AN (FA)-GROUP THAT IS NOT (F$\mathbb{R}$)

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Abstract. An example is given of a finitely generated group L that has a non-trivial stable action on an $\mathbb{R}$-tree but which cannot act, without fixing a vertex, on any simplicial tree. Moreover, any finitely presented group mapping onto L does have a fixed point-free action on some simplicial tree.

1. Introduction

In [20, p. 286] Peter Shalen asked the following:

Question A. Suppose that Γ is a finitely generated group which admits a non-trivial action by isometries on some $\mathbb{R}$-tree. Does it then follow that Γ admits a non-trivial action, by isometries and without inversions, on a $\mathbb{Z}$-tree? Equivalently, does Γ admit a non-trivial decomposition as the fundamental group of a graph of groups?

In this paper we show that the answer to this question is negative by constructing a finitely generated group L that has a non-trivial (i.e., without global fixed points) action on some $\mathbb{R}$-tree T but which has no non-trivial action on any simplicial tree. Recall that a group G is said to satisfy *Serre's property* (FA) if any simplicial action (without edge inversions) of G on a simplicial tree has a global fixed point (see [19, I.6.1]). Similarly, G has *property* (F$\mathbb{R}$) if every isometric action of G on an $\mathbb{R}$-tree fixes a point. The main result of this paper is

Theorem 1.1. *There exists a finitely generated group L which has property* (FA) *but does not have property* (F$\mathbb{R}$). *Moreover, L is not a quotient of any finitely presented group with property* (FA).

In fact, our approach (using Construction 3.2) shows that any finitely generated group G_0 can be embedded in a group L satisfying the claim of Theorem 1.1. Thus there are uncountably many pairwise non-isomorphic groups L with above properties.

The second claim of Theorem 1.1 shows that property (FA) does not define an open subset in the space of marked groups, which answers a question of Yves de Cornulier (see [4] or [21]). This contrasts with the fact that any finitely generated group with property (F$\mathbb{R}$) *is* a quotient of a finitely presented group with this property (this follows from the work of Culler and Morgan [6] and is explicitly stated in [21, Thm. 1.4]).

Recall, that an action of a group Γ on an $\mathbb{R}$-tree is said to be *stable* if there is no sequence of arcs l_i such that l_{i+1} is properly contained in l_i for every i, and for which the stabilizer Γ_i of l_i is properly contained in Γ_{i+1} for every i. In [2] Bestvina and Feighn proved that if a finitely presented group has a non-trivial minimal stable action on an $\mathbb{R}$-tree then it has a non-trivial action on some simplicial tree.

The group L from Theorem 1.1 is not finitely presented and possesses a non-trivial unstable action on a real tree T. It is possible to construct unstable actions of finitely presented groups on $\mathbb{R}$-trees (see [11]), but all of such (known) examples admit non-trivial actions on simplicial trees. I have [12] recently shown that the answer to Question A is positive if the group Γ is finitely presented.

The construction of L uses folding sequences which I have studied in various papers (e.g. [10]). In [9] I used a folding sequence construction to give a negative answer to another question of Shalen [20, Question D, p. 293], by showing that there is a finitely generated group that has a non-trivial action

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on an $\mathbb{R}$-tree with finite cyclic arc stabilizers but which has no non-trivial action on a simplicial tree with small arc stabilizers. This example did, however, have a non-trivial action on a simplicial tree with edge stabilizers which were not small.

In our new examples the arc stabilizers are not small. However the construction depends on finding an ascending sequence of groups G_i that satisfies certain properties (see Section 3). These groups become arc stabilizers in L. It may be possible to construct such a sequence of groups which are small or even finite. The resulting group L would then give a negative answer to both Shalen's questions mentioned above.

The first version of this paper, that appeared on arXiv , was coauthored by A. Minasyan. This version contained errors, one of which was rather subtle. It has taken a substantial rewriting of the paper to correct these errors. It was mistakenly stated in that paper that the $\mathbb{R}$-tree constructed there was a strong limit of the folding sequence given there. In this paper we give two different folding sequences that have the same limit (possibly after some scaling). The $\mathbb{R}$-tree T constructed here is not a strong limit of the first folding sequence given here, which is a corrected version of the folding sequence in the earlier version. In [15] it is shown that a non-geometric action of a finitely generated group G on an $\mathbb{R}$-tree is a strong limit of geometric actions.. Initially I thought that the action here must be a counter-example to this result. However after a correspondence with Gilbert Levitt, in which he displayed commendable patience, I have managed to construct a different folding sequence that does converge strongly to the action. This folding sequence makes it possible to deduce further properties of the limit action. In particular it follows that arc stabilisers are subgroups of conjugates of a G_i. Information is also obtained about the structure of $PLF(L)$, the space of projectivized translation length functions of fixed-point free actions of L on $\mathbb{R}$-trees. It is shown that this space contains a subspace that is the closed interval $[1/2, (\sqrt{5}-1)/2]$ in which $1/2$ is identified $7/12$, the action of our first folding sequence being this identified point.

Ashot Minasyan has indicated that he no longer wishes to be a coauthor of this paper. I regret this and acknowledge his important contribution. In particular he was responsible for the construction of the good sequences of groups. He also pointed out the relevance of [6].

2. The Folding Sequence

The definition below describes the families of groups we will be working with.

Definition 2.1. Let $G_0 < G_1 < G_2 < G_3 < \dots$ be a strictly ascending sequence of groups. We will say that this sequence is *good* if for every $i \in \mathbb{N}$ there is an element $a_i \in G_{i+3}$ such that all of the following hold:

(i) a_i centralizes G_{i-1} in G_{i+3};
(ii) $G_{i+3} = \langle G_i, a_i G_i a_i^{-1} \rangle$;
(iii) if G_{i+1} acts on a simplicial tree T then the restriction of this action to G_i fixes a vertex of T.

Suppose that a sequence of groups $G_0 < G_1 < \dots$ satisfies conditions (i) and (ii) of Definition 2.1. Associated with this sequence of groups and elements $a_i \in G_{i+3}$ is an infinite folding sequence, as in Figure 1.

