Isometries and Morphisms of Real Trees

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Introduction

Tits [T] introduced the idea of an **R**-tree, which is a non-empty metric space in which any two points are joined by a unique arc, and in which every arc is isometric to a closed interval in the real line **R**. Alternatively an **R**-tree is a 0-hyperbolic space. A tree in the combinatorial sense can be regarded as a 1-dimensional simplicial complex. The polyhedron of this complex will be an **R**-tree - called a simplicial **R**-tree. However not every **R**-tree is like this. A point p of an **R**-tree T is called *regular* if $T - p$ has two components. If the points of T which are not regular form a discrete subspace of T then the **R**-tree is simplicial . It is fairly easy to construct examples of **R**-trees where the set of non-regular points is not discrete. There are good introductory accounts of groups acting on **R**-trees in [Be], and [Sh], and of the more general theory of groups acting on Λ-trees in [C]. We assume that all our actions are by isometries. It is a classical result that a group is free if and only if it has a free action on a simplicial tree. As the real line **R** is an **R**-tree and **R** acts on itself freely by translations, any free abelian group has a free action on an **R**-tree. Harrison (see [C]) proved that a 2-generator group that acts freely on an **R**-tree is free or free abelian. Morgan and Shalen [MS] showed that the fundamental group of any compact surface other than the projective plane and the Klein bottle has a free action on an **R**-tree. Rips showed that the only finitely generated groups that act freely on an **R**-tree are free products of free abelian groups and surface groups. Rips never published his proof, but there are proofs of more general results by Bestvina -Feighn [BF] and by Gaboriau-Levitt-Paulin (see [P] or [C]).

In his seminal work [St] Stallings showed that a finitely generated group with more than one end splits over a finite subgroup. In [D1] I showed that a finitely presented group is accessible. This means that a finitely presented group G has a decomposition as the fundamental group of a graph of groups in which vertex groups are one ended and edge groups are finite. This decomposition provides information about every action of G on a simplicial tree with finite edge groups. Thus, let S be the Bass-Serre G-tree associated with the decomposition described and let T be an arbitrary G-tree with finite edge stabilizers, then there is a G-morphism $\theta : S \to T$. We say that any action is *resolved* by the action on S. In [D2, D3] I gave examples of inaccessible groups. These are finitely generated groups - but not finitely presented - for which there is no such G-tree S. These groups do have actions on a special sort of **R**-tree (a realization of a protree) but there appears to be no such action which resolves all the other actions. In [D3] it is shown that there are inaccessible groups with two generators. In these examples the action of each generator on the **R**-tree is elliptic, i.e. each generator fixes a point.

In this paper we investigate two generator group actions on **R**-trees in which each generator induces a hyperbolic isometry. We also require that the axes intersect and the two hyperbolic lengths are independent over the rationals. The axes will intersect in a closed segment of length Δ which may be ∞. We are able to classify such actions, and the groups G which act in this way. If the action is faithful, i.e. the identity element 1 is the only element fixing every point of the tree, then G is either free or has a presentation

$$G = < a, b | (a^{-1}b^{-1}ab)^n = 1 >$$

for some $n = 1, 2, \ldots$. If $n = 1$ then G is free abelian and the action is by translation on the real line $\mathbf{R}$. If the action has small arc stabilizers, i.e. the stabilizer does not contain a non-abelian free subgroup, then in fact arc stabilizers are trivial and either the group is not free and there is exactly one such action for each set of values of $n, \ell(a), \ell(b)$ with $\Delta = \ell(a) + \ell(b)$. or G is free, and $\Delta \leq \ell(a) + \ell(b)$ and the action is uniquely determined by the values $\Delta, \ell(a), \ell(b)$. The action is free and simplicial if and only if $\Delta < \ell(a) + \ell(b)$. This means, of course, that if a group G acts on an $\mathbf{R}$-tree with small arc stabilizers and a, b are hyperbolic elements with rationally independent hyperbolic lengths, then if the axes of a, b have a long intersection ($> \ell(a) + \ell(b)$), then in fact the axes coincide. In [BF] this result is important in the proof, and is proved only for stable actions. In [D4] the result is proved in the case when arc stabilizers are slender. Skora [Sk] showed that every action of an orbifold group on an $\mathbf{R}$-tree with small arc stabilizers is geometric, i.e. it is associated with a measured lamination on the appropriate surface. In the case of 2-generator orbifold groups this can be deduced from our results.

The results involve a construction of an action of a free group $F = F_n$ on an $\mathbf{R}$-tree via a Morse function on a simplicial tree. We are mainly interested when F has rank $n = 2$. However the construction works for all n. In the final section some results are obtained when $n = 3$. Thus we investigate actions of F_3 on an $\mathbf{R}$-tree for which there are three hyperbolic isometries whose axes share a common segment. In particular we investigate the case when the axes of each pair of the three isometries intersect in the same common segment, and the three hyperbolic lengths are independent over the rationals.

Pairs of isometries

Let S be a G-tree, i.e. an $\mathbf{R}$-tree on which the group G acts by isometries and let T be an H-tree. A *morphism* $\theta : S \to T$ is a map $S \to T$ (also denoted θ) and a homomorphism $G \to H$ (also denoted θ) with the following properties:-

(i) Every segment $[s, s']$ of S can be subdivided into finitely many subsegments $[s, s'] = [s, s_1] \cup [s_1, s_2] \cup \ldots \cup [s_{k-1}, s_k]$ so that on each subsegment θ restricts to an isometry.

