HOMOTOPY TYPES OF MOMENT-ANGLE COMPLEXES

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Abstract. We study the homotopy types of moment-angle complexes, or equivalently, of complements of coordinate subspace arrangements. The overall aim is to identify the simplicial complexes K for which the corresponding moment-angle complex $\mathcal{Z}_K$ has the homotopy type of a wedge of spheres or a connected sum of sphere products. When K is flag, we identify in algebraic and combinatorial terms those K for which $\mathcal{Z}_K$ is homotopy equivalent to a wedge of spheres, and give a combinatorial formula for the number of spheres in the wedge. This extends results of Berglund–Jöllenbeck on Golod rings and homotopy theoretical results of the first and third authors. We also establish a connection between minimally non-Golod rings and moment-angle complexes $\mathcal{Z}_K$ which are homotopy equivalent to a connected sum of sphere products. We go on to show that for any flag complex K the loop spaces $\Omega\mathcal{Z}_K$ and $\Omega DJ(K)$ are homotopy equivalent to a product of spheres and loops on spheres when localised away from 2, and investigate how the homotopy class of the map from $\mathcal{Z}_K$ to $DJ(K)$ is determined by Whitehead products.

1. Introduction

Moment-angle complexes are key players in the emerging field of *toric topology*, which lies on the borders between topology, algebraic and symplectic geometry, and combinatorics [8]. The moment-angle complex $\mathcal{Z}_K$, a space with a torus action, appeared first in work of Davis and Januszkiewicz [11] as a topological generalisation of the smooth, projective toric varieties that were being studied intensively in algebraic geometry. The homotopy orbit space of $\mathcal{Z}_K$ is the Davis-Januszkiewicz space $DJ(K)$, which is a cellular model for the Stanley-Reisner ring $\mathbb{Z}[K]$, while the genuine orbit space of $\mathcal{Z}_K$ is the simplicial complex K. Buchstaber and the second author [7] introduced homotopy theoretical models of both the moment-angle complex $\mathcal{Z}_K$ and the Davis-Januszkiewicz space $DJ(K)$ as a homotopy colimit construction of the product functor on the topological pairs (D^2, S^1) and $(\mathbb{C}P^\infty, pt)$ respectively, with the colimit taken over the face category of the simplicial complex K. Recently, homotopy theoretical generalisations of moment-angle complexes and related spaces under the unifying umbrella of polyhedral products (see, for example,[15], [16], [1]) have brought stable and unstable decomposition techniques to bear, and are leading to an improved understanding of toric spaces.

The homotopy theory of moment-angle complexes and polyhedral products in general has far reaching applications in combinatorial and homological algebra,

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in particular, in the study of *face rings* (or *Stanley–Reisner rings*) of simplicial complexes and more general monomial ideals.

In this paper we consider the following related homotopy theoretical and algebraic problems:

- identifying the homotopy type of a moment-angle complex $\mathcal{Z}_K$ for certain simplicial complexes K;
- describing the multiplication and higher Massey products in the Tor-algebra $H^*(\mathcal{Z}_K) = \mathrm{Tor}_{\mathbf{k}[v_1,\ldots,v_m]}(\mathbf{k}[K],\mathbf{k})$ of the face ring $\mathbf{k}[K]$;
- describing the Yoneda algebra $\mathrm{Ext}_{\mathbf{k}[K]}(\mathbf{k},\mathbf{k})$ in terms of generators and relations;
- describing the structure of the Pontryagin algebra $H_*(\Omega DJ(K))$ and its commutator subalgebra $H_*(\Omega\mathcal{Z}_K)$ via iterated and higher Whitehead (Samelson) products;
- identifying the homotopy type of the loop spaces $\Omega DJ(K)$ and $\Omega\mathcal{Z}_K$.

The main objects and constructions are introduced in Section 2, together with some known preliminary results. In Section 3 we give topological interpretations of the Golod property of the face ring $\mathbf{k}[K]$. This ring is *Golod* if the multiplication in the Tor-algebra $H^*(\mathcal{Z}_K) = \mathrm{Tor}_{\mathbf{k}[v_1,\ldots,v_m]}(\mathbf{k}[K],\mathbf{k})$ is trivial, together with all higher Massey products (cf. [17], [18]). The topological interpretations are in terms of $H_*(\Omega\mathcal{Z}_K)$ being a free graded associative algebra, $H^*(\mathcal{Z}_K)$ having a trivial multiplication, and a certain identity holding for the Poincaré series of $H_*(\Omega\mathcal{Z}_K)$.

In Section 4 we concentrate on the case when K is a flag complex. Our techniques allow for a complete solution of the problems above in the case of flag complexes. A flag complex K is determined by its 1-skeleton K^1. The Yoneda algebra $\mathrm{Ext}_{\mathbf{k}[K]}(\mathbf{k},\mathbf{k}) \cong H_*(\Omega DJ(K))$ has a simple presentation as a graph product algebra. In Theorem 4.3 we explicitly describe the minimal generating set of its commutator subalgebra $H_*(\Omega\mathcal{Z}_K)$ and the basis of the corresponding iterated commutators.

From the homotopy-theoretic point of view, particularly important moment-angle complexes $\mathcal{Z}_K$ are those which have the homotopy type of a wedge of spheres. In this case the associative graded algebra $H_*(\Omega\mathcal{Z}_K)$ is free, and the multiplication in the Tor-algebra $H^*(\mathcal{Z}_K) = \mathrm{Tor}_{\mathbf{k}[v_1,\ldots,v_m]}(\mathbf{k}[K],\mathbf{k})$ is trivial, together with all higher Massey products, so the face ring $\mathbf{k}[K]$ is Golod. In Theorem 4.6 we show that for flag complexes K the Golodness of K is the precise algebraic criterion for $\mathcal{Z}_K$ being homotopy equivalent to a wedge of spheres. Using a result of Berglund and Jöllenbeck [4], this can be reformulated entirely in terms of the cup product: for a flag complex K, the moment-angle complex $\mathcal{Z}_K$ is homotopy equivalent to a wedge of spheres if and only if the cup product in $H^*(\mathcal{Z}_K)$ is trivial. Most importantly, there is a purely combinatorial description of the class of flag complexes K for which $\mathcal{Z}_K$ is homotopy equivalent to a wedge of spheres: the 1-skeleton of such K must be a *chordal graph.* This is an important concept in applied combinatorics and optimisation; the vertices in a chordal graph admit a *total elimination ordering* [14].

For general K, the Golod property of $\mathbf{k}[K]$ does do not guarantee that $\mathcal{Z}_K$ is homotopy equivalent to a wedge of spheres. The reason is that for some Golod complexes K, the cohomology ring $H^*(\mathcal{Z}_K;\mathbb{Z})$ may contain non-trivial torsion (see Example 3.3). Especially intriguing is that for all known examples of Golod complexes K, the moment-angle complex $\mathcal{Z}_K$ is a co-H-space (and even a suspension), and this may as well be true in general (see Question 3.4).

The next homotopy type of $\mathcal{Z}_K$ which we consider is a connected sum of sphere products, with two spheres in each product. Such a $\mathcal{Z}_K$ is obtained by attaching a cell to a wedge of spheres along one commutator relation. The corresponding face ring $\mathbf{k}[K]$ is *minimally non-Golod*, and the commutator subalgebra $H_*(\Omega\mathcal{Z}_K)$ in the Yoneda algebra $\mathrm{Ext}_{\mathbf{k}[K]}(\mathbf{k},\mathbf{k}) \cong H_*(\Omega DJ(K))$ is a one-relator algebra. In the case of a flag simplicial complex K the previous statement classifies minimally non-Golod Stanley-Reisner rings $\mathbf{k}[K]$, that is, $\mathbf{k}[K]$ is minimally non-Golod if and only if the moment-angle complex $\mathcal{Z}_K$ is homotopy equivalent to a connected sum of sphere products. It is an open question whether this classification criteria holds for a general simplicial complex (see Question 3.5).

In Section 5 we address the last problem in the list above. Our main result there is Theorem 5.3, which shows that for a flag K, both $\Omega\mathcal{Z}_K$ and $\Omega DJ(K)$ are homotopy equivalent to products of spheres and loops of spheres when localised away from 2. We also show that the integral Pontryagin algebra $H_*(\Omega\mathcal{Z}_K)$ is torsion-free (Corollary 5.2).

In the last Section 6 we go on to identify the iterated commutators of Theorem 4.3 that multiplicatively generate the algebra $H_*(\Omega\mathcal{Z}_K)$ with the Samelson products of the canonical generators of $H_*(\Omega DJ(K))$. In both Golod and minimally non-Golod flag cases, this leads to a canonical identification of the spheres in the wedge or connected sum $\mathcal{Z}_K$ with iterated Whitehead products arising from the inclusion of the bottom 2-spheres into $DJ(K)$, lifted to $\mathcal{Z}_K$ (see Propositions 6.1 and 6.2).

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2. Preliminaries

Let K be a finite simplicial complex on the set $[m] = \{1, 2, \ldots, m\}$, i.e. a collection of subsets $I = \{i_1, \ldots, i_k\} \subset [m]$ closed under inclusion. We refer to $I \in K$ as *simplices* or *faces* of K, and always assume that $\varnothing \in K$.

