THE KROPHOLLER CONJECTURE

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Abstract. A proof is given of the Kropholler Conjecture which gives a relative version of Stallings' Ends Theorem. A generalisation of the Almost Stability Theorem is also obtained.

1. Introduction

In [2] I attempted to give a proof of the long standing Kropholler Conjecture. Unfortunately there was an inadequate argument for a key part of the proof, as was pointed out by Akex Margolis. I am very grateful to Alex for his careful analysis of that paper. In this paper I attempt a new complete solution.

2. Proof of the Kropholler Conjecture

We follow the notation and terminology of [2], which contains a historical account of the conjecture and the results already obtained.

Let G be a group and let H be a subgroup. A subset $S \subset G$ is said to be H-finite if S is contained in finitely many right cosets of H, i.e. for some finite subset F of G, we have $S \subset HF$. A subgroup K of G is H-finite, if and only if $H \cap K$ has finite index in K. Two subsets $R, S \subset G$ are said to be almost H-equal if the symmetric difference $R \cup S - R \cap S$ is H-finite. We write $R =_a S$. A set A is H-almost invariant if $A =_a Ag$ for every $g \in G$. If A is almost invariant, then so is $A^* = G - A$. We say that A is proper if neither A not A^* is H-finite.

Let A be H-almost invariant and let $HA = A$, this means that $A + Ag$ is a union of finitely many right H-cosets. Let K be the right stabiliser of A. Suppose that H is the left stabiliser of A and that $H \leq K$, so that $A = HAH = HAK$.

Let X be the 2-complex in which the 1 skeleton is the complete graph on $VX = AG$, and there is a 2-simplex for each triple of distinct vertices in VX. Thus X is the complete 2-complex on the set $AG = \{Ag | g \in G\}$.

As in [2] we can define a metric on VX in which $d(u, v)$ is the number of right H-cosets in $u + v$.

Consider a 2-simplex σ of X with vertices u, v, w. We know that $d(u, v) \leq d(u, w) + d(w, v)$ and so, as in [1] p224 there is a *pattern* P in $|X|$ for which

(i) For each 2-simplex σ of X $P \cap |\sigma|$ is a union of finitely many disjoint straight lines joining distinct faces of σ.

(ii) If $\gamma = [u, v]$ is a 1-simplex of X, then $|\gamma| \cap P$ consists of $d(u, v)$ points.

A *track* is a connected pattern.

Let EX be the set of 1- simplexes of X. Let $j_P : EX \to \mathcal{Z}$, where $j_P(\gamma)$ is the number of points in $|\gamma| \cap P$. Two patterns P, Q are *equivalent* if $j_P = j_Q$. Two disjoint tracks are equivaent if and only if they bound a band, a closed subset of X that contains no vertex or central region (coloured red) in Figure 1.

Let $\gamma = [u, v]$ and $\rho = [w, z]$ be a 1-simplexes of X and let $F(\gamma)$ be the set of cosets in $u + v$.

Theorem 2.1. *The set of intersections of the* 1*-skeleton* $X^{(1)}$ *of* X *with* P *can be labelled by the set of right cosets of* H *in sucb a way that*

(i) *The labels on* $|\gamma| \cap P$ *are the cosets in* $u + v$*.*
(ii) *Two points of* $X^{(1)} \cap P$ *have the same label if and only if they belong to the same track.*

Proof. Firstly we note some properties that the labelling must have.

(a) If $\gamma = [u, v]$ and $\rho = [u, w]$ have a vertex in common, and $I = (u+v) \cap (u_w) \neq \emptyset$ then in the 2 simplex σ with vertices u, v, w there are $|I|$ lines across the corner u, in whih each line is labelled twice by the same coset of I and for coset of I there is one corresponding line.
(b) If $\gamma = [u, v]$ and $\rho = [w, z]$ have no vertex in common, and $J = (u + v) \cap (w + z) \neq \emptyset$ then after a possibly transposing w and z, the labelling and the intersection of P with the two 2-simplexes $(u, v, w), (v, w, z)$ must be as in Fig 2 in which there $|J|$ distinct tracks joining (u, v) and (v, w) which are labelled with different cosets of J.

These properties become clear by studying Figure 2.

Note that if $u+v$ and $w+z$ is not empty as in Figure 2, then $[v+z] \cap [u+w] = \emptyset$. Also the intersection of the pattern with the square $uvzw$ is determined by the intersections with the four corners. Thus if we split the square into two triangles using the other diagonal we would end up with the same pattern intersection.

For every 2-simplex σ with verticce u, v, w the set $u + v$ is partitioned into those cosets which are also in $u + w$ and those which are also in $v + w$.

For $Hx, Hy \in u + v$ we write $Hx < Hy$ if Hx must occur before Hy in the labelling of $|\gamma| \cap P$ going from u to v. Clearly $Hx < Hy$ if for any w we have $Hx \in u + w, Hy \in v + w$.

In fact for $Hx, Hy \in F(\gamma) = u + v$ we have

(*) $Hx < Hy$ if for every vertex $w \neq u, w \neq v$ we have $Hx \in u+w$ if $Hy \in u+w$ and for some w we have $Hx \in u + w, Hy \in v + w$.

