by two 2-cells with the appropriate attaching maps. Intuitively we are shrinking any leaf which is a loop based at a vertex to that point.

If $e_1, e_2$ have the same length but different terminal vertices the number of edges is reduced by a Type I fold. If they have the same terminal vertices then the 2-cell is attached by a word of length two where $w = e_1, w' = \bar{e}_2$. If $w = e_1, w' = \bar{e}_2$ the whole 2-cell is removed by a Type III fold, and replaced by a single 1-cell, and the number of 1-cells is reduced by one. If $e_1$ is shorter than $e_2$ then subdivide $e_2$ so that the initial part has the same length as $e_1$ and then fold the corner by a Type I fold. If $e_2$ is shorter than $e_1$ then we subdivide $e_1$ and then fold the corner. Now repeat the process. This process may terminate when all the 2-cell is folded away, by a Type III fold, or there is an infinite folding sequence. In general the process of subdivision and Type I fold leaves the number of edges unchanged, while a Type III fold reduces the number of edges by one. Examine each 2-cell. If folding from either end results in a Type III fold after finitely many folds then carry out that folding and no more. If any leaf now begins and ends at the same vertex then contract that leaf creating new two 2-cells. Then again see if folding from a new end of a 2-cell results in a Type III fold. and again repeat the process.

At the end of this process, no folding of a 2-cell results in a Type III fold and no leaf begins and ends at the same vertex. Any remaining 2-cell must correspond to an infinite folding sequence, consisting entirely of alternate subdivisions and Type I folds.

We will show that this happens if and only if the attaching word $w \cup w'$ is quadratic.

First we give an example making it easier to understand the following general explanation. This example is Example 6 of [D]. I have expanded the explanation given there.

Let the complex $X_1$ have four vertices $U, V, Y, Z$ and three oriented edges $a, b, c$. Let $\iota a = U, \tau a = V, \iota b = V, \tau b = Z, \iota c = U, \tau c = Y$. Let the groups of $U, V, Y, Z$ be finite cyclic of order 3 and generated by $u, v, y, z$ respectively. Let the 6-sided 2-cell be attached via the word $w \cup w' = a(vb)(z\bar{b})a(uc)(y\bar{c})$. Here $w = ab(z\bar{b})$ and $w' = \bar{a}(uc)(y\bar{c})$. In this case $X_1$ is a 2-sphere with 4 cone points. We now discuss how the connecting elements occur in $w \cup w'$.

In this case the 1-skeleton of $X_1$ is a tree. We choose a particular lift of this tree in $X_1$ to the universal orbifold cover $\tilde{X}_1$, which is the (hyperbolic) plane tessellated by 6-gons. The attaching word traces out a loop in $\tilde{X}_1$, which is the boundary of a fundamental region. Note that although the image of the path backtracks in $X_1$, it is not allowed to backtrack in $\tilde{X}_1$. This means that there must be non-trivial connecting elements where the image backtracks. In $\tilde{X}$ each fundamental region has 6 vertices, including one point (incident with 3 edges in $\tilde{X}_1$) from the orbits corresponding to $Y$ and $Z$ and two vertices (incident with 6 edges in $\tilde{X}_1$) from each of $U$ and $V$. At one of the vertices corresponding to $U$ (or $V$) we have to use a non-trivial connecting element. We can choose where this is. This corresponds to choosing a lift of the 1-skeleton of $X_1$ to $\tilde{X}_1$. In the account here we choose to do this at points which are not at the end of the foliated disc. This has the advantage that $w, w'$ do not start or end with connecting elements. We get a presentation for $G$ in which the generators are $u, v, y, z$ and a relation obtained by deleting the edges in
the attaching word. This is because the 1-skeleton of $X_1$ is a tree. Thus there is a relation $vzuy = 1$. There are also relations $u^3 = v^3 = y^3 = z^3 = 1$.