Assume we are given a set of m pairs of spaces

$$(\boldsymbol{X}, \boldsymbol{A}) = \{(X_1, A_1), \ldots, (X_m, A_m)\},$$

where $A_i \subset X_i$. For each simplex $I \in K$ we set

$$(\boldsymbol{X}, \boldsymbol{A})^I = \big\{(x_1, \ldots, x_m) \in \prod_{i=1}^m X_i \colon\ x_i \in A_i \quad \text{for } i \notin I\big\}.$$

The *polyhedral product* of $(\boldsymbol{X}, \boldsymbol{A})$ corresponding to K is the following subset in $\prod_{i=1}^m X_i$:

$$(\boldsymbol{X}, \boldsymbol{A})^K = \bigcup_{I \in \mathcal{K}} (\boldsymbol{X}, \boldsymbol{A})^I = \bigcup_{I \in \mathcal{K}} \Big(\prod_{i \in I} X_i \times \prod_{i \notin I} A_i\Big).$$

In the case when all the pairs (X_i, A_i) are the same, i.e. $X_i = X$ and $A_i = A$ for $i = 1, \ldots, m$, we use the notation $(X, A)^K$ for $(\boldsymbol{X}, \boldsymbol{A})^K$.

The main example of the polyhedral product is the *moment-angle complex* $\mathcal{Z}_K = (D^2, S^1)^K$ [6], which is the key object of study in toric topology. The space $\mathcal{Z}_K$ has a natural coordinatewise action of the torus T^m, and it is a manifold whenever K is a triangulation of a sphere. Other important cases of polyhedral products include $DJ(K) = (\mathbb{C}P^\infty, *)^K$, which is referred to as the *Stanley–Reisner space* [6] or the *Davis–Januszkiewicz space* [21], and the complement of the complex *coordinate subspace arrangement* corresponding to K:

$$U(K) = (\mathbb{C}, \mathbb{C}^*)^K = \mathbb{C}^m \setminus \bigcup_{\{i_1,\dots,i_k \notin K\}} \{z_{i_1} = \dots = z_{i_k} = 0\}.$$

According to [6, Th. 5.2.5], there is a T^m-equivariant deformation retraction $U(K) \to \mathcal{Z}_K$. The spaces $\mathcal{Z}_K$ and $(\mathbb{C}P^\infty, *)^K$ are related by the following result.

Proposition 2.1 ([6, Cor. 3.4.5])**.** *There is a homotopy fibration*

$$\mathcal{Z}_K \longrightarrow DJ(K) \longrightarrow (\mathbb{C}P^\infty)^m,$$

i.e. $\mathcal{Z}_K$ *is the homotopy fibre of the canonical inclusion* $DJ(K) \to (\mathbb{C}P^\infty)^m$.

This fibration splits after looping:

$$\Omega DJ(K) \simeq \Omega\mathcal{Z}_K \times T^m,$$

but this is not an H-space splitting. One can think of $\Omega\mathcal{Z}_K$ as the "commutator subgroup" of $\Omega DJ(K)$, although this can be made precise only after passing to Pontryagin (loop homology) algebras.

Proposition 2.2 ([20, (8.2)])**.** *There is an exact sequence of (noncommutative) algebras*

$$1 \longrightarrow H_*(\Omega\mathcal{Z}_K; \mathbf{k}) \longrightarrow H_*(\Omega DJ(K); \mathbf{k}) \longrightarrow \Lambda[u_1, \dots, u_m] \longrightarrow 1 \tag{2.1}$$

where $\mathbf{k}$ *is field or* $\mathbb{Z}$, *and* $\Lambda[u_1, \dots, u_m]$ *is the exterior algebra on* m *generators of degree one.*

In what follows we shall often omit the coefficient ring $\mathbf{k}$ in the notation of (co)homology.

The exterior algebra $\Lambda[u_1, \dots, u_m]$ can be thought of as the abelianisation of a largely noncommutative algebra $H_*(\Omega DJ(K))$ (we expand on this below), so that $H_*(\Omega\mathcal{Z}_K)$ is its commutator subalgebra.

The *face ring* of K (also known as the *Stanley–Reisner ring*) is defined as the quotient of the polynomial algebra $\mathbf{k}[v_1, \dots, v_m]$ by the square-free monomial ideal generated by non-simplices of K:

$$\mathbf{k}[K] = \mathbf{k}[v_1, \dots, v_m] / \bigl(v_{i_1} \cdots v_{i_k} \colon \{i_1, \dots, i_k\} \notin K\bigr).$$

We make it graded by setting $\deg v_i = 2$.

Theorem 2.3 ([11], [6, Prop. 3.4.3])**.** *There is an isomorphism of graded commutative algebras*

$$H^*\bigl(DJ(K); \mathbf{k}\bigr) \cong \mathbf{k}[K]$$

for any coefficient ring $\mathbf{k}$.

The cohomology ring $H^*(\mathcal{Z}_K; \mathbf{k})$ and the Pontryagin algebra $H_*(\Omega DJ(K); \mathbf{k})$ recover different homological invariants of the face ring $\mathbf{k}[K]$, as is stated next.

Theorem 2.4 ([6, Th. 5.3.4])**.** *If* $\mathbf{k}$ *is a field, then there is an isomorphism of graded noncommutative algebras*

$$H_*\bigl(\Omega DJ(K); \mathbf{k}\bigr) \cong \operatorname{Ext}_{\mathbf{k}[K]}(\mathbf{k}, \mathbf{k}),$$

where $\operatorname{Ext}_{\mathbf{k}[K]}(\mathbf{k}, \mathbf{k})$ *is the Yoneda algebra of* $\mathbf{k}[K]$.

This is proved by applying Adams' cobar spectral sequence to the loop fibration $\Omega DJ(K) \to pt \to DJ(K)$ and using the formality of $DJ(K)$.

Theorem 2.5 ([6], [2], [12]). *If $\mathbf{k}$ is a field or $\mathbb{Z}$, then there are isomorphisms of (bi)graded commutative algebras*

$$\begin{aligned} H^*(\mathcal{Z}_K) &\cong \operatorname{Tor}_{\mathbf{k}[v_1,\dots,v_m]}\big(\mathbf{k}[K],\mathbf{k}\big) \\ &\cong H\big[\Lambda[u_1,\dots,u_m]\otimes \mathbf{k}[K],d\big] \\ &\cong \bigoplus_{I\subset[m]} \widetilde{H}^*(K_I). \end{aligned}$$

Here, the second row is the cohomology of the differential bigraded algebra with $\operatorname{bideg} u_i = (-1,2)$, $\operatorname{bideg} v_i = (0,2)$ *and* $du_i = v_i$, $dv_i = 0$ *(the Koszul complex). In the third row, $\widetilde{H}^*(K_I)$ denotes the reduced simplicial cohomology of the full subcomplex $K_I \subset K$ (the restriction of K to $I \subset [m]$). The last isomorphism is the sum of isomorphisms*

$$H^p(\mathcal{Z}_K) \cong \sum_{I\subset[m]} \widetilde{H}^{p-|I|-1}(K_I),$$

and the ring structure (the Hochster ring) is given by the maps

$$H^{p-|I|-1}(K_I)\otimes H^{q-|J|-1}(K_J) \to H^{p+q-|I|-|J|-1}(K_{I\cup J})$$

which are induced by the canonical simplicial maps $K_{I\cup J} \to K_I * K_J$ for $I\cap J=\varnothing$ and zero otherwise.

In [16] several classes of complexes K have been identified for which $\mathcal{Z}_K$ has homotopy type of a wedge of spheres. These include all skeleta of simplices, and the so-called *shifted* complexes. One special case which we will refer to several times later is when K is a wedge of disjoint points.

Theorem 2.6 ([15]). *Let K be the disjoint union of m points. Then there is a homotopy equivalence*

$$\mathcal{Z}_K \simeq \bigvee_{\ell=2}^{m} (S^{\ell+1})^{\vee(\ell-1)\binom{m}{\ell}}.$$

Further, in [16] it was shown that there is a way to build new complexes K whose corresponding $\mathcal{Z}_K$ is a wedge of spheres from existing ones.

Theorem 2.7 ([16, Th. 10.1]). *Assume that $\mathcal{Z}_{K_1}$ and $\mathcal{Z}_{K_2}$ both have homotopy type of a wedge of spheres, and K is obtained by attaching K_1 to K_2 along a common face. Then $\mathcal{Z}_K$ also has homotopy type of a wedge of spheres.*

Corollary 2.8. *Assume that there is an order $I_1,\dots,I_s$ of the maximal faces of K such that $\big(\bigcup_{j<k} I_j\big)\cap I_k$ is a single face for each $k=1,\dots,s$. Then $\mathcal{Z}_K$ has homotopy type of a wedge of spheres.*

3. The Golod property

In this section we give topological interpretations of the Golod property. The face ring $\mathbf{k}[K]$ is called *Golod* (cf. [18]) if the multiplication and all higher Massey operations in $\operatorname{Tor}_{\mathbf{k}[v_1,\dots,v_m]}\big(\mathbf{k}[K],\mathbf{k}\big)$ are trivial. Several combinatorial criteria for Golodness were given in [19]. We say that the simplicial complex K is Golod if $\mathbf{k}[K]$ is a Golod ring. In view of Theorem 2.5, the Golod property is an algebraic approximation to the property of $\mathcal{Z}_K$ being homotopy equivalent to a wedge of spheres, although this approximation is not exact as Example 3.3 below shows.

Our main result is Theorem 3.2, but before stating this we give a more general result which is of independent interest.

