RANK $p-1$ MOD-p H-SPACES

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Abstract. Different constructions by Cooke, Harper and Zabrodsky and by Cohen and Neisendorfer produce torsion free finite p-local H-spaces of rank $l < p-1$. The first construction goes through when $l = p-1$ and we show the second does as well. However, the space produced need not be an H-space. We give a criterion for when an H-space is obtained. In the special case of rank 2 mod-3 H-spaces, we also give a practical test for when the criterion holds, and use this to give many new examples of finite H-spaces.

1. Introduction

Finite H-spaces have been of abiding interest for more than fifty years. One driving problem has been to classify all finite H-spaces, in a manner hopefully analogous to the classification of simple, compact Lie groups. To introduce more flexibility, spaces are usually p-localized or p-completed at a fixed prime p. A major recent development was the complete classification of all finite p-complete loop spaces [AGMV, AG]. In the case of finite H-spaces which are not loop spaces, much remains to be done. This paper sheds new light on a class of torsion free p-local finite H-spaces, and produces many new examples of such spaces at the prime 3.

Let p be an odd prime and localize at p. The most basic example of an H-space is an odd dimensional sphere S^{2n-1}. It makes sense to then ask whether a sphere bundle over a sphere, $S^{2m-1} \longrightarrow B \longrightarrow S^{2n-1}$, is an H-space. Hagelgans [Ha] classified such H-spaces, provided $p \geq 5$, where the determining factor depended on the attaching map for the top cell in B. The case when $p = 3$ remains open, and this is one question we will address in this paper. More generally, it can be asked whether a space which is spherically resolved by odd dimensional spheres is an H-space. To adequately address this, it is first useful to have a body of examples.

Constructions of spherically resolved finite p-local H-spaces were given by Cooke, Harper and Zabrodsky [CHZ] and by Cohen and Neisendorfer [CN]. The techniques were different, and we will follow in the footsteps of Cohen and Neisendorfer. Let $\mathbb{Z}_{(p)}$ be the p-local integers, and take homology with $\mathbb{Z}_{(p)}$ coefficents. Both constructions state the following. Let A be a CW-complex consisting of l cells, all in odd dimensions. Localize at p. If $l < p-1$ then there is a spherically resolved H-space B such that $H_*(B) \cong \Lambda(\widetilde{H}_*(A))$, where $\Lambda(\ )$ is the exterior algebra on the indicated generating set, and there is a map $A \longrightarrow B$ which induces the inclusion of the generating set in

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homology. The Cooke-Harper-Zabrodsky construction goes through when $l = p-1$ but the space B produced need not be an H-space. Our first result is to show the same is true using Cohen and Neisendorfer's methods.

Proposition 1.1. *Fix a prime p. Let A be a CW-complex with $p-1$ cells, all in odd dimensions. Then there is a spherically resolved space B such that $H_*(B) \cong \Lambda(\widetilde{H}_*(A))$ as coalgebras and a map $A \longrightarrow B$ which induces the inclusion of the exterior algebra generators in homology.*

One goal of this paper is to give a criterion for when the space B in Proposition 1.1 is an H-space. To describe this we introduce some notation.

Suppose $l \leq p-1$ and let $V = \widetilde{H}_*(A)$. Then $H_*(\Omega\Sigma A) \cong T(V)$, where $T(\ )$ is the free tensor algebra functor. The Cohen-Neisendorfer construction aims to produce a map $\Omega\Sigma A \longrightarrow B$ which induces the abelianization $T(V) \longrightarrow \Lambda(V)$ in homology. In Lie algebraic terms, it is well known that $T(V) \cong UL\langle V\rangle$, where $L\langle V\rangle$ is the free Lie algebra on V and U is the universal enveloping algebra functor. As well, $\Lambda(V) \cong UL_{ab}\langle V\rangle$, where $L_{ab}\langle V\rangle$ is the free abelian Lie algebra generated by V, that is, the Lie algebra in which the bracket is identically zero. The abelianization $T(V) \longrightarrow \Lambda(V)$ is equivalent to $UL\langle V\rangle \xrightarrow{U\pi} UL_{ab}\langle V\rangle$, where π is a map of Lie algebras. The kernel of π is the Lie algebra $[L\langle V\rangle, L\langle V\rangle]$ generated by the brackets in $L\langle V\rangle$. An explicit basis can be calculated as $W = \oplus_{k=2}^{l+1} W_k$, where W_k consists of homogeneous brackets of length k. We say that W_k can be geometrically realized if there is a space Q_k and a map $\phi_k \colon Q_k \longrightarrow \Sigma A$ such that $H_*(Q_k) \cong \Sigma W_k$ and $(\Omega\phi_k)_*$ induces the inclusion of $UL\langle W_k\rangle$ in $UL\langle V\rangle$. The Cohen-Neisendorfer construction shows that if $l < p-1$ then W_k can be geometrically realized for each $2 \leq k \leq l+1$, and this leads to the existence of a space B with $H_*(B) \cong \Lambda(V)$ and a map $\Omega\Sigma A \longrightarrow B$ which has a right homotopy inverse. The retraction of B off $\Omega\Sigma A$ implies that B is an H-space. The geometric realization of W_k depends on the existence of a certain idempotent on $\Sigma A^{(k)}$, where $A^{(k)}$ is the k-fold smash product of A with itself. This idempotent requires that k be invertible in $\mathbb{Z}_{(p)}$. Thus the construction breaks down when $l = p-1$ as one tries to geometrically realize W_p.

Nevertheless, the geometric realization of W_p may exist, but for different reasons. We show that the existence of this geometric realization is equivalent to B being an H-space.

Theorem 1.2. *Fix a prime p. Let A be a CW-complex with $p-1$ cells, all in odd dimensions. Localize at p. Then the following are equivalent:*

(a) *the space B in Proposition 1.1 is an H-space;*

(b) *W_p can be geometrically realized.*

We go on to study the case $l = 2$ and $p = 3$ more closely. It is worth emphasizing that in this case the space A has two cells, both in odd dimensions, and B is a sphere bundle over a sphere, $S^{2m-1} \longrightarrow B \longrightarrow S^{2n-1}$, where $m < n$. We give a criterion for when W_3 is geometrically realizable, which by Theorem 1.2 is equivalent to showing that B is an H-space. To state this, let $\alpha\colon S^{2n-2} \longrightarrow S^{2m-1}$ be the attaching map for the top cell of A. Let $j\colon S^{2m-1} \longrightarrow A$ be the inclusion of the bottom cell and let $s_m\colon S^{6m-3} \longrightarrow S^{2m-1}$ represent the least dimensional nonvanishing homotopy class in the kernel of the double suspension. The geometric realization of W_2 is a space Q_2, and there is a homotopy cofibration sequence $S^{4m-1} \longrightarrow Q_2 \longrightarrow \Sigma^{2n}A \xrightarrow{d(\alpha)} S^{4m}$ where $d(\alpha)$ is the connecting map. Let $D(\alpha)\colon \Sigma^{2n+2m-3}A \longrightarrow S^{6m-3}$ be the $(2m-3)$-fold suspension of $d(\alpha)$.

Theorem 1.3. *Fix $p = 3$. Let A be a CW-complex with two cells, both in odd dimension. Localize at 3. The following are equivalent:*

(a) *the space B in Proposition 1.1 is an H-space;*

(b) *$j \circ s_m \circ D(\alpha)$ is null homotopic;*

(c) *$s_m \circ D(\alpha) \simeq \alpha \circ x$ for some map $\Sigma^{2m+2n-3}A \xrightarrow{x} S^{2n}$.*

Theorem 1.3 is practical. Part (b) can be used in tandem with Toda's calculations [T] of the 3-primary homotopy groups of spheres to produce several families of new 3-local H-spaces, and to show that several families of 3-local sphere bundles over spheres are not H-spaces. The power of this is illustrated by the fact that, prior to this result, essentially the only case that had been worked out was for $S^{2m-1} \longrightarrow B \longrightarrow S^{2m+3}$ where the attaching map for the $2m+3$-cell in B is α_1. Zabrodsky [Z] showed that B cannot be an H-space if $m > 2$ and $m \not\equiv 0 \pmod 3$ and the third author [Har1] showed that B is an H-space if $m \equiv 0 \pmod 3$. We can now use Theorem 1.3 to go much further: for example, almost all S^3-bundles over S^{2n-1} for $n \leq 42$ and entirely all S^5-bundles over S^{2n-1} for $n \leq 43$ are H-spaces. A comprehensive list of examples is given in Theorem 7.1.

Little is known about the more general case of rank $p-1$ mod-p H-spaces for $p > 3$. Hemmi [He] generalized Zabrodsky's negative results to primes larger than 3 by showing that there is no mod-p H-space with cohomology isomorphic to $\Lambda(x_{2n-1}, \mathcal{P}^1 x_{2n-1}, \ldots, \mathcal{P}^{p-2}x_{2n-1})$ unless $n = p-1$ or $n \equiv 0 \bmod p$. Harper [Har3] later proved that if $n = p-1$ or $n \equiv 0 \bmod p$ then such an H-space does exist.

Finally, now equipped with a wide range of examples, we return to the question of when a spherically resolved space is an H-space.

Theorem 1.4. *Let B be a spherically resolved space with $H_*(B) \cong \Lambda(x_1, \ldots, x_k)$ where $k < p-1$ and each $|x_i|$ is odd. Suppose there is a space A such that $H_*(A) \cong \{x_1, \ldots, x_k\}$ and a map $A \xrightarrow{i} B$ which induces the inclusion of the generating set in homology. Then B is an H-space if and only if Σi has a left homotopy inverse.*

In particular, suppose $k = 2$. Then B is a sphere bundle over a sphere, $S^{2m-1} \longrightarrow B \longrightarrow S^{2n-1}$ and A always exists - let A be the $(2n-2)$-skeleton of B and let $A \xrightarrow{i} B$ be the skeletal inclusion. Let $f\colon S^{2n+2m-3} \longrightarrow A$ be the attaching map for the top cell of B. Then Σi has a left homotopy inverse if and only if Σf is null homotopic. Thus Theorem 1.4 implies that if $p \geq 5$ then B is an H-space if and only if Σf is null homotopic. This reproduces Hagelgans' result in [Ha]. So Theorem 1.4 can be regarded as a generalization of Hagelgans' result to higher ranks.

Collecting our results so far, we obtain a classification of when a sphere bundle over a sphere $S^{2m-1} \longrightarrow B \longrightarrow S^{2n-1}$ is an H-space. If $p \geq 5$ we just saw this occurs if and only if Σf is null homotopic. If $p = 3$ we show in Proposition 6.3 that if B is an H-space then it is homotopy equivalent to the space labelled B in Proposition 1.1. Thus B is an H-space if and only if W_3 can be geometrically realized.

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2. Cohen and Neisendorfer's construction of finite H-spaces

In this section we review Cohen and Neisendorfer's [CN] construction of finite p-local H-spaces. The results are stated in Theorem 2.1. But the methods can do more, as they knew, and we present one generalization in Theorem 2.3 which will be used later in Section 3.

We repeat our standing assumptions that p is an odd prime and homology is taken with coefficients in $\mathbb{Z}_{(p)}$.

Theorem 2.1. *Let A be a CW-complex consisting of l cells, all in odd dimensions. Localize at p. If $l < p-1$ then there is a homotopy fibration $B \longrightarrow Q \longrightarrow \Sigma A$ satisfying:*

- (a) $\Omega\Sigma A \simeq B \times \Omega Q$;
- (b) *there is a Hopf algebra isomorphism $H_*(B) \cong \Lambda(\widetilde{H}_*(A))$;*
- (c) *the composite $A \xrightarrow{E} \Omega\Sigma A \longrightarrow B$ includes $\widetilde{H}_*(A)$ into $H_*(B)$ as the generating set of the exterior algebra.*

Further, all of these statements are functorial for maps $f : A \longrightarrow A'$ between spaces A and A' satisfying the hypotheses. □

Theorem 2.1 (a) implies B is an H-space. The functorial property implies that B is spherically resolved. For if the bottom cell of A is S^{2m+1} then the homotopy cofibration

$$S^{2m+1} \longrightarrow A \longrightarrow A'$$

results in a homotopy fibration

$$S^{2m+1} \longrightarrow B \longrightarrow B'.$$

This can then be iterated with respect to A' and B'.

For later purposes, we need more explicit information about the construction of the fibration $B \longrightarrow Q \longrightarrow \Sigma A$. To motivate what is to come, observe that because A has cells only in odd dimensions, it is torsion free. The James construction [J] implies that $\Sigma\Omega\Sigma A \simeq \bigvee_{i=1}^{\infty} \Sigma A^{(i)}$, where

$A^{(i)}$ is the i-fold smash product of A with itself. In particular, as A is torsion free so is each $A^{(i)}$, and therefore so is $\Sigma\Omega\Sigma A$. But as $H_*(\Omega\Sigma A)$ is a desuspension of $H_*(\Sigma\Omega\Sigma A)$, we also have that $\Omega\Sigma A$ is torsion free. Thus the Bott-Samelson Theorem applies, and states that $H_*(\Omega\Sigma A) \cong T(\widetilde{H}_*(A))$, where $T(\ )$ is the free tensor algebra. It is well known that $T(\widetilde{H}_*(A))$ is isomorphic to the universal enveloping algebra of the free graded Lie algebra generated by $\widetilde{H}_*(A)$. It is this Lie algebraic point of view that is crucial, so we give some general constructions.

Everything that follows is from [CN]. Let V be a graded vector space over $\mathbb{Z}_{(p)}$. Let $L = L\langle V\rangle$ be the free graded Lie algebra generated by V and let UL be the universal enveloping algebra. Let $L_{ab} = L_{ab}\langle V\rangle$ be the free graded abelian Lie algebra generated by V, that is, the bracket in L_{ab} is identically zero. Let $[L, L]$ be the kernel of the quotient map $L \longrightarrow L_{ab}$. The short exact sequence of graded Lie algebras

$$0 \longrightarrow [L, L] \longrightarrow L \longrightarrow L_{ab} \longrightarrow 0$$

results in a short exact sequence of Hopf algebras

$$0 \longrightarrow U[L, L] \longrightarrow UL \longrightarrow UL_{ab} \longrightarrow 0.$$

When the elements of V are all of odd dimension, an explicit Lie basis for $[L, L]$ is given by the following.

Lemma 2.2. *Suppose $V = \{u_1, \ldots, u_l\}$ where each u_i is of odd dimension and l is a positive integer. Let $L = L\langle V\rangle$. Then a Lie basis for $[L, L]$ is given by the elements*

$$[u_i, u_j], [u_{k_1}, [u_i, u_j]], [u_{k_2}, [u_{k_1}, [u_i, u_j]]], \ldots$$

where $1 \leq j \leq i \leq l$ and $1 \leq k_t < k_{t-1} < \cdots < k_2 < k_1 < i$. In particular, the basis elements have bracket lengths from 2 through $l+1$. $\square$

We now turn to topology. Let A be a CW-complex consisting of l cells, all in odd dimensions. Localize at p. Let $V = \widetilde{H}_*(A)$ and $L = L\langle V\rangle$. We would like to geometrically realize the Lie basis elements of $[L, L]$ in Lemma 2.2 as certain Whitehead products. We will see that this can always be done if $l < p-1$ but obstructions arise if $l = p-1$.

