# ENDS AND ACCESSIBILITY

M.J. DUNWOODY

Abstract. An account is given of how I became involved with Stallings' Theorem on the ends of groups and subsequent developments relating to Wall's Conjecture on the accessibility of finitely generated groups.

## 1. Introduction

Stallings' Theorem is a major result in infinite group theory. As corollaries, two well knwn conjectures were resolved : Serre's conjecure that a torsion-free group with a free subgroup of finite index is itself free and the Eilenberg-Ganea Conjecture that a group of cohmological dimension one is free. Stallings' result is contained in two publications in 1969 and 1971. At that time most (but not all) group theorists favoured a very algebraic approach to group theory. Proofs usually involved studying the subgroup structure of the group and establishing normal forms for the elements of the group. Stallings instead obtained his exciting new results by studying the action of the group on a topological object, either a graph or a 2-dimensional simplicial complex. This started what might be seen as a paradigm shift in infinite group theory that was reinforced by the emergence of the Bass-Serre theory of groups acting on trees very soon afterwards.

I am proud of the part I played in the work relating to Stallings' Theorem from 1967 onwards. Here I give a personal perspective on the developments from 1967 to 1991 when I managed to construct a finiitely generated inaccessible group, indicating that I had been wasting rather a lot of research time trying to show that no such groups existed. Terry Wall had conjectured in 1969 that there were were no such groups. and I firmly believed that this was the case up to 1990.

## 2. Stallings and his theorem

John Stallings was Professor at Berkely from 1967 to his death at age 73 in 2005. He was very well liked by his students and fellow mathematicians. I attended a few conferences at which he gave talks. I did not find him a very good speaker. I was somewhat in awe of him as I very much admired his work on the Ends Theorem, and though I would have liked to have developed an easy relationship with him, this never happened. Several of his students and former colleagues have contributed to "Remembering John Stallings" [4]. The contribution by Peter Shalen rang very true. I include part of it. I am afraid some of this - the shyness bit - could be applied to me, though I did not usually try to have a beer before a talk.

Peter Shalen writes:- John was shy and had few really close friends. I never succeeded in getting close to him personally and always had to settle for admiring him from a certain distance. We never had a conversation that lasted for more than a few minutes. I think he was especially shy about talking mathematics. When I brought up connections between his work and mine, he certainly said kind things about my work but never seemed eager to pursue the discussion. Still, his low-key humour always came through even in a brief chat. The last time I saw him was at a conference, possibly at the University of Arkansas, his alma mater. I saw him when I was on the way back from lunch and started walking with him to try to strike up a conversation. He was walking pretty fast and explained that he was trying to get a beer before the afternoon talks.

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*Key words and phrases.* Structure trees, tree decompositions, group splittings.

First we need some definitions before discussing the first Stallings paper which was published in the Annals of Mathematics in 1969.

We follow the notation and terminlology of [14]. Let $X$ be a connected graph. Let $G$ be a group which has a left action on $X$, so that if $e$ is an edge with vertices $u, v$, and $g \in G$, then $ge$ is an edge with vertices $gu$ and $gv$. It is not usually required that $X$ is locally finite, i.e. that each vertex is incident with only finitely many edges. However this is the case if $X$ is the Cayley graph of $G$ with respect to a finite generating set $S$, i.e. the graph $X = X(G, S)$ with vertex set $VX = G$ and edge set $E = \{(g, gs)|g \in G, s \in S\}$.

A subset $A \subseteq VX$ is a cut if $\delta A = \{e \in EX, u \in A, v \in A^* = VX \setminus A\}$ is finite. The set $BX$ of all cuts is an algebra (closed under intersection, union, taking complemements). If $X$ admits a $G$-action, then so does $BX$.

A ray in $X$ is an infinite sequence $x_1, x_2 \dots$ of distinct vertices, where, for each $i$, $x_i, x_{i+1}$ are the vertices of an edge. We say that two rays $R_1, R_2$ are separated by a cut $A$ if $R_1$ is eventually in $A$ and $R_2$ iis eventually in $A^* = VX \setminus A$ or vice versa.

For rays $R.R'$ we write $R \sim R'$ if $R, R'$ cannot be separated by any cut. The relation $\sim$ is an equivalence relation. The ends of $X$ are the equivalence classes. If $X$ is locally finite, then it has more than one end if and only if there is a cut $A$ for which both $A$ and $A^*$ are infinite. More generally one can say that $X$ has more than one end if there is a cut $A$ for which both $A$ and $A^*$ are infinite.

For a finitely generated group $G$ the number of ends $e(G)$ is defined to be the number of ends of a Cayley graph with respect to a finite generating set $S$. This does not depend on $S$.

Stallings, in [23], proved

**0.1)** *If $G$ is a torsion-free, finitely presented group with infinitely many ends, then $G$ is a non-trivial free product.*

In his 1971 monograph Stallings obtained the stronger result

**0.2)** *A finitely generated group with more than one end is either virtually infinite cyclic or is a non-trivial free product $G = A *_C B$ with $C$ finite or $G$ is an HNN=group $G = A*_C$ where $C$ is finite.*

With the benefit of Bass-Serre theory, not available at that time, but shortly afterwards, this somewhat awkward statement becomes the followiing:-

**0.2)** *A finitely generated group with more than one end has a non-trivial action on a tree with finite edge stabilizers.*

An action on the tree $T$ is *trivial* if there is a vertex or a mid-point of an edge that is fixed by all of $G$.