Lemma 3.1. *Let X be a simply-connected cell complex such that $H_*(\Omega X;\mathbf{k})$ is a graded free algebra, where $\mathbf{k}$ is a field. Then $H^*(X;\mathbf{k})$ has trivial multiplication.*

Proof. Let $Q = H_{>0}(\Omega X)/\bigl(H_{>0}(\Omega X)\cdot H_{>0}(\Omega X)\bigr)$ be the space of indecomposable elements, so that $H_*(\Omega X) = T\langle Q\rangle$ by assumption, where $T\langle Q\rangle$ denotes the free associative algebra on the graded $\mathbf{k}$-module Q.

Consider the Rothenberg–Steenrod (bar) spectral sequence, which has E_2-term $E_2^{\mathrm{b}} = \mathrm{Tor}_{H_*(\Omega X)}(\mathbf{k},\mathbf{k})$ and converges to $H_*(X)$. By assumption,

$$E_2^{\mathrm{b}} = \mathrm{Tor}_{T\langle Q\rangle}(\mathbf{k},\mathbf{k}) \cong \mathbf{k}\oplus Q$$

as a $\mathbf{k}$-module. We therefore obtain the following inequalities for the Poincaré series:

$$P\bigl(\Sigma^{-1}\widetilde{H}_*(X);t\bigr) = P(E_\infty^{\mathrm{b}};t) - 1 \leqslant P(E_2^{\mathrm{b}};t) - 1 = P(Q;t). \qquad (3.1)$$

Now consider the Adams (cobar) spectral sequence, which has E_2-term $E_2^{\mathrm{c}} = \mathrm{Cotor}_{H_*(X)}(\mathbf{k},\mathbf{k})$ and converges to $H_*(\Omega X)$. We have a series of inequalities:

$$P\bigl(H_*(\Omega X);t\bigr) = P(E_\infty^{\mathrm{c}};t) \leqslant P(E_2^{\mathrm{c}};t) \leqslant P\bigl(T\langle\Sigma^{-1}\widetilde{H}_*(X)\rangle;t\bigr) \leqslant P\bigl(T\langle Q\rangle;t\bigr),$$

where the second-to-last inequality follows from the cobar construction (it turns to equality when all differentials in the cobar construction on $H_*(X)$ are trivial), and the last inequality follows from (3.1). Now, $P\bigl(H_*(\Omega X);t\bigr) = P\bigl(T\langle Q\rangle;t\bigr)$ by assumption, so all inequalities above turn into equalities, and both spectral sequences collapse at the E_2-term. It follows from the collapse that the homology map

$$\widetilde{H}_*(\Sigma\Omega X) = \Sigma\widetilde{H}_*(\Omega X) \to \widetilde{H}_*(X)$$

induced by the evaluation $\Sigma\Omega X \to X$ is onto. Consider the commutative diagram

$$\begin{array}{ccc}
\widetilde{H}_*(\Sigma\Omega X) & \longrightarrow & \widetilde{H}_*(X) \\
\Big\downarrow{\scriptstyle\Delta} & & \Big\downarrow{\scriptstyle\Delta} \\
\widetilde{H}_*(\Sigma\Omega X)\otimes\widetilde{H}_*(\Sigma\Omega X) & \longrightarrow & \widetilde{H}_*(X)\otimes\widetilde{H}_*(X)
\end{array}$$

in which the vertical arrows are comultiplications, and the horizontal ones are surjective. Since $\Sigma\Omega X$ is a suspension, the left arrow is zero, hence, the right arrow is also zero. By duality, the multiplication in $H^*(X)$ is trivial. □

The *Poincaré series* of a graded $\mathbf{k}$-module $A = \bigoplus_{i\geqslant 0} A^i$ is given by $P(A;t) = \sum_{i\geqslant 0}\dim A^i$. The Golod property of K has the following topological interpretations.

Theorem 3.2. *Let $\mathbf{k}$ be a field. The following conditions are equivalent:*

(a) *$H_*(\Omega\mathcal{Z}_K)$ is a graded free associative algebra;*
(b) *the multiplication in $H^*(\mathcal{Z}_K)$ is trivial;*
(c) *there is the following identity for the Poincaré series:*

$$P\bigl(H_*(\Omega\mathcal{Z}_K);t\bigr) = \frac{1}{1 - P\bigl(\Sigma^{-1}\widetilde{H}^*(\mathcal{Z}_K);t\bigr)},$$

where Σ^{-1} denotes the desuspension of graded $\mathbf{k}$-module.

Proof. The implication (a)⇒(b) holds by Lemma 3.1.

To prove the implication (b)⇒(c) we use the result of Berglund and Jöllenbeck [4, Th. 5.1], according to which if the product in $H^*(\mathcal{Z}_K) = \mathrm{Tor}_{\mathbf{k}[v_1,\ldots,v_m]}\bigl(\mathbf{k}[K],\mathbf{k}\bigr)$ is trivial, then all higher Massey operations are also trivial, i.e. $\mathbf{k}[K]$ is Golod. By the alternative definition of the Golod property [18], $\mathbf{k}[K]$ is Golod if and only if the following identity for the Poincaré series holds:

$$P\bigl(\mathrm{Ext}_{\mathbf{k}[K]}(\mathbf{k},\mathbf{k});t\bigr) = \frac{(1+t)^m}{1-\sum_{i,j>0}\dim\mathrm{Tor}^{-i,2j}_{\mathbf{k}[v_1,\ldots,v_m]}\bigl(\mathbf{k}[K],\mathbf{k}\bigr)t^{-i+2j-1}}.$$

Using Theorems 2.4 and 2.5 we rewrite this as

$$P\bigl(H_*(\Omega DJ(K));t\bigr) = \frac{P\bigl(H_*(T^m);t\bigr)}{1-P\bigl(\varSigma^{-1}\widetilde{H}^*(\mathcal{Z}_K);t\bigr)}.$$

Since $\Omega DJ(K)\simeq\Omega\mathcal{Z}_K\times T^m$, the above identity is equivalent to that of (c).

To prove the implication (c)⇒(a) we observe that

$$\frac{1}{1-P\bigl(\varSigma^{-1}\widetilde{H}^*(\mathcal{Z}_K);t\bigr)} = P\bigl(T\langle\varSigma^{-1}\widetilde{H}_*(\mathcal{Z}_K)\rangle;t\bigr),$$

so the identity from (c) is equivalent to $P\bigl(H_*(\Omega\mathcal{Z}_K);t\bigr) = P\bigl(T\langle\varSigma^{-1}\widetilde{H}_*(\mathcal{Z}_K)\rangle\bigr)$. Hence, all differentials in the cobar construction on $H_*(\mathcal{Z}_K)$ are trivial, which implies that $H_*(\Omega\mathcal{Z}_K)$ is a free associative algebra on $\varSigma^{-1}\widetilde{H}_*(\mathcal{Z}_K)$. □

The conditions of Theorem 3.2 do not guarantee that $\mathcal{Z}_K$ is homotopy equivalent to a wedge of spheres. The reason is that $H^*(\mathcal{Z}_K;\mathbb{Z})$ may contain arbitrary torsion. This follows easily from Theorem 2.5: since $\widetilde{H}^*(K)$ is a direct summand in $H^*(\mathcal{Z}_K)$, one may take K to be a triangulation of a space with torsion in cohomology. The simplest example is the 6-vertex triangulation of $\mathbb{R}P^2$.

Example 3.3. Let K be the simplicial complex shown in Fig. 3.1, where the vertices with the same labels are identified, and the boundary edges are identified according to the orientation shown. A calculation using Theorem 2.5 shows that

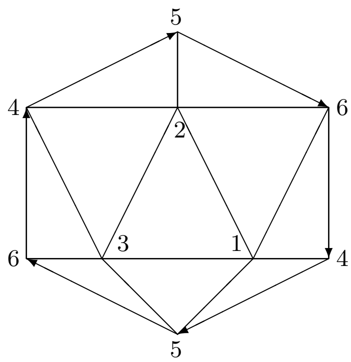

FIGURE 3.1. 6-vertex triangulation of $\mathbb{R}P^2$.

the nontrivial cohomology groups of $\mathcal{Z}_K$ are given by

$$H^0=\mathbb{Z},\quad H^5=\mathbb{Z}^{10},\quad H^6=\mathbb{Z}^{15},\quad H^7=\mathbb{Z}^6,\quad H^9=\mathbb{Z}/2.$$

Therefore, all products and Massey products vanish for dimensional reasons, so K is Golod (over any field). Nevertheless, $\mathcal{Z}_K$ is not homotopy equivalent to a wedge of spheres because of the torsion. In fact, in this example, we have

$$\mathcal{Z}_K \simeq (S^5)^{\vee 10} \vee (S^6)^{\vee 15} \vee (S^7)^{\vee 6} \vee \Sigma^7 \mathbb{R}P^2 \tag{3.2}$$

where $X^{\vee k}$ denotes the k-fold wedge of X. For, if we regard $\mathcal{Z}_K$ as a CW-complex built up by attaching k-cells to the $(k-1)$-skeleton for $6 \leqslant k \leqslant 9$, then the attaching maps are all in the stable range. But stably these attaching maps are all null homotopic since, by [1], the homotopy equivalence in (3.2) holds after one suspension. Therefore the attaching maps are null homotopic, and so (3.2) holds without having to suspend.

Question 3.4. *Assume that $H^*(\mathcal{Z}_K)$ has trivial multiplication, so that K is Golod, over any field. Is it true that $\mathcal{Z}_K$ is a co-H-space, or even a suspension, as in the previous example?*

Denote by $K_{\widehat{i}}$ the full subcomplex of K corresponding to $[m]\setminus\{i\}$; one may think of $K_{\widehat{i}}$ as obtained by deleting the ith vertex from K. It follows from the description of the product in $H^*(\mathcal{Z}_K)$ in Theorem 2.5 that if K is Golod, then $K_{\widehat{i}}$ is also Golod. Following [4], we refer to K as a *minimally non-Golod complex* if it is not Golod, but $K_{\widehat{i}}$ is Golod for each i.