Using (*) as a relation on $u+v$, it is transitive. To see that (*) is anti-symmetric, let $Hx < Hy$ and $Hy < Hx$.then for some w we have $Hx \in u + w, Hy \in v + w$ and for some z we have $Hy \in u + z, Hx \in v + w$. But then. we would have Hx in opposite sides of a square and Hy in the other two sides, which we have seen annot happen.

if neither $Hx < Hy$ nor $Hy < Hx$, then for every

We now show that if $Hx \not\leq Hy$ then in fact $Hy \leq Hx$. If $Hx \not\leq Hy$ then for some vertex w we have $Hx \in v + w$, but $Hy \in u + w$. But now for any other vertex $z, z \neq u, z \neq v$ we have if $Hx \in u + z$, then $Hy \in u + z$, as in Figure 3 Hence $Hy \leq Hx$. For any pair Hx, Hy we have either $Hx \leq Hy$ or $Hy \leq Hx$. If $Hx \leq Hy$ and $Hy \leq Hx$, then for every 1-simplex γ, $Hx \in F(\gamma)$ if and only if $Hy \in F(\gamma)$, and we can alter the orderings so that Hx and Hy represent parallel tracks.

□

The cosets attached to parallel tracks are interchangeable. If t is the track corresponding to the coset H, then the union of the cosets corresponding to the

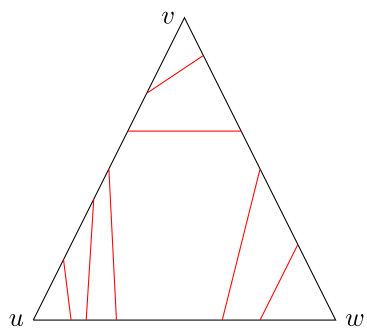

FIGURE 1.

tracks parallel to t form a subgroup containing H as a subgroup of finite index.. Every track separates X. Two vertices u, v of X are separated by t if and only if H is one of the cosets in $u + v$. The fact that tracks separate means that there is tree T in which ET is the set of equivalence classes of tracks. The tree T is a G-tree., with a right G-action, corresponding to the right action on the right cosets of H.

Each vertex v determines a vertex of T, but there may be other vertices, each of which will corespond to a component of $|X| - \bigcup(|e| \in ET)$ that does not contain a vertex of T. It will contain a red region as in Figure 1.

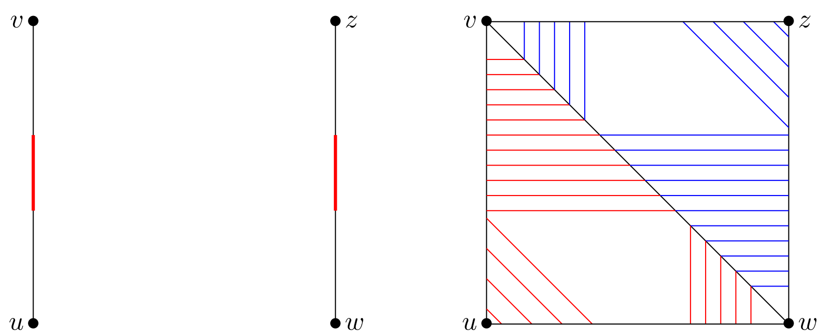

FIGURE 2.

We have proved the following theorem, which proves the Kropholler Conjecture.

Theorem 2.2. *Let G be a group with subgroup H. If there is a proper almost H-invariant subset A, such that $A = HAK$ as above, with $H \leq K$, then G splits over an H-finite subgroup of K. In fact there is a G-tree T such that for every edge $e \in ET$ the stabiliser G_e contains a conjugate of H as a subgroup of finite index.*

Also included in the above proof is a proof of a generalisation of the Almost Stability Theorem [1]. Although the vertex set of T may be larger than AG, since it may include components of $|X| - P$ that do not contain a vertex, the vertex set of T can be regrded as a subset of $M = \{B|B =^a A\}$. Thus $u = A$ is a set of right cosets of H and any other vertex v can be reached by a finite geodesic $[u, v]$ where $u + v = F$ is a finite set of cosets; so $v = A + F \in M$. We have the following H-Almost Stability Theorem (Theorem 4.1 of [2]).

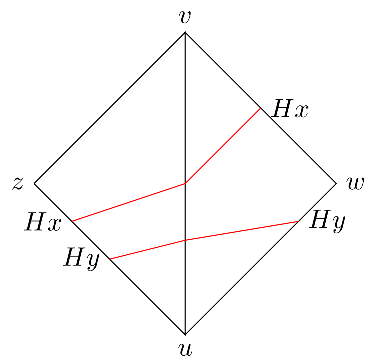

Figure 3.

Theorem 2.3. *Let G be a group with subgroup H. If there is a proper almost H-invariant subset A, such that $A = HAK$ as above, and $M = \{B | B =^a A\}$ is the metric space in which $d(u, v) = |u + v|$, then there is a G-tree T in which VT is a G-subset of M and the metric on M restricts to a geodesic metric on VT. If $e \in ET$ then some edge in the G-orbit of e contains H as a subgroup of finite index*

Note that we also have (yet another) proof of Stallings' Theorem.

References

[1] Warren Dicks and M.J.Dunwoody. *Groups acting on graphs.* Cambridge studies in advanced mathematics **17** (1989).

[2] M.J.Dunwoody. *Structure trees, networks and almost invariant sets. Groups graphs and random walks* LMS lecture Notes **436** (2017) 137-175.