More precisely we define $L_1 = K_2 *_{K_0 = H_0} H_3$, where K_2 is an isomorphic copy of G_2, H_3 is an isomorphic copy of G_3. the respective isomorphisms taking G_0 to $K_0 = H_0$ and G_1 to K_1 and H_1 and taking H_2 to G_2. Here $K_2 \cap H_3 = H_0$. We take T_1 to be the Bass-Serre tree corresponding to this decomposition. We define inductively the group L_i, an L_i-tree T_i, an epimorphism $\phi_i : L_i \to L_{i+1}$ and a morphism $\phi_i : T_i \to T_{i+1}$ of trees, such that $\phi_i(gt) = \phi_i(g)\phi_i(t)$ for each $g \in L_i, t \in T_i$. If i is even, then $\phi_{i+1}\phi_i$ is as in Figure 1, while if i is odd then $\phi_{i+1}\phi_i$ is as in Figure 1, but with the roles of H_k andK_k transposed. Assume that i is even and $L_i = K_{i+1} *_{K^*_{i-1} = H^8_{i-1}} H_{i+2}$, where in the induction step we assume that for the isomorphisms $\alpha_{i+1} : G_{i+1} \to K_{i+1}$ and $\beta_{i+2} : G_{i+2} \to H_{i+2}$ we have $\alpha_{i+1}(G_i) = K_i, \alpha_{i+1}(G_{i-1}) = K^*_{i-1}, \beta_{i+2}(G_i) := H^*_i, \beta_{i+2}(G_{i-1}) := H^*_{i-1}, \beta_{i+2}(G_{i+1}) = H_{i+1}$.

The centre group in the third graph of groups in Figure 1 is the free product with amalgamation $K_i *_{H^*_{i-1}} H^*_i$. From property (ii) above there is an epimorphism from this group to K_{i+3} for which there is an isomorphism $\alpha_{i+3} : G_{i+3} \to K_{i+3}$. We also have $\alpha_{i+3}(G_i) = H_i^* = K_i^* = a_i^{-1} K_i a_i$. Here a_i is an element of K_{i+3}. The isomorphism also means that we can define K_{i+2} to be $\alpha_{i+3}(G_{i+2})$.

The next stage of the folding sequence, labelled conjugating to the left of centre vertex, does not change the tree or the action. It just changes the lift of the spanning tree to give the change of labels indicated. The two edges now have the same labels and so if we carry out a Type I fold we obtain a single edge with that same label. The right hand vertex has label $a_i^{-1} K_{i+1} a_i *_{H_i^*} H_{i+2}$. There is an isomorphism of $a_i^{-1} K_{i+1} a_i$ with H_{i+1} that restricts to the identity on H_i^*. There is then an epimorphism from $a_i^{-1} K_{i+1} a_i *_{H_i^*} H_{i+2}$ to H_{i+2} that restricts to the identity on the right hand factor. This gives the second vertex morphism of the part of the folding sequence illustrated in Figure 1. A further iteration then gives the final graph of groups in which each suffix is two more than in the original graph of groups.

In going from T_i to T_{i+1} for i we define new groups $K_{i+2} < K_{i+3}$ if i is even and new groups $H_{i+2} < H_{i+3}$ if i is odd. It is not the case that $K_{i+1} < K_{i+2}$ if i is even or that $H_{i+1} < H_{i+2}$ if i is odd.

Although we have $a_i \in G_{i+3}$, at the risk of ambiguity we identify these elements with their images in the first group of the folding sequence in which their isomorphic images occur. Thus $a_i \in K_{i+3}$ if i is even and $a_i \in H_{i+3}$ if i is odd. We have $a_i^{-1} K_{i+1} a_i = H_{i+1}$ for i even and $a_i^{-1} H_{i+1} a_i = K_{i+1}$ if i is odd.

Repeating this procedure for $i+1, i+2, \dots$ gives an infinite folding sequence and, as described in [10] and [9]. there is a limit group L which is the direct limit of the sequence of ϕ_i's. In fact, we will show that one can put metrics on the trees T_i so that they become simplicial $\mathbb{R}$-trees (also denoted) T_i in such a way that there is a limit $\mathbb{R}$-tree T of the trees T_i, and so that there is no point of T is fixed by all of L.

Thus the limit group L possesses a non-trivial action on an $\mathbb{R}$-tree. On the other hand we will show that L has (FA). Some properties of L can be deduced from properties of the G_i. For example, if the G_1 and G_2 are finitely generated, then so is L; if G_i are torsion-free for all i then L is torsion-free too.

3. Constructing good sequences

In this section we suggest two approaches for constructing strictly ascending sequences of finitely generated groups $G_0 < G_1 < \dots$ satisfying conditions (i)-(iii) of Definition 2.1. The first method will use R. Thompson's group V, and the second method will be based on small cancellation theory over HNN-extensions.

3.1. **Construction using Thompson's group V.** R. Thompson's group V can be defined as the group of all piecewise linear right continuous bijections of the interval $[0, 1)$ which map dyadic rational numbers to dyadic rational numbers, are differentiable in all but finitely many dyadic rational numbers and such that on every maximal interval, where the function is linear, its slope is a power of 2. We refer the reader to [3] for a a good introduction to the group V.

The group V is finitely presented and simple [3]. Let every G_i be an isomorphic copy of V, $i = 0, 1, \dots$. To explain how G_i is embedded in G_{i+1}, consider the function $f : [0, 1] \to [0, 15/16]$ defined as follows:

$$f(x) := \begin{cases} x & \text{if } x \in \left[0, \frac{1}{4}\right) \\ \frac{x}{2} + \frac{1}{8} & \text{if } x \in \left[\frac{1}{4}, \frac{3}{4}\right) \\ 2x - 1 & \text{if } x \in \left[\frac{3}{4}, \frac{15}{16}\right) \\ x - \frac{1}{16} & \text{if } x \in \left[\frac{15}{16}, 1\right] \end{cases}.$$

Clearly f is continuous, increasing and piecewise linear on $[0, 1]$. Also note that f induces a bijection between dyadic numbers on $[0, 1]$ and $[0, 15/16]$, and

$$f(1) = 15/16,\ f(15/16) = 7/8,\ f(7/8) = 3/4 \text{ and } f(3/4) = 1/2. \tag{1}$$