The condition for K to be minimally non-Golod is an "algebraic approximation" of the topological condition for $\mathcal{Z}_K$ to be homeomorphic to a connected sum of sphere products, with two spheres in each product. In what follows, whenever we say that $\mathcal{Z}_K$ is a connected sum of sphere products, we mean that there are two spheres in each product. (In fact, there is no known example of $\mathcal{Z}_K$ which is homeomorphic to a nontrivial connected sum of sphere products with more than two spheres in at least one product.)

To justify the term "algebraic approximation", the following question needs to be positively answered.

Question 3.5. *Is it true that if $\mathcal{Z}_K$ is a connected sum of sphere products, then K is minimally non-Golod?*

Examples of minimally non-Golod complexes include the boundary complexes of polygons and, more generally, stacked polytopes different from simplices [4, Th. 6.19]. For all these cases it is known that $\mathcal{Z}_K$ is homeomorphic to a connected sum of sphere products, due to a result of McGavran (cf. [5, Th. 6.3]).

4. The case of a flag complex

A *missing face* of K is a subset $I \subset [m]$ such that $I \notin K$, but every proper subset of I is a simplex of K. A simplicial complex K is called a *flag complex* if each of its missing faces has two vertices. Equivalently, K is flag if any set of vertices of K which are pairwise connected by edges spans a simplex.

In the case of flag complexes K we will show that the "algebraic approximations" from the previous section are precise criterions for the appropriate topological properties: $\mathcal{Z}_K$ is a wedge of spheres precisely when K is Golod, and $\mathcal{Z}_K$ is a connected sum of sphere products if and only if K is minimally non-Golod.

There is the following description of $H_*(\Omega DJ(K)) = \mathrm{Ext}_{\mathbf{k}[K]}(\mathbf{k},\mathbf{k})$ for flag K.

Theorem 4.1 ([20, Th. 9.3]). *For any flag complex K, there is an isomorphism*

$$H_*\big(\Omega DJ(K)\big) \cong T\langle u_1,\ldots,u_m\rangle/\big(u_i^2=0,\ u_iu_j+u_ju_i=0 \text{ for } \{i,j\}\in K\big) \tag{4.1}$$

where $T\langle u_1,\ldots,u_m\rangle$ denotes the free algebra on m generators of degree 1.

Remark. Algebra (4.1) may be viewed as a colimit (in the category of noncommutative algebras) of a diagram of algebras over the face category of K, which assigns to each face $I\in K$ the exterior algebra $\Lambda[u_i\colon i\in I]$. Another way to see this algebra is to assign a generator u_i satisfying $u_i^2=0$ to each vertex of K, and think of each edge of K as a commutativity relation between the corresponding u_i's. The resulting algebra is determined by the 1-skeleton (graph) of K, which is not surprising since K is flag. In the non-flag case higher brackets appear, corresponding to higher Samelson products in $\Omega DJ(K)$, and the colimit above has to be replaced by a homotopy colimit, see [20, §8] for the details.

Algebra (4.1) is also known as the *graph product algebra* corresponding to the 1-skeleton of K. Its group-theoretic analogues are *right-angled Artin* and *Coxeter groups*; in fact the polyhedral products of the form $(\mathbb{R}P^\infty)^K$ and $(S^1)^K$ respectively are the classifying spaces of these groups in the flag case (cf. [21, §4]).

The *f-vector* of K is given by $\boldsymbol{f}(K)=(f_0,\ldots,f_{n-1})$ where f_i is the number of i-dimensional faces and $n-1=\dim K$. The h-vector $\boldsymbol{h}(K)=(h_0,h_1,\ldots,h_n)$ is defined from the relation

$$h_0t^n+h_1t^{n-1}+\cdots+t_n=(t-1)^n+f_0(t-1)^{n-1}+\cdots+f_{n-1}.$$

The h-vector is symmetric for sphere triangulations K; the equations $h_i=h_{n-i}$ are known as the *Dehn–Sommerville relations.*

The proof of Theorem 4.1 consists of applying a result of Fröberg [13] on quadratic duality. As another application of this theory, the Poincaré series of $H_*(\Omega\mathcal{Z}_K)$ can be calculated explicitly in terms of the face numbers of K in the flag case.

Proposition 4.2 ([20, Prop. 9.5]). *For any flag complex K, we have*

$$P\big(H_*(\Omega\mathcal{Z}_K);t\big) \;=\; \frac{1}{(1+t)^{m-n}(1-h_1t+\ldots+(-1)^nh_nt^n)}\;.$$

We now go further by identifying a minimal set of multiplicative generators in $H_*(\Omega\mathcal{Z}_K)$ as a specific set of iterated commutators of the u_i.

Theorem 4.3. *Assume that K is flag. The algebra $H_*(\Omega\mathcal{Z}_K)$, viewed as the commutator subalgebra* (4.1) *via exact sequence* (2.1)*, is multiplicatively generated by $\sum_{I\subset[m]}\dim\widetilde{H}^0(K_I)$ iterated commutators of the form*

$$[u_j,u_i],\quad [u_{k_1},[u_j,u_i]],\quad \ldots,\quad [u_{k_1},[u_{k_2},\cdots[u_{k_{m-2}},[u_j,u_i]]\cdots]]$$

where $k_1<k_2<\cdots<k_p<j>i$, $k_s\neq i$ for any s, and i is the smallest vertex in a connected component not containing j of the subcomplex $K_{\{k_1,\ldots,k_p,j,i\}}$. Furthermore, this multiplicative generating set is minimal, i.e. the commutators above form a basis in the submodule of indecomposables in $H_*(\Omega\mathcal{Z}_K)$.

Proof. We observe that, for a given $I=\{k_1,\ldots,k_p,j,i\}$, the number of the commutators containing all $u_{k_1},\ldots,u_{k_p},u_j,u_i$ in the set above is equal to $\dim\widetilde{H}^0(K_I)$ (one less the number of connected components in K_I), so there are indeed $\sum_{I\subset[m]}\dim\widetilde{H}^0(K_I)$ commutators in total.

We first prove a particular case of the statement, corresponding to K consisting of m disjoint points. This result may be of independent algebraic interest, as it is an analogue of the description of a basis in the commutator subalgebra of a free algebra, given by Cohen and Neisendorfer [10].

Lemma 4.4. *Let A be the commutator subalgebra of $T\langle u_1,\dots,u_m\rangle/(u_i^2=0)$, i.e. the algebra defined from the exact sequence*

$$1\longrightarrow A\longrightarrow T\langle u_1,\dots,u_m\rangle/(u_i^2=0)\longrightarrow \Lambda[u_1,\dots,u_m]\longrightarrow 1$$

where $\deg u_i=1$*. Then A is a free associative algebra minimally generated by the iterated commutators of the form*

$$[u_j,u_i],\quad [u_{k_1},[u_j,u_i]],\quad \dots,\quad [u_{k_1},[u_{k_2},\cdots[u_{k_{m-2}},[u_j,u_i]]\cdots]]$$

where $k_1<k_2<\cdots<k_p<j>i$ and $k_s\neq i$ for any s. Here, the number of commutators of length ℓ is equal to $(\ell-1)\binom{m}{\ell}$.

Proof. Let S be the set of commutators in the statement of the lemma. Let B denote the commutator algebra of a free algebra on m generators, i.e. the algebra kernel of the map $T\langle u_1,\dots,u_m\rangle\to\Lambda[u_1,\dots,u_m]$. By [10], B is a free algebra generated the commutators of the same form $[u_{k_1},[u_{k_2},\cdots[u_{k_p},[u_j,u_i]]\cdots]]$, but with the conditions $k_1<k_2<\cdots<k_p<j\geqslant i$ only. We therefore get a larger set T of commutators, in which u_k may repeat. However, note that the inequalities on the indices imply that if u_k repeats within a specified commutator, it does so only once. We have $S\subseteq T$ and wish to show that any commutator in $T-S$ is excluded from the multiplicative generating set of the quotient $T\langle u_1,\dots,u_m\rangle/(u_i^2=0)$. To see this, induct on the length of the commutators, beginning with $[u_k,u_k]=2u_k^2=0$. Suppose the commutators of length $<n$ in T have had any commutator with a repeating u_k excluded from the generating set of $T\langle u_1,\dots,u_m\rangle/(u_i^2=0)$. Choose a commutator of length n with some u_k repeating. Observe that it suffices to consider commutators of the form $[u_k,[u_{k_2},\cdots[u_{k_p},[u_j,u_k]]\cdots]]$, which we write as $[u_k,[u_{k_2},c]]$ for $c=[u_{k_3},\cdots[u_{k_p},[u_j,u_k]]\cdots]$. By the Jacobi identity, $[u_k,[u_{k_2},c]]=\pm[c,[u_k,u_{k_2}]]\pm[u_{k_2},[c,u_k]]$. Rewriting to conform to the restrictions on the indices in the basis for B, we obtain $[u_k,[u_{k_2},c]]=\pm[c,[u_{k_2},u_k]]\pm[u_{k_2},[u_k,c]]$. The first term on the right is a commutator of two elements of lower length in S. The second term on the right has $[u_k,c]$ excluded from the multiplicative generating set of $T\langle u_1,\dots,u_m\rangle/(u_i^2=0)$ by inductive hypothesis, since u_k appears in c. Therefore $[u_k,[u_{k_2},c]]$ is not a multiplicative generator of $T\langle u_1,\dots,u_m\rangle/(u_i^2=0)$.

