Finitely presented groups acting on trees

Chiswell Symposium

Queen Mary, November 2009

M. J. Dunwoody

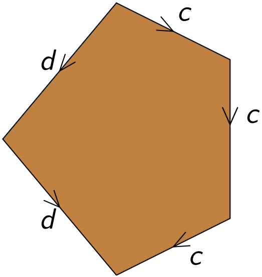
University of Southampton

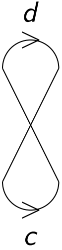
joint with Andrew Bartholomew

November 8, 2009

Group presentations and tracks

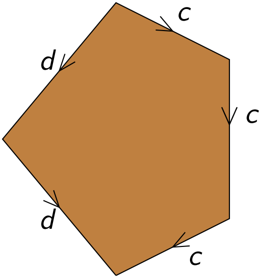
The cell complex for the trefoil group $G = \langle c, d | c^3 = d^2 \rangle$

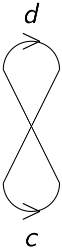




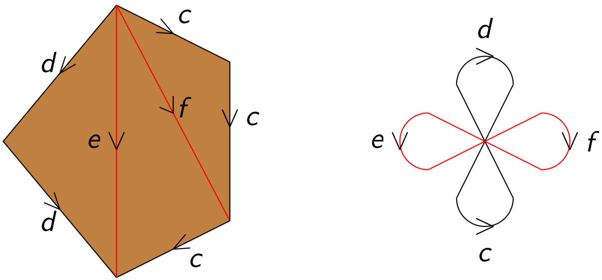
Attach the 5-sided disc to the figure eight as specified by the letters and arrows. The space X has $\pi_1(X) = G$.

A presentation can be changed so that every relation has length at most three.

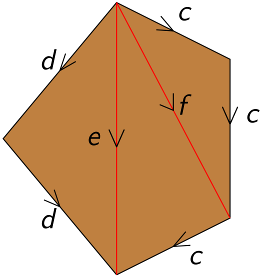


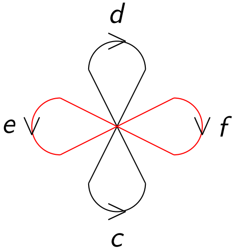


A presentation can be changed so that every relation has length at most three.



A presentation can be changed so that every relation has length at most three.



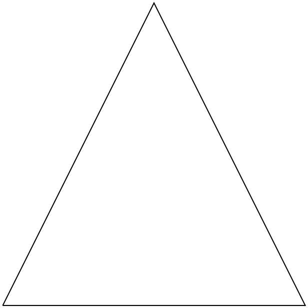


Thus $G = \langle c, d, e, f | d^2 = e, e = fc, f = c^2 \rangle$.

The cell complex X consists of three 3-sided 2-cells attached to a 4-leaved rose.

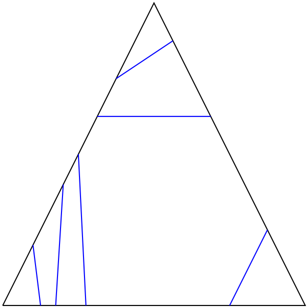
Let X be a cell complex in which each 2-cell is 3-sided.

A pattern is a subset of X which intersects each 2-cell in a finite number of disjoint lines each of which intersects the boundary of the 2-cell in its two end points which lie in distinct edges.

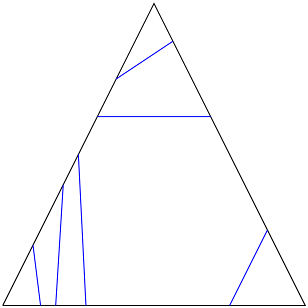


Let X be a cell complex in which each 2-cell is 3-sided.

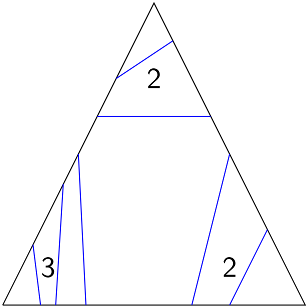
A pattern is a subset of X which intersects each 2-cell in a finite number of disjoint lines each of which intersects the boundary of the 2-cell in its two end points which lie in distinct edges.