Consider the folding sequence corresponding to the marking with $\alpha = 1, \beta = \gamma = \sqrt{2}/2$, which are assigned to the edges $a, b, c$ respectively. Initially we have the 2-cell attached along $a(vb)(zb)\bar{a}(uc)(yc)$, where the left hand vertex is $U$ and the right hand vertex is $V$ in the quotient space $X_1$. The attaching word is quadratic - its image in $X_1$ is $ab\bar{b}ac\bar{c}$ - and we will see that there is an infinite folding sequence in which $\bar{w} = a(vb)(zb)$ is folded against $\bar{w}' = c(y^{-1}\bar{c})(u^{-1}a)$. The total length along top or bottom is $1 + \sqrt{2}$. After the first subdivision and fold we have a new complex $X_2$ with the same vertices $U, V, Y, Z$ and with edges $b, c$ and a new edge $d$ with length $1 - \sqrt{2}/2$ with $\iota d = Y, \tau d = V$ and the attaching word has become $(yd)(vb)(zd)d(uc)(uc)$. Note now that there is a connecting
element at the start vertex \( Y \). We can move this connecting element so that it is at the other visit of the path to \( Y \). When we do this we have to conjugate \( y \) by the product of connecting elements traversed between the two visits to \( Y \). In this case the conjugating element is \( vz = y^{-1}u^{-1} \) and so the connecting element becomes \( uyu^{-1} \). The new attaching word is \( d(vb)(zb)d((uyu^{-1})e)(uc) \). The change of the position of the connecting element corresponds to changing the lift of \( X_2 \) to \( \tilde{X}_2 \). Note that deleting the edges in the attaching word gives the same presentation as before.

The 2-cell has \( w = d(vb)(zb), \bar{w} = \bar{e}(u^{-1}e)((uyu^{-1})d) \). After the next subdivision and fold we have a new complex \( X_3 \) with the same vertex set but with edges \( b, d, e \) where \( ie = U, \tau e = V \) and \( e \) has length \( \sqrt{2}/2 - (1 - \sqrt{2}/2) = \sqrt{2} - 1 \) and the attaching word has become \( b(zb)d((uyu^{-1})d)e(ue) \). Note that in this graph there is a vertex \( V \) of valency 3 whereas previously no vertex had valency more than 2. The attaching word visits the vertex \( V \) three times. As before we move the non-trivial connecting element so that it is not at the start or end point of \( w = (vb)(zb) \). The attaching word becomes \( b(zb)d((uyu^{-1})d(uzu)1)e(ue) \), with \( w = (zb), \bar{w} = \bar{e}(u^{-1}e)(uzu)1 \). We can translate the whole lift by \( u \) giving an attaching word \( b((u^{-1}zu)b)d(yd)(ve)(ue) \). All we have done here is conjugate all the connecting elements by \( u \) to make the connecting elements shorter. The next subdivision and fold starts at \( V \) and folds \( e \) the shorter edge against \( b \), so that we then have a new edge \( g \) with length \( 1 - \sqrt{2}/2 \) replacing \( b \). Here \( \nu g = U, \tau g = Z \) and the attaching word is \( (ug)((u^{-1}zu)g)e(dyd)(ve) \). Again by changing the position of where the connecting element for \( U \) is non-trivial we have attaching word \( g(u^{-1}zu)g(u^{-1}zu)e(d)yd(ve) \). Now note that \( (1 - \sqrt{2})\sqrt{2}/2 = 1 - \sqrt{2}/2 \) and the situation we have reached is similar to the initial situation scaled by \( \sqrt{2} - 1 \). In fact the positions of \( Y \) and \( Z \) have been transposed from the original position. To get an exact scaling carry out the next 3 folds to get the initial position scaled by \((\sqrt{2} - 1)^2\).

We return to the general case. Suppose the folding sequence is infinite and that the 2-complexes in the sequence are \( X_n, n = 1, 2, \ldots \).

We have assigned lengths to the edges (1-cells) of \( X_n \). Traversing the top semi-circular boundary of the 2-cell \( \sigma \) determines to a path (or rather walk) \( w \) in the 1-skeleton of \( X_1 \) and \( w' \) is the path corresponding to the lower semi-circular boundary. These paths are usually not segments - they can even backtrack. Although if the path backtracks (as in the rectangle group case) the backtracking occurs at a vertex where there is a non-trivial “connecting element”. Let \( \ell_n \) be the total length of edges of \( X_n \). It is clear that \( \ell_n \geq \ell_{n+1} \geq 0 \). We have \( \ell_{n+1} < \ell_n \) if the fold is a type I fold, and \( \ell_{n+1} = \ell_n \) for a subdivision. In going from \( X_n \) to \( X_{n+1} \) an arc \( [y_n, y_{n+1}] \) of the upper semicircular boundary of \( \sigma \) is identified with an arc \( [y_n', y_{n+1}'] \) of the lower semicircular boundary. Each such arc is identified with a 1-cell of \( X_n \) and so has the same length. In going from \( X_1 \) to \( X_n \) the folding has identified \( [x, y_n] \) with \( [x, y_n'] \), which will be of the same length.