To see how this comes about, let

$$w_k : \Sigma A^{(k)} \longrightarrow \Sigma A$$

be the k-fold Whitehead product of the identity map on ΣA with itself. Observe that if σ is a permutation in the symmetric group Σ_k on k letters then there is a corresponding map $\sigma : \Sigma A^{(k)} \longrightarrow \Sigma A^{(k)}$ defined by permuting the smash factors. Define a map

$$\beta_k : \Sigma A^{(k)} \longrightarrow \Sigma A^{(k)}$$

inductively by letting $\beta_2 = 1 - (1, 2)$ and $\beta_k = (1 - (k, k-1, \ldots, 2, 1)) \circ (1 \wedge \beta_{k-1})$. In homology (ignoring the suspension coordinate), $(\beta_k)_*(x_1 \otimes \cdots \otimes x_k) = [x_1, [x_2, \ldots [x_{k-1}, x_k] \ldots]]$. This map has the property that $(\beta_k)_* \circ (\beta_k)_* \simeq k \cdot (\beta_k)_*$. Thus if we restrict to $k < p$ and define $\overline{\beta}_k = \frac{1}{k}\beta_k$ then $(\overline{\beta}_k)_*$ is an idempotent. However, the image of $(\overline{\beta}_k)_*$ consists of the suspensions of all length k brackets in L rather than those length k brackets in the basis W. So Cohen and Neisendorfer made a refinement by defining another map $a_k \colon \Sigma A^{(k)} \longrightarrow \Sigma A^{(k)}$ with the property that $(a_k)_*$ is an idempotent in homology and the composite $(a_k \circ \overline{\beta}_k)_*$ has image ΣW_k. Now let $b_k = a_k \circ \overline{\beta}_k$ and let Q_k be the mapping telescope of b_k. Then $H_*(Q_k) \cong \mathrm{Im}\ (b_k)_* \cong \Sigma W_k$. Let S_k be the mapping telescope of $1 - b_k$. As $(b_k)_*$ is an idempotent, so is $(1 - b_k)_*$. Thus, since $(b_k)_* + (1 - b_k)_*$ is the identity map, the map

$$\Sigma A^{(k)} \longrightarrow Q_k \vee S_k$$

induces an isomorphism in homology and so is a homotopy equivalence. Let Q be the wedge sum

$$Q \simeq \bigvee_{k=2}^{p-1} Q_k.$$

Define

$$Q \longrightarrow \Sigma A$$

as the wedge sum of the composites

$$Q_k \longrightarrow \Sigma A^{(k)} \xrightarrow{w_k} \Sigma A.$$

Observe that the cells of $H_*(Q) \cong \Sigma W$. Define B by the homotopy fibration

$$B \longrightarrow Q \longrightarrow \Sigma A.$$

Cohen and Neisendorfer [CN] proved the following result. Note that their explicit statement restricted to the case when $l < p - 1$, but their argument held in the generality stated below.

Theorem 2.3. *Let A be a CW complex consisting of l cells, all in odd dimensions. Localize at p. Let $V = \widetilde{H}_*(A)$ and let $L = L\langle V\rangle$. Then the homotopy fibration sequence*

$$\Omega Q \longrightarrow \Omega\Sigma A \longrightarrow B \longrightarrow Q \longrightarrow \Sigma A$$

has the following property. Let t be the least degree of the Lie basis elements in $[L, L]$ of length p. A homological model for the homotopy fibration $\Omega Q \longrightarrow \Omega\Sigma A \longrightarrow B$ in degrees $< t$ is given by the short exact sequence of Hopf algebras

$$0 \longrightarrow U[L, L] \longrightarrow UL \longrightarrow UL_{ab} \longrightarrow 0.$$

In particular, if $l < p - 1$ then no basis element of $[L, L]$ has length p and so $t = \infty$, implying that $H_*(B) \cong UL_{ab} \cong \Lambda(\widetilde{H}_*(A))$. Furthermore, all of these statements are functorial for maps $A \longrightarrow A'$ between spaces A and A' satisfying the hypotheses. $\square$

3. Rank $p-1$ spherically resolved spaces

Cohen and Neisendorfer's construction of H-spaces in Theorem 2.1 works provided the CW-complex A has l cells, all in odd dimensions, where $l < p-1$. The boundary case when $l = p-1$ is the concern of this paper. In this section we prove Proposition 1.1: given a CW-complex A with $p-1$ cells, all in odd dimensions, it is possible to construct a spherically resolved space B such that $H_*(B) \cong \Lambda(\widetilde{H}_*(A))$ as coalgebras. Whether B is an H-space is another question, which will be addressed in Section 6. A proof of Proposition 1.1 can be found in [W], where B is also shown to fit in a certain EHP-style fibration. However, we do not require this extra property, which leads to a much simpler proof.

We begin by constructing a homotopy fibration sequence to which we can apply Theorem 2.3. Let $V = \widetilde{H}_*(A)$ and suppose $V = \{x_1, \ldots, x_{p-1}\}$ where each $|x_i|$ is odd. As in Section 2, let $L = L\langle V\rangle$, $L_{ab} = L_{ab}\langle V\rangle$, and let $[L, L]$ be the kernel of the Lie algebra map $L \xrightarrow{\pi} L_{ab}$ which takes all brackets in L to zero. By Lemma 2.2, a Lie basis W for $[L, L]$ is given by $W = \oplus_{k=2}^{p} W_k$, where W_k consists of homogeneous brackets of length k, and

$$W_p = \{[x_i, [x_1, [x_2, \ldots, [x_{p-2}, x_{p-1}]] \ldots] \mid 1 \leq i \leq p-1\}.$$

As in Section 2, for $2 \leq k \leq p-1$, there are spaces Q_k with $H_*(Q_k) \cong \Sigma W_k$ and maps $\phi_k \colon Q_k \longrightarrow \Sigma A$ such that the image of $(\Omega\phi_k)_*$ is isomorphic to the sub-Hopf-algebra of $UL\langle V\rangle$ generated by $UL\langle W_k\rangle$. Let

$$\phi\colon \bigvee_{k=2}^{p-1} Q_k \longrightarrow \Sigma A$$

be the wedge sum of the maps ϕ_k. Define F as the homotopy fiber of ϕ, so there is a fibration sequence

$$\Omega\left(\bigvee_{k=2}^{p-1} Q_k\right) \xrightarrow{\Omega\phi} \Omega\Sigma A \longrightarrow F \longrightarrow \bigvee_{k=2}^{p-1} Q_k \xrightarrow{\phi} \Sigma A. \tag{1}$$

Observe that the image of $(\Omega\phi)_*$ is $UL\langle \oplus_{k=2}^{p-1} W_i\rangle$.

Proof of Proposition 1.1. Assume the elements $x_1, \ldots, x_{p-1} \in V$ have been ordered so that $|x_1| \leq |x_2| \leq \cdots \leq |x_{p-1}|$. Observe that W_p is m-connected, where $m = (\Sigma_{i=1}^{p-1}|x_i|) + (|x_1| - 1)$. Thus Theorem 2.3 implies that a homological model for the homotopy fibration $\Omega(\bigvee_{k=2}^{p-1} Q_k) \xrightarrow{\Omega\phi} \Omega\Sigma A \longrightarrow F$ in degrees $\leq m$ is given by the short exact sequence of Hopf algebras $U[L, L] \longrightarrow UL \longrightarrow UL_{ab}$. In particular, in dimensions $\leq m$, there is a coalgebra isomorphism $H_*(F) \cong UL_{ab} \cong \Lambda(x_1, \ldots, x_{p-1})$.

Let B be the m-skeleton of F. Observe that the dimension of $\Lambda(x_1, \ldots, x_{p-1})$ is $\Sigma_{i=1}^{p-1}|x_i|$, which is less than m. Thus there is a coalgebra isomorphism $H_*(B) \cong \Lambda(x_1, \ldots, x_{p-1})$. Further, since the dimension of A is less than m, the composite $A \xrightarrow{E} \Omega\Sigma A \longrightarrow F$ factors through the m-skeleton of F to give a map $i\colon A \longrightarrow B$. Since $A \xrightarrow{E} \Omega\Sigma A$ induces the inclusion of the generating set in homology and since $\Omega\Sigma A \longrightarrow F$ induces the abelianization of the universal enveloping algebra in dimensions $\leq m$, we have $A \xrightarrow{i} B$ inducing the inclusion of the exterior algebra generators in homology. Finally, since Theorem 2.3 is functorial, if $A' \longrightarrow A \longrightarrow \bigvee_{j=1}^{t} S^{2k+1}$ is a cofibration which pinches onto the cells of A of highest dimension, then there is a fibration $B' \longrightarrow B \longrightarrow \prod_{j=1}^{t} S^{2k+1}$ where $H_*(B') \cong \Lambda(\widetilde{H}_*(A'))$ as coalgebras, and there is a map $A' \longrightarrow B'$ which induces the inclusion of the exterior algebra generators in homology. Inductively, B' is spherically resolved, and so B is spherically resolved. $\square$

We now record two additional properties of the construction of B which will be used subsequently in Section 4.

Lemma 3.1. *The homotopy fibration* $\Omega(\bigvee_{k=2}^{p-1} Q_k) \xrightarrow{\Omega\phi} \Omega\Sigma A \longrightarrow F$ *splits as* $\Omega\Sigma A \simeq F \times \Omega(\bigvee_{k=2}^{p-1} Q_k)$.

Proof. This is a consequence of a much more general construction by Selick and the sixth author in [SW1, SW2]. They show that if X is a space, there is a homotopy fibration sequence $\Omega(\bigvee_{k=1}^{\infty} \overline{Q}_k) \xrightarrow{\Omega\overline{\phi}} \Omega\Sigma X \longrightarrow A^{\min}(X) \longrightarrow \bigvee_{k=2}^{\infty} \overline{Q}_k \xrightarrow{\overline{\phi}} \Sigma X$ and a decomposition $\Omega\Sigma X \simeq A^{\min}(X) \times \Omega(\bigvee_{k=2}^{\infty} \overline{Q}_k)$, where $A^{\min}(X)$ is the minimal functorial homotopy retract of $\Omega\Sigma A$. In our case, with $X = A$, the definition of $\overline{Q}_k$ for $2 \leq k \leq p-1$ is precisely the same as that for Q_k, and the definition of $\phi|_{\overline{Q}_k}$ is precisely the same as that for ϕ_k. Thus Selick and Wu's decomposition implies that $\Omega\phi$ has a left homotopy inverse. The lemma now follows. $\square$

Lemma 3.2. *The map* $\Sigma A \xrightarrow{\Sigma i} \Sigma B$ *has a left homotopy inverse.*

Proof. The proof is modelled on [CN, 4.1]. By [J], there is a homotopy equivalence $\Sigma\Omega\Sigma A \simeq \bigvee_{k=1}^{\infty} A^{(k)}$, and $H_*(\Sigma A^{(k)})$ is the suspension of the submodule of length k tensors in $H_*(\Omega\Sigma A) \cong T(\widetilde{H}_*(A))$. By permuting smash factors, there is a map $s_k : \Sigma A^{(k)} \longrightarrow \Sigma A^{(k)}$ which corresponds to the sum of all permutations in the symmetric group on k letters. So $(s_k)_* \circ (s_k)_* = k!(s_k)_*$. Thus if $k < p$ the map $\bar{s}_k = \frac{1}{k!} s_k$ induces an idempotent in homology. Let S_k be the mapping telescope of $\bar{s}_k$ and let T_k be the mapping telescope of $1 - \bar{s}_k$. Then the sum $\Sigma A^{(k)} \longrightarrow S_k \vee T_k$ induces an isomorphism in homology and so is a homotopy equivalence.

Note that $H_*(S_k) \cong \Sigma M_k$ where M_k is the submodule of length k symmetric tensors in $T(\widetilde{H}_*(A))$. Note also that the abelianization $T(\widetilde{H}_*(A)) \longrightarrow \Lambda(\widetilde{H}_*(A))$ induces an isomorphism $\theta\colon \oplus_{k=1}^{p-1} M_k \hookrightarrow T(\widetilde{H}_*(A)) \longrightarrow \Lambda(\widetilde{H}_*(A))$. After suspending, this can be geometrically realized. Consider the composite $f\colon \bigvee_{k=1}^{p-1} S_k \longrightarrow \bigvee_{k=1}^{p-1} \Sigma A^{(k)} \longrightarrow \bigvee_{k=1}^{\infty} \Sigma A^{(k)} \simeq \Sigma\Omega\Sigma A \longrightarrow \Sigma F$. Let d be the dimension of B. Recall from the proof of Proposition 1.1 that $\Omega\Sigma A \longrightarrow F$ induces in homology the abelianization of the tensor algebra $T(\widetilde{H}_*(A))$ in degrees $\leq d$. Thus $f_* = \Sigma\theta$ in dimensions $\leq d+1$. But B was defined as the d-skeleton of F and each S_k has dimension $\leq d+1$, so for dimensional reasons f factors through a map $\bigvee_{k=1}^{p-1} S_k \xrightarrow{f'} \Sigma B$, and $(f')_* = \Sigma\theta$. As $\Sigma\theta$ is an isomorphism, f' is a homotopy equivalence.

Finally, observe that $S_1 = \Sigma A$. By definition of i, in homology i_* is the inclusion of the generating set. Thus the composite $\Sigma A \xrightarrow{\Sigma i} \Sigma B \longrightarrow S_1$ induces an isomorphism in homology and so is a homotopy equivalence. $\square$

4. A characterization of rank $p-1$ torsion free H-spaces

We continue to assume that the underlying CW-complex A has $p-1$ cells, all in odd dimensions. In this section we prove Theorem 1.2, which states that, starting with A, the space B produced in Proposition 1.1 is an H-space if and only if the W_p can be geometrically realized. Recall that W_p consists of the homogeneous brackets of length p in the Lie basis $W = \oplus_{k=2}^{p} W_k$ of $[L, L]$. Recall as well what is meant by being geometrically realized.

Definition 4.1. We say that the module W_k can be *geometrically realized* if there is a space Q_k and a map $\phi_k \colon Q_k \longrightarrow \Sigma A$ such that $H_*(Q_k) \cong \Sigma W_k$ and $(\Omega\phi_k)_*$ induces the inclusion of $UL\langle W_k\rangle$ in UL.

To prove the equivalence in Theorem 1.2, we begin with the easier direction.