(ii) The map $\theta : S \to T$ is equivariant with respect to the group actions, i.e.

$$\theta(gs) = \theta(g)\theta(s), g \in G, s \in S.$$

Let F be a free group with free generating set $\{x_1, x_2, \ldots . x_n\}, n \geq 2$. Let $\phi : F \to \mathbf{R}$ be a homomorphism with dense image which has rank n. (Here we regard $\mathbf{R}$ as a group under addition.) We also assume $\phi(x_i) > 0, i = 1, 2, \ldots, n$. Let T be the simplicial F-tree with one orbit of vertices and n orbits of edges. Choose a vertex $v \in VT$ and edges $e_1, e_2, \ldots, e_n$ representing the distinct orbits of edges. Suppose also that e_i has vertices v, v_i where $v_i = x_i v$ and that e_i is isometric with the closed interval $[0, \phi(x_i)]$. We extend ϕ to a map $\phi : T \to \mathbf{R}$, by putting $\phi(v) = 0, \phi(fv) = \phi(f)$ and extending ϕ linearly to edges. Then ϕ is a morphism of $\mathbf{R}$-trees. We regard ϕ as a height function on T.

Let $\Delta \in \mathbf{R}, \Delta > 0$. Define a relation $\sim$ on T as follows.

Let $s, t \in T$ and let $[s,t]$ be the geodesic joining them. Then $s \sim t$ if $\phi(s) = \phi(t)$ and $\phi(s) - \Delta \leq \phi(x) \leq \phi(s)$ for each $x \in [s,t]$.

It is easy to see that $\sim$ is an equivalence relation. We will show that we can put a metric on $T/\sim$ so that it is an $\mathbf{R}$-tree and there is a morphism $T \to T/\sim$. Let $s, t \in T$ be points such that $s \sim t$. We show that the image of $[s,t]$ is a finite simplicial tree and describe its structure. Consider the function ϕ on $[s,t]$. We know that $\phi(s) - \Delta \leq \phi(x) \leq \phi(s) = \phi(t)$ for each $x \in [s,t]$. Clearly s, t will be local maxima for ϕ. Let $s = M_1 < m_1 < M_2 < m_2 < \ldots < m_{k-1} < M_k = t$ be the sequence of local maxima and minima as one traverses $[s,t]$. Now ϕ is monotone on each subinterval and so no two points of a subinterval lie in the same equivalence class. We use induction on k to show that the image of $[s,t]$ is a tree with a root r which is the image of s, t and any other local maxima M for which $\phi(M) = \phi(s)$. The local minima map to vertices of valency one. If $k = 2$, then m_1 is the midpoint of $[s,t]$ the map $[s,t] \to [s,t]/\sim$ is a folding of $[s,t]$ onto $[m_1, t]$. Thus the result is true for $k = 2$. If $k > 2$ then there is at least one internal local maxima. Let M_j be an internal local maxima for which ϕ takes its largest value. There are unique points s', t' in $[s, m_1]$ and $[m_{k-1}, t]$ respectively for which $\phi(s') = \phi(t') = \phi(M_j)$. By induction, the image of $[s', M_j]$ and $[M_j, t']$ are the folded trees as described above. If $s = s'$ so that $t = t'$ and $\phi(s) = \phi(M_j)$ then the two trees are joined by their roots to give the folded tree for $[s,t]$. If $s \neq s'$ then again join the two trees by their roots to give a tree with root r' representing the folded tree for $[s', t']$ then add an extra segment $[r, r']$ isometric with both $[s, s']$ and $[t, t']$ to give the folded tree for $[s,t]$. This folding is illustrated in Fig 1.

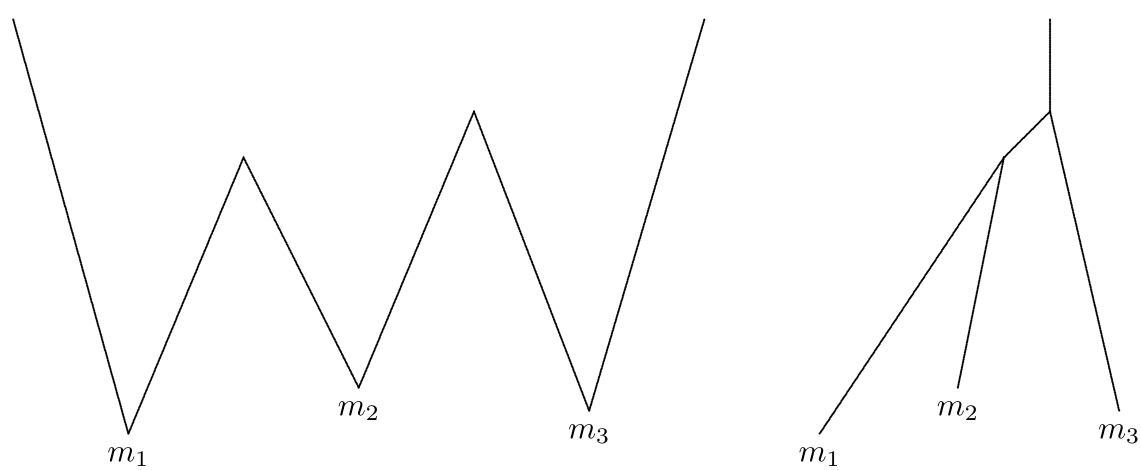

Fig1

Note the relation $\sim$ satisfies the following:-

If $s \sim t$ and $x \sim y$, but $s \not\sim x$, and s, t, x, y are all points of a segment which is ordered as a closed interval then either $[x,y] \subset [s,t]$ or $[x,y] \cap [s,t] = \emptyset$ or $[s,t] \subset [x,y]$.