We assume that \( y = \lim y_n \), and that \( y' = \lim y_n' \). We will show that \( y = y' \) is the end point of \( \sigma \). Let \( \lambda_n \) be the length of the arc \( [y_n, y] \). Thus \( \lambda_n \) is the length of the arc which remains to be folded. Consider the subspace of \( X_1 \) which is the union of the images of the paths corresponding to \( [x, y] \) and \( [x, y'] \). If this is not a subgraph of \( X_1 \), then one of the paths corresponding to \( [x, y], [x, y'] \) in \( X_1 \) must end in part of an edge not visited by the other path. It is not hard to see that this will not produce an infinite folding sequence.
Thus we assume that this subgraph is all of the 1-skeleton of $X_1$.

If a fold at the $n$-th stage produces an edge which is not in the subgraph $X'_{n+1}$ determined by the remaining folding sequence, then folding this edge away will result in a Type III move which is not allowed. Thus we know that each folded edge is in the subgraph determined by the remaining folding sequence.

For a Type I fold $\ell_n$ and $\lambda_n$ are reduced by the same amount.

Since we are assuming that each folded edge is in the subgraph determined by the remaining folding sequence, it is clear that $\ell_n$ tends to zero as $\lambda_n$ tends to zero. Since $\ell_n - \lambda_n$ is constant, it follows that $\ell_n = \lambda_n$.

Let $w_y, w'_y$ be the directed paths in $X_1$ which are the images of $[x, y], [x, y']$ respectively. Clearly they are initial parts of the paths $w, w'$, so they begin at the same point. In [D] it is claimed that $y$ and $y'$ are always vertices in the original graph, which must therefore be the end point of $\sigma$, but I doubt if this is the case. I think that $y, y'$ may be vertices that are created in the subdivisions of the finite sequence that precedes a Type III fold. However after the last Type III fold the limit point of the infinite sequence - which involves no Type III fold must be a vertex and it will have the same position in both $w$ and $w'$, i.e. it will be at the same distance from the initial point of $w$.

From length considerations every edge of $X_1$ occurs exactly twice in $w \cup w'$ or at least one edge occurs only once. If the latter occurs we will arrive at a contradiction by showing that the folding sequence must have contained a Type III fold. Let $e$ the edge which occurs only once in $w \cup w'$. Without loss of generality suppose it is in $w$. In fact we can assume that it is the first edge of $w$, since we can fold away any edges which precede it. This folding will not affect the edge $e$, and we arrive at the situation considered previously, which we know produces a Type III fold.

An infinite folding sequence therefore occurs only when there is a 2-cell in which the attaching map is quadratic.

It remains to show that the standard complex obtained resolves any action of $G$ on an $\mathbb{R}$-tree.

Consider the effect of folding on $p$ and the $u_i$’s. A subdivision followed by a Type I fold results in an elementary operation on $p$. If $x, y$ are the lengths of the edges at the relevant corner and $x < y$, then after the subdivision and fold the edge of length $y$ has been replaced by one of length $y - x$

Thus if the vectors $u_i, i = 1, 2, \ldots, 3n + m$ and $p$ is arranged as the rows of a matrix then the fold corresponds to an elementary column operation on the matrix. All the information about this operation can be obtained from the row corresponding to $p$ as the coefficients of the sum for $p$ are linearly independent over the rationals. If a Type III fold can be made then two columns must be equal as this happens if and only if the two column entries in the row corresponding to $p$ are the same. Note that this means that in any other linear combination of the $u_i$’s the two column entries will be the same. This means that if instead of $p$ we started with a marking corresponding to another internal point of the solution space then we would reach a Type III move at precisely the same point of the folding process.

Deleting one of the two equal columns will induce a linear bijection on the solution space.
In the folding process at the \( n \)-th stage the row corresponding to \( p \) will always correspond to a marking of the space \( X_n \), but the other rows need not correspond to a marking in that some coefficients may become negative.

We see then that there is a linear bijection between the solution space of the matching equations for one term of the folding sequence and the next term.