If X has m 2-cells then a pattern is specified (up to an obvious equivalence) by a $3m$-vector in which there are three coefficients for each 2-cell which record the number of lines joining the two edges at each corner. Thus for previous 2-cell

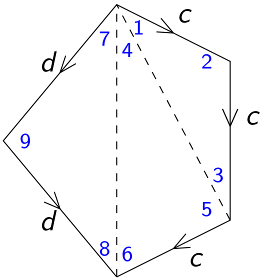


If X has m 2-cells then a pattern is specified (up to an obvious equivalence) by a $3m$-vector in which there are three coefficients for each 2-cell which record the number of lines joining the two edges at each corner. Thus for previous 2-cell



the coefficients $2, 2, 3$ record the intersection of the pattern with that particular 2-cell.

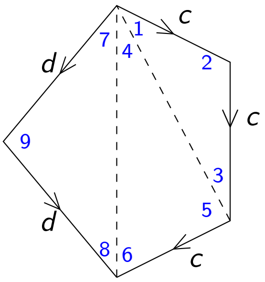
For the complex X for the trefoil group G a pattern is specified by a 9-vector. The i-th coefficient corresponds to the number of lines crossing the i-th corner as indicated below



An example is as follows. The 9-vector

$t_1 = (1, 1, 1, 0, 2, 0, 0, 0, 0)$

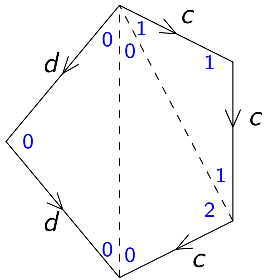
corresponds to the pattern



An example is as follows. The 9-vector

$t_1 = (1, 1, 1, 0, 2, 0, 0, 0, 0)$

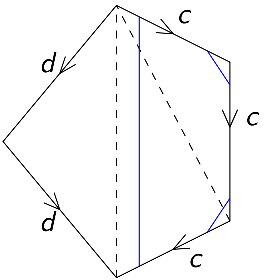
corresponds to the pattern



An example is as follows. The 9-vector

$t_1 = (1, 1, 1, 0, 2, 0, 0, 0, 0)$

corresponds to the pattern



This pattern is connected. A connected pattern is called a track.

A $3m$-vector corresponds to a pattern, if and only if

- (i) Each entry is a non-negative integer.
- (ii) It is a solution vector to a finite set of linear equations called the matching equations.

For the trefoil complex X a vector of non-negative integers $x = (x_1, x_2, \ldots, x_9)$ is a pattern if it satisfies the matching equations

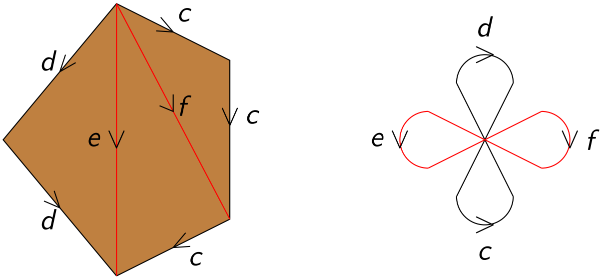
$x_1 + x_2 = x_2 + x_3 = x_5 + x_6$, (number of intersection points with edge c)

$x_1 + x_3 = x_4 + x_5$ (number of intersection points with edge f)

$x_4 + x_6 = x_7 + x_8$ (number of intersection points with edge e)

$x_7 + x_9 = x_8 + x_9$ (number of intersection points with edge d)

The universal cover $\tilde{X}$ is a 2-dimensional cell complex. Its 1-skeleton is the Cayley graph of G with respect to the generating set corresponding to the 1-cells of X.