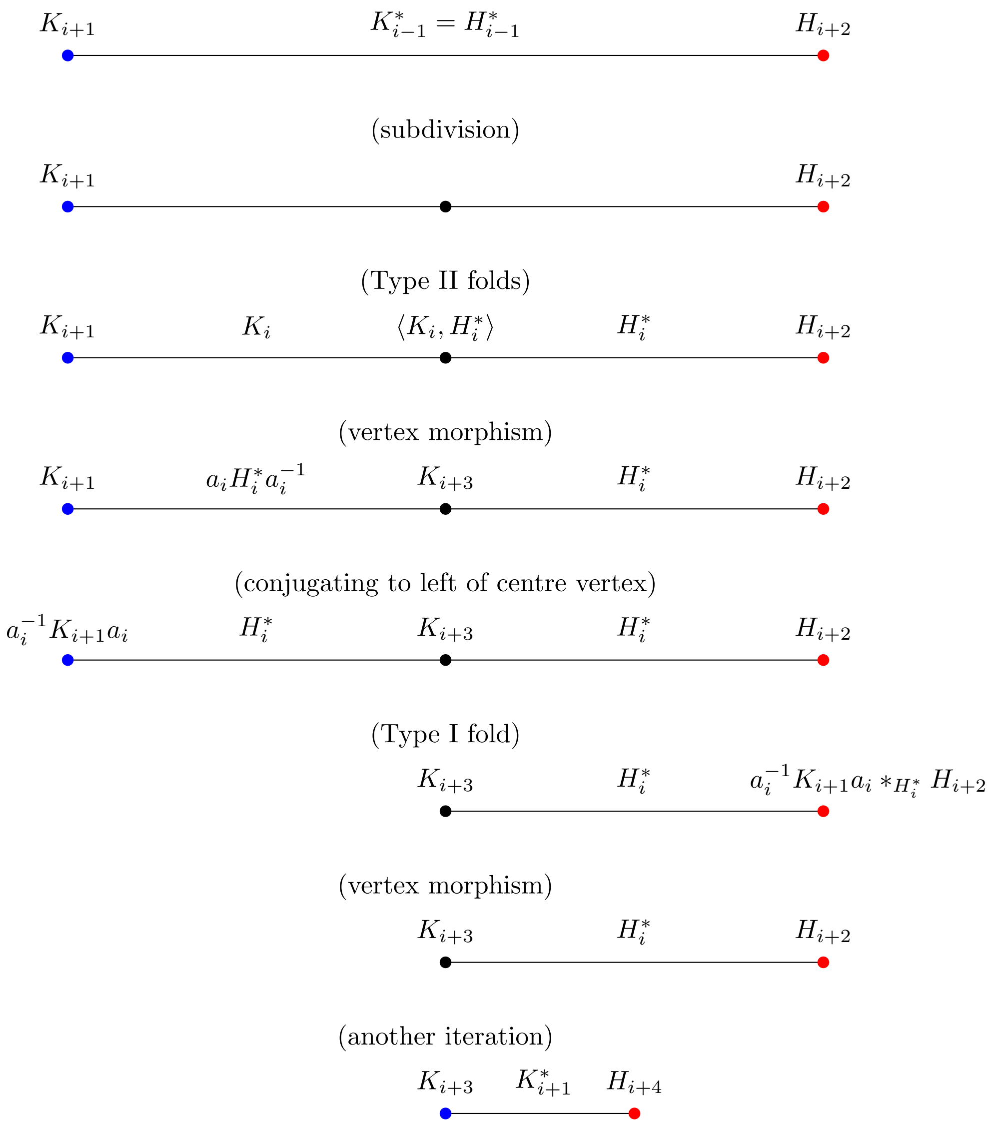

FIGURE 1. Folding sequence of graph of groups

For each $i \in \mathbb{N} \cup \{0\}$ we define the embedding $\gamma_i : G_i \to G_{i+1}$ as follows: for any function $g \in V \cong G_i$ set

$$\gamma_i(g)(x) := \begin{cases} (f \circ g \circ f^{-1})(x) & \text{if } x \in \left[0, \frac{15}{16}\right) \\ x & \text{if } x \in \left[\frac{15}{16}, 1\right) \end{cases} .$$

Clearly $\gamma_i(V) \leqslant V_{[0,15/16)}$, where $V_S := \{h \in V \mid \mathrm{supp}(h) \subseteq S\} \leqslant V$ for any subset $S \subseteq [0,1)$ (where $\mathrm{supp}(h) := \{x \in [0,1) \mid h(x) \neq x\}$). It is also clear that γ_i is invertible and $\gamma_i^{-1} : V_{[0,15/16)} \to V$. Hence every γ_i is an isomorphism between V and $V_{[0,15/16)}$. Thus one can regard G_i inside of G_{i+1} as $V_{[0,15/16)}$ inside of V. Similarly, since we picked the function f to satisfy (1), G_{i-1} and G_i in G_{i+3} will correspond to $V_{[0,1/2)}$ and $V_{[0,3/4)}$ in V respectively.

Define $a_i \in G_{i+3}$ to be the element of V exchanging the intervals $[1/2, 3/4)$ and $[3/4, 1)$, which can be given by the following formula:

$$a_i(x) := \begin{cases} x & \text{if } x \in \left[0, \frac{1}{2}\right) \\ x + \frac{1}{4} & \text{if } x \in \left[\frac{1}{2}, \frac{3}{4}\right) \\ x - \frac{1}{4} & \text{if } x \in \left[\frac{3}{4}, 1\right) \end{cases} .$$

Since $\operatorname{supp}(a_i) \cap [0, 1/2) = \emptyset$, a_i will centralize $V_{[0,1/2)}$ in V, thus the condition (i) from Definition 2.1 is satisfied. Also observe that $a_i V_{[0,3/4)} a_i^{-1} = V_{[0,1/2)\cup[3/4,1)}$ in V and in order to establish (ii) we need to check that V is generated by $V_{[0,3/4)}$ and $V_{[0,1/2)\cup[3/4,1)}$. We will do this by showing that some generating set of V is contained in $\langle V_{[0,3/4)}, V_{[0,1/2)\cup[3/4,1)} \rangle$.

From [3, Lemma 6.1] we know that V is generated by its elements A, B, C and π_0, where

$$A(x) := \begin{cases} \frac{x}{2} & \text{if } x \in \left[0, \frac{1}{2}\right) \\ x - \frac{1}{4} & \text{if } x \in \left[\frac{1}{2}, \frac{3}{4}\right) \\ 2x - 1 & \text{if } x \in \left[\frac{3}{4}, 1\right) \end{cases} , \quad B(x) := \begin{cases} x & \text{if } x \in \left[0, \frac{1}{2}\right) \\ \frac{x}{2} + \frac{1}{4} & \text{if } x \in \left[\frac{1}{2}, \frac{3}{4}\right) \\ x - \frac{1}{8} & \text{if } x \in \left[\frac{3}{4}, \frac{7}{8}\right) \\ 2x - 1 & \text{if } x \in \left[\frac{7}{8}, 1\right) \end{cases} ,$$

$$C(x) := \begin{cases} \frac{x}{2} + \frac{3}{4} & \text{if } x \in \left[0, \frac{1}{2}\right) \\ 2x - 1 & \text{if } x \in \left[\frac{1}{2}, \frac{3}{4}\right) \\ x - \frac{1}{4} & \text{if } x \in \left[\frac{3}{4}, 1\right) \end{cases} , \quad \pi_0(x) := \begin{cases} \frac{x}{2} + \frac{1}{2} & \text{if } x \in \left[0, \frac{1}{2}\right) \\ 2x - 1 & \text{if } x \in \left[\frac{1}{2}, \frac{3}{4}\right) \\ x & \text{if } x \in \left[\frac{3}{4}, 1\right) \end{cases} .$$