Proposition 4.2. *Let B be as in Proposition 1.1. If W_p can be geometrically realized then B is an H-space.*

Proof. As in (1), each W_k for $2 \leq k \leq p-1$ is geometrically realized by a map $Q_k \xrightarrow{\phi_k} \Sigma A$. By assumption, W_p is also geometrically realized by a map $Q_p \xrightarrow{\phi_p} \Sigma A$. Let $\phi' \colon \bigvee_{k=2}^{p} Q_k \longrightarrow \Sigma A$ be the wedge sum of the maps ϕ_k for $2 \leq k \leq p$. Observe that $(\Omega\phi')_*$ induces the inclusion of $U[L, L]$ in UL. Define the space F' by the homotopy fibration sequence

$$\Omega\left(\bigvee_{k=2}^{p} Q_k\right) \xrightarrow{\Omega\phi'} \Omega\Sigma A \longrightarrow F' \longrightarrow \bigvee_{k=2}^{p} Q_k \xrightarrow{\phi'} \Sigma A. \tag{2}$$

Consider the Eilenberg-Moore spectral sequence for the fibration $\Omega(\bigvee_{k=2}^{p} Q_k) \xrightarrow{\Omega\phi'} \Omega\Sigma A \longrightarrow F'$ which converges to $H_*(F')$. This has E^2-term $\mathrm{Tor}^{U[L,L]}(\mathbb{Z}_{(p)}, UL)$. Since the sequence of Hopf algebras $U[L, L] \longrightarrow UL \longrightarrow UL_{ab}$ is short exact, there is an isomorphim $UL \cong UL_{ab} \otimes U[L, L]$ of right $U[L, L]$-modules. Therefore the Eilenberg-Moore spectral sequence collapses at E^2, and we obtain a coalgebra isomorphism $H_*(F') \cong UL_{ab}$.

By Lemma 3.2, the map $\Sigma A \xrightarrow{\Sigma i} \Sigma B$ has a left homotopy inverse $\Sigma B \longrightarrow \Sigma A$. Adjointing, we obtain a map $B \longrightarrow \Omega\Sigma A$ with the property that the composite $A \xrightarrow{i} B \longrightarrow \Omega\Sigma A$ is homotopic to the suspension E. Thus the composite $B \longrightarrow \Omega\Sigma A \longrightarrow F'$ induces in homology a self-map of $UL_{ab} \cong \Lambda(x_1, \ldots, x_{p-1})$ which acts as the identity map on each x_i. Dualizing, this implies that the map $B \longrightarrow F'$ induces an isomorphism in cohomology and so is a homotopy equivalence. In particular, we obtain a decomposition $\Omega\Sigma A \simeq B \times \Omega\left(\bigvee_{k=2}^{p} Q_k\right)$, implying that B is an H-space. $\square$

To prove the converse in Theorem 1.2 we require several preliminary lemmas and constructions. We may assume that B is an H-space. Let $\overline{\mu}\colon B * B \longrightarrow \Sigma B$ be the Hopf construction, which has homotopy fibre B. Define the space R and the map r by the homotopy pullback

$$\begin{tikzcd} B \arrow[r] \arrow[d, equal] & R \arrow[r, "r"] \arrow[d] & \Sigma A \arrow[d, "i"] \\ B \arrow[r] & B * B \arrow[r, "\overline{\mu}"] & \Sigma B. \end{tikzcd}$$

From this we obtain a homotopy commutative diagram of fibration connecting maps

$$\begin{tikzcd} \Omega\Sigma A \arrow[r, "\gamma"] \arrow[d, "\Omega\Sigma i"] & B \arrow[d, equal] \\ \Omega\Sigma B \arrow[r, "\partial"] & B \end{tikzcd}$$

which defines the maps γ and ∂. One property of the Hopf construction is that ∂ has a right homotopy inverse. The next lemma shows that γ also has a right homotopy inverse.

Lemma 4.3. *The map $\Omega\Sigma A \xrightarrow{\gamma} B$ has a right homotopy inverse $s\colon B \longrightarrow \Omega\Sigma A$ which can be chosen so that $(s \circ i)_* = E_*$.*

Proof. Since B is an H-space, Lemma 3.2 implies that $\Sigma A \xrightarrow{\Sigma i} \Sigma B$ has a left homotopy inverse $t\colon \Sigma B \longrightarrow \Sigma A$. Consider the diagram

$$\begin{tikzcd} A \arrow[r, "E_A"] \arrow[d, "i"] & \Omega\Sigma A \arrow[d, "\Omega\Sigma i"] \arrow[dr, equal] & \\ B \arrow[r, "E_B"] & \Omega\Sigma B \arrow[r, "\Omega t"] & \Omega\Sigma A \end{tikzcd}$$

where E_A and E_B are the suspension maps. The left square homotopy commutes by the naturality of the suspension map, and the right triangle homotopy commutes since t is a left homotopy inverse of Σi. Let $s\colon B \longrightarrow \Omega\Sigma A$ be the composite $s = \Omega t \circ E_B$. Then the outer perimeter of the diagram implies that $E_A \simeq s \circ i$.

We claim that the composite $B \xrightarrow{s} \Omega\Sigma A \xrightarrow{\gamma} B$ is a homotopy equivalence, which would complete the lemma. We have just seen that $s \circ i \simeq E_A$. Suppose that $(\gamma \circ E_A)_* = i_*$. Then $(\gamma \circ s \circ i)_* = i_*$, so $(\gamma \circ s)_*$ is a self-map of $H_*(B) \cong \Lambda(\widetilde{H}_*(A))$ which sends $\widetilde{H}_*(A)$ to itself. Dualizing, $(\gamma \circ s)^*$ is a self-map of $H^*(B) \cong \Lambda(\widetilde{H}^*(A))$ which induces an isomorphism of the generating set. As $(\gamma \circ s)^*$ is an algebra map, it is therefore an isomorphism. Hence $\gamma \circ s$ is a homotopy equivalence.

It remains to show that $(\gamma \circ E_A)_* = i_*$. Consider the homotopy commutative diagram

$$\begin{tikzcd} A \arrow[r, "E_A"] \arrow[d, "i"] & \Omega\Sigma A \arrow[r, "\gamma"] \arrow[d, "\Omega\Sigma i"] & B \arrow[d, equal] \\ B \arrow[r, "E_B"] & \Omega\Sigma B \arrow[r, "\partial"] & B. \end{tikzcd}$$

The Hopf construction $B*B \xrightarrow{\overline{\mu}} \Sigma B$ has the property that $\widetilde{H}_*(\Sigma A) \hookrightarrow \widetilde{H}_*(\Sigma B)$ is in the complement of $\mathrm{Im}(\overline{\mu})_*$. Thus $\Omega(B*B) \xrightarrow{\Omega\overline{\mu}} \Omega\Sigma B$ has the property that $\widetilde{H}_*(A) \hookrightarrow \widetilde{H}_*(B) \hookrightarrow H_*(\Omega\Sigma B) \cong T(\widetilde{H}_*(B))$ is in the algebra cokernel of $(\Omega\overline{\mu})_*$. The Hopf construction has the property that the fibration $\Omega(B*B) \xrightarrow{\Omega\overline{\mu}} \Omega\Sigma B \xrightarrow{\partial} B$ splits as $\Omega\Sigma B \simeq B \times \Omega(B*B)$. Thus the algebra cokernel of $(\Omega\overline{\mu})_*$ is $H_*(B)$. Therefore the composite $\widetilde{H}_*(A) \hookrightarrow \widetilde{H}_*(B) \hookrightarrow H_*(\Omega\Sigma B) \cong T(\widetilde{H}_*(B)) \xrightarrow{\partial^*} H_*(B) \cong \Lambda(\widetilde{H}_*(A))$ is the inclusion of the generating set. But the sequence of inclusions $\widetilde{H}_*(A) \hookrightarrow \widetilde{H}_*(B) \hookrightarrow H_*(\Omega\Sigma B) \cong T(\widetilde{H}_*(B))$ is the map induced in homology by the composite $A \xrightarrow{i} B \xrightarrow{E_B} \Omega\Sigma B$. Hence the composite $A \xrightarrow{i} B \xrightarrow{E_B} \Omega\Sigma B \xrightarrow{\partial} B$ induces the inclusion of the exterior algebra generators in homology. The homotopy commutativity of the previous diagram therefore implies that the composite $A \xrightarrow{E_A} \Omega\Sigma A \xrightarrow{\gamma} B$ induces the inclusion of the exterior algebra generators in homology. That is, $(\gamma \circ E_A)_* = i_*$, as required. □

Corollary 4.4. *The homotopy fibration $\Omega R \xrightarrow{\Omega r} \Omega\Sigma A \xrightarrow{\gamma} B$ splits as $\Omega\Sigma A \simeq B \times \Omega R$.* □

We will need one more property.

Lemma 4.5. *In the homotopy fibration sequence $\Omega\Sigma A \xrightarrow{\gamma} B \longrightarrow R \xrightarrow{r} \Sigma A$ the space R is a co-H-space.*

Proof. In [Gr2] it was shown that for a homotopy fibration $F \longrightarrow E \longrightarrow \Sigma X$ there is a homotopy equivalence $E/F \simeq \Sigma A \rtimes F$. In our case, $R/B \simeq \Sigma A \rtimes B$. Note that $\Sigma A \rtimes B \simeq \Sigma A \vee (\Sigma A \wedge B)$, so $\Sigma A \rtimes B$ is a suspension. On the other hand, by Lemma 4.3, γ has a left homotopy inverse. Thus the map $B \longrightarrow R$ is null homotopic, implying that $R/B \simeq R \vee \Sigma B$. Hence R retracts off the suspension $\Sigma A \rtimes B$, implying that it is a co-H-space. □

We can now compare the two fibrations

$$\Omega R \xrightarrow{\Omega r} \Omega\Sigma A \xrightarrow{\gamma} B$$

$$\Omega(\bigvee_{k=2}^{p-1} Q_k) \xrightarrow{\Omega\phi} \Omega\Sigma A \longrightarrow F.$$

By Corollary 4.4 there is a homotopy equivalence $\Omega\Sigma A \simeq B \times \Omega R$, so Ωr has a left homotopy inverse

$$s\colon \Omega\Sigma A \longrightarrow \Omega R.$$

By Lemma 3.1 there is a homotopy equivalence $\Omega\Sigma A \times F \times \Omega(\bigvee_{k=2}^{p-1} Q_k)$, so $\Omega\phi$ has a left homotopy inverse

$$t\colon \Omega\Sigma A \longrightarrow \Omega(\bigvee_{k=2}^{p-1} Q_k).$$

In general, if X is a co-H-space, then [Ga2] proved that there is a map $\sigma\colon X \longrightarrow \Sigma\Omega X$ which is a right homotopy inverse of the canonical evaluation map $ev\colon \Sigma\Omega X \longrightarrow X$. In our case, by construction each Q_i is a co-H-space so there is a map $\sigma_1\colon \bigvee_{k=2}^{p-1} Q_k \longrightarrow \Sigma\Omega(\bigvee_{k=2}^{p-1} Q_k)$, giving a composite

$$f\colon \bigvee_{k=2}^{p-1} Q_k \xrightarrow{\sigma_1} \Sigma\Omega(\bigvee_{k=2}^{p-1} Q_k) \xrightarrow{\Sigma\Omega w} \Sigma\Omega\Sigma A \xrightarrow{\Sigma s} \Sigma\Omega R \xrightarrow{ev} R. \tag{3}$$

As well, by Lemma 4.5, R is a co-H-space so there is a map $\sigma_2\colon R \longrightarrow \Sigma\Omega R$, giving a composite

$$g\colon R \xrightarrow{\sigma_2} \Sigma\Omega R \xrightarrow{\Sigma\Omega r} \Sigma\Omega\Sigma A \xrightarrow{\Sigma t} \Sigma\Omega(\bigvee_{k=2}^{p-1} Q_k) \xrightarrow{ev} \bigvee_{k=2}^{p-1} Q_k. \tag{4}$$

We aim towards Proposition 4.15, which states that the composite $\bigvee_{k=2}^{p-1} Q_k \xrightarrow{f} R \xrightarrow{g} \bigvee_{k=2}^{p-1} Q_k$ is a homotopy equivalence. Recall that we are assuming that all spaces and maps have been localized at p. We begin by showing that $g \circ f$ is a p-local equivalence provided it is a rational equivalence. For a space X or a map f, let $X_{(0)}$ and $f_{(0)}$ be their rationalizations.

Lemma 4.6. *The composite $\bigvee_{i=2}^{p-1} Q_i \xrightarrow{f} R \xrightarrow{g} \bigvee_{i=2}^{p-1} Q_i$ is a p-local homotopy equivalence if and only if the composite $\bigvee_{i=2}^{p-1} (Q_i)_{(0)} \xrightarrow{f_{(0)}} R_{(0)} \xrightarrow{g_{(0)}} \bigvee_{i=2}^{p-1} (Q_i)_{(0)}$ is a homotopy equivalence.*

Proof. In general, rationalization preserves homotopy equivalences. That is, if $Y \xrightarrow{a} X \xrightarrow{b} Y$ is a homotopy equivalence in the p-local category, then the rationalization $Y_{(0)} \xrightarrow{a_{(0)}} X_{(0)} \xrightarrow{b_{(0)}} Y_{(0)}$ is a homotopy equivalence in the rational category. The converse is not true in general. However, it is true if X and Y are simply-connected, of finite type, and torsion free. For then the finite type and torsion free conditions imply that the induced maps $H_*(Y;\mathbb{Z}_{(p)}) \longrightarrow H_*(Y_{(0)};\mathbb{Q})$ and $H_*(X;\mathbb{Z}_{(p)}) \longrightarrow H_*(X_{(0)};\mathbb{Q})$ in turn induce isomorphisms between the $\mathbb{Z}_{(p)}$ and rational Euler-Poincaré series. Since $g_{(0)} \circ f_{(0)}$ is a homotopy equivalence, it too induces an isomorphism of Euler-Poincaré series. Thus $(g \circ f)_*$ induces an isomorphism of Euler-Poincaré series. Therefore, $(g \circ f)_*$ is a self-map of $H_*(Y;\mathbb{Z}_{(p)})$ which induces an isomorphism on Euler-Poincaré series, implying that it is an isomorphism. Since X and Y are simply-connected, $g \circ f$ is therefore a homotopy equivalence.

In our case, both A and B are assumed to be path-connected. Also, A is assumed to have cells only in odd dimensions, and B is constructed so that $H_*(B;\mathbb{Z}_{(p)}) \cong \Lambda(\widetilde{H}_*(A;\mathbb{Z}_{(p)}))$. So both A and B are torsion free and of finite type. This implies that $\Sigma A^{(i)}$ for $2 \leq i \leq p-1$ and $\Sigma A \rtimes B \simeq \Sigma A \vee (\Sigma A \wedge B)$ are torsion free and of finite type. Note that the suspension in each case implies that these spaces are also simply-connected. By its construction, Q_i is a retract of $\Sigma A^{(i)}$ and by Proposition 4.5, R is a retract of $\Sigma A \rtimes B$. Hence each Q_i for $2 \leq i \leq p-1$ and R are simply-connected, torsion free, and of finite type. The statement of the lemma now follows from the general argument in the first paragraph. □

We are reduced to showing that $g_{(0)} \circ f_{(0)}$ is a homotopy equivalence. We begin with two observations. First, since B is an H-space, it is rationally homotopy equivalent to a product of Eilenberg-MacLane spaces, $B_{(0)} \simeq \prod_{i=1}^{p-1} K(\mathbb{Q}, 2n_i - 1)$. This implies that $\prod_{i=1}^{p-1} K(\mathbb{Q}, 2n_i)$ is a classifying space $B(B_{(0)})$ for $B_{(0)}$. In particular, there is an evaluation map $ev\colon \Sigma B_{(0)} \simeq \Sigma\Omega B(B_{(0)}) \longrightarrow B(B_{(0)})$.