In the autumn of 1967 I heard that Stallings had an important new result that provided a proof that a finitely generated group of codimension one (over $\mathbb{Z}$) was free, and that Peter Neumann had a copy. This sounded very interesting as I had thought about this a bit and come to the conclusion that it was completely intractible. I wrote to Peter and asked if he could send me a copy, which he did. I had been looking for a new direction in which to take my research and I decided very quickly that this was the way I wanted to go. There had been a big expansion of universities in the U.K. in the 60's. As a result of which I had a permanent job at Sussex, one of the new universities, and so had the luxury of being able to begin a new research topic that interested me, where it might not yield very many publications. During the 70's and 80's the situation was completely different. There were hardly any job vacancies and the pressure to publish greatly increased.

I found Stallings' paper very exciting but quite hard to follow due to my lack of geometric background. I had done undergraduate courses on topology at Manchester which included some general topology and the fundamental group of a space, At the ANU I had read a lot of group theory, and not much else. At the time most group theorists felt that anything important in group theory should have a purely algebraic proof. Of course there were exceptions to this, for example people working on Fuchsian or Kleinian groups. Thus in 1967 I had not studied 2-manifolds or covering spaces. Fortunately for me, Sussex had just appointed Roger Fenn, who had a first class geometric intuition, and he was very helpful.

It took me a couple of months to get a good understanding of Stallings' paper. I made a bit of a breakthrough when I realised that I could use the Cayley graph of the group rather than the universal cover of a 2-complex with the right fundamental group. Then after a few more months I managed to use a graph theoretic argument to show that Stallings' Theorem worked for finitely generated groups and not just almost finitely presented ones. I was very excited by this and wrote it up quite quickly and submitted it to Sandy Green for the Journal of Algebra. I also sent a copy to Stallings. Stallings replied as shown below.

July 17, 1968

Mr. M. J. Dunwoody
The University of Sussex
School of Mathematical and Physical Sciences
Falmer, Brighton, Sussex, England

Dear Mr. Dunwoody:

Thank you for the copy of your paper "Ends of Finitely Generated Groups." The conjecture you resolve has also been solved by George Bergman of the University of California, Berkeley, and by Danny Cohen of the University of London. Bergman's paper is due to come out at the same time as mine in the Annals of Mathematics.

Both Bergman and Cohen manage to define a "complexity" for the set P (page 9 of your paper); this complexity has values in a well-ordered set (in Bergman's version this is the main difficulty); then if you choose P to have minimal complexity, it is provable that one of $E \cap g E$, etc., must be finite (the difficulty in Cohen's version is here).

It appears, from scanning your paper, that your idea is rather different in that you prove that a naively minimal P works, by using the idea of a Schreier system mod the subgroup generated by g. I feel that your idea is the best one, although I have not digested it yet.

If you have your paper published, perhaps you should refer to Specker's paper for Theorem 1, and to Hopf's paper for Theorem 2.

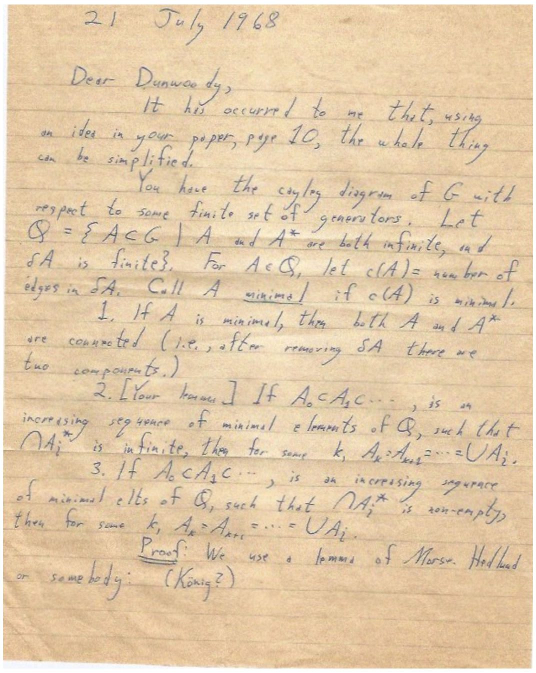
I am writing a paper to take care of groups which are not assumed to be torsion-free, but I have been stuck at the point where I have to use Bergman's or Cohen's idea; and if I understand your idea and it seems neater, perhaps I shall use it instead, with due credit to you.

Sincerely yours,

John R. Stallings

John R. Stallings

I was a bit downcast as I was hoping to be the first with my result and did not really take in Stallings comment that my approach was the best of the three solutions. A few days later, I got the handwritten letter (shown below, followed by a transcript) from Stallings, containing a much nicer proof of the finitely generated, torsion free case. This left me unable to decide what to do. I wrote to Sandy Green, who I knew as an undergraduate and as a colleague at Sussex, before he moved to Warwick. He suggested I rewrite my paper using Stallings' argument, acknowledging that half of it it was Stallings' argument. This is what I did, and my paper [6] eventually appeared in the Journal of Algebra.



21 July 1968

Dear Dunwoody,

It has occurred to me that, using an idea in your paper, page 10, the whole thing can be simplified.