One easily sees that $\pi_0 \in V_{[0,3/4)}$, $C \circ \pi_0 \in V_{[0,1/2)\cup[3/4,1)}$ and $\pi_0 \circ B \circ \pi_0^{-1} \in V_{[0,1/2)\cup[3/4,1)}$. Note that $(A^{-1} \circ B)(x) = x$ for all $x \in [7/8, 1)$, so taking any element $D \in V_{[0,1/2)\cup[3/4,1)}$ with $D([3/4, 1)) = [7/8, 1)$, we have $(D^{-1} \circ A^{-1} \circ B \circ D)(x) = x$ for all $x \in [3/4, 1)$. Thus $D^{-1} \circ A^{-1} \circ B \circ D \in V_{[0,3/4)}$ and we have proved that $A, B, C, \pi_0 \in \langle V_{[0,3/4)}, V_{[0,1/2)\cup[3/4,1)} \rangle$.

Hence the condition (ii) from Definition 2.1 holds for the sequence $G_0 < G_2 < \ldots$. Finally, the condition (iii) from Definition 2.1 holds because V has property (FA), as shown by D. Farley in [13] (based on the notes of K. Brown).

3.2. **Construction using small cancellation methods.** Let us start with the following well-known observation (see [7, Remark on p. 680]).

Lemma 3.1. *Any countable group G can be embedded into a finitely generated group F with property* (FA). *If G is finitely presented then one can take F also to be finitely presented.*

Proof. Take any non-elementary word hyperbolic group H with property (FA) (e.g., a hyperbolic triangle group or a hyperbolic group with Kazhdan's property (T) if one looks for a torsion-free example). The group H is SQ-universal (as proved by T. Delzant [7, Thm. 3.5], and independently, by A. Olshanskii [17, Thm. 1]), thus G can be embedded into some quotient F of H. Since property (FA) passes to quotients, this proves the first part of the lemma.

Now, suppose that G is finitely presented. This means that $G \cong \mathbb{F}_n/N$, where $\mathbb{F}_n$ is the free group of rank $n \geq 2$ and $N \lhd \mathbb{F}_n$ is a normal subgroup which the normal closure of finitely many elements $f_1, \ldots, f_k \in \mathbb{F}_n$ for some $k \in \mathbb{N} \cup \{0\}$. According to [17, Theorems 2,3] the word hyperbolic group H contains a copy of $\mathbb{F}_n$ with the congruence extension property. Abusing the notation, let us identify $\mathbb{F}_n$ with this copy of it. The congruence extension property for $\mathbb{F}_n$ in H implies that $M \cap \mathbb{F}_n = N$, where M is the normal closure of $f_1, \ldots, f_k$ in H. Therefore there is a natural embedding of $G \cong \mathbb{F}_n/N = \mathbb{F}_n/(\mathbb{F}_n \cap M)$ into $F := H/M$. As before, F will have (FA); F will also be finitely presented because it is a quotient of the finitely presented group H by the normal closure of finitely many elements. □

Remark 3.2. A more technical argument, still based on the methods from [17], would allow one to ensure in Lemma 3.1 that F is torsion-free if one starts with a torsion-free group G .

The construction of good sequences we suggest here employs the theory of small cancellation quotients of HNN-extensions, which was developed by G. Sacerdote and P. Schupp in [18]. We refer the reader to [18] or [16, V.11] for the details of this theory. This method is quite flexible and allows us to start with an *arbitrary* finitely generated group G_0. By Lemma 3.1 we can embed G_0 into a finitely generated group F_0 with property (FA), and we can let G_1 to be the free product of F_0 with an infinite cyclic group: $G_1 := F_0 * \langle u_1 \rangle$. For $i = 2, 3$ we proceed similarly: first embed G_{i-1} into a finitely generated group F_{i-1} with (FA) and then set $G_i := F_{i-1} * \langle u_i \rangle$. Now, suppose $G_1, G_2, \dots, G_{i+2}$ have already been built, with $i \in \mathbb{N}$. To simplify the notation we will identify each G_j, F_j and u_{j+1} with their canonical images in G_i, whenever $j < i$. Again, we embed G_{i+2} into a finitely generated group F_{i+2} with property (FA), and consider the HNN-extension E_{i+2}, of $F_{i+2} * \langle u_{i+3} \rangle$, defined by the presentation

$$E_{i+2} := \langle F_{i+2}, u_{i+3}, t_i \mid t_i g t_i^{-1} = g \text{ for all } g \in G_{i-1} \rangle. \tag{2}$$

Let $\{v_1, \dots, v_l\}$ be a finite generating set of $F_{i+2} * \langle u_{i+3} \rangle$. By construction the subgroup that G_{i-1} and $\langle u_i \rangle$ generate in F_{i+2} is isomorphic to their free product. Therefore

$$u_i^p G_{i-1} u_i^q \cap G_{i-1} = \emptyset \text{ whenever } p, q \in \mathbb{Z} \text{ and } p \neq -q. \tag{3}$$

Hence we can replace u_i with its power, if necessary, to assume that

$$u_i^{100j+100} v_j^{-1} u_i^{100j+1} \notin G_{i-1} \text{ for every } j = 1, \dots, l. \tag{4}$$

Now, consider the words $r_0, r_1, \dots, r_l$ defined by

$$r_0 := t_i^{-1} u_i t_i u_i^2 t_i^{-1} \dots u_i^{98} t_i u_i^{99} t_i^{-1} u_i^{100}, \text{ and}$$

$$r_j := v_j^{-1} u_i^{100j+1} t_i u_i^{100j+2} t_i^{-1} \dots u_i^{100j+98} t_i u_i^{100j+99} t_i^{-1} u_i^{100j+100}$$

for $j = 1, \dots, l$.