In a standard complex the markings of the free edges and each \( A \)-complex and \( B \)-complex are independent and the subspace is a direct sum of the subspaces for each subcomplex together with a one dimensional subspace corresponding to each free edge.

We now analyse the space of markings of an \( A \)-complex. Such a complex is given by a single quadratic word \( w \cup w' \), and a marking is an assignment of positive lengths to the letters in such a way that the total length of \( w \) is the same as that of \( w' \). Let \( a_1, a_2, \ldots, a_r \) be the letters which lie both in \( w \), and \( w' \). Let \( b_1, b_2, \ldots, b_s \) be the letters which occur twice in \( w \) (and so not in \( w' \)) and let \( c_1, c_2, \ldots, c_t \) be the letter which occur twice in \( w' \), then the \( a_i b_j c_k \) can be assigned arbitrary positive lengths \( \alpha_i, \beta_j, \gamma_k \) subject only to the single constraint \( \beta_1 + \beta_2 + \ldots + \beta_s = \gamma_1 + \gamma_2 + \ldots + \gamma_t \). The space, therefore, has dimension \( r + s + t - 1 \).

Although it is not too relevant here one can also say which are the vertex solutions (extreme fundamental solutions in \([B]\)). In fact there are \( r + st \) vertex solutions. There are \( r \) solutions \( y_i, i = 1, 2, \ldots, r \) where in \( y_i, \alpha_i = 1 \) and all other coefficients are zero, and there are \( st \) solutions \( z_{j,k}, j = 1, 2, \ldots, s, t = 1, 2, \ldots, t \), where in \( z_{j,k}, \beta_j = \gamma_k = 1 \) and all other coefficients are zero.

Consider now the space of markings of a \( B \)-complex. The initial marking on \( X \) will give a marking on each \( B \)-complex. I claim that the space of markings cannot be one dimensional. For the solution space would be the set of scalar multiples of a vector with rational coefficients, being solutions to a set of linear equations with integer coefficients. But if the edges of a the \( B \)-complex are all positive rationals then there can be no infinite folding sequence. The complex will collapse to a graph.

We can extend the theory of patterns and tracks in simplicial complexes (see \([DD]\)) to \( A \)-complexes and \( B \)-complexes.

Let \( X \) be either an \( A \)-complex or a \( B \)-complex, and let \( G \) be the fundamental group of the complex of groups. Suppose we have an integer solution to the matching equations. This assigns a non-negative integer \( f(\gamma) \) to each 1-cell \( \gamma \), and not all numbers assigned are zero. Choose \( f(\gamma) \) points in the interior of each 1-cell. In each 2-cell there is a unique way of joining these points by disjoint straight lines going from the upper semi-circle to the lower semi-circle. This is because we have the same number along the top and along the bottom and one joins the \( i \)-th point along the top to the \( i \)-th point along the bottom. The union of all these straight lines (closed intervals) is called a pattern. A track is a connected pattern. Clearly each component of a pattern is a track as it gives a solution to the matching equations. If a track is non-separating then there is a non-trivial homomorphism from \( G \) to \( \mathbb{Z} \). A separating track will give a two colouring of \( X \) and it will induce a two colouring of each 2-cell. A track will intersect each 2-cell in a pattern - which may be empty. If the original track is separating then no component of the induced pattern in a 2-cell can be non-separating, as there would not be a two colouring of that cell.

Choose a basis for the solution space where each of the basis elements comes from
one of the smaller subspaces making the direct sum decomposition. A key fact is that this basis will also be a basis for the solution space regarded as a vector space over $\mathbb{R}$.