We now show that an arbitrary segment is folded into a finite tree. We have already proved this if the end points are in the same equivalence class. Let $[s, t]$ be a segment. If $x \not\sim y$ for every distinct x, y of $[s, t[$ then the map $[s, t] \to [s, t]/\sim$ is a bijection. Suppose then that $x \sim y$ for some distinct pair x, y. We can choose x closest to s which has this property, and choose y furthest from x so that it is in the same equivalence class. The property of $\sim$ stated above means that $[x, y]/\sim$ injects into $[s, t]/\sim$ and the only intersection with $[s, t]/\sim$ is at the identified image of x and y. The result follows easily again by induction on the number of minima in the image of $[s, t]$. If $f \in F$ then $fx \sim fy$ if and only if $x \sim y$. It follows that there is an action of F on $T/\sim$. If we allow $\Delta = \infty$, then the action of F on $T/\sim$ is that presented in [B] Example 3. In this case the action is abelian, in the terminology of [C], and F fixes a unique end of $T/\sim$. Note though that the action of F on $T/\sim$ is faithful, i.e. if $f \neq 1$, then there is exists a point $z \in T/\sim$ for which $fz \neq z$. If $\phi(f) \neq 0$, then this is true for any $z \in T/\sim$. If $\phi(f) = 0$, let $\alpha = \alpha(f) = \max\{\phi(u) | u \in [v, fv]\}$. If $\alpha > 0$ then we can take $z = v$. If $\alpha = 0$, then choose $n > 1$ so that $x_1^{-n} f x_1^n$ when written in reduced form has x_1 as its final letter. For this $n, \alpha(x_1^{-n} f x_1^n)) \geq \phi(x_1) > 0$. Thus $v \not\sim x_1^{-n} f x_1^n v$ and so $x_1^n v \not\sim f x_1^n v$, and we can take $z = x_1^n v$.

We now consider the case $n = 2$. Let S be an $\mathbf{R}$-tree on which the group G acts. Suppose G is generated by a, b and that the action of G is faithful, and T is minimal. We assume a, b induce hyperbolic isometries of lengths $\ell(a), \ell(b)$ which are independent over the rationals. We also require that the axes intersect coherently (i.e. if $[d, u]$ is the intersection of the axes, then orienting this segment from d to u is the orientation induced by positive translation by both a and b). Let Δ be the length of the segment $[d, u]$. We have seen that there is a morphism $\phi : T \to \mathbf{R}$ in which $\phi(x_1) = \ell(a), \phi(x_2) = \ell(b)$. There is also a morphism $\theta : T \to S$ in which $\theta(x_1) = a, \theta(x_2) = b$. Clearly there is a homomorphism of groups $\theta : F \to G$ in which $\theta(x_1) = a, \theta(x_2) = b$.. We can get a morphism of trees by mapping v to d, then mapping fv to $\theta(f)d$ and mapping the orbits of the edges e_1, e_2 equivariantly and isometrically.

In this case ($n = 2$) in Fig 1 the subsegments are also initial subsegments of segments corresponding to subwords in which the indices of x_1 and x_2 are all positive or all negative and these are mapped isometrically by θ. This is because there is folding at a subword $a^{-1}b$ at d, i.e. a segment corresponding to the word $x_1^{-1}x_2$ has a minima at v and it folds in the morphism. There can be no folding at subwords aa, bb, ab or ba as, for example, if there is folding at an ab subword, then since there is folding at $a^{-1}b$, which is the same as folding at $b^{-1}a$, and there will be folding at $aa = (ab)(b^{-1}a)$, and folding at aa does not occur as it is mapped to part of a translate of the axis of a.

Thus any folding that occurs in the morphism θ must be at a maxima or minima for the height function ϕ. It follows that the morphism ϕ factors through θ. In particular there is a homomorphism from G to a free abelian rank 2 group. This seems to be of sufficient interest to warrant recording as the conclusion of a theorem.

Theorem 1. *Let S be a G-tree and suppose G is generated by hyperbolic isometries a, b with lengths $\ell(a), \ell(b)$ whose axes intersect in a segment with more than one element. Let T be the simplicial F-tree with two orbits of edges of lengths $\ell(a), \ell(b)$ and one orbit of edges. The morphism $\phi : T \to \mathbf{R}$ in which $\phi(x_1) = \ell(a), \phi(x_2) = \ell(b)$ factors through S. If*

$\ell(a), \ell(b)$ are independent over the rationals, then G made abelian is free abelian of rank two.