Observe that (3) and (4) imply that the words $r_0, \dots, r_l$ are cyclically reduced in the HNN-extension E_{i+2}. Let R be the set of all cyclically reduced conjugates of $r_0^{\pm 1}, \dots, r_l^{\pm 1}$. It is straightforward to check (using (3)) that for any two words $w_1, w_2 \in R$, representing distinct elements of E_{i+2}, at most three pairs of t_i-letters can cancel in the product $w_1 w_2$. It follows that the length of every piece relative to R is at most 7 and so R satisfies the small cancellation condition $\mathrm{C}'(1/6)$ (see [18]). Let $N \lhd E_{i+2}$ be the normal closure of R in E_{i+2} and let $G_{i+3} := E_{i+2}/N$. Then [18, Cor. 1] states that the natural epimorphism $\nu : E_{i+2} \to G_{i+3}$ is injective on $\langle F_{i+2}, u_{i+3} \rangle$. Letting $a_i := \nu(t_i) \in G_{i+3}$ and identifying G_{i-1} and G_i with their images in G_{i+3} we see that a_i centralizes G_{i-1} in G_{i+3} (by (2)). Moreover, since $r_j = 1$ in G_{i+3}, for every $j = 0, 1, \dots, l$, we see that the generating set $\{t, v_1, \dots, v_l\}$ of E_{i+2} is mapped inside of $\langle u_i, a_i u_i a_i^{-1} \rangle$ in G_{i+3}. Thus $G_{i+3} = \langle u_i, a_i u_i a_i^{-1} \rangle \leqslant \langle G_i, a_i G_i a_i^{-1} \rangle$.

Evidently, continuing this way we will obtain a strictly ascending sequence of finitely generated groups $G_0 < G_1 < \dots$ that satisfies the properties (i) and (ii) from Definition 2.1. It is also clear that this sequence satisfies property (iii) because for each $i \in \mathbb{N}$, $G_i < F_i < G_{i+1}$ and F_i has (FA).

4. Proof of the main result

Proof of Theorem 1.1. Using any of the procedures from the previous section we can construct a strictly ascending sequence $G_0 < G_1 < \dots$ of finitely generated groups with properties (i),(ii) and (iii). Section 2 tells us how to produce a sequence of finitely generated groups $L_1, L_2, \dots$ together with epimorphisms $\phi_i : L_i \to L_{i+1}$, $i \in \mathbb{N}$. Let $L := \lim_{i \to \infty}(L_i, \phi_i)$ be the direct limit of this sequence.

Observe that each L_i, $i \in \mathbb{N}$, splits as a non-trivial amalgamated free product, and so L_i has a fixed point-free action on the corresponding simplicial Bass-Serre tree T_i (cf. [19, Thm. I.4.7]).

We now show that L does not have (F$\mathbb{R}$) using a general theorem of Culler and Morgan [6, Thm. 4.5], see also [21, Thm. 4.7]). We will also give a specific proof that L does not have (F$\mathbb{R}$) by describing the image of a edge of T_1 in the limit $\mathbb{R}$-tree T.

An action of a group on an $\mathbb{R}$-tree is determined be a translation length function, which is a function α defined on the conjugacy classes of G. Thus if C is a conjugacy class and $g \in C$, then the value of α on C is 0 is the minimal distance $d(x, gx)$ for $x \in G$. This value is zero if g fixes a point of G and, for a finitely generated group, is the value taken when x is on the axis of g. Let $PLF(G)$ be the set of projective classes of translation length functions. Culler and Morgan show that this space is compact.

The action of L_i on T_i gives an action of L_1 on T_i and so we have a sequence in $PLF(L_1)$ which has a convergent subsequence. The limit of this subsequence in fact will give a non-trivial action of L on an $\mathbb{R}$-tree.

In an earlier version of this paper it was incorrectly stated that the limit action was the strong limit of the actions on the T_i. What this would mean is that if one puts a natural metric on T_i so that the length of an edge in T_i is the sum of the lengths of its subdivided parts in T_{i+1}, then if x_i, y_i are the images in T_i of points $x = x_1, y = y_1$ in T_1, then $d(x_i, y_i)$ is constant for i large enough. For a strong limit one can deduce information about arc stabilizers of the limit action from the arc stabilizers of the T_i. In our case the limit action is not a strong limit. The whole sequence for the T_i's does converge in $PLF(L_1)$, but one has to scale the metrics to get a non-trivial metric on the limit tree. This metric is unique up to scaling.

We now give an analysis of what happens to an edge of T_1 in the folding sequence.

Let T_n be the tree formed at stage n. The tree T_n is formed from T_{n-1} by subdivision and folding as in Figure 1. Let x, y be adjacent vertices in T_1, and $x_i = \phi_{i-1}\phi_{i-2}\ldots\phi_1(x), y_i = \phi_{i-1}\phi_{i-2}\ldots\phi_1(y)$. There is an obvious metric on T_1 so that $d(x, y) = 1$.

Lemma 4.1. *If a vertex appears for the first time in T_n as a result of subdivision, then no folding occurs at that vertex in T_{n+1} or T_{n+2}. Folding does occur at that vertex in T_{n+3}.*

Proof. The folding is illustrated in Fig 2 for $n = 1$. It is only necessary to consider $n = 1$, as the situation for T_n is obtained by adding $n-1$ to each suffix. The label of vertex indicates the stabilizer of that vertex. The centre vertex x is created by subdivision. We have $a_1 x = x$. Also if e' is the edge to the right of x then $a_1 e'$ is the subdivided edge to the left of x. The only folds that can affect the centre vertex are Type II folds. In T_2, folding occurs between two edges if they are in the same K_2-orbit . In T_3, folding occurs if two edges are in the same K_3-orbit. In T_4 folding occurs if they are in the same $a_3 H_4 a_3^{-1}$-orbit. Since $K_4 = a_3 H_4 a_3^{-1}$ and $a_1 \in K_4$ the two incident edges to the centre vertex are folded in T_4 but not before. ☐

Let I be the unit interval in which each point is specified by its binary expansion. We regard I as a simplicial tree with two vertices. Let $I_n,\ n = 0, 1, 2, \ldots$ be the finite trees that are defined by induction as follows. Let $I_0 = I$. Suppose I_n has been defined. Then I_{n+1} is the tree obtained from I_n by subdividing each edge (introducing a single mid-point to each edge), and then if $v \in VI_n$ was formed by subdivision in I_{n-2}, then a fold of the two incident edges takes place. Clearly every vertex of I_n can be labelled by a binary expansion with n terms after the point. A vertex can have more than one label. Thus .0111 and .1001 label the same point in I_4.

Lemma 4.2. *Let I_n be the tree as defined above.*

(i) *In I_n each vertex, apart from* 1 *is uniquely specified by a label of length n that contains no subsequence* 111.
(ii) *In I_n each internal vertex in the geodesic path $[0, 1]_n$ joining* 0 *and* 1 *is specified uniquely by a label of length n that contains no subsequence* 000 *or* 111.