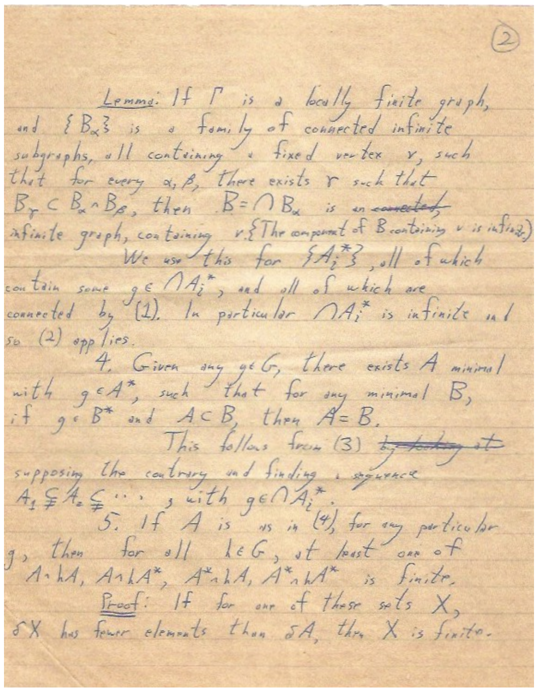
You have the cayley diagram of $G$ with respect to some finite set of generators. Let $\mathcal{Q} = \{A \subset G \mid A$ and $A^*$ are both infinite, and $\delta A$ is finite$\}$. For $A \in \mathcal{Q}$, let $c(A) =$ number of edges in $\delta A$. Call $A$ minimal if $c(A)$ is minimal.

1. If $A$ is minimal, then both $A$ and $A^*$ are connected (i.e., after removing $\delta A$ there are two components.)

2. [Your lemma] If $A_0 \subset A_1 \subset \cdots$, is an increasing sequence of minimal elements of $\mathcal{Q}$, such that $\bigcap A_i^*$ is infinite, then for some $k$, $A_k = A_{k+1} = \cdots = \bigcup A_i$.

3. If $A_0 \subset A_1 \subset \cdots$, is an increasing sequence of minimal elts of $\mathcal{Q}$, such that $\bigcap A_i^*$ is non-empty, then for some $k$, $A_k = A_{k+1} = \cdots = \bigcup A_i$.

Proof: We use a lemma of Morse-Hedlund or somebody: (König?)



(2)

Lemma: If $\Gamma$ is a locally finite graph, and $\{B_\alpha\}$ is a family of connected infinite subgraphs, all containing a fixed vertex $v$, such that for every $\alpha, \beta$, there exists $\gamma$ such that $B_\gamma \subset B_\alpha \cap B_\beta$, then $B = \bigcap B_\alpha$ is an ~~connected,~~ infinite graph, containing $v$. {The component of $B$ containing $v$ is infinite.)

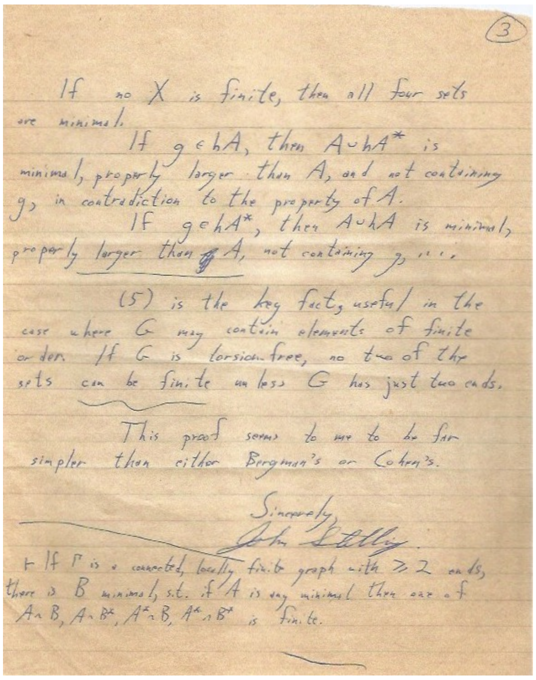
We use this for $\{A_i^*\}$, all of which contain some $g \in \bigcap A_i^*$, and all of which are connected by (1). In particular $\bigcap A_i^*$ is infinite and so (2) applies.

4. Given any $g \in G$, there exists $A$ minimal with $g \in A^*$, such that for any minimal $B$, if $g \in B^*$ and $A \subset B$, then $A = B$.

This follows from (3) ~~by looking at~~ supposing the contrary and finding a sequence $A_1 \subsetneqq A_2 \subsetneqq \cdots$, with $g \in \bigcap A_i^*$.

5. If $A$ is as in (4), for any particular $g$, then for all $h \in G$, at least one of $A \cap hA$, $A \cap hA^*$, $A^* \cap hA$, $A^* \cap hA^*$ is finite.

Proof: If for one of these sets $X$, $\delta X$ has fewer elements than $\delta A$, then $X$ is finite.



(3)

If no $X$ is finite, then all four sets are minimal.

If $g \in hA$, then $A \cup hA^*$ is minimal, properly larger than $A$, and not containing $g$, in contradiction to the property of $A$.

If $g \in hA^*$, then $A \cup hA$ is minimal, properly larger than $A$, not containing $g$, ...

(5) is the key fact, useful in the case where $G$ may contain elements of finite order. If $G$ is torsion-free, no two of the sets can be finite unless $G$ has just two ends.

This proof seems to me to be far simpler than either Bergman's or Cohn's.

Sincerely,
John Stallings

$\vdash$ If $\Gamma$ is a connected, locally finite graph with $\geqslant 2$ ends, there is $B$ minimal, s.t. if $A$ is any minimal then one of $A \cap B$, $A \cap B^*$, $A^* \cap B$, $A^* \cap B^*$ is finite.