By [Ga1], the evaluation map fits in a homotopy fibration $B_{(0)} * B_{(0)} \xrightarrow{\overline{\mu}_{(0)}} \Sigma B_{(0)} \xrightarrow{ev} B(B_{(0)})$. Thus the definition of R as the pullback of $\overline{\mu}$ and Σi implies that, rationally, there is a homotopy pullback diagram

$$\begin{tikzcd} R_{(0)} \arrow[r, "r_{(0)}"] \arrow[d] & \Sigma A_{(0)} \arrow[r, "a"] \arrow[d, "\Sigma i_{(0)}"] & B(B_{0)}) \arrow[d, equal] \\ B_{(0)} * B_{(0)} \arrow[r, "\overline{\mu}_{(0)}"] & \Sigma B_{(0)} \arrow[r, "ev"] & B(B_{0)}) \end{tikzcd}$$

where a is defined as $ev \circ \Sigma i_{(0)}$. Consequently, we have the following.

Lemma 4.7. *When the p-local homotopy fibration sequence $B \longrightarrow R \xrightarrow{r} \Sigma A$ is rationalized, the connecting map can be chosen to be the loop map Ωa.* □

For the second observation, By (1) there is a p-local homotopy fibration sequence

$$\Omega\Sigma A \xrightarrow{\partial} F \longrightarrow \bigvee_{k=2}^{p-1} Q_k \xrightarrow{\phi} \Sigma A \tag{5}$$

where Q_k retracts off $\Sigma A^{(k)}$, $\phi = \bigvee_{k=2}^{p-1} \phi_k$, and each w_k factors through the k-fold iterated Whitehead product w_k of the identity map on ΣA with itself. The space Q_k is defined functorially as the mapping telescope of an idempotent e_k on $\Sigma A^{(k)}$, where e_k depends on k being invertible in $\mathbb{Z}_{(p)}$. In particular, e_p does not exist functorially, so the space Q_p does not exist functorially. However, when the fibration sequence (5) is rationalized, the idempotent e_p does exist on $\Sigma A^{(p)}_{(0)}$, so we can define $\widehat{Q}_p$ as the mapping telescope of e_p, and $\widehat{\phi}_p$ as the composite

$$\widehat{\phi}_p \colon \widehat{Q}_p \longrightarrow \Sigma A^{(p)}_{(0)} \xrightarrow{w_p} \Sigma A_{(0)}.$$

Therefore the Cohen-Neisendorfer construction can be completed rationally. Let

$$\widehat{Q}_2^{p-1} = \bigvee_{k=2}^{p-1} (Q_k)_{(0)}$$

and let

$$\widehat{\phi} \colon \widehat{Q}_2^{p-1} \vee \widehat{Q}_p \longrightarrow \Sigma A_{(0)}$$

be the wedge sum of $\phi_{(0)}$ and $\widehat{\phi}$. We immediately obtain the following.

Lemma 4.8. *There is a homotopy fibration sequence*

$$\Omega\Sigma A_{(0)} \xrightarrow{\widehat{\partial}} B_{(0)} \longrightarrow \widehat{Q}_2^{p-1} \vee \widehat{Q}_p \xrightarrow{\widehat{\phi}} \Sigma A_{(0)}$$

where $\widehat{\partial}$ has a right homotopy inverse, $\widehat{Q}_2^{p-1} \vee \widehat{Q}_p$ geometrically realizes $(\oplus_{k=2}^{p} W_k) \otimes \mathbb{Q}$, and $\widehat{Q}_p$ geometrically realizes $W_p \otimes \mathbb{Q}$. □

Now let us compare the fibration sequences in Lemmas 4.7 and 4.8. Consider the composite $\widehat{Q}_2^{p-1} \vee \widehat{Q}_p \xrightarrow{\widehat{\phi}} \Sigma A_{(0)} \xrightarrow{a} B(B_{(0)})$. Since $B(B_{(0)})$ is a product of Eilenberg-MacLane spaces, it is an H-space. Therefore, as $\widehat{\phi}$ factors thorugh Whitehead products the composite $a \circ \widehat{\phi}$ is null homotopic. Thus there is a lift

$$\begin{tikzcd} & \widehat{Q}_2^{p-1} \vee \widehat{Q}_p \arrow[dl, "\lambda"'] \arrow[d, "\widehat{\phi}"] \\ R_{(0)} \arrow[r, "r_{(0)}"] & \Sigma A_{(0)} \end{tikzcd}$$

for some map λ. This lift induces a homotopy fibration diagram

$$\begin{tikzcd} \Omega\Sigma A_{(0)} \arrow[r, "\widehat{\partial}"] \arrow[d, equal] & B_{(0)} \arrow[r] \arrow[d, "\theta"] & \widehat{Q}_2^{p-1} \vee \widehat{Q}_p \arrow[r, "\widehat{\phi}"] \arrow[d, "\lambda"] & \Sigma A_{(0)} \arrow[d, equal] \\ \Omega\Sigma A_{(0)} \arrow[r, "\Omega a"] & B_{(0)} \arrow[r] & R_{(0)} \arrow[r, "r_{(0)}"] & \Sigma A_{(0)} \end{tikzcd} \tag{6}$$

for some map θ of fibres.

Lemma 4.9. *The maps θ and λ in (6) are homotopy equivalences.*

Proof. It suffices to show that θ is a homotopy equivalence, for then the five-lemma implies that λ is as well. To see that θ is a homotopy equivalence, recall that $H_*(B_{(0)};\mathbb{Q}) \cong \Lambda(\widetilde{H}_*(A_{(0)};\mathbb{Q}))$. Precompose the leftmost square in (6) with the suspension map $A_{0)} \xrightarrow{E_{(0)}} \Omega\Sigma A_{(0)}$. One property of the Cohen-Neisendorfer construction is that the composite $\widehat{\partial} \circ E_{(0)}$ is homotopic to $i_{(0)}$. On the other hand, Lemmas 4.7 and 4.3 imply that the composite $\Omega a \circ E_{(0)}$ has the property that in rational homology $(\Omega a \circ E_{(0)})_*$ equals $(E_{(0)})_*$. Thus the commutativity of the leftmost square in (6) implies that θ_* is a self-map of $H_*(B_{(0)};\mathbb{Q})$ which is the identity map on the generating set. Dualizing to cohomology, θ^* is an algebra map which is the identity on the generating set and so is an isomorphism in all degrees. Hence θ is a homotopy equivalence. $\square$

Lemma 4.9 shows that the two homotopy fibrations in (6) derived from the Cohen-Neisendorfer construction and the Hopf construction can be identified rationally.

Next, we use the fact that λ is a homotopy equivalence to identify certain composites as loop maps. By Lemma 4.8, the map $\Omega(\widehat{Q}_2^{p-1} \vee \widehat{Q}_p) \xrightarrow{\Omega\widehat{\phi}} \Omega\Sigma A_{(0)}$ has a left homotopy inverse $t' \colon \Omega\Sigma A_{(0)} \longrightarrow \Omega(\widehat{Q}_2^{p-1} \vee \widehat{Q}_p)$.

Lemma 4.10. *The composite $\Omega R_{(0)} \xrightarrow{\Omega r_{(0)}} \Omega\Sigma A_{(0)} \xrightarrow{t'} \Omega(\widehat{Q}_2^{p-1} \vee \widehat{Q}_p)$ is homotopic to $\Omega(\lambda^{-1})$.*

Proof. Consider the diagram

$$\begin{tikzcd} & \Omega(\widehat{Q}_2^{p-1} \vee \widehat{Q}_p) \arrow[dl, "\Omega\lambda"'] \arrow[d, "\Omega\widehat{\phi}"] \arrow[dr, equal] & \\ \Omega R_{(0)} \arrow[r, "\Omega r_{(0)}"] & \Omega\Sigma A_{(0)} \arrow[r, "t"] & \Omega(\widehat{Q}_2^{p-1} \vee \widehat{Q}_p). \end{tikzcd}$$

The left triangle homotopy commutes by definition of λ as a lift through $r_{(0)}$. The right triangle homotopy commutes by definition of t. The outer perimeter of the diagram implies that $t\circ\Omega r_{(0)}\circ\Omega\lambda\simeq 1$, where 1 is the identity map on $\Omega(\widehat{Q}_2^{p-1}\vee\widehat{Q}_p)$. On the other hand, λ is a homotopy equivalence so λ^{-1} exists, and we have $\Omega(\lambda^{-1})\circ\Omega\lambda\simeq 1$. Thus $t\circ\Omega r_{(0)}\circ\Omega\lambda\simeq\Omega(\lambda^{-1})\circ\Omega\lambda$. Composing on the right with $\Omega(\lambda^{-1})$ we obtain $t\circ\Omega r_{(0)}\simeq\Omega(\lambda^{-1})$. $\square$

Similarly, if $s\colon\Omega\Sigma A_{(0)}\longrightarrow\Omega R_{(0)}$ is a left homotopy inverse of $\Omega r_{(0)}$, then we obtain the following.

Lemma 4.11. *The composite* $\Omega(\widehat{Q}_2^{p-1}\vee\widehat{Q}_p)\xrightarrow{\Omega\widehat{\phi}}\Omega\Sigma A_{(0)}\xrightarrow{s}\Omega R_{(0)}$ *is homotopic to* $\Omega\lambda$.

Proof. Argue as in Lemma 4.10. $\square$

In what follows, we wish to choose the map t' somewhat more precisely. Start with a random left homotopy inverse $\bar{t}\colon\Omega\Sigma A_{(0)}\longrightarrow\Omega(\widehat{Q}_2^{p-1}\vee\widehat{Q}_p)$ of $\Omega\widehat{\phi}$. Recall that the p-local map $\Omega(\bigvee_{k=2}^{p-1}Q_k)\longrightarrow\Omega\Sigma A$ has a left homotopy inverse $t\colon\Omega\Sigma A\longrightarrow\Omega(\bigvee_{k=2}^{p-1}Q_k)$ and that, by definition, $\widehat{Q}_2^{p-1}=\bigvee_{k=2}^{p-1}(Q_k)_{(0)}$. In general, for any simply-connected spaces there is a homotopy equivalence $\Omega(X\vee Y)\simeq\Omega X\times\Omega Y\times\Omega(\Omega X*\Omega Y)$. In our case, we have a homotopy equivalence $\Omega(\widehat{Q}_2^{p-1}\vee\widehat{Q}_p)\xrightarrow{e}\Omega\widehat{Q}_2^{p-1}\times\Omega\widehat{Q}_p\times\Omega(\Omega\widehat{Q}_2^{p-1}*\Omega\widehat{Q}_p)$. Define t'' as the composite $t''\colon\Omega\Sigma A_{(0)}\xrightarrow{\bar{t}}\Omega(\widehat{Q}_2^{p-1}\vee\widehat{Q}_p)\longrightarrow\Omega\widehat{Q}_p\times\Omega(\Omega\widehat{Q}_2^{p-1}*\Omega\widehat{Q}_p)$, where the right map is the projection. Define $t'\colon\Omega\Sigma A_{(0)}\longrightarrow\Omega(\widehat{Q}_2^{p-1}\vee\widehat{Q}_p)$ by taking the product of $t_{(0)}$ and t'' and applying e^{-1}. Notice that $t_{(0)}$ is then homotopic to the composite $\Omega\Sigma A_{(0)}\xrightarrow{t'}\Omega(\Omega\widehat{Q}_2^{p-1}*\Omega\widehat{Q}_p)\xrightarrow{\Omega q}\Omega Q_{(0)}$, where q is the pinch map.

Now we identify the maps $f_{(0)}$ and $g_{(0)}$. Consider the diagram

$$\begin{tikzcd}
R_{(0)} \arrow[r, "(\sigma_2)_{(0)}"] \arrow[dr, equal] & \Sigma\Omega R_{(0)} \arrow[r, "\Sigma\Omega r_{(0)}"] \arrow[d, "ev"] & \Sigma\Omega\Sigma A_{(0)} \arrow[r, "\Sigma t'"] & \Sigma\Omega\widehat{Q}_2^{p-1}\vee\widehat{Q}_p) \arrow[r, "\Sigma\Omega q"] \arrow[d, "ev"] & \Sigma\Omega(\widehat{Q}_2^{p-1} \arrow[d, "ev"] \\
 & R_{(0)} \arrow[rr, "\lambda^{-1}"] & & \widehat{Q}_2^{p-1}\vee\widehat{Q}_p \arrow[r, "q"] & \widehat{Q}_2^{p-1}
\end{tikzcd}$$

The left square homotopy commutes since σ_2 is a right inverse of ev. By Lemma 4.10, $\Sigma t'\circ\Sigma\Omega r_{(0)}\simeq\Sigma\Omega(\lambda^{-1})$. So the middle rectangle and the right square homotopy commute by the naturality of the evaluation map. By the choice of t' in the paragraph preceding the lemma, we have $\Omega q\circ t'\simeq t_{(0)}$. Therefore the upper direction around the diagram is the rationalization of the composite (4) defining the map g. The homotopy commutativity of the diagram therefore implies the following.

Lemma 4.12. *There is a homotopy* $g_{(0)}\simeq q\circ\lambda^{-1}$. $\square$

Next, consider the diagram

$$\begin{tikzcd}
\widehat{Q}_2^{p-1} \arrow[r, "(\sigma_1)_{(0)}"] \arrow[dr, equal] & \Sigma\Omega\widehat{Q}_2^{p-1} \arrow[r, "\Sigma\Omega\iota"] \arrow[d, "ev"] & \Sigma\Omega(\widehat{Q}_2^{p-1}\vee\widehat{Q}_p) \arrow[r, "\Sigma\Omega\widehat{\phi}"] \arrow[d, "ev"] & \Sigma\Omega A_{(0)} \arrow[r, "\Sigma s_{(0)}"] & \Sigma\Omega R_{(0)} \arrow[d, "ev"] \\
 & \widehat{Q}_2^{p-1} \arrow[r, "\iota"] & \widehat{Q}_2^{p-1}\vee\widehat{Q}_p \arrow[rr, "\lambda"] & & R_{(0)}
\end{tikzcd}$$

where ι is the inclusion of the wedge summand. The left square homotopy commutes since σ_1 is a right homotopy inverse of ev. By Lemma 4.11, $\Sigma s_{(0)} \circ \Sigma\Omega w \simeq \Sigma\Omega\lambda$. So the middle square and the right rectangle homotopy commute by the naturality of the evaluation map. By the definition of $\widehat{\phi}$, the composite $\widehat{\phi} \circ \iota$ is $\vee_{i=2}^{p-1}\phi_i$. Thus the upper direction around the diagram is the rationalization of the composite (3) defining the map f. The homotopy commutativity of the diagram therefore implies the following.

Lemma 4.13. *There is a homotopy* $f_{(0)} \simeq \lambda \circ \iota$. □

Finally, we prove that $g_{(0)} \circ f_{(0)}$ is a homotopy equivalence.