Proof. The proof for (i) is by induction on n. The statement is true for $n = 0, 1, 2$. Suppose $n \geq 2$ and that the lemma is true for n. Thus every vertex is specified by a label without any subsequence 111. The vertices that arose by subdivision in T_{n-2} have labels that end 100. A fold takes place at such a vertex labelled x and so in I_{n+1} there is a vertex with two labels corresponding to the mid-points of the edges incident with x one of which ends in 1001 and the other in 111. We choose the label ending in 1001 to represent this vertex.

The proof of (ii) is also by induction on n. The statement is true for $n = 0, 1, 2$. Suppose $n \geq 2$ and that the statement is true for n. Thus every vertex in $[0, 1]_n$ is specified by a label without any subsequence 000 or 111. Any vertex of I_n that is not in $[0, 1]_n$ will still not be in $[0, 1]_{n+1}$ after subdivision and folding, nor will any vertex that is the mid-point of an edge that is not in $[0,1]_n$. If $v \in [0,1]_n$, then adding a 0 to its label gives the same point in I_{n+1} and this will be in $[0,1]_{n+1}$ unless the label for v ended in 100 in which case it is not in $[0,1]_{n+1}$. In all cases adding a 1 to the label of v will be the label of a vertex in $[0,1]_{n+1}$. The statement is proved for $n+1$. □

We can see from the last two Lemmas what happens to an edge in the folding sequence. We define a metric in T_i so that the sequence $d(x_i, y_i) = 1$ for every i. As is shown in Figure 2 there are paths with 2 edges joining x_2, y_2, 4 edges joining x_3, y_3 and 8 edges joining x_4, y_4. However, because of folding there are only 14 edges in the path joining x_5, y_5 and so the length of an edge in T_5 is 1/14. After n iterations there will be a path with 2^n edges joining x_n, y_n, but this path can back-track as shown in Fig 2. In fact what happens is that after a new vertex is created by subdivision, then after three more iterations that vertex is moved off the geodesic. If we stick with the natural metric so that an edge in T_n has length $1/2^n$ then $d(x_n, y_n)$ tends to zero, and so when we make the space Hausforff by identifying points zero apart the limit tree will have only one point. However one can linearly expand edges so that the limit is non-zero. If t_n is the number of edges in the geodesic path joining x_n, y_n, then this squence will tell one what the scaling has to be. Thus if δ_n is the length of an edge in T_n and we put $t\delta_n = 1\backslash t_n$ then $d(x_n, y_n) = 1$ for every n.

If we take the binary expansion for points in the unit interval I as representing the points on the edge joining x, y, then the points that are always on the geodesic path joining x_n, y_n are given by those with a binary expansion that has no subsequence 0111 or 1000. This is clear from the last two Lemmas.

Let ϵ_n be the real number with binary expansion $.00...01001001001001\overline{001}$ in which there are initially n zero 's.

Let $z \in I$ be a point with a finite binary expansion of length n ending in a 1 and which contains no subsequence 1000 or 0111, then every w such that $z < w < z + \epsilon_n$ has an expansion that contains 0001 and every w such that $z - \epsilon_n < w < z$ has an expansion containing 0111. In the limit space for each such z the point $z + \epsilon_n$ is identified with $z - \epsilon_n$, when the limit space is made Hausdorff. Every point of I with an expansion that contains a subsequence 0111 or 1000 must occur in a sub-interval $[z - \epsilon_n, z + \epsilon_n$. The points of I with binary expansion containing no subsequence 1000 or 0111 form a Cantor set. If we scale as above and identify the end points of subintervals as indicated then we obtain the geodesic $[x, y]$ joining the limit points x, y which is homeomorphic to the unit interval I.

In the limit space a_1 fixes the point $.1 - \epsilon_1 = .1 + \epsilon_1$. This is the only point of the geodesic joining $\bar{x}, \bar{y}$ fixed by a_1, since a_1 maps the second half of the geodesic to the first half. In a similar way each a_i fixes a single point on the geodesic and so $a_1 a_2$ will be a hyperbolic element. This means that the action is non-trivial. The translation length function takes non-zero value on the conjugacy class of $a_1 a_2$.

In an action that is the limit of a folding sequence of simplicial actions, each of which has one orbit of edges and two orbits of vertices, there are only a limited number of possibilities for how the folding affects an edge. If the limit is a strong limit, then no folding will affect the edge and at each iteration subdivision is the only operation that affects the edge. If a folding operation does result in folding of an edge, and each iteration involves a single subdivision as in our example, then the limit tree will not be a strong limit and the folding of the edge will be as in our example in that the first fold in the edge is at the centre vertex and it takes place after n iterations after that centre vertex was created, and n is at least 3.

We now show that L does have property (FA). We assume that L contains the sequences of subgroups H_i, K_i as described in Section 2, and that $a_i \in K_{i+3}$ if i is even and $a_i \in H_{i+3}$ is i is odd. Suppose L acts on a simplicial tree T. We want to show that the action is trivial, i.e. there is a vertex fixed by L. Let x_i be a vertex fixed by K_i and let y_i be a vertex fixed by H_i. We know by property (iii) that such vertices must exist. Choose x_i, y_{i+1} so that they are a minimal distance d_i apart. Here we take distance as being the number of edges in the geodesic path joining the two vertices. If $x_i = y_{i+1}$ then this vertex is fixed by $\langle H_i, K_{i+1} \rangle = L$ and we are done. Let m be such that d_j is constant for $j \geq m$. We suppose $x_m \neq y_{m+1}$.