21st July 1968

Dear Dunwoody,

It has occurred to me that, using an idea in your paper, page 10, the whole thing can be simplified.

You have the cayley diagram of $G$ with respect to some finite set of generators.

Let $Q = \{A \subset G | \ A$ and $A^*$ are both infinite, and $\delta A$ is finite$\}$. For $A \in Q$ let $c(A) =$ number of edges in $\delta A$.. Call $A$ minimal if $c(A)$ is minimal .

*1. If $A$ is minimal, then both $A$ and $A^*$ are connected (i.e. after removing $\delta A$ there are two components.)*

*2. [Your lemma] If $A_0 \subset A_1 \subset \dots,$ is an increasing. sequence of minimal elements of $Q$ such that $\bigcap A_i^*$ is infinite, then for some $k,, A_k = A_{k+1} = \dots = \bigcup A_i$.*

*3. If If $A_0 \subset A_1 \subset \dots,$ is an increasing. sequence of minimal elements of $Q$ such that $\bigcap A_i^*$ is non-empty, then for some $k,, A_k = A_{k+1} = \dots = \bigcup A_i$.*

Proof We use a Lemma of Morse, Hedlund or somebody (Konig ?).

Lemma *If $\Gamma$ is a locally finite graph and $\{B_\alpha\}$ is a family of connected infinite subgraphs, all containing a fixed vertex $v$, such that for every $\alpha, \beta$, there exists $\gamma$ such that $B_\gamma \subset B_\alpha \cap B_\beta$, then $B = \bigcap B_\alpha$ is an infinite graph, containing $v$. (The component of $B$ containing $v$ is infinite.)*

We use this for $\{A_i^*\}$, all of which contain some $g \in \bigcap A_i^*$, and all of which are connected by (1). In particular $\bigcap A_i^*$ is infinite and so (2) applies.

4. Given any $g \in G$, there exists $A$ minimal with $g \in A^*$, such that for any minimal $B$, if $g \in B^*$ and $A \subset B$, then $A = B$.

This follows from (3) supposing the contrary and finding a sequence of distinct minimal $A_i$'s $A_1 \subset A_2 \subset \dots,$ with $g \in \bigcap A_i^*$.

5. If $A$ is as in (4), for any particular $g$, then for all $h \in G$ , at least one of $A \cap hA, A \cap hA^*, \mathcal{A}^* \cap A, A^* \cap hA^*$ is finite.

Proof: If for one of these sets $X$, $\delta X$ has fewer elements than $\delta A$, then $X$ is finite. If no $X$ is finite, then all four sets are minimal.

If $g \in hA,$, then $A \cup hA^*$ is minimal, properly larger than $A$, and not containing $g$, in contradiction to the property of $A$.

If $g \in hA^*$, then $A \cup hA$ is minimal, properly larger than $A$, not containing $g$, $\dots$.

(5) is the key fact useful in the case where $G$ may contain elements of finite order. If $G$ is torsion-free, no two of the sets can be finite unless $G$ has just two ends.

This proof seems to me to be far simpler then either Bergman's or Cohen's.

Sincerely,

John Stallings

If $\Gamma$ is a locally finite graph with 2 ends, then there is $B$ minimal, such that if $A$ is any minimal, then one of $A \cap B, A \cap B^*, A^* \cap B, A^* \cap B^*$is finite

I did not write to Stallings again about his theorem. I wish I had. When his monograph containing the proof for all finitely generated groups appeared, Stallings did not mention my contribution, which was disappointing. He must have seen my paper and thought that that registered my contribution. That is reasonable, but it would have been kind to put my name in his monograph.

## 3. Accessibility

Stallings' Theorem says that certain groups "factorise" in a certain way. It is a natural question to ask if every finitely generated group has a complete factorisation in which the factors are finite or one-ended, just as any positive integer other than 0 or 1 has a unique factorisation as a product of positive prime numbers. A finitely generated group with this property is said to be accessible. I eventually showed this did not happen, with a counter-example twenty-four years later. In 1968 I was convinced that there were no such counter-examples. Terry Wall reinforced my belief by including the statement as a conjecture in his 1969 paper [26]. The approach I tried for a number of years was to show that the algebra $B\Gamma$ of cuts for the Cayley graph was finitely generated as a $G$-module, and that this would mean that $G$ was accessible. In fact the statements

(i) *The finitely generated group $G$ is accessible.*

(ii) *The algebra $B\Gamma$ is finitely generated as a $G$-module*

are equivalent, as I eventually managed to show in my 1979 paper, which is probably my best paper. Before that I got some partial results, one with my student Carl Bamford [1], but nothing substantial until a train journey in 1976.

As we saw above, the statement of Stallings' Theorem is simplified by using Bass Serre theory, which I first heard about at a wonderful plenary address by Serre at a very well attended British Mathematical Colloquium (BMC) in York in 1970.

When this became clear, it was also clear that there ought to be a proof that directly constructed the tree rather than going via normal form arguments. At the time there were quite a lot of infinite group theorists who preferred using normal forms rather than a Bass-Serre tree. I could never understand that. I thought finding such a tree in this case should be easy and had some vague idea how to do it. I should have written down a careful construction much earlier than I did.