Lemma 4.14. *The composite* $\widehat{Q}_2^{p-1} \xrightarrow{f_{(0)}} R_{(0)} \xrightarrow{g_{(0)}} \widehat{Q}_2^{p-1}$ *is a homotopy equivalence.*

Proof. The identifications of $g_{(0)}$ and $f_{(0)}$ in Lemmas 4.12 and 4.13 show that $g_{(0)} \circ f_{(0)} \simeq q \circ \lambda^{-1} \circ \lambda \circ \iota \simeq q \circ \iota$. But ι is the inclusion of $\widehat{Q}_2^{p-1}$ in $\widehat{Q}_2^{p-1} \vee \widehat{Q}_p$, while q is the pinch map onto the same summand. Thus $q \circ \iota$ is the identity map on $\widehat{Q}_2^{p-1}$, so in fact $g_{(0)} \circ f_{(0)}$ is homotopic to the identity map on $\widehat{Q}_2^{p-1}$. □

Combining Lemmas 4.6 and 4.14 gives the following.

Proposition 4.15. *The composite* $\bigvee_{k=2}^{p-1} Q_k \xrightarrow{f} R \xrightarrow{g} \bigvee_{k=2}^{p-1} Q_k$ *is a homotopy equivalence.* □

Define Q_p by the cofibration

$$\bigvee_{k=2}^{p-1} Q_k \xrightarrow{f} R \longrightarrow Q_p.$$

By Proposition 4.15, f has a left homotopy inverse, which implies that there is a homotopy equivalence

$$R \simeq (\bigvee_{k=2}^{p-1} Q_k) \vee Q_p.$$

Proposition 4.16. *The space R geometrically realizes $\oplus_{k=2}^{p} W_k$ and the space Q_p geometrically realizes W_p.*

Proof. Again using $\widehat{Q}_2^{p-1} = \bigvee_{k=1}^{p-1}(Q_i)_{(0)}$, consider the diagram

$$\begin{tikzcd}
\bigvee_{k=2}^{p-1} Q_k \arrow[r, "f"] \arrow[d] & R \arrow[r, "r"] \arrow[d] & \Sigma A \arrow[d] \\
\widehat{Q}_2^{p-1} \arrow[r, "f_{(0)}"] \arrow[d, equal] & R_{(0)} \arrow[r, "r_{(0)}"] \arrow[d, "\lambda^{-1}"] & \Sigma A_{(0)} \arrow[d, equal] \\
\widehat{Q}_2^{p-1} \arrow[r, "\iota"] & \widehat{Q}_2^{p-1} \vee \widehat{Q}_p \arrow[r, "\widehat{\phi}"] & \Sigma A_{(0)}.
\end{tikzcd} \tag{7}$$

The middle row is the rationalization of the top row so the two upper squares homotopy commute. The lower left square homotopy commutes by Lemma 4.13 and the lower right square homotopy commutes by (6). By Lemma 4.8, $\widehat{Q}_2^{p-1} \vee \widehat{Q}_p$ geometrically realizes $(\oplus_{k=2}^{p} W_k) \otimes \mathbb{Q}$. Since λ^{-1} is a homotopy equivalence, we equivalently have that $r_{(0)}$ geometrically realizes $(\oplus_{k=2}^{p} W_k) \otimes \mathbb{Q}$. That is, the map $(\Omega r_{(0)})_*$ has image isomorphic to the sub-Hopf-algebra $U\langle \oplus_{k=2}^{p} W_k\rangle \otimes \mathbb{Q} \cong U[L,L] \otimes \mathbb{Q}$.

Since A has cells only in odd dimensions, it is torsion free. Therefore so is $\Sigma A^{(k)}$ for each $k \geq 1$. Hence so is $\Sigma\Omega\Sigma A \simeq \bigvee_{k=1}^{\infty} \Sigma A^{(k)}$. Since $H_*(\Sigma\Omega\Sigma A; \mathbb{Z}_{(p)}) \cong \Sigma H_*(\Omega\Sigma A; \mathbb{Z}_{(p)})$, we see that $\Omega\Sigma A$ is torsion free as well. Thus the loop of the rationalization, $\Omega\Sigma A \longrightarrow \Omega\Sigma A_{(0)}$, induces an isomorphism between the $\mathbb{Z}_{(p)}$ and rational Euler-Poincaré series. This implies that $(\Omega r)_*$ has image isomorphic to the sub-Hopf-algebra $U\langle \oplus_{k=2}^{p} W_k\rangle \cong U[L,L]$. That is, R geometrically realizes $\oplus_{k=2}^{p} W_k$.

Similarly, since the restriction of $\widehat{\phi}$ to $\widehat{Q}_2^{p-1}$ geometrically realizes $(\oplus_{k=2}^{p-1} W_k) \otimes \mathbb{Q}$, the homotopy commutativity of (7) implies that, via $r \circ f$, $\bigvee_{k=2}^{p-1} Q_k$ geometrically realizes $\oplus_{k=2}^{p-1} W_k$. Therefore the complement of $\bigvee_{k=2}^{p-1} Q_k$ in R – that is, Q_p – has the property that the image of the map $\Sigma^{-1}\widetilde{H}_*(Q_p; \mathbb{Z}_{(p)}) \hookrightarrow H_*(\Omega Q_p; \mathbb{Z}_{(p)}) \longrightarrow H_*(\Omega R; \mathbb{Z}_{(p)}) \xrightarrow{(\Omega r)_*} H_*(\Omega\Sigma A; \mathbb{Z}_{(p)})$ is isomorphic to W_p. Therefore the composite $\Omega Q_p \longrightarrow \Omega R \longrightarrow \Omega\Sigma A$ has image isomorphic to the sub-Hopf-algebra $U\langle W_p\rangle$. In other words, Q_p geometrically realizes W_p. □

At last, we can prove Theorem 1.2.

Proof of Theorem 1.2. Proposition 4.2 shows that if Q_p can be geometrically realized then B is an H-space. Conversely, if B is an H-space then Proposition 4.16 states that Q_p can be geometrically realized. □

5. A condition for producing H-spaces at the prime 3

In this section we specialize to the prime 3 and prove Theorem 1.3. Now A is a two-cell complex, the space B from Theorem 1.1 is a sphere bundle over a sphere, and there is an inclusion $A \longrightarrow B$ of the bottom two cells which induces the inclusion of the generating set in homology. In terms of the related algebra, suppose that $\widetilde{H}_*(A) \cong \{u, v\}$ where u and v both have odd degree. Then $H_*(\Omega\Sigma A) \cong UL\langle u, v\rangle$. Letting $L = L\langle u, v\rangle$ there is a short exact sequence of Lie algebras $0 \longrightarrow [L,L] \longrightarrow L \longrightarrow L_{ab} \longrightarrow 0$. and by Lemma 2.2, a Lie basis for $[L,L]$ is $W = W_2 \oplus W_3$ where

$$W_2 = \{[u,u], [u,v], [v,v]\}$$

$$W_3 = \{[u,[u,v]], [u,[v,v]]\}.$$

By Theorem 2.3, W_2 can be geometrically realized. By Theorem 1.2, B is an H-space if and only if W_3 can also be geometrically realized.

We will use the simple structure of W_2 and W_3 to produce a condition which guarantees that W_3 can be realized. The condition is practical in the sense that it reproduces known examples of rank 2 mod 3 H-spaces and also produces many more new examples.

We begin by recalling the geometric realization of W_2, and then relating it to an obstruction for the geometric realization of W_3. We have $\widetilde{H}_*(A) \cong \{u, v\}$. Suppose $|u| = 2m-1$ and $|v| = 2n-1$. Let $\alpha\colon S^{2m-2} \longrightarrow S^{2n-1}$ be the attaching map for A. As in Section 2, there is a homotopy decomposition $\Sigma A^{(2)} \simeq Q_2 \vee T_2$, where Q_2 is the mapping telescope of $\Sigma A^{(2)} \xrightarrow{b_2} \Sigma A^{(2)}$ and T_2 is the mapping telescope of $1 - b_2$. It is worth noting for simplicity's sake that $b_2 = \frac{1}{2}(1 - \Sigma T)$ where T is the map interchanging the factors of $A^{(2)}$. We have $\widetilde{H}_*(Q_2) \cong \Sigma W_2$ and the composite $\phi_2\colon Q_2 \longrightarrow \Sigma A^{(2)} \xrightarrow{w_2} \Sigma A$ has the property that $(\Omega\phi_2)_*$ is the inclusion of the sub-Hopf-algebra $UL\langle W_2\rangle$ in $UL\langle u, v\rangle$.

Lemma 5.1. *There is a homotopy cofibration $S^{4m-1} \longrightarrow Q_2 \longrightarrow \Sigma^{2n}A$ where the left map is the inclusion of the bottom cell.*

Proof. Let $i_1\colon S^{4m-1} \longrightarrow Q_2$ be the inclusion of the bottom cell and let C be its homotopy cofibre. We wish to identify the homotopy type of C. There is a homotopy equivalence $\Sigma A^{(2)} \simeq Q_2 \vee T_2$ where $H_*(Q_2) \cong \Sigma W_2$. This implies that $H_*(T_2) \cong H_*(S^{2m+2m-1})$, and therefore $T_2 \simeq S^{2m+2n-1}$. Let $i_2\colon S^{4m-1} \longrightarrow \Sigma A^{(2)}$ be the inclusion of the bottom cell and let X be its homotopy cofibre. Since the complementary wedge summand T_2 of Q_2 in $\Sigma A^{(2)}$ is $(2m+2n-2)$-connected, i_1 factors through i_2 and vice-versa. Thus there is a cofibration diagram

$$\begin{tikzcd}
S^{4m-1} \arrow[r, equal] \arrow[d] & S^{4m-1} \arrow[r, equal] \arrow[d] & S^{4m-1} \arrow[d] \\
Q_2 \arrow[r] \arrow[d] & \Sigma A^{(2)} \arrow[r] \arrow[d] & Q_2 \arrow[d] \\
C \arrow[r] & X \arrow[r] & C
\end{tikzcd}$$

in which the columns are cofibrations. Since the middle row is a homotopy equivalence, the five-lemma implies that the bottom row is as well. Thus $X \simeq C \vee T_2$.

On the other hand, the inclusion i_2 factors as a composite $S^{4m-1} \simeq \Sigma S^{2m-1} \wedge S^{2m-1} \longrightarrow \Sigma A \wedge S^{2m-1} \longrightarrow \Sigma A \wedge A$. This determines a homotopy pushout diagram

$$\begin{tikzcd}
\Sigma S^{2m-1} \wedge S^{2m-1} \arrow[r] \arrow[d] & \Sigma A \wedge S^{2m-1} \arrow[r] \arrow[d] & \Sigma S^{2n-1} \wedge S^{2m-1} \arrow[d] \\
\Sigma S^{2m-1} \wedge S^{2m-1} \arrow[r] & \Sigma A \wedge A \arrow[r] \arrow[d, "c"] & X \arrow[d, "d"] \\
 & \Sigma A \wedge S^{2n-1} \arrow[r, equal] & \Sigma A \wedge S^{2n-1}
\end{tikzcd}$$

which defines the maps c and d. We claim that the composite $C \xrightarrow{a} X \xrightarrow{d} \Sigma A \wedge S^{2n-1} \simeq \Sigma^{2n}A$ induces an isomorphism in homology, in which case it is a homotopy equivalence, and thereby proves the lemma.

We have $\widetilde{H}_*(A^{(2)}) = \{u\otimes u, u\otimes v, v\otimes u, v\otimes v\}$ and $\widetilde{H}_*(Q_2)\cong \Sigma W_2$ for $W_2 = \{[u,u],[u,v],[v,v]\}$. Removing the bottom cell from Q_2 we obtain $H_*(C)\cong\{\sigma[u,v],\sigma[v,v]\}$. Note that in $\widetilde{H}_*(A^{(2)})$, we have $[u,u] = u\otimes u-(-1)^{|u|\cdot|u|}u\otimes u = 2u\otimes u$, similarly $[v,v]=2v\otimes v$, and $[u,v] = u\otimes v-(-1)^{|u|\cdot|u|}v\otimes u = u\otimes v+v\otimes u$. As well, $c_*(\sigma u\otimes v)=0$, $c_*(\sigma v\otimes u)=\sigma v\otimes u$ and $c_*(\sigma v\otimes v)=\sigma v\otimes v$. Since d factors through c, we obtain $(d\circ a)*([u,v]) = d_*(u\otimes v+v\otimes u) = v\otimes u$ and $(d\circ a)_*([v,v]) = d_*(2v\otimes v) = 2v\otimes v$. Thus $(d\circ a)_*$ is an isomorphism, as required. □

Recall that the attaching map for the top cell in A is $S^{2n-2}\xrightarrow{\alpha}S^{2m-1}$. Let $\Sigma^{2n}A\xrightarrow{d(\alpha)}S^{4m}$ be the connecting map for the homotopy cofibration in Lemma 5.1. Define the map $D(\alpha)$ by suspending $d(\alpha)$ enough times so the target sphere is S^{6m-3} -

$$D(\alpha)\colon \Sigma^{2m+2n-3}A\xrightarrow{\Sigma^{2m-3}d(\alpha)}S^{6m-3}.$$

The reason for introducing S^{6m-3} is to connect $D(\alpha)$ to the double suspension. Let $E^2\colon S^{2m-1}\longrightarrow \Omega^2S^{2m+1}$ be the double suspension. Let W_m be its homotopy fibre. It is well known that W_m is $(6m-4)$-connected. Let s_m be the composite

$$s_m\colon S^{6m-3}\longrightarrow W_m\longrightarrow S^{2m-1}$$

where the left map is the inclusion of the bottom cell. It is well known (see [T], for example) that s_m generates an element of order 3 in $\pi_{6m-3}(S^{2m-1})$, and that $S^{6m-2}\xrightarrow{\Sigma s_m}S^{2m}$ is homotopic to the iterated Whitehead product $[\iota,[\iota,\iota]]$ where ι is the identity map on S^{2m}.

Now consider the composition

$$\Sigma^{2m+2n-3}A\xrightarrow{D(\alpha)}S^{6m-3}\xrightarrow{s_m}S^{2m-1}\xrightarrow{j}A$$

where j is the inclusion of the bottom cell. Understanding this composition is the heart of proving Theorem 1.3 and producing examples of sphere bundles over spheres which are H-spaces (or are non-H-spaces). Before proving this, we require a lemma concerning the double suspension. This is stated at the prime 3, but is easily modified to any odd prime.