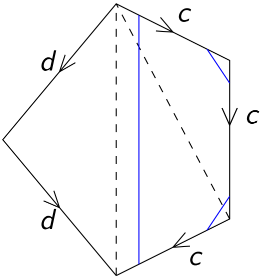
In this case $\tilde{X}$ has vertex set which can be identified with G, four orbits of 1-cells and three orbits of two cells. There is a free action of G on $\tilde{X}$ (on the left) and $X = G\backslash\tilde{X}$

A pattern P in X lifts to a pattern $\tilde{P}$ in $\tilde{X}$. Each component track t of $\tilde{P}$ separates $\tilde{X}$, i.e. $\tilde{X} - t$ has two components. This means that the dual graph to $\tilde{P}$ is a tree $T = T(P)$. The action of G on $\tilde{X}$ induces an action of G on T. By Bass-Serre theory we obtain a decomposition of G as the fundamental group of a graph of groups. This graph of groups has underlying graph $G \backslash T$, the edges and vertices of which are labelled by appropriate stabilizers of T.

Thus a pattern in X determines a decomposition of G. If the pattern consists of a single track then the decomposition is as a free product with amalgamation if the track in X is separating and it is as an HNN-group if the track is untwisted and non-separating.

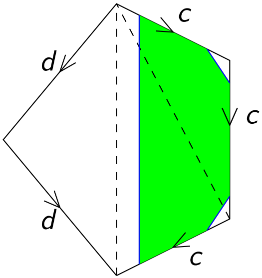
The track shown in blue is separating and gives a decomposition of G as a free product with amalgamation

$G = \langle d \rangle *_{\langle d^2 = c^3 \rangle} \langle c \rangle$.

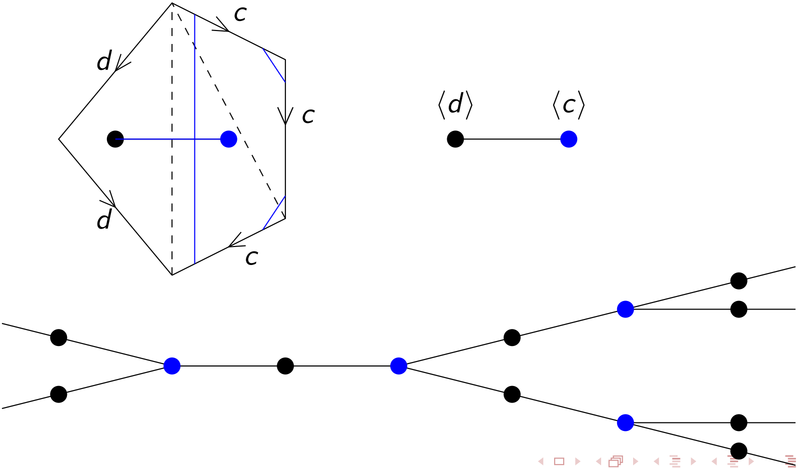


The track shown in blue is separating and gives a decomposition of G as a free product with amalgamation

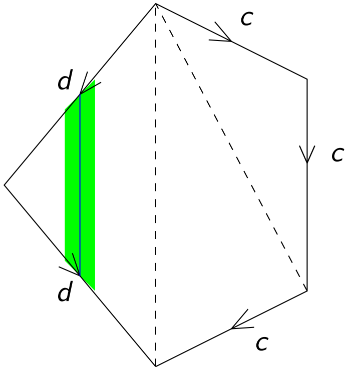
$G = \langle d \rangle *_{\langle d^2 = c^3 \rangle} \langle c \rangle$.



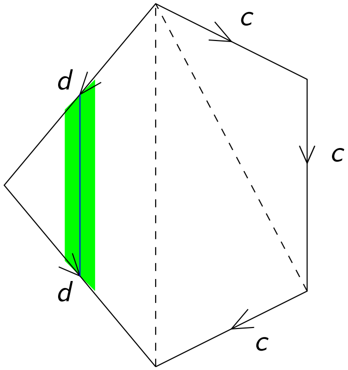
The decomposition of the group $G = \langle d \rangle *_{\langle d^2 = c^3 \rangle} \langle c \rangle$ corresponding to this track corresponds to an action of G on a tree T for which $G \backslash T$ has one orbit of edges and two orbits of vertices.