I gave a seminar in Liverpool in 1976. In it I said that one could construct a tree. Peter Scott asked how it could be done and I gave a hand wavy reply. On the train back to Brighton I realised what I had said was wrong and what was the right way to construct the tree and that this would have some exciting new applications. R.G.Swan [25] had extended to torsion-free groups Stallings' result that finitely generated groups of cohomological dimension one are free. In Stallings result the coefficient ring is the integers. For Swan it was any commutative ring. I was able to classify all groups of cohomological dimension one over any commutative ring of coefficiients.

Danny Cohen liked my tree construction a lot. As Danny said, it was clear what the edges of the tree had to be. The problem was the vertices.

In the proof of Stallings' Theorem one shows that the algebra of cuts contains an almost nested $G$-set of cuts.

Two cuts $C, D$ are nested if one of $C \cap D, C \cap D^*, C^* \cap D, C^* \cap D^*$ is empty, or alternatively if one of $C \subseteq D^*, C \subseteq D, C^* \subseteq D, C^* \subseteq D^*$ holds. Thus $C, D$ are nested if one of the regions of the standard Venn diagram is empty. Two infinite cuts are almost nested if one of the four regions is finite.

A nested set of cuts has an obvious order relation, namely inclusion $C \subset D$. This order carries over to almost nested sets. Thus we have an order:- $C \subset^a D$ if $C$ is almost contained in $D$, i.e. if $C \cap D^*$ is finite. The cuts $C, D$ are said to be almost equal if both $C \subset^a D$ and $D \subset^a C$, in which case $C$ and $D$ differ by a finite set of vertices and we write $C =^a D$. The relation $=^a$ is an equivalence relation. The directed edges of the tree are the equivalence classes of cuts. If there is an upper bound on $|\delta A|$ for a set $J$ of almost nested cuts then $J$ satisfies a finite interval condition,

If $u, v \in VX$, then the set of cuts $\{C \in J | u \in C, v \in C^*\}$ is finite.

We now have a set, with an order as above, which we would like to be the directed edge set of a tree. To get the vertices we proceed as follows.

Each edge $e$ has an arrow pointing from $u = \iota e$ to $v = \tau e$. If you have a finite interval condition then there is an equivalence relation on $J$ in which $e \cong f$ if $e^* < f$, and for no $g \in J$ is $e^* < g < f$. The equivalence classes for this relation are the vertices of the tree. If $v$ is an equivalence class then the edges in the class are those for which $\iota e = v$.

I submitted my paper to the Proceedings of the London Math. Soc. hoping it would win one of their prizes, which would have helped me get promoted at Sussex. I did not win a prize but did get promoted from lecturer to reader fairly soon after..

My 1979 paper shows that for a finitely generated group $G$, $BG$ has an almost nested $G$-set of generators. Reflecting on this result, it seemed to me that the proof did not use all the properties of a Cayley graph, namely it did not need a free action, nor that the graph be locally finite. I was able to show that for any connected graph $X$, $BX$ has a nested set of generators, invariant under the action of a group $G$ acting on the graph. This full result did not appear in print until 1989 as part of Chapter II of [5]. What happened was, I submitted it to Inventiones and it got rejected. Looking back on it I can see this was reasonable. The area of research is not really that of the journal and I did not write it up at all well. I did get a partial result published however, as explained below.

In early 1981 I gave a seminar in Oxford (I think in their algebra series). I talked, I think, about this result and almost invariant sets. Curiously it got very little reaction at the time. However shortly afterwards I had a letter from Peter Cameron. He explained that he had been at the talk and had been thinking (at the seminar) about a problem on distance transitive graphs that his student Dugald Macpherson was working on and had realised that what I had been talking about might be useful. I find this a very unlikely coincidence, but it happened. He encouraged me to submit a paper on my result to Combinatorica which was very new. I should have tidied up my paper and submitted that. Instead I submitted a short paper in which I showed that if $X$ had more than one end, then there was a cut $C$ such that $C$ and $C^*$ are both infinite and for which $C$ and $gC$ are nested for every $g \in G$. As noted earlier, the set $\{gC, gC^* | g \in G\}$ is the edge set of a tree. The action of $G$ is non-trivial unless there is a vertex fixed by $G$, in which case the tree is starlike with diameter 2. Thus we have a significant generalisation of Stallings' Theorem, which has now become a result in graph theory.

Macpherson was able to use my result to classify infinite, locally finite, distance transitive graphs.

In the 70's I had a rather low opinion of graph theory and I was not alone in thinking that. For me this view came from hearing talks on graph theory at the BMC and at other meetings and realising the problems discussed were of no great interest even to other graph theorists. Unfortunately for graph theorists, their subject is very easy for other mathematicians to understand, so that when they give a talk they cannot shelter behind little understood technicalities. As I subsequently discovered there are some excellent mathematicians, like Thomassen and Woess, that work in graph theory.

But Stallings' Theorem is a deep result and in its most general form is a result in infinite graph theory. In Stallings' first paper, his main tool was 2-complexes and the result he obtains is for almost finitely presented groups. He obtains a stronger result by considering a graph, the Cayley graph of a finitely generated group.

It turns out, then, that for any connected graph $X$, the algebra $BX$ of cuts has a nested set $\mathcal{E}$ of generators. In fact, much later on, probably due to a German postgraduate Armin Weiss, it emerged that this set is uniquely determined (canonical) so that $\mathcal{E}$ admits any group acting on $X$. If $\mathcal{E}$ satisfies the finite interval condition, then there is a uniquely determined tree $T$ with edge set $\mathcal{E}$. However there are inaccessible graphs which do not satisfy the finite interval condition. In this case there will be a uniquely determined infinite sequence of trees.