Also the morphism $\theta : T \to S$ factors through $T/\sim$, where Δ, the length of the intersection of the axes, is the Δ used to define the relation $\sim$. To prove this we need to show that if $s \sim t, s, t \in T$, then $\theta(s) = \theta(t)$. We refer back to our consideration of $\sim$ and Fig 1. We use induction on k, where $k-1$ is the number of minima for ϕ restricted to $[s, t]$. If $k = 2$, then the two segments correspond to words with positive indices in x_1 and x_2 If one segment was a positive power of x_1 and the other a positive power of x_2, then $\theta(s) = \theta(t)$, since they map to a translate of the shared part of the axis of a and the axis of b. If the first segment, say, is an initial subsegment of a segment corresponding to a product of more than one positive power of x_1 and x_2, then we use induction on the number of different positive powers. One end (the initial part) starts at a translate of v. If the first part of the segment corresponds to a power of x_1 then the first part of the other segment correspond to power of x_2. Suppose the last change in the first segment is from a power of x_1 to a power of x_2 and it occurs at the point z of the segment. Then $z = fv$ for some v. The segment $[z.s]$ in T is the f-translate of $[v, f^{-1}s]$ which is part of the axis of x_1 in T. Its length is less that Δ. Let $[v, f^{-1}s']$ be the segment of the same length, which is part of the axis x_2 in T. Then $\theta(f^{-1}s) = \theta(f^{-1}s')$ and so both $s \sim s'$ and $\theta(s) = \theta(s')$. Thus we can replace s by s' and the first segment corresponds to a word with fewer different positive powers of x_1 and x_2.

If $\Delta < \ell(a) + \ell(b)$ then the action is free and simplicial. This is proved in [C]. We give a proof here. We will show that in this case the F-morphism $T/\sim \to S$ is an isomorphism. This proves Harrison's Theorem (see [C]). Since it shows that if we have two hyperbolic isometries a, b of an $\mathbf{R}$-tree whose axes intersect in a segment of length Δ where $\Delta < \ell(a)+\ell(b)$, then a, b generate a free group of isometries. First consider the case when $\Delta < \min(\ell(a), \ell(b))$. We construct a simplicial tree $\tilde{X}$ with the right properties.

Let X be the graph as shown in Fig 2. Make this into an $\mathbf{R}$-graph by giving α the topology of a segment of length $\ell(a)-\Delta$, β the topology of a segment of length $\ell(b)-\Delta$ and δ the topology of a segment of length Δ. The fundamental group G of X acts on the $\mathbf{R}$-tree which is the universal cover $\tilde{X}$ of X and is generated by two hyperbolic isometries $\tilde{a}, \tilde{b}$ whose axes intersect in a segment of length Δ. Also if we have two hyperbolic isometries a, b of an $\mathbf{R}$-tree S whose axes intersect in a segment of length Δ, and such that S is minimal for the group G generated by a, b, then there will be a morphism $\rho : \tilde{X} \to S$, in which $\rho(\tilde{a}) = a$ and $\rho(\tilde{b}) = b$. It is also clear that there will be no folding in this morphism, i.e. segments are mapped isometrically, and so, since S is minimal, ρ is bijective and G is freely generated by a, b. Also the obvious morphism $T \to \tilde{X}$ factors through $T/\sim$ and the morphism $(T/\sim) \to \tilde{X}$ must be an isometry.

Consider now the case when $\ell(a) + \ell(b) > \Delta \geq \min(\ell(a), \ell(b))$. Assume $\ell(a) > \ell(b)$ so that $\Delta \geq \ell(b)$. In [C] Lemma 3.3.4 it is shown that, in this situation, ab^{-1} is a hyperbolic isometry which meets the axis of b coherently in a segment whose right hand end point is the same as that of the intersection of the axes of a and b. $\ell(ab^{-1}) = \ell(a) - \ell(b)$, and $\Delta(ab^{-1}, b) = \Delta(a, b) - \ell(b)$. Notice, as is pointed out in [D], that $\ell(a) + \ell(b) - \Delta(a, b) = \ell(ab^{-1}) + \ell(b) - \Delta(ab^{-1}, b)$. Conversely if a, ab^{-1} satisfy these derived conditions stated, then a, b will satisfy the original conditions. It follows that by carrying out a sequence of

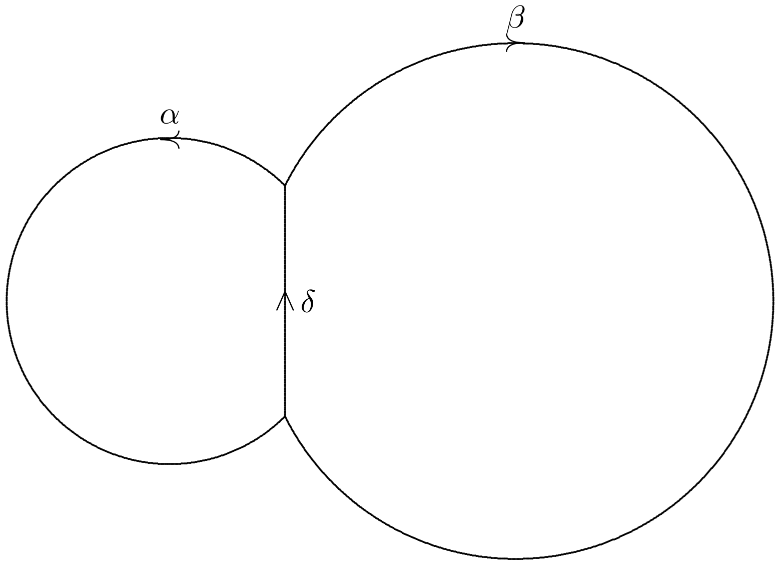

Fig 2

Nielsen operations on the pair a, b one will eventually arrive at a pair of generators a', b' for G for which $\Delta(a', b') < \min(\ell(a'), \ell(b'))$. By the case already discussed, a', b' freely generate G and so a, b freely generate G. It can also be seen from this analysis that the action when $\Delta < \ell(a) + \ell(b)$ is unique, and it is the action on $T/\sim$ described in the first paragraph.