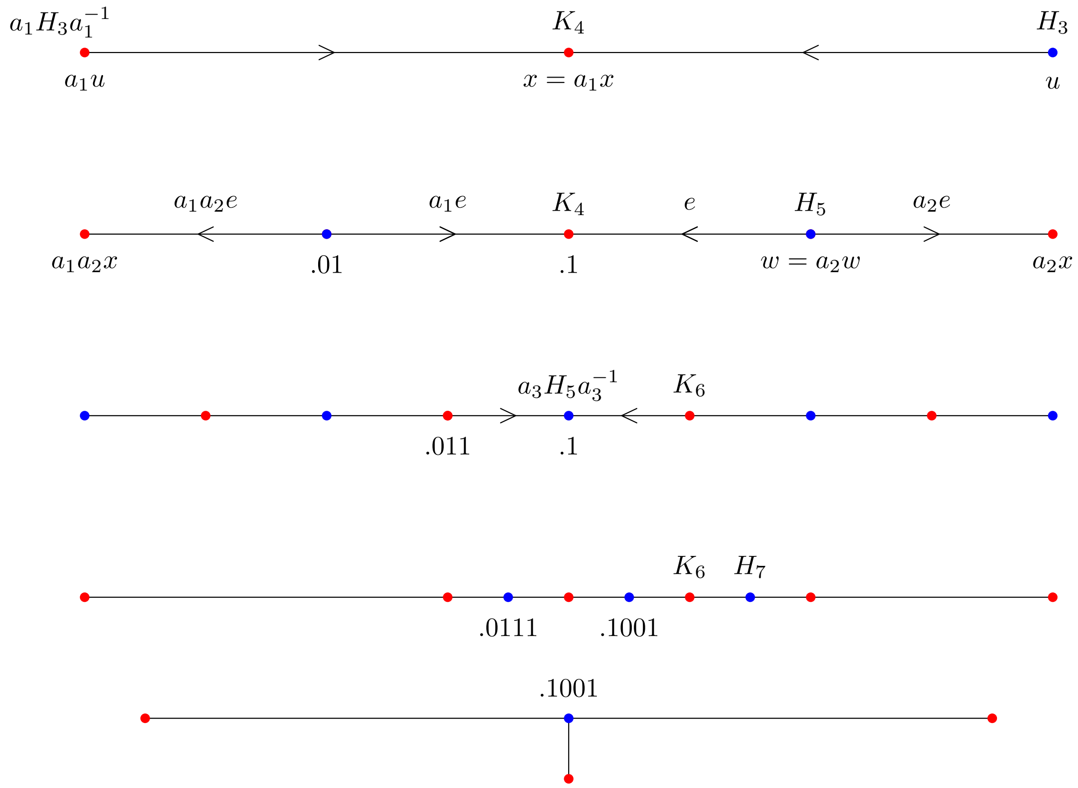

FIGURE 2. Image of an edge in four iterations

We know that H_{i+2} is generated by subgroups of H_i and K_{i+1}. If follows that H_{i+2} will fix a vertex of the geodesic $[x_i, y_{i+}]$. We therefore have a contradiction unless $x_m = x_{m+2}$ Since $H_{m+1} < H_{m+2}$ we have $x_m = x_{m+1} = x_{m+2}$. It follows easily that $x_m = x_j$ for all $j > m$ and similarly that $y_j = y_{m+1}$ for all $j > i+1$. We saw in Section 2 that $a_i \in K_{i+3}$ if i is even and $a_i \in H_{i+3}$ if i is odd. Also $a_i^{-1}K_{i+1}a_i = H_{i+1}$ for i even and $a_i^{-1}H_{i+1}a_i = K_{i+1}$ if i is odd. Thus for any $i > m$ either $a_i^{-1}H_{i+1}a_i = K_{i+1}$ or $a_iH_{i+1}a_i^{-1} = K_{i+1}$. It follows easily that x_m is fixed by both $K_{i+1}$and H_{i+1} for every $i > m$, which means that x_m is fixed by L and we have a contradiction. Thus L has property (FA).

For the second claim of the theorem, suppose that P is a finitely presented group that maps onto L. By a standard argument (see [5, Lemma 3.1]), there is $n \in \mathbb{N}$ such that P maps onto L_n. The idea is simple: P is a quotient of some free group $\mathbb{F}_m$ modulo a normal subgroup $N \lhd \mathbb{F}_m$, which is normally generated by finitely many elements. Any epimorphism from P to L gives rise to an epimorphism $\zeta : \mathbb{F}_m \to L$, which factors through each L_i. Let $N_i \lhd \mathbb{F}_m$ denote the kernel of the corresponding homomorphism $\zeta_i : \mathbb{F}_m \to L_i$. It follows that $N_i \leqslant N_{i+1}$ and $\ker \zeta = \bigcup_{i \in \mathbb{N}} N_i$. Evidently $N \leqslant \ker \zeta$, hence there is $l \in \mathbb{N}$ such that $N \leqslant N_i$ for all $i \geq l$ because N is the normal closure of finitely many elements of $\mathbb{F}_m$. Consequently, the homomorphism $\zeta_i : \mathbb{F}_m \to L_i$ factors through the natural homomorphism from $\mathbb{F}_m$ to $\mathbb{F}_m/N \cong P$ whenever $i \geq l$. Finally, since L_1 is finitely generated and $\zeta : \mathbb{F}_m \to L$ is surjective there is $k \in \mathbb{N}$ such that the homomorphisms $\zeta_i : \mathbb{F}_m \to L_i$ are surjective for all $i \geq k$. Thus one can take $n = \max\{k, l\}$. Therefore P will act non-trivially on the Bass-Serre tree T_n and so it does not have (FA). This concludes the proof. $\square$

5. The action as a strong limit

In [15] Levitt and Paulin show that any action of a group on an $\mathbb{R}$-tree is a strong limit of geometric actions. As I could not see how the action just described was a strong limit, I thought that this result must be incorrect. However after more consideration I was able to construct a strongly limiting folding sequence which did have this action as its limit: or at least this action with some scaling applied.

The new folding sequence is shown in Fig 3. The edges which are incompressible, i.e. the edge label is a proper subgroup of each incident vertex label, are thick black. If one contracts the compressible edges, one obtains a folding sequence similar to the one in Fig 1. The distances of the vertices in the first graph of groups going from the left hand vertex are

$$a, \frac{1}{\lambda}a, \frac{(2\lambda^3 - 2\lambda + 1)}{\lambda(2\lambda - 1)}a, \frac{(2\lambda^2 - 2\lambda + 1)}{(2\lambda - 1)}a, \frac{\lambda}{(2\lambda - 1)}a.$$

Here $1/2 < \lambda \leq (\sqrt{5}-1)/2$. In Fig 3 $\lambda = 0.6$. In the final graph of groups the suffixes have increased by one and the distances are scaled by λ. For $\lambda = (\sqrt{5}-1)/2$, the middle edge disappears, but one still has a folding sequence as in Fig 4.

It follows from viewing the action as a strong limit that we can see that the arc stabilizers in the limit action are conjugates of the G_i's. Every point in the limit tree has stabilizer that is a union of conjugates of the G_i. Thus if a point of T is One point in the orbit of points in the limit tree corresponding to the image of the left hand vertex in each graph of groups has stabilizer that is the union of the G_i's. It can be seen that there is just one direction at that point, since there is just one direction at each vertex of the Bass-Serre tree corresponding to the left hand vertex of the graph of groups at each stage of the iteration. If one removes these limit points from the limit $\mathbb{R}$-tree T, one obtains a connected space T' that is an $\mathbb{R}$-tree, but which is not complete.