We know that any action of $G$ on an $\mathbb{R}$-tree is resolved by a marking on $X_1$. This gives a solution to the matching equations. We have found a basis for the solution space and so our solution can be expressed uniquely as a sum of solutions in the subspaces. If the action being resolved is a simplicial action then the resolving marking has integer values and corresponds to a pattern $P$ in $X_1$ (see [DD] or [D1]). The components of $P$ are tracks. Each track will also correspond to a solution of the matching equations and $P$ will be a linear combination of these solutions with positive integral coefficients. Suppose we express a track as a sum of solutions in the subspaces. Each term in the sum will have rational coefficients. In each subspace there is a basis each term of which corresponds to a track in a component complex or an edge not belonging to any component complex. A track in an edge is just a point in that edge. Tracks in different components are compatible. To get a track as a sum of solutions in the subspaces, all the terms must be from the same subspace. Thus any track in $X_1$ determines a track in one of the subcomplexes. It follows that any simplicial action of $G$ is resolved by a pattern of tracks in the component complexes. More generally any action of $G$ on an $\mathbb{R}$-tree determines a foliation on $X_1$ and a solution of the matching equations. We have seen that the solution space is a direct sum of solution spaces for the individual complexes together with one-dimensional subspaces corresponding to the free edges. There is a problem that writing the solution as a sum one may not get positive coefficients for the summands. The foliation is determined by a system of isometries and as such (see [C] p266) decomposes into finitely many subsystems, some of which correspond to foliations which are thickened tracks and the others correspond to foliations in which leaves are dense. Each of these subsystems gives a solution to the matching equations and each subsystem must lie in one of the subcomplexes. We get a positive solution to the matching equation for the subcomplex by recording the transverse measure of the intersection of the subsystem with each 1-cell.

To complete the proof of Theorem 1 it remains to show that every marking of a standard complex resolves an action on some tree. It clearly suffices to do this for a $\mathcal{A}$-complex and a $\mathcal{B}$-complex.

**Theorem 2.** The fundamental group of an $\mathcal{A}$-complex or a $\mathcal{B}$-complex is an infinite group. It is either a finitely generated subgroup of the isometry group of $\mathbb{R}$ or it resolves incompatible decompositions over subgroups that are not small. It does not resolve a non-trivial action on a tree with finite edge stabilizers. Every marking resolves a non-trivial action on an $\mathbb{R}$-tree.

**Proof.** Let $X$ be either an $\mathcal{A}$-complex or a $\mathcal{B}$-complex, and let $G$ be the fundamental group of the complex of groups.

In [D5] I constructed rectangle groups and parallelepiped groups. These groups admitted actions on $\mathbb{R}$-trees corresponding to an assignment of lengths to the generators. For a particular parallelepiped group $S$ this assignment is essentially a marking of a $\mathcal{B}$-complex $P$ whose fundamental group is $S$. We show in [D5] that an assignment of lengths to the edges of $P$ lifts to a foliation on $\tilde{P}$ (denoted $X$ in [D5]) for which the leaves are the points of an $\mathbb{R}$-tree $T$. For any $\mathcal{B}$-complex $X$ with fundamental group $G$, there is a simply connected space $\tilde{X}$ on which $G$ acts. A marking of $X$, induces a foliation of $X$
which lifts to a foliation of $\tilde{X}$. I conjecture that the leaf space of this foliation is an $\mathbb{R}$-tree on which $G$ acts and this is resolved by the marking on $X$. We will show that there is a homomorphism of $G$ into a parallelepiped group $S$ and a marking of the corresponding parallelepiped $\mathcal{B}$-complex $P$ which is induced by the marking on $X$ and the action of $G$ on the corresponding $\mathbb{R}$-tree $T$ corresponding to the leaf space in $\tilde{P}$ which factors through $S$ is resolved by the marking on $X$.

We describe $P$. Suppose then we have a marking of $X$, which is either an $\mathcal{A}$-complex or a $\mathcal{B}$-complex with fundamental group $G$. Let $J$ be the subgroup of $\mathbb{R}$ which is generated by numbers assigned to the edges of $X$ in the marking. Let $b_1, b_2, \ldots, b_n$ be a basis for $A$ so that every coefficient is a linear sum with integer coefficients of the $b_i$'s. The parallelepiped space $P$ will be $n$-dimensional. Thus $S$ will have many subgroups which are copies of $J$ and $\tilde{P}$ will have subspaces which are $n$-dimensional Euclidean spaces. Choose one of these subspaces $E$ and the corresponding subgroup (also denoted $J$) of $S$. The vertices in $E$ will be an integral lattice. Put a metric on $E$ so that the vertices on the $i$-th axis are distance $b_i$ apart. The attaching map of each 2-cell of $X$ will determine a closed loop in $E$. It the attaching map is given by $w \cup w'$, the words $w, w'$ determine paths in $E$ from any common start point. Then the matching equation for the 2- cell says that these paths must have the same end point. This is what happens with the rectangle group as described in the introduction.