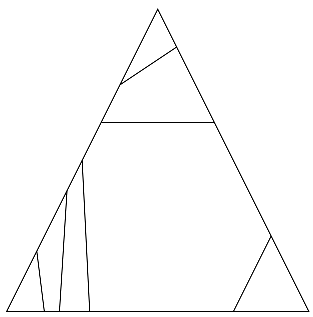
It does seem rather extraordinary to me that for such a general object as a graph, that there should be such a precise description of the generators of the algebra of cuts, in what became known as the theory of structure trees.

The key event that led me to prove that finitely presented groups are accessible was reading Hempel's book on 3-manifolds. I find 3-manifolds a very attractive theory, with its connections with big problems like the Poincaré Conjecture. In [22] Stallings says that he made his breakthrough by thinking about the Sphere Theorem of Papakyriakopoulos [20]. Many of the main results in our story here are algebraic analogues of theorems on 3-manifolds. Stallings' Theorem could be called the Algebraic Sphere Theorem. He gives a new proof in [23] of the Sphere Theorem using his new approach. Stallings was also thinking about the Poincaré Conjecture when he proved his theorem, which he suggests might lead to a new approach. Hempel's book was published in 1976. I started reading it but got stuck on the section in the first chapter on general position and gave up, thinking the rest of the book would be like that. Fortunately I resumed my study of Hempel's book, though after a gap I cannot remember how long, probably a couple of years. I found the later chapters much more readable. The result discussed in Chapter III, that I found of particular interest, is Kneser's result that there is a bound on the number $n$ of disjoint 2-spheres $S_1, S_2, \dots, S_n$ that can be embedded in a compact 3-manifold $M$ before the closure of at least one component of $M - \bigcup S_i$ is a punctured 3-ball. This then leads to showing that every compact 3-manifold has a factorisation as a connected sum of prime factors.

Kneser's proof involves using a triangulation of $M$ which is in general position with respect to a set of 2-spheres and working with the intersection of the polyhedron with the union of the 2-spheres. It can be assumed that the intersection with the 2-skeleton is what I defined to be a *pattern.* It seems that my earlier efforts to understand general position must have had some success. A *track* is a connected pattern. My breakthrough on accessibility for almost finitely presented groups came from considering patterns in finite 2-complexes. In particular, if $G$ is an almost finitely presented group, then $G$ acts freely on a 2-complex $K$ which satisfies $H^1(K, \mathbb{Z}_2) = 0$ and for which $L = G\backslash K$ is also a 2-complex with fundamental group $G$. A pattern $P$ in such a complex $L$ lifts to a $G$-pattern in $K$ which determines a $G$-tree in which edges correspond to tracks and vertices to the components of $K - P$. The action on such a tree might be thought of as a 'geometric' action. Not every action of an almost finitely presented group on a tree is geometric, but Roger Fenn and I in [12] show that an action of such a group on a $G$-tree is 'resolved' by a geometric action (see also [5]).

Kneser's argument works just as well for $L$, which means that if a pattern in $L$ has more than a fixed integer $n(L)$ of component tracks then at least two tracks are parallel, i.e. they have the same number of intersection points with each 1-simplex.

When I gave talks on the accessibility of finitely presented groups, the talks would go better than my earlier talks about ends. This was probably due to a number of factors. I had improved as a lecturer. The subject matter was more appealing and I found ways of presenting it that made it enjoyable to lecture on and to listen to. A pattern on the 2-complex $X$ is uniquely determined by the function $j_P$, where for each 1-simplex $\gamma$, $j_P(\gamma)$ is the number of intersection points of $P$ with $\gamma$. For a 2-simplex $\sigma$ with faces $\gamma_1, \gamma_2, \gamma_3$, $j_P$ would take three positive integer values $n_1, n_2, n_3$ where each value is at most the sum of the other two values. I would invite the audience to propose three values, hoping that there was no awkward person who would suggest numbers that did not work, and then draw the unique pattern.



The diagram above is for the triple $(3, 4, 5)$.

We now come to the end of the story. One might think that having shown that finitely presented groups are accessible, I would have spent a lot of time trying to show that there were examples of finitely generated groups that were not accessible. Sadly this was not the case. I remained strongly of the opinion that all finitely generated groups were accessible. However Bestvina and Feighn iin [3] published an example of a finitely generated group that was inaccessible if one weakened the definition of accessible to allow splitting over "small" subgroups. They used a folding sequence to construct their example.

In September 1991 Graham Niblo and Martin Roller organised a conference on Geometric Group Theory at the Isle of Thorns Conference Centre which is twenty miles north of Brighton in lovely countryside. Nearly all the important people in the field attended, including Stallings, Bestvina, Gromov and Rips. Somewhat inspired by this conference, I was thinking about accessibility one evening, a few weeks later. Fortunately there was nothing that I liked on TV. I realised that if you have a lattice of finite groups as shown in Figure 1 then it would be possible to construct a finitely generated inaccessible group. In the diagram lines represent proper inclusions. It is also required that $K_i$ and $H_{i+1}$ together generate $G_{i+1}$.