Now observe that the set of commutators S generates A multiplicatively, since A is a quotient of $B/(u_i^2=0)$. To show that A is a free algebra, and the given generator set is minimal, we use a topological argument. We have that $A=H_*(\Omega\mathcal{Z}_K)$ where K is a disjoint union of m points. By Theorem 2.6, $\mathcal{Z}_K$ is homotopy equivalent to the wedge of spheres $\bigvee_{\ell=2}^m(S^{\ell+1})^{\vee(\ell-1)\binom{m}{\ell}}$. The Bott–Samelson Theorem implies that $A=H_*(\Omega\mathcal{Z}_K)$ is a free algebra, and the number of generators in each degree ℓ agrees with the number of given commutators of length ℓ. □

To complete the proof of Theorem 4.3 we must deal with how the remaining relations in (4.1), those of the form $u_iu_j+u_ju_i=0$ if $\{i,j\}\in K$, affect the iterated commutators listed in Lemma 4.4. Note that $u_iu_j+u_ju_i=[u_i,u_j]$ and that no u_k repeats in any of the iterated commutators listed in Lemma 4.4.

Assume that i, i' are vertices in the same connected component of K. Then there are vertices $i_1 = i, i_2, \dots, i_{k-1}, i_k = i'$ for some k with the property that the edges $\{i_1, i_2\}, \dots, \{i_{k-1}, i_k\}$ are all in K. Arguing inductively as in the proof of Lemma 4.4, the Jacobi identity implies that any iterated commutator of length l involving all $u_{i_1}, \dots, u_{i_k}$ can be rewritten as a sum of iterated commutators formed from iterated commutators of lengths $< l$. In particular, if K is connected (with m vertices) then any iterated commutator of length m is zero modulo commutators of lesser length.

Continuing, suppose that we are given an index set $I = \{k_1, \dots, k_p, j, i\}$ with $k_1 < k_2 < \cdots < k_p < j > i$ and $k_s \neq i$ for any s. Consider iterated commutators of length $p + 2$ involving one occurrence of u_k for each $k \in I$. One example is $[u_{k_1}, [u_{k_2}, \cdots [u_{k_p}, [u_j, u_i]] \cdots]]$. Observe that the restrictions on the order of the indices imply that the only other examples occur by interchanging u_i and u_{k_l} provided $k_{l-1} < i < k_{l+1}$. Now if i, j are in the same connected component of K_I then $[u_{k_1}, [u_{k_2}, \cdots [u_{k_p}, [u_j, u_i]] \cdots]] = 0$ modulo iterated commutators of lesser length, by the argument in the previous paragraph applied to K_I. So to obtain nontrivial commutators we require that i, j appear in different components. Also, if $\{k_{l_1}, \dots, k_{l_r}\}$ is the subset of $\{k_1, \dots, k_p\}$ which lie in the same connected component of K_I as i, then the iterated commutators $[u_{k_1}, [u_{k_2}, \cdots, u_{k_{l_t-1}}, [u_i, [u_{k_{l_t+1}}, \cdots [u_{k_p}, [u_j, u_{k_{l_t}}]] \cdots]]$ and $[u_{k_1}, [u_{k_2}, \cdots [u_{k_p}, [u_j, u_i]] \cdots]]$ can be identified modulo iterated commutators of lesser lengths. So to enumerate the one independent iterated commutator, we use the convention of writing $[u_{k_1}, [u_{k_2}, \cdots [u_{k_p}, [u_j, u_i]] \cdots]]$ where i is the smallest vertex in its connected component within K_I. This leaves us with precisely the set of iterated commutators in the statement of the theorem.

At this point, we have shown that the set of iterated commutators in the statement of the theorem multiplicatively generates $H_*(\Omega\mathcal{Z}_K)$. It remains to show that this is a minimal generating set. To see this, it suffices to show that if $I = \{k_1, \dots, k_p, j, i\}$ where $k_1 < \cdots < k_p < j > i$, then the remaining iterated commutators on this index set are algebraically independent. Let $\{k_{l_1}, \dots, k_{l_r}\}$ be the subset of $\{k_1, \dots, k_p\}$ whose elements lie in the same connected component of K_I as i. Let $K_{\widehat{I}}$ be the full subcomplex of K_I on the vertex set $I - \{k_{l_1}, \dots, k_{l_r}\}$. There is a projection $K_I \to K_{\widehat{I}}$. Observe that the connected component of $K_{\widehat{I}}$ containing the vertex i is precisely the singleton $\{i\}$, and there is a one-to-one correspondence between the remaining iterated commutators of the form $[u_{k_1}, [u_{k_2}, \cdots [u_{k_p}, [u_j, u_i]] \cdots]]$ in $H_*(DJ(K_I))$ and the iterated commutators of length $(p + 2) - r$ in $H_*(\Omega DJ(K_{\widehat{I}}))$ formed by deleting the elements u_{k_l} whenever $k_l \in \{k_{l_i}, \dots, k_{l_r}\}$. The latter set is algebraically independent since, topologically, $DJ(K_{\widehat{I}})$ is the wedge $\mathbb{C}P^\infty \vee DJ(K_{\widehat{I}} - \{i\})$, and the iterated commutators correspond to independent Whitehead products in $\Sigma\Omega\mathbb{C}P^\infty \wedge \Omega D \simeq \Sigma S^1 \wedge \Omega D$, where $D = DJ(K_{\widehat{I}} - \{i\})$. Hence the former set is algebraically independent, as required. □

We now come to identifying the class of flag complexes K for which $\mathcal{Z}_K$ has homotopy type of a wedge of spheres.

Let Γ be a graph on the vertex set $[m]$. A *clique* of Γ is a subset I of vertices such that every two vertices in I are connected by an edge. Obviously, each flag complex K is the *clique complex* of its one-skeleton $\Gamma = K^1$, i.e. the simplicial complex formed by filling in each clique of Γ by a face.

A graph Γ is called *chordal* if each of its cycles with $\geqslant 4$ vertices has a chord (an edge joining two vertices that are not adjacent in the cycle). Equivalently, a chordal graph is a graph with no induced cycles of length more than three.

The following result gives an alternative characterisation of chordal graphs.

Theorem 4.5 ([14])**.** *A graph is chordal if and only if its vertices can be ordered in such a way that, for each vertex i, the lesser neighbours of i form a clique.*

Such an order of vertices is called a *perfect elimination ordering.*

Theorem 4.6. *Let K be a flag complex and $\mathbf{k}$ a field. The following conditions are equivalent:*

(a) *$\mathbf{k}[K]$ is a Golod ring;*
(b) *the multiplication in $H^*(\mathcal{Z}_K)$ is trivial;*
(c) *$\Gamma = K^1$ is a chordal graph;*
(d) *$\mathcal{Z}_K$ has homotopy type of a wedge of spheres.*

Proof. (a)⇒(b) This is by definition of the Golod property and Theorem 2.5.

(b)⇒(c) Assume that K^1 is not chordal, and choose an induced chordless cycle I with $|I| \geqslant 4$. Then the full subcomplex K_I is the same cycle (the boundary of an $|I|$-gon), and therefore $\mathcal{Z}_{K_I}$ is a connected sum of sphere products. Hence, $H^*(\mathcal{Z}_{K_I})$ has nontrivial products (this can be also seen directly by using Theorem 2.5). Then, by Theorem 2.5, the same nontrivial products appear in $H^*(\mathcal{Z}_K)$.

(c)⇒(d) Assume that the vertices of K are in total elimination order. We assign to each vertex i the clique I_i consisting of i and the lesser neighbours of i. Each maximal face of K (i.e., each maximal clique of K^1) is obtained in this way, so we get an induced order on the maximal faces: $I_{i_1}, \ldots, I_{i_s}$. Then, for each $k = 1, \ldots, s$, the simplicial complex $\bigcup_{j<k} I_{i_j}$ is flag (since it is the full subcomplex $K_{\{1,2,\ldots,i_{k-1}\}}$ in a flag complex). The intersection $\bigl(\bigcup_{j<k} I_{i_j}\bigr) \cap I_{i_k}$ is a clique, so it is a face of $\bigcup_{j<k} I_{i_j}$. Therefore, $\mathcal{Z}_K$ has homotopy type of a wedge of spheres by Corollary 2.8.

(d)⇒(a) This is by definition of the Golod property and Theorem 2.5. □

Remark. The equivalence of (a), (b) and (c) was proved in [4, Th. 6.5].

All the implications in the above proof except (c)⇒(d) are valid for arbitrary K, with the same arguments. However, (c)⇒(d) fails in the non-flag case; Example 3.3 is a counterexample.