Perhaps the most interesting case for our construction $T/\sim$ occurs when $\phi(x_1) + \phi(x_2) = \Delta$. This is illustrated in Fig 3. The bottom dotted line indicates points at height $\phi(x_1) + \phi(x_2)$. The points intersecting this line on the left of the diagram are identified to give the point p on the right of the diagram. Thus p is fixed by the element $x_1 x_2 x_1^{-1} x_2^{-1}$. Arc stabilizers in $T/\sim$ are trivial. In $T/\sim$, there are infinitely many directions at p lying above p, on which $x_1 x_2 x_1^{-1} x_2^{-1}$ acts transitively. In our diagram $x_1 x_2 x_1^{-1} x_2^{-1} y = x$. (In the diagram $\phi(x_2) < \phi(x_1)$ and the x_1 edges point upwards to the left.) There are also infinitely many directions lying below p , on which $x_1 x_2 x_1^{-1} x_2^{-1}$ acts transitively. If we have two hyperbolic isometries a, b of an $\mathbf{R}$-tree S whose axes intersect in a segment of length $\phi(a) + \phi(b) = \Delta$, and such that S is minimal for the group G generated by a, b, then there will be a morphism $\theta : \tilde{X} \to S$. We will see that this morphism need not be an isometry, but it will be if arc stabilizers in S are trivial. If $\phi(a) + \phi(b) < \Delta$, then there will be a morphism $\theta : \tilde{X} \to S$ and arc stabilizers in both $\tilde{X}$ and S are not small.

If $\phi(x_1) + \phi(x_2) < \Delta$, then we show that any arc in $T/\sim$ has an infinitely generated stabilizer. Note that a direction above p will now be fixed by $x_1 x_2 x_1^{-1} x_2^{-1}$ as x and y are identified if the height above p is less than $\Delta - \phi(x_1) - \phi(x_2)$. But note that we have also shown that by using Nielsen moves on the pair x_1, x_2 we can produce infinitely many hyporbolic elements h for which $\ell(h) < \Delta - \phi(x_1) - \phi(x_2)$. Also the axis of h will share

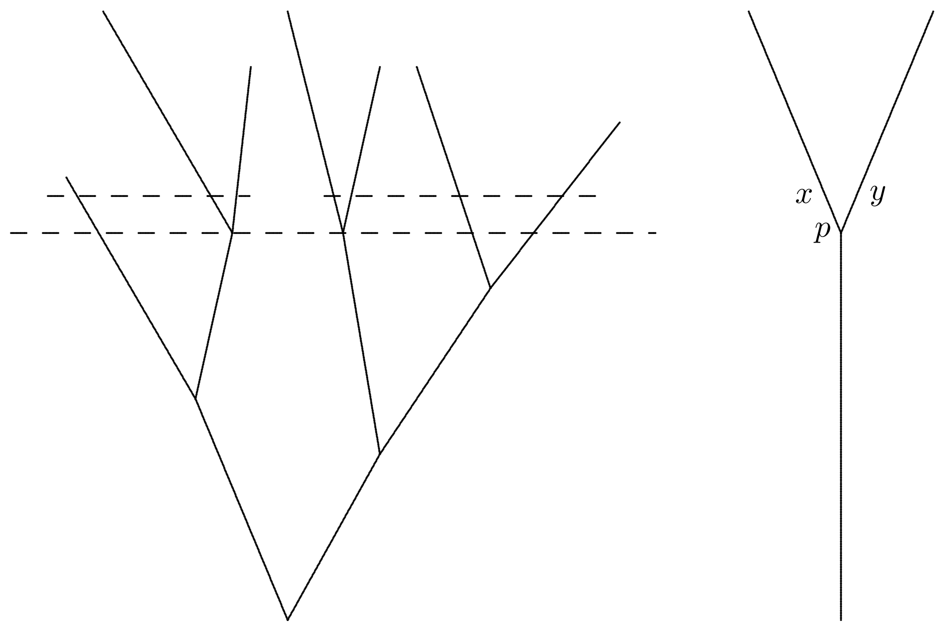

Fig 3

the segment of length $\Delta - \phi(x_1) + \phi(x_2)$ lying above p also lying in the axis of x_1 and x_2. But then $hx_1x_2x_1^{-1}x_2^{-1}h^{-1}$ will also fix the direction above p fixed by $x_1x_2x_1^{-1}x_2^{-1}$.