Lemma 5.2. *Let X be a space of dimension $\leq 6m-3$ and suppose there is a map $X\longrightarrow\Omega^2S^{2m+1}$. Then X lifts through E^2 to a map $X\longrightarrow S^{2m-1}$.*

Proof. By [Gr2], the fibre W_m of the double suspension has a classifying space BW_m and there is a homotopy fibration $S^{2m-1}\xrightarrow{E^2}\Omega^2S^{2m+1}\longrightarrow BW_m$. Since BW_m is $(6m-3)$-connected and X is of smaller dimension, the composite $X\longrightarrow\Omega^2S^{2m+1}\longrightarrow BW_m$ is null homotopic. Thus the asserted lift exists. □

Proof of Theorem 1.3. We begin with some preliminary work. The composite $Q_2\longrightarrow\Sigma A^{(2)}\xrightarrow{w_2}\Sigma A$ geometrically realizes W_2. Now take the Whitehead product with Σj to obtain a composite

$$S^{2m-1}\wedge Q_2\longrightarrow S^{2m-1}\wedge\Sigma A^{(2)}\xrightarrow{[\Sigma j,w_2]}\Sigma A.$$

Observe that $\Sigma^{2m-1}Q_2$ has its bottom cell in degree $6m-2$. Including this, we obtain a homotopy commutative diagram

$$\begin{tikzcd} S^{6m-2} \arrow[rr, "\pm[\iota,[\iota,\iota]]"] \arrow[d] & & S^{2m} \arrow[d, "\Sigma j"] \\ \Sigma^{2m-1}Q_2 \arrow[r] & S^{2m-1}\wedge\Sigma A^{(2)} \arrow[r, "{[\Sigma j, w_2]}"] & \Sigma A \end{tikzcd} \tag{8}$$

where $[\iota,[\iota,\iota]]$ is the three-fold Whitehead product of the identity map on S^{2m} with itself. The sign of $[\iota,[\iota,\iota]]$ may depend on how Q_2 is included into $\Sigma A^{(2)}$, but this is not essential.

Let $\varphi_3\colon \Sigma^{2m-2}\wedge Q_2 \longrightarrow \Omega\Sigma A$ be the adjoint of the map in the bottom row of (8). Since $[\iota,[\iota,\iota]] \simeq \Sigma s_m$, taking adjoints in (8) gives a homotopy commutative diagram

$$\begin{tikzcd} S^{6m-3} \arrow[r, "s_m"] \arrow[d] & S^{2m-1} \arrow[r, "j"] & A \arrow[d, "E"] \\ \Sigma^{2m-2}Q_2 \arrow[rr, "\varphi_3"] & & \Omega\Sigma A. \end{tikzcd} \tag{9}$$

Since Q_2 geometrically realizes W_2 and algebraically, $W_3=[u,W_2]$ and $[u,[u,u]]=0$, the definition of φ_3 as the adjoint of a Whitehead product implies that $\operatorname{Im}(\varphi_3)_*=W_3$.

Now we show that (b) implies (a). Consider the diagram

$$\begin{tikzcd} \Sigma^{2m+2n-3}A \arrow[r, "D(\alpha)"] & S^{6m-3} \arrow[r] \arrow[d, "j\circ s_m"] & \Sigma^{2m-2}Q_2 \arrow[d, "\varphi_3"] \\ & A \arrow[r, "E"] & \Omega\Sigma A. \end{tikzcd}$$

The square homotopy commutes by (9) and the top row is a cofibration. By assumption, $j\circ s_m\circ D(\alpha)$ is null homotopic, so there is a map $\bar{s}\colon \Sigma^{2m-2}Q_2 \longrightarrow A$ which extends $j\circ s_m$. Define $f\colon \Sigma^{2m-2}Q_2 \longrightarrow \Omega\Sigma A$ by $f=\varphi_3 - E\circ\bar{s}$. Then the restriction of f to S^{2m-3} is null homotopic, so it extends to a map $g\colon \Sigma^{2m+2n-2}A \longrightarrow \Omega\Sigma A$. Since $(\varphi_3)_*=W_3$, we also have $g_*=W_3$. Thus if we define $Q_3=\Sigma^{2m+2n-1}A$ and define $\phi_3\colon Q_3\longrightarrow \Sigma A$ as the adjoint of g, then Q_3 geometrically realizes W_3. Therefore by Theorem 1.2, B is an H-space.

Next, we show that (a) implies (c). Since B is an H-space, by Lemma 4.3 there is a map $\Omega\Sigma A \xrightarrow{\gamma} B$ which has a right homotopy inverse. Consider the diagram

$$\begin{tikzcd} & \Sigma^{2m+2n-3}A \arrow[r, "D(\alpha)"] \arrow[dl, dashed, "x"'] \arrow[dd, dashed, "y"] & S^{6m-3} \arrow[r] \arrow[d, "s_m"] & \Sigma^{2m-2}Q_2 \arrow[d, "\varphi_3"] \\ & & S^{2m-3} \arrow[r, "E\circ j"] \arrow[d, equal] & \Omega\Sigma A \arrow[d, "\gamma"] \\ S^{2n-2} \arrow[r, "E"] & \Omega S^{2n-1} \arrow[r, "\partial"] & S^{2m-1} \arrow[r, "j'"] & B \end{tikzcd} \tag{10}$$

where j' is the inclusion of the bottom sphere and the maps y and x will be defined momentarily. The upper right square homotopy commutes by (9). The lower right square homotopy commutes since $j\circ E$ is the inclusion of the bottom cell in $\Omega\Sigma A$ and γ_* is degree one in H_{2m-3}. Notice that ∂ is the homotopy fibre of j' and that the composite $\partial\circ E$ is homotopic to the attaching map α. Observe that the top row is null homotopic since it is two consecutive maps in a homotopy cofibration. The homotopy commutativity of the two righthand squares then implies that the composite $j'\circ s_m\circ D(\alpha)$ is null homotopic. Therefore $s_m\circ D(\alpha)$ lifts to a map y. Since A has dimension $2n-1$, the dimension of $\Sigma^{2m+2n-3}A$ is $2m+4n-4$. As we have assumed $m<n$, we obtain $2m+4n-4\leq 6n-4<6n-2$. So Lemma 5.2 implies that y lifts through E to a map x. Now the homotopy commutativity of (10) as a whole implies that $s_m\circ D(\alpha)\simeq\partial\circ E\circ x$ for some map x. But $\partial\circ E\simeq\alpha$, so $s_m\circ D(\alpha)\simeq\alpha\circ x$, proving part (c).

Finally, we show that (c) implies (b). As there is a homotopy cofibration $S^{2n-2}\xrightarrow{\alpha}S^{2m-1}\xrightarrow{j}A$, the composite $j\circ\alpha$ is null homotopic. Therefore the assumption that $s_m\circ D(\alpha)\simeq\alpha\circ x$ implies that $j\circ s_m\circ D(\alpha)\simeq j\circ\alpha\circ x\simeq *$, and so part (b) holds. $\square$

6. H-structures on spherically resolved spaces

Before using Theorem 1.3 to produce examples of rank 2 mod-3 H-spaces, we need to deal a uniqueness property. One of the equivalent conditions in Theorem 1.3 for the existence of a rank 2 mod-3 H-space involves a particular construction of a spherically resolved space B. In Section 7 it is more convenient to use a different construction. So we need to know that the two constructions produce spaces which are homotopy equivalent. More generally, we can ask for conditions which determine the homotopy type of a spherically resolved space. This can be asked at any rank, so we return momentarily to the more general case where the rank is $\leq p-1$. A set of conditions is given in Proposition 6.3.

We begin with the following question.

Question. Let B be a spherically resolved space such that $H_*(B)\cong\Lambda(x_1,\ldots,x_{p-1})$ as coalgebras, where each $|x_i|$ is odd. When is B an H-space?

We will consider a special class of such spherically resolved spaces, which we call retractile.

Definition 6.1. Let B be a spherically resolved space with $H_*(B)\cong\Lambda(x_1,\ldots,x_k)$ where each $|x_i|$ is odd. Then B is *retractile* if there is a space A such that $H_*(A)\cong\{x_1,\ldots,x_k\}$, a map $i\colon A\longrightarrow B$ which induces the inclusion of the exterior algebra generators in homology, and Σi has a left homotopy inverse. We also say that (A,i,B) is a *retractile triple.*

Note that if $p=3$ and B is any sphere bundle over a sphere, then the space A and the map i always exist. For if $H_*(B)\cong\Lambda(x_1,x_2)$ with $|x_1|\leq|x_2|$ then take A to be the $|x_2|$-skeleton of B and i to be the skeletal inclusion. Note also that if $S^{|x_1|+|x_2|-1}\xrightarrow{f}A\xrightarrow{i}B$ is a cofibration where f

attaches the top cell to B, then the condition that B is retractile is equivalent to Σf being null homotopic. For suspending this cofibration, we see that Σi has a left homotopy inverse if and only if Σf is null homotopic.

Our first result shows that the retractile property is closely linked to the existence of an H-structure.

Lemma 6.2. *Let B be a spherically resolved space with $H_*(B) \cong \Lambda(x_1, \ldots, x_k)$ where each $|x_i|$ is odd and $k \leq p-1$. Suppose there is a space A such that $H_*(A) \cong \{x_1, \ldots, x_k\}$ and there is map $i\colon A \longrightarrow B$ which induces the inclusion of the exterior algebra generators in homology. If B is an H-space, then (A, i, B) is a retractile triple.*

Proof. Since B is an H-space, the map i extends to a map $\bar{i}\colon \Omega\Sigma A \longrightarrow B$. We have $\bar{i} \circ E \simeq i$, and i_* is the inclusion of the generating set in homology. So as $k \leq p-1$, by [CN] there is a homotopy equivalence $\Sigma B \simeq \bigvee_{j=1}^{k} S_j$ where $H_*(S_j)$ is isomorphic to the submodule of symmetric tensors of length j in $H_*(\Omega\Sigma A) \cong T(\widetilde{H}_*(A))$. In particular, $H_*(S_1) \cong \widetilde{H}_*(\Sigma A)$, and this isomorphism is induced by the composite $\Sigma A \xrightarrow{\Sigma i} \Sigma B \longrightarrow S_1$. Hence $S_1 \simeq \Sigma A$, implying that (A, i, B) is a retractile triple. $\square$

To go further, we link the retractile condition to our earlier work in Section 3 where, starting with a space A consisting of l odd dimensional cells, we constructed a spherically resolved space B such that $H_*(B) \cong \Lambda(\widetilde{H}_*(A))$. This requires a slight change of notation, as the letter B is being used two ways. Let A be a space such that $\widetilde{H}_*(A) \cong \{x_i, \ldots, x_k\}$ where each $|x_i|$ is odd. If $k < p-1$ then by Theorem 2.1 there is a homotopy fibration sequence

$$\Omega\Sigma A \xrightarrow{r} \overline{B} \longrightarrow \bigvee_{i=2}^{k} Q_i \longrightarrow \Sigma A \tag{11}$$

where $H_*(\overline{B}) \cong \Lambda(\widetilde{H}_*(A))$ and r_* is the abelianization of the tensor algebra. If $k = p-1$ then by (1) and Proposition 1.1, there is a homotopy fibration sequence

$$\Omega\Sigma A \xrightarrow{r} F \longrightarrow \bigvee_{i=2}^{p-1} Q_i \longrightarrow \Sigma A \tag{12}$$

where if $d = \Sigma_{i=1}^{p-1}|x_i|$ and $\overline{B}$ is the d-skeleton of F, then there is a coalgebra isomorphism $H_*(\overline{B}) \cong \Lambda(\widetilde{H}_*(A))$ and the restriction of r_* to dimensions $\leq d$ is the abelianization of the tensor algebra.

Proposition 6.3. *Let B be a spherically resolved space such that (A, i, B) is a retractile triple. Suppose the rank k of B satisfies $k < p$. Let $\overline{B}$ be the space constructed from A in (11) if $k < p-1$ or in (12) if $k = p-1$. Then there is a homotopy equivalence $B \simeq \overline{B}$. Further, if $k < p-1$ then B and $\overline{B}$ are H-spaces.*

Proof. Since (A,i,B) is a retractile triple, the map $\Sigma A \xrightarrow{\Sigma i} \Sigma B$ has a left homotopy inverse $s\colon \Sigma B \longrightarrow \Sigma A$. Adjointing, we obtain a map $t\colon B \longrightarrow \Omega\Sigma A$. Consider the diagram

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d, "E"] & B \arrow[d, "E"] \arrow[dr, "t"] & \\ \Omega\Sigma A \arrow[r, "\Omega\Sigma i"] & \Omega\Sigma B \arrow[r, "\Omega s"] & \Omega\Sigma A. \end{tikzcd}$$

The square homotopy commutes by the naturality of the suspension map E. The triangle homotopy commutes by the definition of t as the adjoint of s. The bottom row is homotopic to the identity map on $\Omega\Sigma A$ since s is a left homotopy inverse of Σi. Thus the diagram as a whole shows that the composite $A \xrightarrow{i} B \xrightarrow{t} \Omega\Sigma A$ is homotopic to the suspension map E.

Now suppose $k < p-1$ and consider the composite $B \xrightarrow{t} \Omega\Sigma A \xrightarrow{r} \overline{B}$. We have $H_*(B) \cong \Lambda(\widetilde{H}_*(A)) \cong H_*(\overline{B})$. Since $t\circ i \simeq E$ and $(r\circ E)_*$ is the inclusion of the generating set, the restriction of $(r\circ t)_*$ to $\widetilde{H}_*(A)$ is the identity map. Dualizing to cohomology, we obtain a map $H^*(\overline{B}) \xrightarrow{(r\circ t)^*} H^*(B)$ which is an isomorphism on the generating set, and therefore an isomorphism in all degrees. Hence $r\circ t$ is a homotopy equivalence. This implies that B and $\overline{B}$ are retracts of $\Omega\Sigma A$, and so they are both H-spaces.

If $k = p-1$, consider the composite $B \xrightarrow{t} \Omega\Sigma A \xrightarrow{r} F$. Since the B has dimension d, the map t factors through the d-skeleton of F, which by definition is $\overline{B}$. Thus we obtain a map $B \longrightarrow \overline{B}$, whose behavior in homology is determined by $(r\circ t)_*$. Now arguing as in the previous case, we obtain a homotopy equivalence $B \simeq \overline{B}$. $\square$

Corollary 6.4. *Let B and $\overline{B}$ be as in Lemma 6.3. Then B is an H-space if and only $\overline{B}$ is an H-space.* $\square$

Thus, up to homotopy equivalence, Theorem 2.1 and Proposition 1.1 produce all possible H-spaces which are resolved by odd dimension spheres, of rank $\leq p-1$, and have a generating set in homology which can be geometrically realized. In particular, if $p=3$ then, up to homotopy equivalence, Proposition 1.1 produces all possible H-spaces which are sphere bundles over a sphere, $S^{2m-1} \longrightarrow B \longrightarrow S^{2n-1}$.

Finally, we prove Theorem 1.4, which states that a triple (A,i,B) with the rank of B less than $p-1$ is an H-space if and only if it is retractile. Consequences, especially for $p=3$, were given in the Introduction.

Proof of Theorem 1.4. Combine Proposition 6.3 with Lemma 6.2. $\square$

7. Examples of rank 2 mod-3 H-spaces

Let $\alpha \in \pi_{2n-2}(S^{2m-1})$ and write $B_m(\alpha)$ for the space introduced in Proposition 1.1. This section is concerned with determining whether $B_m(\alpha)$ is an H-space at the prime 3. By Theorem 1.3, this question is equivalent to determining whether the composite $j\circ s_m\circ D(\alpha)$ is null homotopic.

Before engaging in computations, we first give a classical construction for $B_m(\alpha)$ that works for any odd prime. After this, we present two general properties of $D(\alpha)$, based on its construction, that are aids in computation. We then make several computations. The principal results are presented in the following summary. Our notation is Toda's as found in the memoir [T].