A tracks t is twisted if the pattern $2t$ is also a track. A track is untwisted if and only if a small neighbourhood of it is homeomorphic to $t \times I$

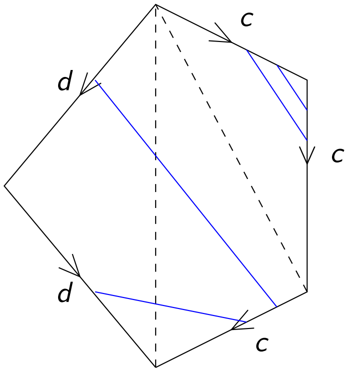


A track t is twisted if the pattern $2t$ is also a track. A track is untwisted if and only if a small neighbourhood of it is homeomorphic to $t \times I$. For this track a small neighbourhood is a Möbius band.



A separating track is always untwisted. It t is twisted, then $2t$ is separating and hence untwisted.

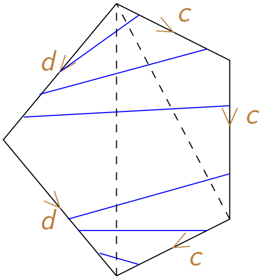
The track t is twisted so the pattern $2t$ is also a track. The separating track $2t$ gives the trivial decomposition $G = G *_H H$ where H has index two in G



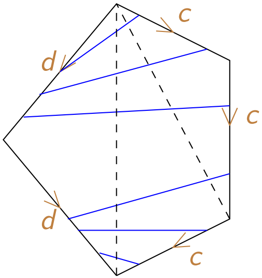
The trivial decomposition $G = G *_H H$ where H has index two in G corresponds to an action of G on the tree

In this case G fixes the centre vertex and H is the stabilizer of each of the other vertices. If a track corresponds to a trivial decomposition then the corresponding tree has diameter two. It may not be possible to decide if a decomposition is trivial or not. One has to be able to decide if a given a set of elements of a group whether it generates a proper subgroup.

The track shown in blue is non-separating and untwisted, and gives a decomposition of G as an HNN-group.

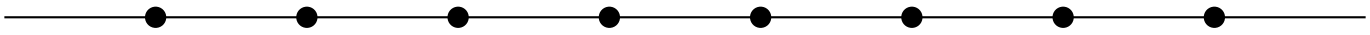


The track shown in blue is non-separating and untwisted, and gives a decomposition of G as an HNN-group.



Such a track is always associated with a homomorphism $G \to \mathbb{Z}$.
In this case $c \mapsto 2, d \mapsto 3$.

A homomorphism $G \to \mathbb{Z}$ gives an action on the tree

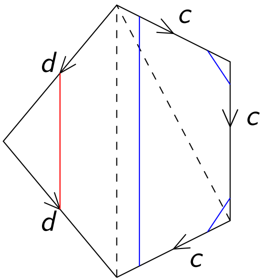


Not every action of a group on a tree corresponds to a pattern. An action corresponding to a pattern with untwisted component tracks is called geometric. However every action on a tree (without involutions) is resolved by a geometric action. What this means is that if S is a tree with a G-action then there is a G-morphism $T \to S$ where T a tree with a geometric G-action.

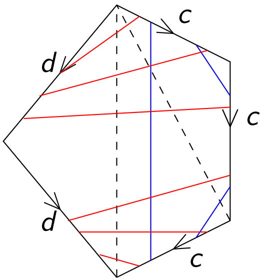
In particular this means that if G has a non-trivial action on a tree then it has a non-trivial geometric action on a tree. Thus there is an untwisted track for which the corresponding action is non-trivial.

A pattern is specified by a vector p. If p, q are patterns then so is $p + q$. Two tracks s, t are compatible if the pattern $s + t$ has component tracks s, t.