Recall that we have two hyperbolic isometries a, b of an $\mathbf{R}$-tree S whose axes intersect in a segment of length Δ, and such that S is minimal for the group G generated by a, b. We also assume that the normal subgroup of G consisting of all elements fixing every point of S is trivial, i.e. the natural map of G into the isometry group of S is faithful. If $\ell(a) + \ell(b) < \Delta$, then $c = aba^{-1}b^{-1}$ is an elliptic element and in fact will fix a segment of the axis A_a of a of length $\Delta - \ell(a) - \ell(b)$. Let C be the subgroup of G generated by c Let S' be the subtree of S which is the minimal C-subtree containing A_a. The action of C on S' is simplicial and trivial (i.e. there is a point fixed by all of C) and the orbit space $Y = C\backslash S'$ is homeomorphic to $\mathbf{R}$. Thus we can subdivide Y into finitely many segments, two of which will be non-compact. Two points in the interior of a particular segment correspond to orbits of points with the same stabilizers in C. Bass-Serre theory tells us that the action on S' is specified by this division into segments and the labelling of the segments with the corresponding subgroups of C. There will be one segment labelled by C and moving up (or down) from this segment the labels will give a finite decreasing chain of subgroups of C. A subgroup is conveniently specified by its index in C. Thus one obtains a sequence $1 = \upsilon_0 < \upsilon_1 < \upsilon_2, \ldots < \upsilon_r$ where $\upsilon_1, \upsilon_2, \ldots, \upsilon_{r-1}$ are positive integers and u_r is either a positive integer or ∞ and υ_i divides υ_{i+1} for $i = 1, 2, \ldots, r-1$. We assume that every integer divides ∞. Similarly moving down we have a sequence $1 = \delta_0 < \delta_1 < \delta_2 < \ldots < \delta_s$ with similar properties. If we attach lengths to the compact segments then we get a complete description of the tree S'. This is illustrated in Fig 4.

We choose a point p in the intersection of the two axes. Choose p to be the point with smallest height, fixed by c, unless $s = 0$, in which case no such p exists, in which case we take p to be the point of largest height. Note that p will be at height $\phi(a) + \phi(b)$ above the point of lowest height where the two axes intersect. We assign the segment with index υ_i the length u_i. and the segment with index δ_i the length d_i. Note that by our choice of $d_0 = 0$ unless $s = 0$, in which case $u_0 = 0$ and $d_0 = \infty$.

We show now that there is an F-tree in which F is freely generated by a, b with $C =< c >, c = aba^{-1}b^{-1}$and the C-subtree S' has the structure given by any such set of data.

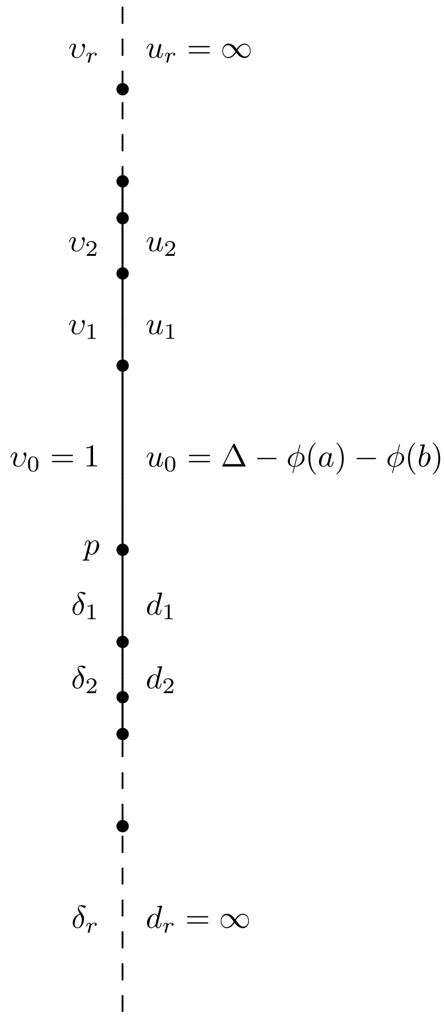

Fig 4

First take the F-tree $T_1 = T/\sim$ where $\sim$ is for $\Delta = \phi(a) + \phi(b)$ Since each $\sim$-equivalence class consists of points with the same values under ϕ, it is clear that we can regard ϕ as defined also on T_1. Also segments in T_1 can be divided into finitely many subsegments - just as in T - in which ϕ is monotone. Consider a point at which ϕ changes from being decreasing to increasing. This point will be a translate r of p, and will be fixed by a conjugate c_1 of c. The two incident segments (on which ϕ is decreasing and then increasing) represent directions at r and there will be a smallest positive power c_1^u which takes one of these directions to the other. We say that the change in direction has index u.

We define a relation $\approx$ on T_1 as follows.

Let $s, t \in T_1$ and let $[s, t]$ be the geodesic joining them. Then $s \approx$ if $\phi(s) = \phi(t)$ and each internal local maxima M_i for which $\phi(M_i) > \phi(s)$ satisfies

if $\phi(M_i) - \phi(s) > u_0 + u_1 + \ldots u_{j-1}$, then the index of M_i is divisible by v_j,

and each internal local minima m_i satisfies if $\phi(s) - \phi(m_i) \geq d_0 + d_1 + \ldots d_j$, then the index of m_i is divisible by δ_j,

In a similar way to that of $\sim$ one can show that $T_2 = T_1/\approx$ is an F-tree. In T_1, consider the axis of x_1. Let s be a point lying height $u_0 + u_1 + \ldots + u_j$ below p. Consider the segment $[s, c^{v_j} s]$. This is a path on which ϕ has a unique maximum at p with index v_j. Clearly $\phi(p) - \phi(s) = u_0 + u_1 + \ldots + u_j > u_0 + u_1 + \ldots + u_{j-1}$ and so the condition is satisfied for j and $s \approx c^{v_j} s$. if $\phi(s') < \phi(s)$, i.e. if s' is a point lying further below p the condition would not be satisfied for $j + 1$ and so $s' \not\approx c^{v_j} s'$. Thus $u_0 + u_1 + \ldots + u_j$ is the length of the largest segment of the x_1-axis fixed by c^{v_j}. It follows easily that the C-subtree S' of T_2 has the required structure.