Choose a particular map for the boundary 2-cell of $X$ so that vertices map to vertices. Thus we choose a lift of the 2-cell to $\tilde{X}$ and then map its boundary into $E$ so that the vertices map into the integer lattice. Take another 2-cell which shares an edge with the first. Map its boundary into $E$. Eventually we have mapped all the boundaries of 2-cells into $E$. We have an induced map of the vertices of $X$ into the vertices of $P$.

The attaching maps for the 2-cells of $X$ give the defining relations for $G$. As they map to loops in $\tilde{P}$ there is a homomorphism of $G$ with the required properties. In order to get a mapping from $X$ to $P$ which induces the homomorphism it is necessary to use a different cell decomposition of $E$ so that edges of $X$ map to edges in $P$.

Theorem 2 follows because the action of $S$ on the tree $T$ has the required properties. Thus it is shown in [D5] that unless $S$ is a group of isometries of $\mathbb{R}$, the action is not small. It is fairly easy to see that $\tilde{P}$ is one ended, since $E$ is one ended. If $G$ had more than one end, then it would have to have a non-trivial action on a tree with more than one end. For this to be resolved by an action on $\tilde{X}$, the space $\tilde{X}$ would have more than one end. But any action on $\tilde{X}$ resolves an action on $\tilde{P}$ and this means that $\tilde{P}$ would have more than one end.

Theorem 1 follows because it follows from Theorem 2 that every marking on an $\mathcal{A}$-complex or a $\mathcal{B}$-complex resolves at least one action on a tree. It would be nice to show that the leaf space in one of these complex is already an $\mathbb{R}$-tree.

It is possible to calculate a standard complex corresponding to a particular presentation. This can be done as in [B] by calculating the vertex solutions (called extreme fundamental solutions in [B]) of the matching equations. It is probably more efficient to follow the approach of our proof.

Recall that we have shown one folds from one end of a foliated 2-cell then either one reaches a Type III fold or one has an infinite sequence of subdivisions and Type I folds
and the limit point is a common vertex in both $w$ and $w'$. In this latter case shrinking the leaf at the common limit point if it is not the end point of the cell one obtains a 2-cell with a quadratic attaching map in which at least 4 edges are involved.

Based on the above statement an algorithm to compute a standard form would be as follows.

**Algorithm**

**Step 1**
Find a basis $u_1, u_2, \ldots, u_n$ for the solution space to the marking equations for $X_1$. Choose the vectors $u_i$ so that they are markings, i.e. so that all the coefficients are non-negative. Let $p = \alpha_1 u_1 + \alpha_2 u_2 + \ldots + \alpha_n u_n$ where $\alpha_1 = 1, \alpha_i = \sqrt{p_i - 1}$, where $p_1, p_2, \ldots, p_{n-1}$ are distinct primes. Then $p$ is a marking for $X_1$.

**Step 2**
Examine each marked 2-cell. If the same vertex occurs in both $w$ and $w'$ and is in the same leaf, i.e. it is the same distance from the end of the cell, then shrink the leaf to a point and create two 2-cells. Eventually each 2-cell will have no leaf which is a loop at a vertex.

**Step 3**
Determine if there are any 2-cells not attached by quadratic words. Stop if there are none.

**Step 4**
Fold a 2-cell from one end that is not attached by a quadratic word until a Type III move occurs. This reduces number of edges.

Andy Bartholomew is hoping to implement this algorithm.

(See http://www.layer8.co.uk/maths/.)

It will usually be the case that some of the decompositions corresponding to one-dimensional subspaces of the solution space that are found in the above process will correspond to trivial decompositions, i.e. one in which $G = A \ast C B$ where $C = A$. The decompositions obtained from an $A$-complex are always non-trivial as they correspond to essential curves in a 2-orbifold. We have seen above that the decompositions resolved by a $B$-complex are non-trivial also and are not over finite subgroups.

To decide if a group has a non-trivial splitting it will suffice to decide if any free edge in the graph of group decomposition corresponds to a non-trivial splitting, and it will have more than one end if and only if there is such splitting over a finite subgroup.

A $B$-complex may resolve actions on trees with small subgroups, but these correspond to tracks which only intersect one of the 2-cells. Thus it is possible to read off information about all splittings over small subgroups. An example of this would be the complex with one vertex and six 1-cells labeled $a, b, c, d, e, f$ and two 2-cells attached via $abcdab\bar{c}d, aef\bar{a}b\bar{e}\bar{f}$, and there are small splittings over tracks that intersect one of $c, d, e$ or $f$.
References


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