Theorem 7.1. *The space $B_m(\alpha)$ is a 3-local H-space in the following cases:*

(a) α *is unstable in stems* ≤ 75;

(b) α *is divisible by* 3;

(c) $\alpha = \alpha_1$ *and either* $m = 2$ *or* $m \equiv 0\ (3)$;

(d) $\alpha = \alpha_2$ *and either* $m = 7$ *or both* $m \not\equiv 7\ (9)$ *and* $m \not\equiv 23\ (27)$ *hold;*

(e) $\alpha = \alpha_1 \circ \gamma$ *and* $m \equiv 0\ (3)$; $\alpha = \alpha_1 \circ \gamma$ *and* $m = 2$, *where* γ *and* β_1 *commute up to sign on* S^6; $\alpha = \alpha_1\beta_1^2\beta_2$ *and* $m \equiv 1\ (3)$;

(f) $\alpha = \beta_1 \circ \gamma$ *and* $m \equiv 0\ (3)$.

The space $B_m(\alpha)$ is not a 3-local H-space in the following cases:

(g) $\alpha = \alpha_1$, $m \not\equiv 0\ (3)$, *and* $m \neq 2$;

(h) $\alpha \in \{\alpha_1\beta_1, \alpha_1\beta_2, \alpha_1\beta_2^2\}$, $m \equiv 2\ (3)$, *and* $m \neq 2$;

(i) $\alpha \in \{\alpha_1\beta_1, \alpha_1\beta_1^2, \alpha_1\beta_2\}$ *and* $m \equiv 1\ (3)$;

(j) $\alpha \in \{\epsilon', \beta_1\epsilon', \mu\}$, $m \not\equiv 0\ (3)$, *and* $m \geq 4$.

Construction on $B_m(\alpha)$.

Regard S^{2m-1} as the unit sphere in $\mathbb{R}^{2m}$. The map

$$\varphi\colon S^{2m-1} \times S^{2m-1} \longrightarrow S^{2m-1}$$

given by $\varphi(x, y) = y - 2\langle x, y\rangle x$ is reflection of y through the hyperplane perpindicular to x. This map has bidegree $(2, -1)$. The Hopf construction on φ yields a map

$$H(\varphi)\colon S^{4m-1} \longrightarrow S^{2m}$$

with classical Hopf invariant equal to -2. Moreover, $H(\varphi)$ is a quasi-fibration. Our model for $B_m(\alpha)$ is the (strict) pullback

$$\begin{array}{ccc} B_m(\alpha) & \longrightarrow & S^{4m-1} \\ \Big\downarrow & & \Big\downarrow{\scriptstyle H(\varphi)} \\ S^{2n-1} & \xrightarrow{\alpha} & S^{2m}. \end{array}$$

To analyze the homotopy type of the space produced by the construction, we observe that $r \circ \varphi = \varphi \circ (r \times r)$, where $r\colon S^{2m-1} \longrightarrow S^{2m-1}$ is the reflection $r(x_1, \ldots, x_{2m}) = (-x_1, x_2, \ldots, x_{2m})$. It follows by naturality for pullbacks that there is a self-map $r^\sharp\colon B_m(\alpha) \longrightarrow B_m(\alpha)$. Let $A = S^{2m-1} \cup_\alpha e^{2n-1}$ be the $(2n-1)$-skeleton of $B_m(\alpha)$ and let $i\colon A \longrightarrow B_m(\alpha)$ be the skeletal inclusion. If (A, i, B) is a retractile triple then $r^\sharp$ induces a map in integral homology with the property that $\widetilde{H}_*(A; \mathbb{Z})$ is the

(-1)-eigenspace in $H_*(B_m(\alpha);\mathbb{Z})$. The retractile property also ensures, by Proposition 6.3, that the rank 2 spherically resolved space $B_m(\alpha)$ is homotopy equivalent to the space B produced from A in Proposition 1.1. Therefore, to check whether $B_m(\alpha)$ is an H-space, we can use the equivalent conditions in Theorem 1.3.

Next, observe that at odd primes $H(\varphi)$ is homotopic to the Whitehead product $-[\iota_{2m},\iota_{2m}]$. To see this, note that the property $r\circ\phi=\phi\circ(r\times r)$ yields $\Sigma r\circ H(\varphi)\simeq H(\varphi)$, and Σr is multiplication by -1 on suspensions. So $2\Sigma H(\varphi)$ is null homotopic. Thus both $H(\varphi)$ and $-[\iota_{2m},\iota_{2m}]$ have Hopf invariant -2 and both suspend trivially at odd primes. Since $\pi_{4m-1}(S^{2m})=\mathbb{Z}\oplus S$, where the $\mathbb{Z}$ generator is determined by the Hopf invariant and S survives the first suspension, we therefore have $H(\varphi)\simeq-[\iota_{2m},\iota_{2m}]$ at odd primes.

As additional information, we compare the case $B_2(\alpha_1)$ (that is, $m=2$ and $\alpha=\alpha_1$) with the classical Lie group $Sp(2)$. These spaces are 3-equivalent, but the respective 3-local fibrations are pullbacks from different maps:

$$\begin{tikzcd} B_2(\alpha) \arrow[r] \arrow[d] & S^7 \arrow[d, "-[\iota_4,\iota_4]"] \\ S^7 \arrow[r, "\alpha_1"] & S^4 \end{tikzcd} \qquad \begin{tikzcd} Sp(2) \arrow[r] \arrow[d] & S^7 \arrow[d, "\nu"] \\ S^7 \arrow[r, "\pm\alpha_1"] & S^4 \end{tikzcd}$$

where ν is the result of the Hopf construction on quaternionic multiplication on S^3.

For the remainder of the section, we assume all spaces and maps are localized at 3.

Two general properties of $D(\alpha)$.

From here on, as notational convenience we give generators of homotopy groups and representative maps the same label. In particular, for a map α we say $\alpha=0$ if α is null homotopic.

It is useful to first observe the following fact about $D(\alpha)$.

Lemma 7.2. *The restriction of $\Sigma^{2m+2n-3}A\xrightarrow{D(\alpha)}S^{6m-3}$ to the bottom cell of $\Sigma^{2m+2n-3}A$ is $\Sigma^{4m-2}\alpha$.*

Proof. By definition, $D(\alpha)=\Sigma^{2m-3}d(\alpha)$ where $d(\alpha)$ is the connecting map for the cofibration $S^{4m-1}\longrightarrow Q_2\longrightarrow\Sigma^{2n}A$ in Lemma 5.1. The space Q_2 is a 3-cell complex, and the restriction to its bottom two cells is homotopy equivalent to $\Sigma^{2m}A$. Thus the restriction of $d(\alpha)$ to the bottom cell of $\Sigma^{2n}A$ is $\Sigma^{2m+1}\alpha$. The lemma now follows. □

Theorem 1.3 states that $j\circ s_m\circ D(\alpha)=0$ if and only if $s_m\circ D(\alpha)=\alpha\circ x'$ for some map $\Sigma^{2m+2n-3}A\xrightarrow{x'}S^{2n}$. The first property of $D(\alpha)$ we prove is a refinement of this equivalence, given the extra condition that α is stably trivial. The argument proving this property in Lemma 7.3 was suggested to the authors by Lucia Fernández.

Lemma 7.3. *Suppose that $\Sigma^\infty\alpha=0$. Then $j\circ s_m\circ D(\alpha)=0$ if and only if $s_m\circ\Sigma^{4m-2}\alpha=\alpha\circ x$ for some map $S^{4m+2n-4}\xrightarrow{x}S^{2n}$ in the stable $4m-2$ stem.*

Proof. If $j \circ s_m \circ D(\alpha) = 0$, then Theorem 1.3 states that $s_m \circ D(\alpha) = \alpha \circ x'$ for some map $\Sigma^{2m+2n-3}A \xrightarrow{x'} S^{2n}$. Restricting to the bottom cell of $\Sigma^{4m+2n-3}A$ and using Lemma 7.2, we obtain $s_m \circ \Sigma^{4m-2}\alpha = \alpha \circ x$ where x is the restriction of x' to $S^{4m+2n-4}$. Note that as $n > m$, x' is in the stable range.

Conversely, suppose that $s_m \circ \Sigma^{4m-2}\alpha = \alpha \circ x$. Arguing as in the (b) implies (c) part of the proof of Theorem 1.3 shows that $j \circ s_m \circ \Sigma^{4m-2}\alpha = 0$. (That is, the restriction of $j \circ s_m \circ D(\alpha)$ to the bottom cell of $\Sigma^{2m+4n-3}A$ is trivial.) Now arguing as in the preliminary part of the proof of Theorem 1.3 shows that the element $[u, [u, v]] \in H_*(\Omega\Sigma A; \mathbb{Z}_{(3)})$ is spherical. We wish to show that the element $[u, [v, v]] \in H_*(\Omega\Sigma A; \mathbb{Z}_{(3)})$ is also spherical. If so, then $W_3 = \{[u, [u, v]], [u, [v, v]]\}$ is geometrically realized by a wedge of two spheres. Therefore $B_m(\alpha)$ is an H-space by Theorem 1.2, which implies by Theorem 1.3 that $j \circ s_m \circ D(\alpha) = 0$.

It remains to show that $[u, [v, v]]$ is spherical. The space $A^{(2)}$ is produced in the following 3×3 diagram by smashing the cofibration sequence $S^{2n-2} \xrightarrow{\alpha} S^{2m-1} \longrightarrow A$ with its terms:

$$\begin{tikzcd}
S^{4n-4} \arrow[r, "\Sigma^{2n-2}\alpha"] \arrow[d, "\Sigma^{2n-2}\alpha"] & S^{2m+2n-3} \arrow[r] \arrow[d, "\Sigma^{2m-1}\alpha"] & A \wedge S^{2n-2} \arrow[d] \\
S^{2m+2n-3} \arrow[r, "\Sigma^{2m-1}\alpha"] \arrow[d] & S^{4m-2} \arrow[r] \arrow[d] & A \wedge S^{2m-1} \arrow[d] \\
S^{2n-2} \wedge A \arrow[r] & S^{2m-1} \wedge A \arrow[r] & A \wedge A.
\end{tikzcd}$$

The space $S^{2m-1} \wedge A^{(2)}$ is obtained by suspending this diagram. In the suspended diagram, the maps out of the top left corner are both equal to $\Sigma^{2m+2n-3}\alpha$. This map is null homotopic because it lies in the stable range. By using the same null homotopy for both maps in the top left square of the suspended diagram, this square is equivalent to the square

$$\begin{tikzcd}
S^{2m+4n-5} \arrow[r, "*"] \arrow[d, "*"] & S^{4m+2n-3} \arrow[d, "\Sigma^{4m-2}\alpha"] \\
S^{4m+2n-3} \arrow[r, "\Sigma^{4m-2}\alpha"] & S^{6m-3}
\end{tikzcd}$$

where equivalence is in the sense of Mather [Ma, p.229]. Thus $S^{2m-1} \wedge A^{(2)}$ is homotopy equivalent to the mapping cone of the map $S^{4m+2n-4} \vee S^{2m+4n-4} \xrightarrow{\Sigma^{4m-2}\alpha \vee *} S^{4m-2} \wedge A$. It follows that the homology class $[u, [v, v]]$ is spherical in dimension $2m + 4n - 3$ of $H_*(\Omega\Sigma A; \mathbb{Z}_{(3)})$. □

Remark 7.4. In what follows, it is useful to observe that, in general, if $s_m \circ \Sigma^{4m-2}\alpha$ cannot be expressed as $\alpha \circ x$ for some $x \in \pi^S_{4m-2}$, then $B_m(\alpha)$ is not a 3-local H-space. For the failure of $s_m \circ \Sigma^{4m-2}\alpha = \alpha \circ x$ implies, by Lemma 7.2, a failure of the equation $s_m \circ D(\alpha) = \alpha \circ x'$ for some map $\Sigma^{2m+2n-3}A \xrightarrow{x'} S^{2n}$. Theorem 1.3 then implies that $B_m(\alpha)$ is not a 3-local H-space.

To develop the second property of $D(\alpha)$ we exploit the functorial nature of its construction to obtain a formula for $D(\alpha)$ when α is a composition, $\alpha = \beta \circ \gamma$. In this paper, both β and γ are maps of spheres. However, our development is more general so that one can analyze $D(\alpha)$ when the factors may not pass through spheres, as in secondary compositions. We keep S^{2m-1} as the target sphere for α.

Suppose L is $(2m-2)$-connected (so as not to interfere with homology in dimension $2m-1$) and α factors through L as in the diagram

$$\begin{tikzcd} S^{2n-2} \arrow[r, "\gamma"] \arrow[d] & L \arrow[d, "\beta"] \\ S^{2n-2} \arrow[r, "\alpha"] & S^{2m-1}. \end{tikzcd}$$

Let $C = S^{2m-1} \cup_\beta CL$ be the mapping cone of β. Recall that $A = S^{2m-1} \cup_\alpha CS^{2n-2}$. Let

$$\widehat{\gamma}\colon A \longrightarrow C$$

extend the identity on S^{2m-1} by forming the cone on γ, $C\gamma\colon CS^{2n-1} \longrightarrow CL$. As in Section 5, for a space X, let $b_2\colon \Sigma X^{(2)} \longrightarrow \Sigma X^{(2)}$ be the map $b_2 = \frac{1}{2}(1 - \Sigma T)$, where T is the map swapping the factors of $X^{(2)}$. Let $\mathrm{Tel}_{b_2}(\Sigma X^{(2)})$ be the mapping telescope of b_2. In our case, by naturality, b_2 commutes with $\Sigma\widehat{\gamma}^{(2)}$. Thus we obtain a cofibration diagram

$$\begin{tikzcd} S^{4m-1} \arrow[r] \arrow[d, equal] & \mathrm{Tel}_{b_2}(\Sigma A^{(2)}) \arrow[r] \arrow[d] & \mathrm{Tel}_{b_2}(\Sigma A^{(2)})/S^{4m-1} \arrow[r, "d(\alpha)"] \arrow[d, "q(\gamma)"] & S^{4m} \arrow[d, equal] \\ S^{4m-1} \arrow[r] & \mathrm{Tel}_{b_2}(\Sigma C^{(2)}) \arrow[r] & \mathrm{Tel}_{b_2}(\Sigma C^{(2)})/S^{4m-1} \arrow[r, "d(\beta)"] & S^{4m} \end{tikzcd} \tag{13}$$

where $d(\alpha)$ and $d(\beta)$ are defined as the cofibration connecting maps, and $q(\gamma)$ is an induced map of cofibres. (Note that as A has two cells, both in odd dimensions, the top row of this diagram can be identified with the cofibration $S^{4m-1} \longrightarrow Q_2(A) \longrightarrow \Sigma^{2n}A$ in Lemma 5.1.) As in Section 5, let $D(\alpha) = \Sigma^{2m-3}d(\alpha)$ and $Q(\gamma) = \Sigma^{2m-3}q(\gamma)$. Then from the rightmost square in (13) we obtain the formula

$$D(\alpha) = D(\beta) \circ Q(\gamma).$$

In case L is a sphere we may say more. If $L = S^{2r-2}$ is an even dimensional sphere, then C has two cells, both in odd dimensions, so the bottom row in (13) can be identified with the cofibration $S^{4m-1} \longrightarrow Q_2(C) \longrightarrow \Sigma^{2r}C$ in Lemma 5.1. Thus $q(\gamma)$ is a map $q(\gamma)\colon \Sigma^{2n}A \longrightarrow \Sigma^{2r}C$. The pointwise construction of (13) says more, that there is a cofibration diagram

$$\begin{tikzcd} S^{4n-2} \arrow[r, "\Sigma^{2n}\alpha"] \arrow[d, "\gamma^{(2)}"] & S^{2m+2n-1} \arrow[r] \arrow[d, "\Sigma^{2m}\gamma"] & \Sigma^{2n}A \arrow[d, "q(\gamma)"] \\ S^{4r-2} \arrow[r, "\Sigma^{2r}\beta"] & S^{2m+2r-1} \arrow[r] & \Sigma^{2r}C. \end{tikzcd} \tag{14}$$