It is interesting - to me at least - to attempt to use the above analysis to understand something about the structure of $PLF(L)$, the space of projectivized translation length functions of fixed-point free actions of L on $\mathbb{R}$-trees. Culler and Morgan [6] show that this space is compact. One gets a different action for each value of $\lambda, 1/2 < \lambda \leq (\sqrt{5}-1)/2$. The obvious question is what action is the limit as λ tends to $\frac{1}{2}$. Let $\ell(\lambda)$ be the length function of L corresponding to the limit of the folding sequence for $a = 1$ and a particular value of λ. Then $(2\lambda - 1)\ell(\lambda)$ is the length function for $a = 2\lambda - 1$. Observe that the vertices in Fig 3 for this length function will be at distances $(2\lambda - 1, (2\lambda - 1)/\lambda, (2\lambda^3 - 2\lambda + 1)/\lambda, 2\lambda - 2\lambda + 1, \lambda$, and as λ tends to $1/2$, these distances tend to $0, 0, 1/2, 1/2, 1/2$. These distances correspond to a folding sequence as in Fig 1. Thus as λ tends to a $1/2$ the sequence of actions tends to the first action described in this paper. This convergence does not correspond to a strongly limiting sequence however. The fact that our original action is a strong limit follows from the fact that it is also the action for $\lambda = 7/12$. To see this note that in the first graph of groups of Fig 3 the right hand vertex labelled K_{i+2} is at distance $\frac{a\lambda}{(2\lambda-1)}$ which we take to be $1 - \epsilon_0$ and the vertex labelled G_{i+2} in the last graph of groups is at distance $a\lambda$ which we take to be $\epsilon_0 = .\overline{001} = 1/7$. It follows from our analysis of what happens to an edge, that if $x \in K_1 - K_0$ and $y \in H_2 - H_1$ then the hyperbolic length of xy is $1 - 2\epsilon_0$. This give the equality $a\frac{\lambda}{(2\lambda-1)} = 1 - \epsilon_0$. Hence $a = 12/49, \lambda = 7/12$. It follow that $PLF(L)$ has a subspace that is the closed interval $[1/2, (\sqrt{5}-1)/2]$ in which $1/2$ is identified with $7/12$. It seems likely that this subspace is all of $PLF(L)$.

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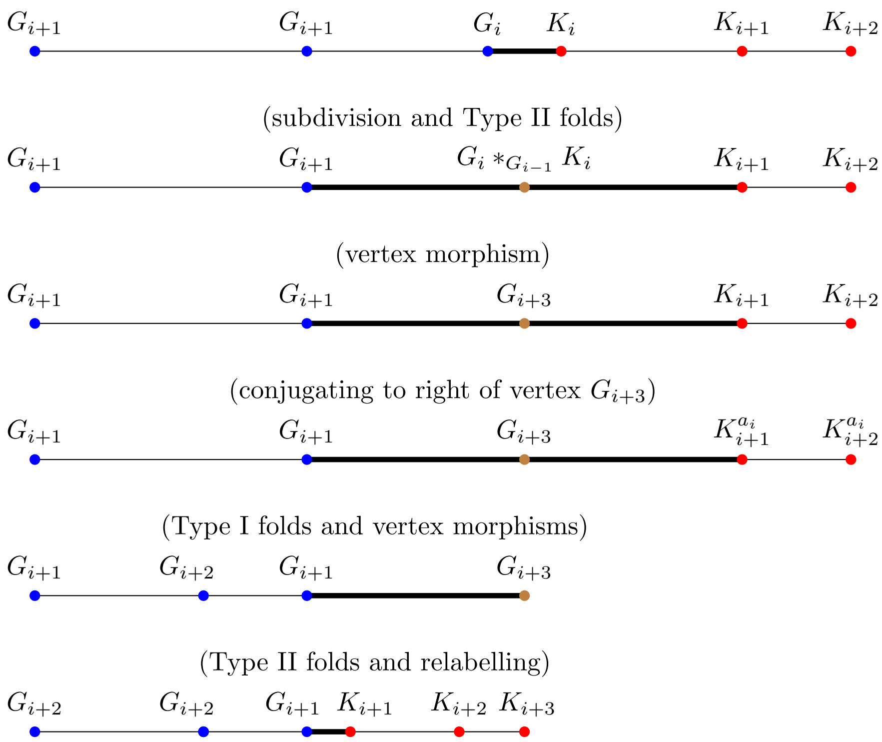

FIGURE 3. Strongly limiting sequence

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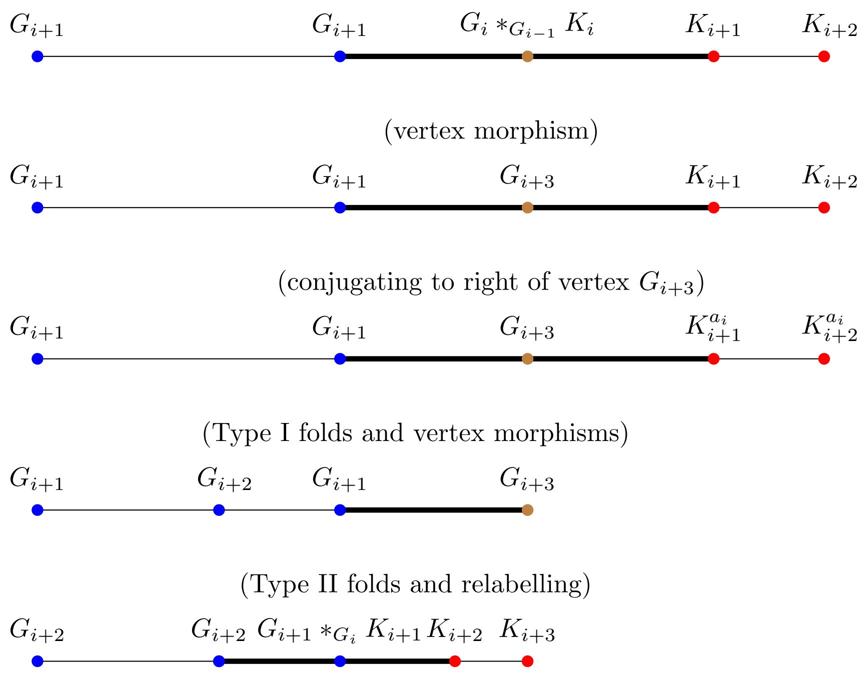

FIGURE 4. The sequence for $\lambda = (\sqrt{5} - 1)/2$

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