The explanation given in [11] is not entirely correct. After this first construction, I used folding sequences to construct other examples of inaccessible groups. This technique, is originally due to Stallings, and had been used by Bestvina and Feighn as noted above. In Figure 2 I give the folding sequence for the group I first constructed. I introduced vertex morphisms to supplement the Stallings folds. A vertex morphism is carried out on a tree $T$ with a group $G$ acting. It acts as a morphism of trees and an epimorphism of groups. It acts as an isomorphism on all except one orbit of vertices and on their stabilisers. On the exceptional vertices, it maps their stabilisers as a homomorphism that restricts to an isomorphism on every edge stabiliser. In our example below a vertex with stabiliser that is the free product of $K_1$ and $H_2$ with $H_1$ amalgamated is mapped to a vertex with stabiliser $G_2$. Since $G_2$ is generated by $K_1$ and $H_2$, there is a unique way of doing this. The group $H$ is a finitely generated group that contains all the groups in the ascending chain of finite groups $H_1 < H_2 < \dots$. It is well known that every countable group can be embedded in a finitely generated group. The limit $L$ of this folding sequence is a finitely generated inaccessible group. It is finitely generated because it is generated by $G_1$ and $H$, each of which are finitely generated. It is inaccessible because it has infinitely many decompositions as a free product with amalgamation over the finite subgroup $K_i$.

It is not hard to find sequences of finite groups $G_n, H_n, K_n$ that fit into the lattice in the required way and so I had constructed an inaccessible group.

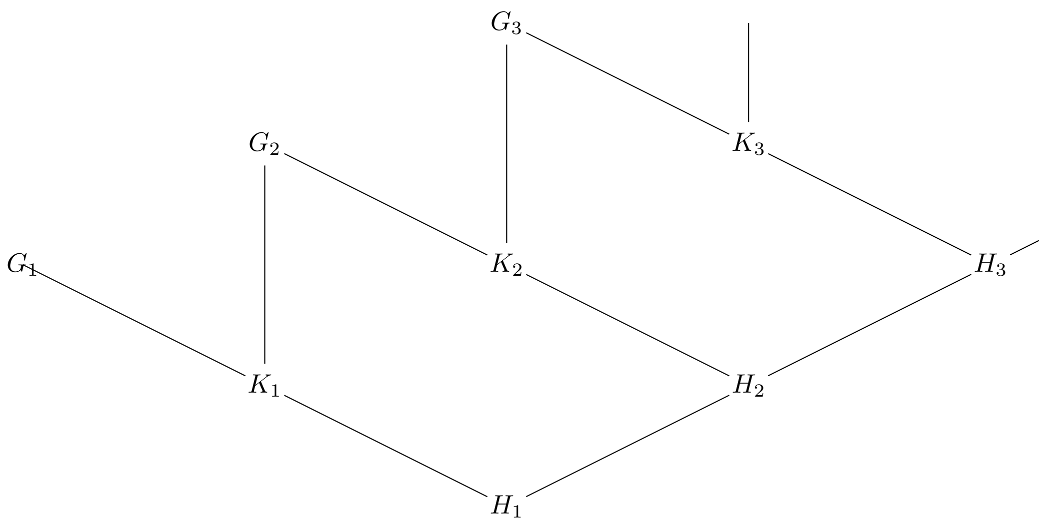



FIGURE 1. A lattice of finite groups

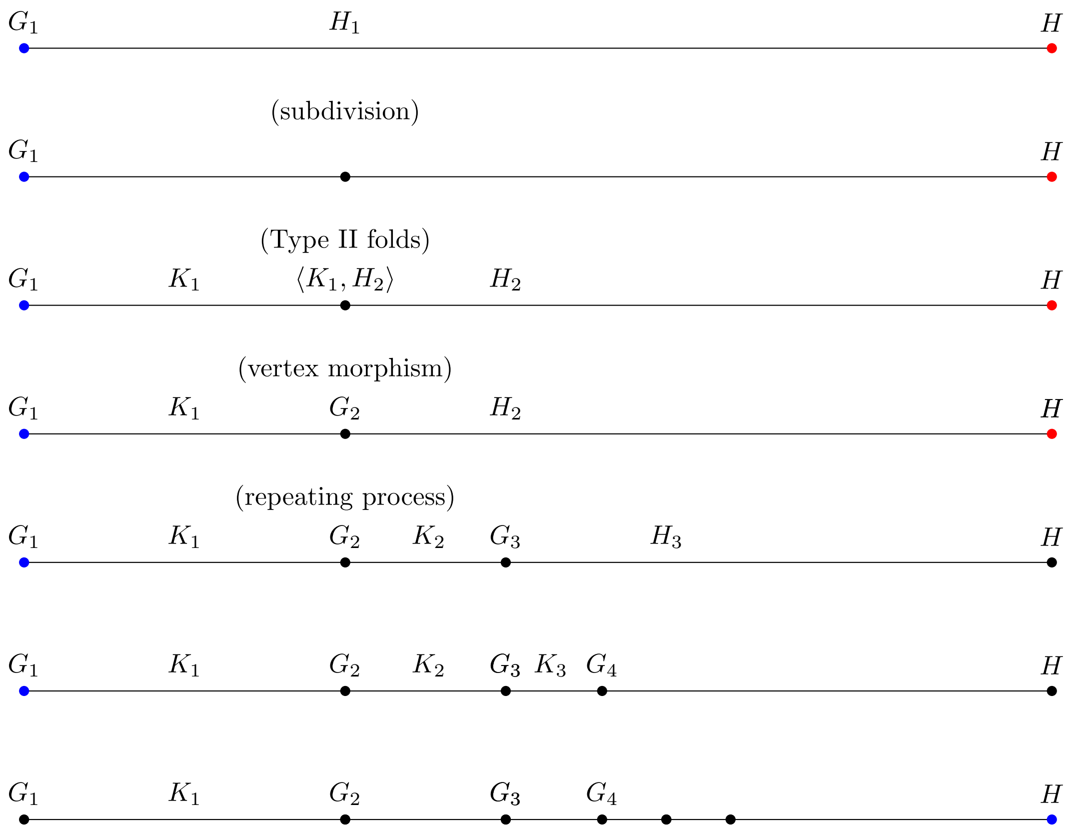



Figure 2. Folding sequence of graphs of groups

## 4. Later Developments

Much of my research has been extending and applying the theory I have described. In particular in our book [5] Warren Dicks and I extend and strengthen the results of [7] in the Almost Stability Theorem. Warren had got interested in the area after reading [7] and had used the theory in proving some nice results on projective augmentation ideals in group rings.