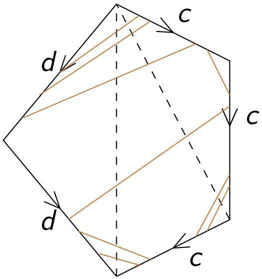
The tracks shown below are compatible



The tracks t_1. t_2 shown below are incompatible

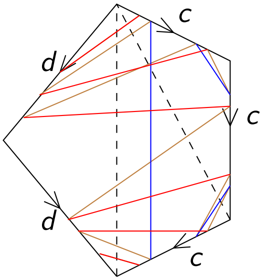


The sum $t_1 + t_2$ (a twisted track) is shown below



It has the same intersection with the 1-skeleton as $t_1 \cup t_2$.

The sum $t_1 + t_2$ (a twisted track) is shown below



It has the same intersection with the 1-skeleton as $t_1 \cup t_2$.

A pattern is a vector solution with non-negative integer coefficients to a set of homogenous linear equations. The set of solutions with non-negative rational coefficients form a cone which in projective space is the convex closure of finitely many extreme solutions. Multiplying by a rational we obtain a track, called an extreme track.

A track t is extreme if and only if for a positive integer n the only patterns p, q for which $nt = p + q$ are $p = kt, q = (n - k)t$ where $k = 1, 2, \ldots, n - 1$. The trefoil presentation has 5 extreme tracks. The Higman presentation $\langle a, b, c, d | a^2 = bab^{-1}, b^2 = cbc^{-1}, c^2 = dcd^{-1}, d^2 = dad^{-1} \rangle$ has 8769 extreme tracks.

Andy Bartholomew in his 1987 thesis wrote programmes which gave the extreme tracks for a presentation, decomposed patterns into tracks and gave the decomposition corresponding to a track. His programmes are available on his homepage
http://www.layer8.co.uk/maths/index.htm

We were hoping to show that if a group has a non-trivial action on a tree, then at least one extreme track gives a non-trivial decomposition.

A track t is minimal if t cannot be written as a sum of two patterns. An extreme track is minimal, but not every minimal track is extreme.

For the trefoil there are 5 extreme tracks and 8 minimal tracks. Thus $t_3 = (0, 2, 0, 0, 0, 2, 1, 1, 0)$, $t_4 = (2, 0, 2, 2, 2, 0, 1, 1, 0)$ are extreme tracks. $t_6 = (1, 1, 1, 1, 1, 1, 1, 1, 0)$ is a minimal track. It is not extreme since $2t_6 = t_3 + t_4$.

There are only finitely many minimal tracks.

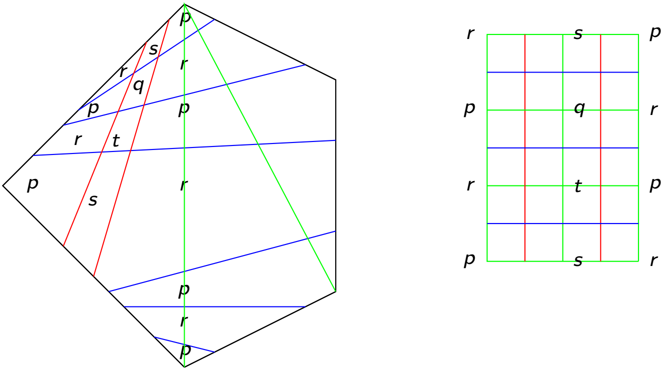
Every pattern p can be written as a sum $p = \alpha_1 t_1 + \alpha_2 t_2 + \cdots + \alpha_r t_r$ where $t_1, t_2, \ldots, t_r$ are extreme and $\alpha_1, \alpha_2, \ldots, \alpha_r$ are positive rationals.

But with minimal tracks, p can be written $p = \beta_1 m_1 + \beta_2 m_2 + \cdots + \beta_r m_s$ where $m_1, m_2, \ldots, m_s$ are minimal and the β_i's are positive integers.