Corollary 4.7. *Assume that K is flag and $\mathcal{Z}_K$ has homotopy type of a wedge of spheres. Then the number of spheres of dimension $\ell + 1$ in the wedge is given by $\sum_{|I|=\ell} \dim \widetilde{H}^0(K_I)$, for $2 \leqslant \ell \leqslant m$. In particular, $H^i(K_I) = 0$ for $i > 0$ and all I.*

Proof. The first statement follows from Theorem 4.3. The second one follows from Theorem 2.5. □

Proposition 4.8. *Assume that K is flag and $\mathbf{k}$ a field. The following conditions are equivalent:*

(a) *K is minimally non-Golod;*
(b) *$\mathcal{Z}_K$ is homeomorphic to a connected sum of sphere products.*

Proof. Indeed, if K is flag and minimally non-Golod, then it is the boundary of an m-gon with $m \geqslant 4$. □

5. The homotopy type of $\Omega\mathcal{Z}_K$ when K is flag

In general, the homotopy type of $\mathcal{Z}_K$ when K is a flag complex may not be easy to determine. We have shown that $\mathcal{Z}_K$ has the homotopy type of a wedge of spheres if K is Golod, and $\mathcal{Z}_K$ has the homotopy type of a connected sum of sphere products if K is minimally non-Golod. Beyond these two classes, it is not clear what the homotopy type of $\mathcal{Z}_K$ may be. However, we will show in Theorem 5.3 that the homotopy type of $\Omega\mathcal{Z}_K$ localised away from 2 is a product of spheres and loops on spheres.

To begin, suppose that K is a flag complex on m vertices. Let $\overline{K}$ be the disjoint union of the m vertices. Then the inclusion

$$i\colon \overline{K} \longrightarrow K$$

induces an inclusion

$$DJ(i)\colon DJ(\overline{K}) = \bigvee_{j=1}^{m} \mathbb{C}P^\infty \longrightarrow DJ(K)$$

and we obtain a homotopy pullback diagram

$$\begin{array}{ccccc} \mathcal{Z}_{\overline{K}} & \xrightarrow{\overline{f}} & DJ(\overline{K}) & \longrightarrow & \prod_{i=1}^m \mathbb{C}P^\infty \\ \downarrow{\scriptstyle \mathcal{Z}(i)} & & \downarrow{\scriptstyle DJ(i)} & & \Vert \\ \mathcal{Z}_K & \xrightarrow{f} & DJ(K) & \longrightarrow & \prod_{i=1}^m \mathbb{C}P^\infty \end{array} \tag{5.1}$$

which defines the maps $\mathcal{Z}(i), \overline{f}$ and f.

We require some initial algebraic information.

Lemma 5.1. *Let $f\colon X \longrightarrow Y$ be a map between two simply-connected spaces. If $H_*(\Omega X;\mathbb{Z})$ is torsion-free and $(\Omega f)_*$ is onto for coefficients in any field, then $H_*(\Omega Y;\mathbb{Z})$ is also torsion-free.*

Proof. Suppose $H_*(\Omega Y;\mathbb{Z})$ is not torsion-free. Then there is a prime p and elements $b, \bar{b} \in H_*(\Omega Y;\mathbb{Z}/p\mathbb{Z})$ such that $\beta^r\bar{b} = b$, where β^r is the r^{th}-Bockstein. As $(\Omega f)_*$ is onto in mod-p homology, there are elements $a, \bar{a} \in H_*(\Omega Y;\mathbb{Z}/p\mathbb{Z})$ such that $(\Omega f)_*(a) = b$ and $(\Omega f)_*(\bar{a}) = \bar{b}$. As β^r commutes with $(\Omega f)_*$, we obtain

$$(\Omega f)_*(\beta^r\bar{a}) = \beta^r(\Omega f)_*(\bar{a}) = \beta^r\bar{b} = b,$$

implying that $\beta^r\bar{a} \neq 0$. This contradicts the fact that $H_*(\Omega X;\mathbb{Z})$ is torsion-free. □

Corollary 5.2. *Let K be a flag complex. Then $H_*(\Omega\mathcal{Z}_K;\mathbb{Z})$ is torsion-free.*

Proof. Observe that $\Omega DJ(\overline{K}) \simeq T^m \times \Omega\mathcal{Z}_{\overline{K}}$ and by Theorem 2.6, $\mathcal{Z}_{\overline{K}}$ is homotopy equivalent to a wedge of spheres. Thus $H_*(\Omega DJ(\overline{K}))$ is torsion-free. By Theorem 4.1 and Lemma 4.4, $(\Omega DJ(i))_*$ is onto for coefficients in any field. So by Lemma 5.1, $H_*(\Omega\mathcal{Z}_K;\mathbb{Z})$ is torsion-free. □

We now show that $\Omega\mathcal{Z}_K$ for K flag is homotopy equivalent to a product of spheres and loops on spheres, when localised away from 2.

Theorem 5.3. *Let K be a flag complex. The following hold when localised away from 2:*

(a) *the map $\Omega DJ(\overline{K}) \xrightarrow{\Omega DJ(i)} \Omega DJ(K)$ has a right homotopy inverse;*

(b) *the map* $\Omega\mathcal{Z}_{\overline{K}} \xrightarrow{\Omega\mathcal{Z}(i)} \Omega\mathcal{Z}_K$ *has a right homotopy inverse;*
(c) $\Omega DJ(K)$ *and* $\Omega\mathcal{Z}_K$ *are homotopy equivalent to products of spheres and loops on spheres.*

Remark. Theorem 5.3 may be true integrally. Corollary 5.2 says there are no obstructions arising from torsion homology classes. When K is Golod, so $\mathcal{Z}_K$ is homotopy equivalent to a wedge of spheres, then the integral statement is a consequence of the Hilton-Milnor Theorem. When K is minimally non-Golod, so $\mathcal{Z}_K$ is homeomorphic to a connected sum of sphere products, then the integral statement holds by [3]. The methods in [3] arise in a different context and may or may not adapt to the case of $\mathcal{Z}_K$ for a general flag complex; at present not enough information is known about $\mathcal{Z}_K$. The methods presented below may possibly be fine tuned to prove the integral case, but more delicate information would have to be known about the commutators in $H_*(\Omega\mathcal{Z}_K)$. In particular, Theorem 4.3 gives a minimal multiplicative basis for $H_*(\Omega\mathcal{Z}_K)$, but we do not know enough about potential relations among them.

Proof. We begin with an integral argument to establish some equivalences between statements in the theorem. After looping (5.1), we obtain a homotopy pullback diagram

$$\begin{array}{ccccc}
\Omega\mathcal{Z}_{\overline{K}} & \xrightarrow{\Omega\overline{f}} & \Omega DJ(\overline{K}) & \longrightarrow & T^m \\
\downarrow{\scriptstyle \Omega\mathcal{Z}(i)} & & \downarrow{\scriptstyle \Omega DJ(i)} & & \| \\
\Omega\mathcal{Z}_K & \xrightarrow{\Omega f} & \Omega DJ(K) & \longrightarrow & T^m.
\end{array}$$

Since the fibration along the top row splits, it induces a splitting of the fibration along the bottom row. Therefore, using the loop structures in $\Omega DJ(\overline{K})$ and $\Omega DJ(K)$ to multiply, we obtain a homotopy commutative diagram of homotopy equivalences

$$\begin{array}{ccc}
T^m \times \Omega\mathcal{Z}_{\overline{K}} & \xrightarrow{\simeq} & \Omega DJ(\overline{K}) \\
\downarrow{\scriptstyle 1\times\Omega\mathcal{Z}(i)} & & \downarrow{\scriptstyle \Omega DJ(i)} \\
T^m \times \Omega\mathcal{Z}_K & \xrightarrow{\simeq} & \Omega DJ(K).
\end{array}$$

Thus $\Omega DJ(i)$ has a right homotopy inverse if and only if $\Omega\mathcal{Z}(i)$ has a right homotopy inverse. Further, as $\Omega DJ(K) \simeq T^m \times \Omega\mathcal{Z}_K$, we see that $\Omega DJ(K)$ is homotopy equivalent to a product of spheres and loops on spheres if and only if $\Omega\mathcal{Z}_K$ is.

Now localise away from 2. It remains to show that $\Omega DJ(i)$ has a right homotopy inverse and $\Omega DJ(K)$ is homotopy equivalent to a product of spheres and loops on spheres. We begin with a reduction. Suppose that there is a map $\varphi\colon S \longrightarrow \Omega DJ(\overline{K})$ which exists when localised away from 2, where S is a product of spheres and loops on spheres, and φ has the property that the composite $S \xrightarrow{\varphi} \Omega DJ(\overline{K}) \xrightarrow{\Omega DJ(i)} \Omega DJ(K)$ induces an isomorphism in rational homology. By Corollary 5.2, $H_*(\Omega\mathcal{Z}_{\overline{K}};\mathbb{Z})$ is torsion-free. So the composite $\Omega DJ(i) \circ \varphi$ inducing an isomorphism in rational homology implies that it induces an isomorphism in $\mathbb{Z}_{(1/2)}$-homology, where $\mathbb{Z}_{(1/2)}$ is the integers localised away from 2. Thus $\Omega DJ(i) \circ \varphi$ is a homotopy equivalence localised away from 2. Since $\Omega DJ(\overline{K})$ is homotopy equivalent to a product of spheres and loops on spheres (see (5.2) below), we are done.