As we have seen Stallings' Theorem is the first result in finding algebraic analogues of results in the theory of 3-manifolds. Sometimes this is helpful in getting a better understanding of the original 3-manifold result. Stallings' Theorem is the algebraic analogue of the Sphere Theorem and the accessibility of finitely presented groups is the analogue of Kneser's result on the decomposition of closed 3-manifolds into prime factors. An extensive algebraic theory analogous to the JSJ-theory of Jaco-Shalen-Johannson for 3-manifolds has been developed starting with Peter Kropholler's papers [17] and [18]. Terry Wall, in [27], gives a good account of many of these developments as do Vincent Guirardel and Gilbert Levitt in [16] . These results usually involve a relative version of Stallings' Theorem in which $G$ is a finitely generated group with subgroup $H$ and the structure of $G$ is determined when the quotient graph $H\backslash X$ has more than one end. In this connection Kropholler made a conjecture in 1988 that I have been trying to prove ever since. See [14] for an account of this conjecture and its history. It also contains a 'proof' that was later shown to be wrong by Alex Margolis. Over the years I have come up with several inadequate proofs. I think I might be getting close. In [15] , which replaces another 'proof' shown to be inadequate by Sam Shepherd, I describe an approach which might work.

After developing the theory of tracks and patterns, it seemed to me that it might be used to provide a group theory analogue of Haken's algorithm for determining if a knot is non-trivial. The algorithm that I and my student Andrew Bartholomew were looking for, from 1984 onwards, would determine if a finitely presented group has more than one end, or more generally if it had a non-trivial action on a tree, for groups that have a solvable word problem. Andy produced some splendid software that from a particular presentation of the group produces a finite list of tracks at least one of which ought to correspond to a non-trivial decomposition. But so far we have been unable to prove that this is the case. This is written up in [2].

in [9] I also give a geometric proof of Stallings' Theorem, using tracks and patterns, for almost finitely presented groups. This is close to Stalling' original proof in [22]. Graham Niblo [19] found a way of extending this to finitely generated groups.

In [10] I gave a combinatorial proof of the Equivariant Sphere Theorem using tracks and patterns in the 2-skeleton of a triangulated 3-manifold. This result is also written up in the final chapter of [5] where a mistake in [10] is corrected and a proof of the Equivariant Loop Theorem is also included. These proofs are a lot more elementary than the original proofs of Meeks and Yau. This tempted me to think that similar methods might give a proof of the Poincare Conjecture, leading me to make a fool of myself. As remarked earlier in [23] Stallings suggested that his result might be of use in a proof of the Poincare Conjecture. He had earlier warned in [24] to be very careful when putting forward prospective proofs. I wish I had heeded that advice. However Stallings had great insight and I feel there may just be a successful approach using Stallings' Theorem possibly with patterned 2-spheres in a triangulated fake 3-ball.

Graph theorists like to cut up graphs using finite sets of vertices instead of finite sets of edges. Thus a vertex cut in a connected graph $X$ is a subset $A$ of $VX$ for which $NA$ is finite, where $NA$ is the set of vertices that are not in $A$ but are adjacent to a vertex of $A$. One then can define an end (or vertex end) by replacing edge cuts by vertex cuts in the definition of end. For a locally finite graph, every edge end is a vertex end and vice versa. However there are graphs with more than one vertex end but only one edge end. Bernhard Krön and I [13] show that there is a structure tree theory to cover vertex cuts instead of edge cuts, though it only works for minimal vertex cuts. Our approach actually produces a non-trivial result for finite graphs, generalising a construction of Bill Tutte, a hero of Bletchley Park. Reflecting on this made me realise that for edge cuts, if you work with edge cuts separating pairs of elements from $VX \cup \Omega X$, one gets a structure tree theory for graphs and networks including finite ones. This gives a nice construction of a Gomory-Hu tree for a finite network. I had taught the Max-Flow Min-Cut Theprem in undergraduate courses for many years without seeing this obvious connection. This is written up in [14].

## 5. Final thoughts

When I was a PhD student in Canberra in the early 60's, Zvonomir Janko was a research fellow at the ANU, just before his discovery of a new sporadic finite simple group. Janko did not like infinite group theory. He told me that any statement about all groups (finite or infinite) is either trivially true or false. Thus infinite group theory is just a collection of counter examples. At the time I could not think of a counter example to Janko's observation. Now, though, I think that Stallings' Theorem does provide such an example.

I think I have been very fortunate in getting into this area of mathematics, which I have enjoyed thinking about for over fifty years. As I have explained, Stallings came up with his major breakthrough at precisely the right time for me, when I was looking for a new research area and did not have to worry about finding a job. I recommend it to young mathematicians in a similar situation.

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