An untwisted track is called minimal if it cannot be written as a non-trivial integer sum of untwisted tracks. There are finitely many minimal untwisted tracks. Every untwisted track can be written as an integer sum of minimal ones. The trefoil has 11 minimal untwisted tracks.

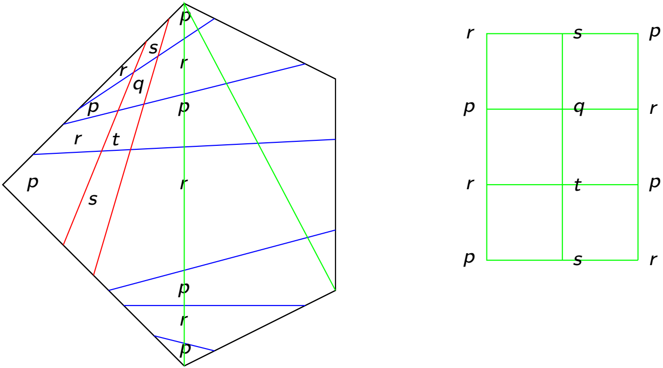
An untwisted track corresponds to a decomposition of the group G either as a free product with amalgamation or an HNN-group. Michah Sageev (1995) described a cubing $\tilde{C}$ associated with a finite number of such decompositions. The space $\tilde{C}$ is a $CAT(0)$ cube complex with a G-action.

A cube complex is similar to a simplicial complex except that the building blocks are n-cubes rather than simplices.

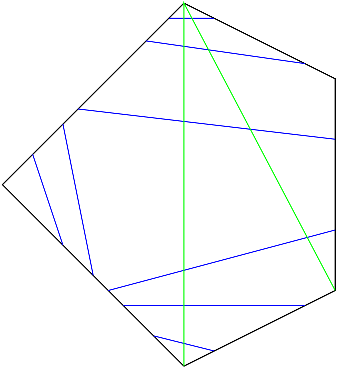


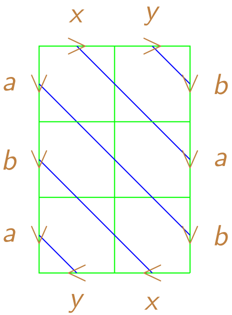
The diagram show the cube complex $C = G\backslash\tilde{C}$ for the two decompositions given by the red and blue tracks t_1, t_2.

Now note that the track $t_1 + t_2$ also corresponds to a “track ” in C. A pattern $\beta t_1 + \beta t_2$ where β_1, β_2 are non-negative integers will correspond to a pattern in C

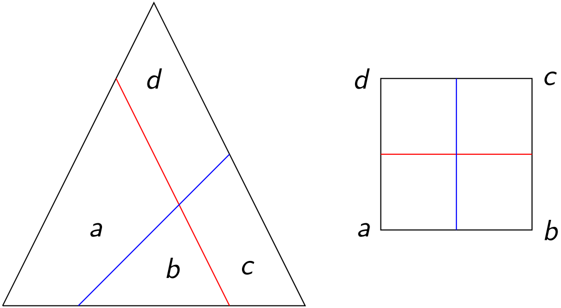


Now note that the track $t_1 + t_2$ also corresponds to a "track " in C. A pattern $\beta t_1 + \beta t_2$ where β_1, β_2 are non-negative integers will correspond to a pattern in C

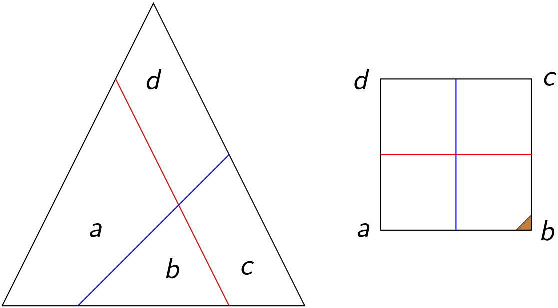




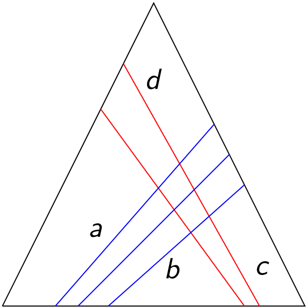
Each crossing point of tracks t_1, t_2 in X corresponds to a 2-cell of C. Let s_1, s_2 be the line segments of the two tracks in the three-sided two cell in X. One of the three sides contains 2 points of $s_1 \cup s_2$. In this case the bottom side.