One can use a similar argument to that for T_1 to show that if S is any G-tree in which the C subtree S' has a structure with a given set of data then the morphism $\theta_1 : T_1 \to S$ factors through T_2 where the data for $\approx$ is that determined by S'. But the morphism $T_2 \to S$ must be an isometry. This is because if it is not injective then there must be folding in some "up-down" path of T_2 which would only be achieved by extra folding than that given by $\approx$ in T_1. But we will show that this would give different data for the subtree S'. Let $\pi : T_1 \to T_2$ be the natural morphism. We know that there is a morphism $\phi : T_2 \to \mathbf{R}$ which factors through S. Consider an "up-down" path in T_2, i.e. a geodesic $[s, t]$ split up into subsegments on which ϕ is monotone. Consider a local maxima M. Either this comes from a local maxima in T_1 or there are points $s', t' \in T_1$ such that $\pi(s') = \pi(t') = M$ and there are small subsegments $[s''.s'], [t', t'']$ of $[s, t]$ which are mapped by π to segments of T_2 which intersect only in M. Since the condition for $\approx$ is satisfied by s', t' but for no other pair chosen one from each of these segments, there must be a local maxima $M_i \in [s', t']$ with index divisible by v_j, but not by v_{j+1}, for which $\phi(M_i) - \phi(s') = u_0 + u_1 + \ldots + u_j$. It is conceivable that this happens for two different $M_1's$ for the same local maxima M. Thus for every maxima M we can associate at least one index v_j. Similarly we can associate at least one index δ_j with each local minima of $[s, t]$. If the morphism $T_2 \to S$ is not injective then there will be folding for some segment $[s, t]$ at some local maxima or minima. But if this is a maxima with associated index v_j then the value of u_j can be increased in the relation $\approx$ and we will still get an $\mathbf{R}$-tree $T_1/\approx$ through which $\theta : T_1 \to S$ factors. When we consider the restriction to the respective C-subtrees we see that this cannot happen.

Thus the morphism $\theta' : T_2' \to S$ is injective. Since S is minimal with respect to the action of G which is generated by a, b, it follows that θ' is an isometry. Also G is isomorphic to F/N where N consists of all elements that fix every element of T'. If we examine the action of C on S we see that N will contain c^m where m is the least common multiple of the labels - provided they are finite - of the non-compact segments of Y. If either of these labels is ∞ then N is trivial. To see this recall that we have shown that if S' has no compact segments, i.e. it consists of two non-compact segments, one of which is stabilized by c and the other has trivial stabilizer, then the action of F on $T/\sim$ is faithful. If the data for T' has either non-compact segment with trivial stabilizer the the morphism $T \to T'$ factors through $T/\sim$ and so the action on T' is faithful. Otherwise N contains c^m where m is the least common multiple of the labels m_u, M_d of the non-compact segments of Y. The action of F on the upper ends of the translates of the axis of a is that of F on the cosets of $< c^{m_u} >$. Similarly the action on F on the lower ends is that of F on the cosets of $< c^{m_d} >$. It follows that if N' is the normal closure of c^m, then F/N' acts faithfully on T and so $N = N'$. Notice that even after factoring out N to make the action on S faithful, arc stabilizers will not be small. This is because the only small subgroups of a Fuchsian group are cyclic.

We summarize our results.

Theorem 2. *Let G be a group generated by elements a, b and let T be an $\mathbf{R}$-tree on which G acts faithfully by isometries. Suppose a, b are hyperbolic with lengths $\ell(a), \ell(b)$ respectively which are independent over the rationals. Suppose the axes of a and b meet coherently in a non-empty segment of length Δ.*

(i) If $\Delta < \ell(a) + \ell(b)$ then G is freely generated by a and b and the action is uniquely determined by the values $\Delta, \ell(a)$ and $\ell(b)$.

(ii) If $\Delta \geq \ell(a) + \ell(b)$ then either G is freely generated by a, b or

$$G =< a, b | (aba^{-1}b^{-1})^n = 1 >$$

for some $n = 1, 2, \ldots$. If $n = 1$ then G is free abelian and the action is by translation on the real line $\mathbf{R}$. In every case $c = aba^{-1}b^{-1}$ is an elliptic element. If arc stabilizers are small, then in fact they are trivial : if G is free or if $n > 1$, then $\Delta = \ell(a) + \ell(b)$ and there is only one such action for each such G and for each pair of values of $\ell(a)$ and $\ell(b)$.

(iii) There are non-small actions for any value of $\Delta > \ell(a)+\ell(b)$. Let $C =< c >$, then such an action is determined by the structure of the minimal C-subtree of T containing the axis of a. This subtree is simplicial and can be specified by a finite set of data.

Triples of isometries

Now we examine the case $n \geq 3$. In fact we consider the case $n = 3$. This can be generalized to $n \geq 3$ fairly easily. Thus F is freely generated by x_1, x_2, x_3. For small values of Δ, $T/\sim$ will be simplicial. For large values it will not be simplicial, and so a natural question is for what value of Δ does the transition takes place.