It now remains to show the existence of the map φ and that $\Omega DJ(i)\circ\varphi$ induces an isomorphism in rational homology. First consider rational homology. By Theorem 4.1, there are isomorphisms

$$H_*\big(\Omega DJ(\overline{K});\mathbb{Q}\big)\cong T\langle u_1,\ldots,u_m\rangle/\big(u_i^2=0\big)$$
$$H_*\big(\Omega DJ(K);\mathbb{Q}\big)\cong T\langle u_1,\ldots,u_m\rangle/\big(u_i^2=0,\ u_iu_j+u_ju_i=0 \text{ for } \{i,j\}\in K\big).$$

The free tensor algebra $T\langle u_1,\ldots,u_m\rangle$ is isomorphic to $UL\langle u_1,\ldots,u_m\rangle$, the universal enveloping algebra of the free Lie algebra on $u_1,\ldots,u_m$. The relations in the two tensor algebras above are induced from relations imposed on the underlying free Lie algebra $L\langle u_1,\ldots,u_m\rangle$. For as 2 is inverted, the relation $u_i^2=0$ is equivalent to the relation $[u_i,u_i]=0$, and as each u_i is of degree 1, we have $u_iu_j+u_ju_i=[u_i,u_j]$. Thus there are isomorphisms

$$T\langle u_1,\ldots,u_m\rangle/\big(u_i^2=0\big)\cong U\big(L\langle u_1,\ldots,u_m\rangle/\big([u_i,u_i]=0\big)\big)$$
$$T\langle u_1,\ldots,u_m\rangle/\big(u_i^2=0,\ u_iu_j+u_ju_i=0 \text{ for } \{i,j\}\in K\big)\cong$$
$$U\big(L\langle u_1,\ldots,u_m\rangle/\big([u_i,u_i]=0,\ [u_i,u_j]=0 \text{ for } \{i,j\}\in K\big)\big).$$

To simplify notation, let

$$\overline{L}=L\langle u_1,\ldots,u_m\rangle/\big([u_i,u_i]=0\big)$$
$$L=L\langle u_1,\ldots,u_m\rangle/\big([u_i,u_i]=0,\ [u_i,u_j]=0 \text{ for } \{i,j\}\in K\big).$$

Observe as well that in passing from loop space homology to universal enveloping algebras of Lie algebras, the map $(\Omega DJ(i))_*$ is modelled by $U(\pi)$, where

$$\pi\colon\overline{L}\longrightarrow L$$

is the quotient map of Lie algebras. As a map of $\mathbb{Q}$-modules, π has a right inverse. Thus if $\widetilde{L}$ is the kernel of π, then by [9] there is an isomorphism of left $U\widetilde{L}$-modules

$$U\overline{L}\cong U\widetilde{L}\otimes UL.$$

Taking associated graded modules if necessary, by the Poincaré-Birkhoff-Witt Theorem we obtain an isomorphism of $\mathbb{Q}$-modules

$$S(\overline{L})\cong S(\widetilde{L})\otimes S(L)$$

where $S(\ )$ is the free symmetric algebra functor.

In this case the Poincaré-Birkhoff-Witt Theorem has a geometric realisation. Since $\overline{K}$ is a disjoint union of points, by Theorem 2.6, there is an integral homotopy equivalence $\mathcal{Z}_{\overline{K}}\simeq\bigvee_{\ell=2}^{m}(S^{\ell+1})^{\vee(\ell-1)\binom{m}{\ell}}$. Therefore there are integral homotopy equivalences

$$\Omega DJ(\overline{K})\simeq T^m\times\Omega\mathcal{Z}_{\overline{K}}\simeq T^m\times\Omega\Big(\bigvee_{\ell=2}^{m}(S^{\ell+1})^{\vee(\ell-1)\binom{m}{\ell}}\Big).$$

The Hilton-Milnor Theorem gives an explicit decomposition of the loops on a wedge of spheres as an infinite product of looped spheres. In our case, we obtain an integral homotopy equivalence

$$\Omega DJ(\overline{K})\simeq T^m\times\prod_{\alpha\in\mathcal{I}}\Omega S_\alpha \tag{5.2}$$

for some index set $\mathcal{I}$, where each S_α is a sphere.

Take homology in (5.2) with rational coefficients. Observe that $H_*(T^m) \cong \Lambda[u_1,\ldots,u_m]$, where each u_i is of degree one. That is, $H_*(T^m)\cong\bigotimes_{i=1}^m S(u_i)$. Next, if the dimension of S_α is odd, say $S_\alpha = S^{2k+1}$, then $H_*(\Omega S_\alpha)\cong\mathbb{Q}[u_\alpha]$, where $|u_\alpha| = 2k$, so $H_*(\Omega S_\alpha)\cong S(u_\alpha)$. If the dimension of S_α is even, say $S_\alpha = S^{2k}$ then the rational splitting $\Omega S^{2k}\simeq S^{2k-1}\times\Omega S^{4k-1}$ implies that $H_*(\Omega S_\alpha)\cong\Lambda[u_\alpha]\otimes\mathbb{Q}[v_\alpha]$, where $|u_\alpha| = 2k-1$ and $|v_\alpha| = 4k-1$, so $H_*(\Omega S_\alpha)\cong S(u_\alpha)\otimes S(v_\alpha)$. Putting all this together, (5.2) implies that there is a coalgebra isomorphism

$$H_*\big(\Omega DJ(\overline{K});\mathbb{Z}\big)\cong\bigotimes_{\alpha'\in\mathcal{I}'} S(u_{\alpha'})$$

where the index set $\mathcal{I}'$ consists of $\{1,2,\ldots,m\}$, every $\alpha\in\mathcal{I}$ where S_α is of odd dimension, and two indices $\alpha_{2k-1},\alpha_{4k-1}$ for every $\alpha\in\mathcal{I}$ where S_α is of dimension $2k$.

We now have two descriptions of $H_*(\Omega DJ(\overline{K}))$ as symmetric algebras, so there is an isomorphism

$$S(\overline{L})\cong\bigotimes_{\alpha'\in\mathcal{I}'} S(u_{\alpha'}).$$

On the other hand, there is a decomposition $S(\overline{L})\cong S(\widetilde{L})\otimes S(L)$, so we can choose a new index set $\mathcal{J}\subseteq\mathcal{I}'$ such that the composite

$$\bigotimes_{\beta\in\mathcal{J}} S(u_\beta)\hookrightarrow\bigotimes_{\alpha'\in\mathcal{I}'} S(u_{\alpha'})\xrightarrow{\cong} S(\overline{L})\xrightarrow{\text{proj}} S(L) \tag{5.3}$$

is an isomorphism. Write $\mathcal{J}=\mathcal{J}_1\sqcup\mathcal{J}_2$ where $\mathcal{J}_1$ (respectively $\mathcal{J}_2$) consists of all those $\beta\in\mathcal{J}$ with $|u_\beta|$ odd (respectively even). Observe that (5.3) is induced in homology by the composite

$$\Big(\prod_{\beta\in\mathcal{J}_1} S_\beta\Big)\times\Big(\prod_{\beta\in\mathcal{J}_2}\Omega S_\beta\Big)\hookrightarrow T^m\times\prod_{\alpha\in\mathcal{I}}\Omega S_\alpha\xrightarrow{\simeq}\Omega DJ(\overline{K})\xrightarrow{\Omega DJ(i)}\Omega DJ(K).$$

The left map exists localised away from 2, since the rational decomposition $\Omega S^{2k}\simeq S^{2k-1}\times\Omega S^{4k-1}$ holds 2-locally. Thus if we take φ to be the composite of the left and middle maps above, then φ has property that $\Omega DJ(i)\circ\varphi$ induces an isomorphism in rational homology. This completes the proof. □

6. Whitehead products

We now consider how $\mathcal{Z}_K$ maps into $DJ(K)$ when K is a flag complex which is Golod or minimally non-Golod. To begin, we geometrically realise the algebra generators of $H_*(\Omega DJ(K))$ by Samelson products. As in Section 5, let $\overline{K}$ be the disjoint union of the m vertices in K. We obtain an inclusion $i\colon\overline{K}\longrightarrow K$ which induces an inclusion $DJ(i)\colon DJ(\overline{K})=\bigvee_{j=1}^m\mathbb{C}P^\infty\longrightarrow DJ(K)$. For $1\leqslant i\leqslant m$ let $\overline{\mu}_i$ be the composite

$$\overline{\mu}_i\colon S^2\longrightarrow\mathbb{C}P^\infty\longrightarrow\bigvee_{j=1}^m\mathbb{C}P^\infty\xrightarrow{DJ(i)} DJ(K)$$

where the left map is the inclusion of the bottom cell and the middle map is the inclusion of the i^{th}-wedge summand. Let

$$\mu_i\colon S^1\longrightarrow\Omega DJ(K)$$

be the adjoint of $\overline{\mu}_i$. Then in the description of $H_*(\Omega DJ(K))$ in (4.1), the Hurewicz image of μ_i is the algebra generator u_i.

Since the Samelson product commutes with the Hurewicz homomorphism, the Hurewicz image of any iterated Samelson product of the μ_i's is the corresponding iterated commutator of the u_i's. As well, in the homotopy fibration $\Omega\mathcal{Z}_K \longrightarrow \Omega DJ(K) \longrightarrow T^m$, since $\pi_k(T^m) = 0$ for $k > 1$, any iterated Samelson product of the μ_i's composes trivially into T^m and so lifts to $\Omega\mathcal{Z}_K$.

As notation, if

$$s\colon S^k \longrightarrow \Omega DJ(K)$$

is an iterated Samelson product of the μ_i's, let

$$t\colon S^k \longrightarrow \Omega\mathcal{Z}_K$$

be its lift to $\Omega\mathcal{Z}_K$. Note that the homotopy class of t is determined by that of s since the homotopy fibration $\Omega\mathcal{Z}_K \longrightarrow \Omega DJ(K) \longrightarrow T^m$ splits. Let

$$\bar{s}\colon S^{t+1} \longrightarrow DJ(K), \quad \bar{t}\colon S^{t+1} \longrightarrow \mathcal{Z}_K$$

be the adjoints of s and t respectively. In particular, $\bar{s}$ is an iterated Whitehead product of the $\overline{\mu}_i$'s and $\bar{t}$ is its lift to $\mathcal{Z}_K$.