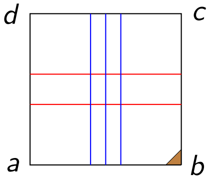


Now mark the corner (b) in the square in C corresponding to that side.

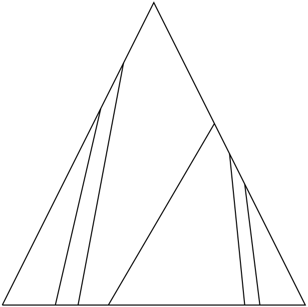


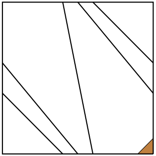
In the track corresponding to $2t_1 + 3t_2$ avoid lines crossing the marked corner.





In the track corresponding to $2t_1 + 3t_2$ avoid lines crossing the marked corner.





An untwisted track in C will determine a decomposition of G, i.e. an action on a tree. Let $\tilde{C}$ be the cubing corresponding to all minimal untwisted tracks. The advantage of using tracks in C rather than tracks in X is that the action of G on $\tilde{C}$ is not usually free. Thus if all the minimal untwisted tracks give trivial decompositions then G fixes a vertex of $\tilde{C}$ which means that any track in C will give a trivial action. Any untwisted track in X will correspond to an untwisted track in $C = G\backslash\tilde{C}$.

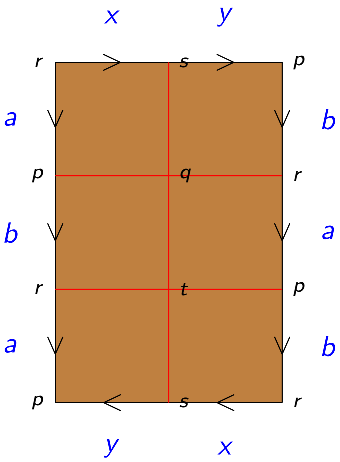
We have the following theorem.

[For any group G a subgroup H of G is *G-unsplittable* if for any action of G on a tree the induced action of H is trivial. i.e. it fixes a vertex.]

Theorem

A finitely presented group G has a finite list of n splittings for which the associated G-cubing $\tilde{C}$ has edge and vertex groups which are G-unsplittable, and every G-unsplittable subgroup of G fixes a vertex of $\tilde{C}$.

The group G has a non-trivial action on a tree if and only if at least one splitting in the list is non-trivial. The list of splittings is computable.

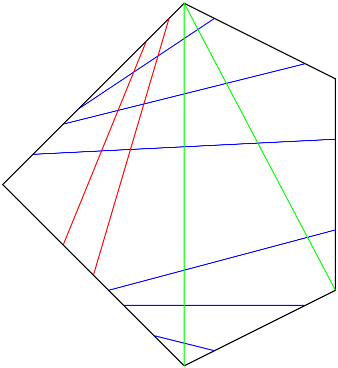


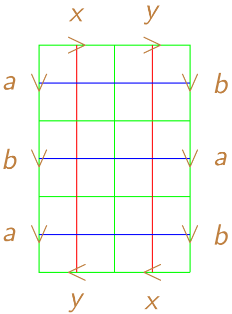
The above is the complex $C = G \backslash \tilde{C}$ for the trefoil group.