Lemma 1. *Let S be an $\mathbf{R}$-tree, and let a, b, c be hyperbolic isometries with axes A, B, C and lengths $\ell(a) > \ell(b) > \ell(c)$. Suppose $A \cap B = A \cap C = B \cap C$ be a segment $[d, u]$ of positive length $\Delta \geq \ell(c)$ and if we orient this segment from d to u, this is the positive translation direction for all three isometries, i.e. they meet coherently. Then $a' = ac^{-1}, b' = bc^{-1}, c' = c$ are also hyperbolic isometries meeting coherently. If A', B', C'*

are the corresponding axes, then $A' \cap B' = A' \cap C' = B' \cap C'$ is a segment $[d', u]$ of length $\Delta' = \Delta - \ell(c)$. Also $\ell(a') = \ell(a) - \ell(c), \ell(b') = \ell(b) - \ell(c), \ell(c') = \ell(c)$ so that $\ell(a') + \ell(b') + \ell(c') - 2\Delta' = \ell(a) + \ell(b) + \ell(c) - 2\Delta$.

Proof. this follows easily from [C] Lemma 3.3.4.

The above lemma suggests a process of repeatedly carrying out Nielsen moves so that the hyerbolic lengths of the triple of isometries and Δ are reduced. What happens to this data is given in the following Lemma.

Lemma 2 . *Let $\mathbf{s}_i = (x_i, y_i, z_i, d_i), i = 1, 2, \ldots$ be a sequence of 4-tuples of real numbers defined inductively as follows:-*

(i) x_1, y_1, z_1, d_1 are positive real numbers with $x_1 > y_1 > z_1$and x_1, y_1, z_1 are independent over the rationals.

(ii) The 3-tuple $(x_{i+1}, y_{i+1}, z_{i+1})$ consists of the real numbers $x_i - z_i, y_i - z_i, z_i$ arranged in decreasing order, and $d_{i+1} = d_i - z_i$.

Then the sequence $\mathbf{s}_i$ tends to a limit $\mathbf{s} = (x, 0, 0, d)$, in which $x \geq 0$. Also $x > 0$ if and only if for some j and every $i \geq j, x_i > y_i + z_i$. If this happens, then for each such i, $x = x_i - y_i - z_i$ and $d = d_i - y_i - z_i$.

If $x = 0$ then $2d = 2d_i - (x_i + y_i + z_i)$ for every i

The proof is left as an exercise.

It follows from Lemma 1 that if we have a triple of isometries (a, b, c) satisfying the hypotheses of the Lemma 1, and for which the lengths are independent over the rationals, then we can use Nielsen moves to get a new triple of isometries (ac^{-1}, bc^{-1}, c), satisfying the same hypotheses, except we may now have $\Delta < \ell(c)$. Of course we rearrange the triple so that lengths are decreasing.

The lengths of two isometries of the triple and Δ are reduced by the same amount. Thus the data $(x_1 = \ell(a), y_1 = \ell(b), z_1 = \ell(b), d_1 = \Delta)$ is changed to (x_2, y_2, z_2, d_2) and this will be the data for the situation of Lemma 1 unless $d_2 < z_2$. If we keep repeating this process then we obtain a sequence of data as in Lemma 2 We will investigate when we end up with $d_k < z_k$. From Lemma 2 this will happen if for some $i, x_i > y_i + z_i > d_i$ or if there is no such i and $2\Delta = 2d_1 < (x_1 + y_1 + z_1)$. If $\Delta < \ell(c)$, then we can adapt the argument for pairs of isometries to show that in this case a, b, c are free generators of a subgroup G of the group of isometries of S. Also G acts simplicially on a subtree of S which is uniquely specified by the data $\ell(a), \ell(b), \ell(c), \Delta$, and it is the tree $T/\sim$. One argues using a space X, as in Fig 2, with an extra 1-cell γ joining the same pair of points as the other 1-cells.

We show in the following example that there are triples of values for $x_1 = \ell(a), y_1 = \ell(b), z_1 = \ell(c)$ for which the sequence never give a 3-tuple in which $x_i > y_i + z_i$.

Example. Let $x_1 = 1 - a + a^2, y_1 = \alpha, z_1 = 1$, in which α is the real root of $x^3 - 2x^2 + 2x - 2$. $x_2 = 1, y_2 = \alpha^2 - a, z_2 = \alpha - 1$. Let $\mathbf{s'}_i$ be the 3-tuple which is the first three terms of $\mathbf{s}_i$, then $(1, \alpha^2 - \alpha, \alpha - 1) = (\alpha - 1)(1 - \alpha + \alpha^2), \alpha, 1)$, and $\mathbf{s'}_2 = (\alpha - 1)\mathbf{s'}_1$ As in the original triple $x_1 < y_1 + z_1$ this will be the case at any stage in the sequence of moves, since $\mathbf{s'}_i = (\alpha - 1)^{i-1}\mathbf{s'}_1$. We know from the Lemma that $2d = 2\Delta - (\ell(a) + \ell(b) + \ell(c) = 2d_i - (x_i + y_i + x_i)$. If $d < 0$ then eventually $z_n > d_n$ and the action is simplicial. If $d = 0$, i.e. if $\Delta = \frac{1}{2}(2 + \alpha^2)$, then one has a non-simplicial action, which seems to be free and a

Levitt type action. This contrasts with the case $n = 2$, when the commutator $aba^{-1}b^{-1}$ was elliptic.

The action given by this example is constructed in a different way in [L] p.661.

If $\Delta > \frac{1}{2}(2+\alpha^2)$, then the action is not free. To see this note that $d > 0$ and for large values of $i, d_i > x_i + y_i$. It follows from our discussion for $n = 2$ that stabilizers are not small.

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