Since we are regarding $H_*(\Omega\mathcal{Z}_K)$ as the commutator subalgebra of $H_*(\Omega DJ(K))$ via exact sequence (2.1), if s is an iterated Samelson product of the μ_i's then we can regard its lift t to $\Omega\mathcal{Z}_K$ as having the same Hurewicz image as s. Therefore, the algebra generators

$$[u_j, u_i], \quad [u_{k_1}, [u_j, u_i]], \quad \ldots, \quad [u_{k_1}, [u_{k_2}, \cdots [u_{k_{m-2}}, [u_j, u_i]] \cdots]]$$

of $H_*(\Omega\mathcal{Z}_K)$ in Theorem 4.3, with restrictions on the indices as stated in the theorem, are the Hurewicz images of the lifts to $\Omega\mathcal{Z}_K$ of the iterated Samelson products

$$[\mu_j, \mu_i], \quad [\mu_{k_1}, [\mu_j, \mu_i]], \quad \ldots, \quad [\mu_{k_1}, [\mu_{k_2}, \cdots [\mu_{k_{m-2}}, [\mu_j, \mu_i]] \cdots]]. \tag{6.1}$$

Our first result is to show that if K is flag and Golod then the homotopy class of the map $\mathcal{Z}_K \longrightarrow DJ(K)$ is determined by Whitehead products of the maps $\overline{\mu}_i$.

Proposition 6.1. *Let K be a flag complex and $\mathbf{k}$ a field. Suppose that K is Golod, or equivalently by Theorem 4.6, that $\mathcal{Z}_K$ is homotopy equivalent to a wedge of spheres. Then each sphere in this wedge maps to $DJ(K)$ by an iterated Whitehead product of the maps $\overline{\mu}_1, \ldots \overline{\mu}_m$.*

Proof. Since $\mathcal{Z}_K$ is homotopy equivalent to a wedge of spheres, $H_*(\Omega\mathcal{Z}_K)$ is a free associative algebra, where each algebra generator of degree d corresponds to a sphere of dimension $d+1$ in the wedge decomposition of $\mathcal{Z}_K$. On the other hand, a minimal generating set for $H_*(\Omega\mathcal{Z}_K)$ is given by the iterated commutators in Theorem 4.3, so each iterated commutator listed in Theorem 4.3 of degree d corresponds to a sphere of dimension $d + 1$ in the wedge decomposition of $\mathcal{Z}_K$. Applying the map $\Omega\mathcal{Z}_K \longrightarrow \Omega DJ(K)$, these iterated commutators are the Hurewicz images of the iterated Samelson products in (6.1). Therefore, adjointing, the spheres in the wedge decomposition of $\mathcal{Z}_K$ map to $DJ(K)$ by the iterated Whitehead products

$$[\overline{\mu}_j, \overline{\mu}_i], \quad [\overline{\mu}_{k_1}, [\overline{\mu}_j, \overline{\mu}_i]], \quad \ldots, \quad [\overline{\mu}_{k_1}, [\overline{\mu}_{k_2}, \cdots [\overline{\mu}_{k_{m-2}}, [\overline{\mu}_j, \overline{\mu}_i]] \cdots]]$$

with restrictions on the indices as in Theroem 4.3. □

Next, consider the case when K is flag and minimally non-Golod. Observe that if M is a connected sum of sphere products then $M - \{*\}$ is homotopy equivalent to a wedge of spheres. That is, if M has dimension n then the $(n-1)$-skeleton of M is homotopy equivalent to a wedge of spheres.

Proposition 6.2. *Let K be a flag complex and $\mathbf{k}$ a field. Suppose that K is minimally non-Golod, or equivalently by Theorem 4.6, that $\mathcal{Z}_K$ is homeomorphic to a connected sum of sphere products. Then $\mathcal{Z}_K - \{*\}$ is homotopy equivalent to a wedge of spheres and the equivalence can be chosen so that each sphere in the wedge maps to $DJ(K)$ by an iterated Whitehead product of the maps $\overline{\mu}_1, \dots \overline{\mu}_m$.*

Proof. Assume that $\mathcal{Z}_K$ has dimension n, so the $(n-1)$-skeleton of $\mathcal{Z}_K$ is homotopy equivalent to a wedge of spheres, say $\mathcal{Z}_K - \{*\} \simeq \bigvee_{i=1}^k S^{n_i}$ where each $n_i < n$. Skeletal inclusion then gives a map $j\colon \bigvee_{i=1}^k S^{n_i} \longrightarrow \mathcal{Z}_K$. Note that for any K, $\mathcal{Z}_K$ is simply-connected (in fact, 2-connected). So in our case, $\bigvee_{i=1}^k S^{n_i}$ is a suspension, implying that j adjoints to a map $\epsilon\colon \bigvee_{i=1}^k S^{n_i-1} \longrightarrow \Omega\mathcal{Z}_K$. Adjointing back we obtain a homotopy between j and the composite

$$\bigvee_{i=1}^k S^{n_i} \xrightarrow{\Sigma\epsilon} \Sigma\Omega\mathcal{Z}_K \xrightarrow{ev} \mathcal{Z}_K$$

where ev is the canonical evaluation map.

Next, recall the map $\mathcal{Z}_{\overline{K}} \xrightarrow{\mathcal{Z}(i)} \mathcal{Z}_K$ from (5.1), where $\overline{K}$ is the disjoint union of the vertices in K and i is the inclusion of the vertices. By [3], $\Omega\mathcal{Z}(i)$ has a right homotopy inverse, $s\colon \Omega\mathcal{Z}_K \longrightarrow \Omega\mathcal{Z}_{\overline{K}}$. (Note that this holds integrally, we do not have to rely on the localised away from 2 statement in Theorem 5.3.) Consider the diagram

$$\begin{array}{ccccccc}
\bigvee_{i=1}^k S^{n_i} & \xrightarrow{\Sigma\epsilon} & \Sigma\Omega\mathcal{Z}_K & \xrightarrow{\Sigma s} & \Sigma\Omega\mathcal{Z}_{\overline{K}} & \xrightarrow{ev} & \mathcal{Z}_{\overline{K}} \\
 & & & \searrow{=} & \downarrow{\scriptstyle \Sigma\Omega\mathcal{Z}(i)} & & \downarrow{\scriptstyle \mathcal{Z}(i)} \\
 & & & & \Sigma\Omega\mathcal{Z}_K & \xrightarrow{ev} & \mathcal{Z}_K.
\end{array}$$

The triangle homotopy commutes since s is a right homotopy inverse for $\Omega\mathcal{Z}(i)$. The square homotopy commutes by the naturality of the evaluation map. So the entire diagram homotopy commutes. The lower direction around the diagram is $ev \circ \Sigma\epsilon$, which we observed is homotopic to j. Let $\gamma = ev \circ \Sigma s \circ \Sigma\epsilon$, the composite along the top row. Then the homotopy commutativity of the diagram implies that γ is a lift of j through $\mathcal{Z}(i)$.

Now consider the diagram

$$\begin{array}{ccccc}
\bigvee_{i=1}^k S^{n_i} & \xrightarrow{\gamma} & \mathcal{Z}_{\overline{K}} & \xrightarrow{\overline{f}} & DJ(\overline{K}) \\
 & \searrow{\scriptstyle j} & \downarrow{\scriptstyle \mathcal{Z}(i)} & & \downarrow{\scriptstyle DJ(i)} \\
 & & \mathcal{Z}_K & \xrightarrow{f} & DJ(K)
\end{array} \qquad (6.2)$$

where the right square is from (5.1). Since $\overline{K}$ is a disjoint union of points, by Theorem 2.6, $\mathcal{Z}_{\overline{K}}$ is homotopy equivalent to a wedge of spheres. Restrict the triangle to $(n-1)$-skeletons. Since j is a skeletal inclusion, the restriction of j is the identity map. Therefore, as $\bigvee_{i=1}^k S^{n_i}$ is of dimension $\leqslant n-1$, the restriction of j shows that $\bigvee_{i=1}^k S^{n_i}$ retracts off $\mathcal{Z}_{\overline{K}}$. We now have $\bigvee_{i=1}^k S^{n_i}$ retracting off $\mathcal{Z}_{\overline{K}}$, which itself is homotopy equivalent to a wedge of spheres. Thus there is a self-equivalence e of $\bigvee_{i=1}^k S^{n_i}$ such that $\gamma \circ e$ is an inclusion into a subwedge of $\mathcal{Z}_{\overline{K}}$. Replace γ and j in (6.2) by $\gamma\circ e$ and $j\circ e$. By Proposition 6.1, each sphere in the wedge decomposition of $\mathcal{Z}_{\overline{K}}$ maps to $DJ(\overline{K})$ by an iterated Whitehead product of the maps $\overline{\mu}_1, \cdots \overline{\mu}_m$. Thus the same is true of $\overline{f} \circ \gamma \circ e$, and therefore of $DJ(i) \circ \overline{f} \circ \gamma \circ e$, the upper direction around (6.2). The homotopy commutativity of (6.2) then implies that there is a choice of a homotopy equivalence $\mathcal{Z}_K - \{*\} \simeq \bigvee_{i=1}^{k} S^{n_i}$ with the property that each sphere in this wedge maps to $DJ(K)$ by an iterated Whitehead product of the maps $\overline{\mu}_1, \cdots \overline{\mu}_m$. □

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