Sketch of Proof

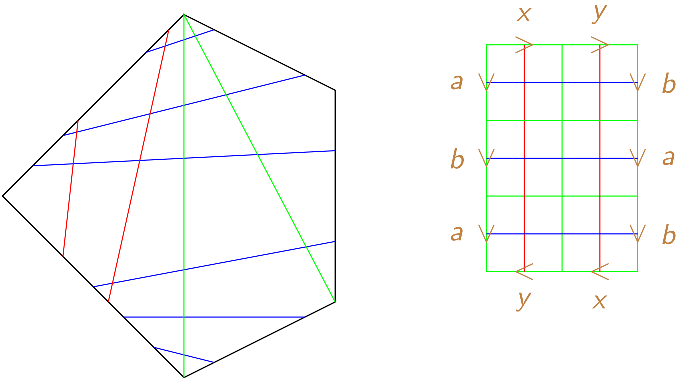
Let $t_1, t_2, \ldots, t_n$ be the minimal unwisted tracks. Consider their union in X. There are different ways of embedding the tracks. We assume that they intersect transversely with intersection points in the interior of the 3-sided 2-cells. For a particular embedding there is a G-map $\tilde{\theta} : \tilde{X} \to \tilde{C}$ (inducing $\theta : X \to C$) so that the lift $\tilde{t}_i$ of a track t_i is the inverse image of an orbit of hyperplanes in $\tilde{C}$.

Here is one embedding of two tracks





Here is another embedding of the two tracks



If t is an untwisted track then $t = \beta_1 t_1 + \beta_2 t_2 + \cdots + \beta_n t_n$ for non-negative integers β_i and we can form a track in $C = G \backslash \tilde{C}$ corresponding to t as described above. Thus each 2-cell of C corresponds to a pair i, j and we replace the β_i line segments in the i-direction and the β_j segments in the j-direction with $i + j$ segments joining the same boundary points as above. Then t will be the inverse image of this track t' in C.

We do the same for all embeddings of the union of the tracks t_i in X. We always get t as the inverse image of a track t' in C. Although the embedding changes the preimage does not. If all the t_i's give trivial decompositions, then G fixes a vertex of $\tilde{C}$. If this happens then the pattern in C corresponding to $\beta_1 t_1 + \beta_2 t_2 + \cdots + \beta_n t_n$ has $\beta_1 + \beta_2 + \cdots + \beta_n$ components and so the only tracks in C are those corresponding to the t_i's.

Thus if all the t_i's give trivial decompositions then every action of G on a tree is trivial.

HAPPY

RETIREMENT

IAN!

Notes and References

Similar results have been obtained when X is the spine of a 3-manifold. In this case a track corresponds to a patterned surface in the 3-manifold (see Dicks and Dunwoody [1989]), which is a generalization of a normal surface.

Jaco-Oertel [1984] andJaco-Tollefson [1995] have shown that extreme normal surfaces carry important information. Thus Jaco and Oertel give an algorithm for deciding if a manifold is Haken, and Jaco and Tollefson show that the the extreme solutions contain a set of 2-spheres giving a complete factorization of a closed 3-manifold. Our algorithm is a generalization of part of Haken's algorithm for deciding the genus of a knot. (See Hemion [1992]).

[Notes and References]

Nicholas Touikan has recently given an algorithm for finding the Grusko decomposition of a finitely presented group using tracks.

References

Warren Dicks and M.J.Dunwoody, Groups acting on graphs, Cambridge University Press, 1989.

M.J.Dunwoody, The accessibility of finitely presented groups, Invent. Math. 81 (1985) 449-457.

G. Hemion, The classification of knots and 3-dimensional spaces, Oxford University Press, 1992.

[Notes and References]

W.Jaco and U.Oertel, An algorithm to decide if a 3-manifold is a Haken manifold, Topology **23** (1984) 195-201.

W.Jaco and J.L.Tollefson, Algorithms for the complete decomposition of a closed 3-manifold, Illinois J. Math. 39 (1995) 358-406.

M.Sageev, Ends of group pairs and non-positively curved cube complexes, Proc. London Math. Soc. (3) 71 (1995) 585-617.

N.W.M.Touikan, Effective Grushko decompositions, arXiv:0906.3902 (2009).