Now consider the diagram

$$
\begin{tikzcd}
\Sigma^{2m+2n-3}A \arrow[r, equal] \arrow[d, "D(\alpha)"] & \Sigma^{2m+2n-3}A \arrow[r, "k'"] \arrow[d, "Q(\gamma)"] & S^{2m+4n-4} \arrow[ddl, "\Sigma^{4n-4r}\gamma^{(2)}"] \\
S^{6m-3} \arrow[d, "s_m"] & \Sigma^{2m+2r-3}C \arrow[l, "D(\beta)"'] \arrow[d, "k"] & \\
S^{2m-1} & S^{2m+4r-4} \arrow[l, "\eta"'] &
\end{tikzcd}
\tag{15}
$$

where the maps k, k' and η are to be defined momentarily. The upper left square homotopy commutes since we already have the formula $D(\alpha) = D(\beta) \circ Q(\gamma)$. The restriction of $D(\beta)$ to the bottom cell of $\Sigma^{2m+2r-4}C$ is $\Sigma^{4m-2}\beta$. Suppose $s_m \circ \Sigma^{4m-2}\beta = 0$. Then $D(\beta)$ factors as $\Sigma^{2m+2r-3}C \xrightarrow{k} S^{2m+4r-4} \xrightarrow{\eta} S^{2m-1}$ where k is the pinch map to the top cell and η is some map, so the lower left square homotopy commutes. The map k' is the pinch map to the top cell and the right triangle homotopy commutes since it is the square of cofibration connecting maps induced by (14). Thus the entire diagram homotopy commutes, given the hypothesis that $s_m \circ \Sigma^{4m-2}\beta = 0$.

If $L = S^{2r-1}$ is an odd dimensional sphere, then $C = S^{2m-1} \cup_\beta e^{2r}$ and a calculation shows that $H_*(\mathrm{Tel}_{b_2}(\Sigma C^{(2)})) = \mathrm{span}\{\sigma[u,u], \sigma[u,v]\}$ where $|u| = 2m-1$ and $|v| = 2r$. Arguing as in Lemma 5.1 shows that $\mathrm{Tel}_{b_2}(\Sigma C^{(2)}) \simeq \Sigma^{2m}C$. Thus in (13) the cofibration sequence $S^{4m-1} \longrightarrow \mathrm{Tel}_{b_2}(\Sigma C^{(2)}) \longrightarrow \mathrm{Tel}_{b_2}(\Sigma C^{(2)}) \xrightarrow{d(\beta)} S^{4m}$ can be identified with the cofibration sequence $S^{4m-1} \xrightarrow{a} \Sigma^{2m}C \xrightarrow{b} S^{2m+2r} \xrightarrow{\Sigma^{2m+1}\beta} S^{4m}$ where a is the inclusion of the bottom cell and b is the pinch map to the top cell. Therefore $D(\beta) = \Sigma^{2m-3}d(\beta) \simeq \Sigma^{4m-2}\beta$.

We summarize all these properties in Lemma 7.5.

Lemma 7.5. *If $\alpha = \beta \circ \gamma$, where β and γ are maps of spheres, then the following hold:*

(a) *if the stem of β is odd and $s_m \circ \Sigma^{4m-2}\beta = 0$, then $s_m \circ D(\alpha)$ factors through a suspension of $\gamma^{(2)}$;*

(b) *if the stem of β is even, then $D(\alpha) = \Sigma^{4m-2}\beta \circ Q(\gamma)$. In particular, if β is multiplication by 3, then $s_m \circ D(\alpha)$ is null homotopic.*

Remark 7.6. Lemma 7.5 (b) yields Theorem 7.1 (b) since s_m has order 3.

Framework for the calculations.

We map the calculation of $s_m \circ D(\alpha)$ into the EHP sequence for the double suspension. Recall from Section 5 the homotopy fibration $W_m \longrightarrow S^{2m-1} \xrightarrow{E^2} \Omega^2 S^{2m+1}$, and the fact that s_m is defined as the composite $S^{6m-3} \longrightarrow W_m \longrightarrow S^{2m-1}$, where the left map is the inclusion of the bottom cell. To label maps, consider the fibration sequence induced by E^2,

$$\Omega S^{2m-1} \xrightarrow{E} \Omega^3 S^{2m+1} \xrightarrow{H} W_m \xrightarrow{P} S^{2m-1}$$

which defines the maps E, H and P. Note that $E \simeq \Omega E^2$. In [Gr2] it was shown that W_m has a classifying space BW_m and there is a homotopy fibration $S^{2m-1} \xrightarrow{E^2} \Omega^2 S^{2m+1} \longrightarrow BW_m$. Further, there is a homotopy fibration (localized at 3) $\Omega S^{6m-1} \longrightarrow BW_m \longrightarrow \Omega^2 S^{6m+1}$, where the composite $\Omega S^{2m+1} \longrightarrow BW_m \longrightarrow \Omega^2 S^{6m+1}$ is homotopic to the loops on the 3^{rd} James-Hopf invariant. To label maps, looping gives a homotopy fibration $\Omega^2 S^{6m-1} \xrightarrow{I} W_m \xrightarrow{J} \Omega^3 S^{6m+1}$. Note that the composite $S^{6m-3} \xrightarrow{E^2} \Omega^2 S^{6m-1} \xrightarrow{I} W_m$ is homotopic to the inclusion of the bottom cell. So $s_m = P \circ I \circ E^2$.

We relate the calculations for $s_m \circ D(\alpha)$ and the EHP sequence in the following *principal diagram*, valid for $m \geq 2$:

$$
\begin{tikzcd}
 & & \Sigma^{2m+2n-3}A \arrow[r, "k'"] \arrow[d, "D(\alpha)"] & S^{2m+4n-4} \arrow[d, "\eta^\sharp"] \\
S^{2m+4n-5} \arrow[r, "\Sigma^{2m+2n-3}\alpha"] \arrow[dd, "\eta"] & S^{4m+3n-4} \arrow[r, "\Sigma^{4m-2}\alpha"] \arrow[dd, "\zeta"] & S^{6m-3} \arrow[r, "s_m"] \arrow[d, "E^2"] & S^{2m-1} \arrow[dd, equal] \\
 & & \Omega^2 S^{6m-1} \arrow[d, "I"] & \\
\Omega S^{2m-1} \arrow[r, "E"] & \Omega^3 S^{2m+1} \arrow[r, "H"] & W_m \arrow[r, "P"] \arrow[d, "J"] & S^{2m-1} \\
 & & \Omega^3 S^{6m+1} &
\end{tikzcd}
\tag{16}
$$

Starting with the lower right square $s_m = P \circ I \circ E^2$, the diagram is constructed based on the assumptions that the composites $s_m \circ \Sigma^{4m-2}\alpha$ and $\Sigma^{4m-2}\alpha \circ \Sigma^{2m+2n-3}\alpha$ are null homotopic. The null homotopy for $s_m \circ \Sigma^{4m-2}\alpha$ implies the existence of a map ζ making the middle square homotopy commute. Note that ζ is an element with Hopf invariant $I \circ E^2 \circ \Sigma^{4m-2}\alpha$. The null homotopy for $\Sigma^{4m-2}\alpha \circ \Sigma^{2m+2n-3}\alpha$ along with the homotopy commutativity of the middle square determines the map η, making the left square homotopy commute. Since the restriction of $D(\alpha)$ to the bottom cell $S^{4m+3n-4}$ of $\Sigma^{2m+2n-3}A$ is $\Sigma^{4m-2}\alpha$, the null homotopy for $s_m \circ \Sigma^{4m-2}\alpha$ implies that $D(\alpha)$ factors as a composite $\Sigma^{2m+2n-3}A \xrightarrow{k'} S^{2m+4n-4} \xrightarrow{\epsilon} S^{2m-1}$, where k' is the pinch map to the top cell and ϵ is some map. By the Peterson-Stein formula (see [Har2, 3.4.2] for example), we may choose ϵ to be $\eta^\sharp$, the adjoint of η. Thus the upper right square homotopy commutes.

Observe that the map $\eta^\sharp$ is a representative for the Toda bracket $\langle s_m, \alpha, \alpha \rangle$ but, because of the way $D(\alpha)$ is constructed, we cannot assume that any null homotopy for the α-composition may be used to construct $\eta^\sharp$. In fact, one of our examples in Theorem 7.1 (i) is a case where the bracket contains 0 but the space is not an H-space.

We use Toda's notation for the maps to and from W_m. The elements discusssed in Theorem 7.1 are:

Toda name	stem	sphere of origin	Hopf invariant
α_1	3	3	1
α_2	7	3	$H(\alpha_2) = \alpha_1$
β_1	10	5	$I(\alpha_1)$
β_2	26	9	$JH(\beta_2) = \beta_1$
ϵ'	37	7	$I(\beta_2)$
μ	75	5	$I(\lambda)$

Comments:

- for α_1 and α_2 the Hopf invariant is the 3^{rd} James-Hopf invariant $\Omega S^3 \xrightarrow{H} \Omega S^7$;
- for ϵ', we have $\epsilon' \in \langle \beta_1, [3], \beta_2 \rangle$ and $\epsilon' \in \langle \alpha_1, \alpha_1, \beta_1^3 \rangle$;
- for μ the element λ is in the 68-stem, is born on S^9, and satisfies $JH(\lambda) = \beta_2^2$.

We begin our calculations with the unstable elements in Theorem 7.1 (a).

Proposition 7.7. *In stems ≤ 79, every unstable element yields a 3-local H-space with the possible exception of an element of order 3 in the 77-stem with $m = 4$.*

Proof. Inspection of Toda's tables reveals that $\Sigma^{4m-2}\alpha = 0$ for all the unstable elements in this range, except for $\alpha_1\beta_1^3$ when $m = 2$, and an element of order 3 in the 77-stem when $m = 3, 4, 5$. The sphere of origin for this element, which we call ρ, is either S^5 or S^7, but it is not known which. Moreover, ρ suspends to 0 on S^{29}. We assume ρ is born on S^5.

For the elements satisfying $\Sigma^{4m-2}\alpha = 0$, Lemma 7.3 implies that $j \circ s_m \circ D(\alpha) = 0$. Theorem 1.3 therefore implies that $B_m(\alpha)$ is a 3-local H-space.

For the remaining elements, observe that $s_2 \circ \alpha_1\beta_1^3 = 0$ because $s_2 = \alpha_1^2$ and $\alpha_1^3 = 0$. Note too that $\alpha_1\beta_1^3$ is a 6-fold suspension here. Next, we have $s_3 \circ \Sigma^{10}\rho = 0$ since $s_3 = \beta_1 \circ [3]$ and ρ has order 3. As well, we have $s_5 \circ \Sigma^{22}\rho = P \circ I \circ E^2 \circ \Sigma^{22}\rho = P \circ I \circ \Sigma^{24}\rho = 0$, where the equality with zero is due to the fact that the target of $\Sigma^{24}\rho$ is S^{29} and ρ has the property that it suspends to 0 on S^{29}. In all three cases, we have $s_m \circ \Sigma^{4m-2}\alpha = 0$. So Lemma 7.3 implies that $j \circ s_m \circ D(\alpha) = 0$, which implies by Theorem 1.3 that $B_m(\alpha)$ is a 3-local H-space.

The case of $B_4(\rho)$ remains open. □

Next, we turn our attention to the case of α_1. The original impetus to this work appears in [Mi]. In [Z], Zabrodsky extended the negative results by Mimura and Toda for the cases $m = 5, 8$ to the cases of Theorem 7.1 (g). The positive result in Theorem 7.1 (c) first appeared in [Har1] and suggested a possible pattern. Here we obtain these results by means of our formulas.

By [T, Proposition 4.4], the Hopf invariant on s_m satisfies

$$H(s_m) = \chi I(\alpha_1) \qquad \text{with } \chi \equiv 0 \ (3) \Leftrightarrow m \equiv 0 \ (3).$$

On S^5, $H(\beta_1) = I(\alpha_1)$ up to sign for $m = 3$ and $s_3 = 3\beta_1$. On S^3, $s_2 = \alpha_1^2$.

Proposition 7.8. *If $m \equiv 2\ (3)$, $m \neq 2$, then $B_m(\alpha_1)$ is not a 3-local H-space.*

Proof. Toda's calculation in [T, chart 5.1.11, first column p.46] shows that there is an element $\xi \in \pi_{6m-2}(S^{2m-3})$ such that $E\xi = s_m \circ \alpha_1 \neq 0$ and $H(\xi) = I(\beta_1) \neq 0$. In fact, $\xi \in \langle s_{m-1}, \alpha_1, \alpha_1 \rangle$, but we do not need this here. We will show that the equation $s_m \circ \alpha_1 = \alpha_1 \circ x$ cannot hold. Theorem 1.3 then implies that $B_m(\alpha_1)$ is not a 3-local H-space.

If $s_m \circ \alpha_1 = \alpha_1 \circ x$ holds, then x is at least a 4-fold suspension, so $\alpha_1 \circ x = \Sigma^4 y$. Then $E(\xi - \Sigma^2 y) = 0$ so $\xi - \Sigma^2 y = P(z)$ for some z. For dimensional reasons, $z = I(\alpha_2)$ is the only nonzero possibility. Thus $\xi - \Sigma^2 y = PI(\alpha_2)$. Now, the Hopf invariant of the double suspension $\Sigma^2 y$ is zero, so $H(\xi - \Sigma^2 y) = H(\xi) = I(\beta_1) \neq 0$. On the other hand, $H(\xi - \Sigma^2) = HPI(\alpha_2) = H(s_{m-1} \circ \alpha_2) = \alpha_1\alpha_2 = 0$, a contradiction. □

Proposition 7.9. *If $m \equiv 1\ (3)$, then $B_m(\alpha_1)$ is not a 3-local H-space.*

Proof. We wish to show that $j \circ s_m \circ D(\alpha_1) \neq 0$, which implies by Theorem 1.3 that $B_m(\alpha_1)$ is not a 3-local H-space. In this case $s_m \circ \alpha_1 = 0$ and, up to sign, $H(s_{m+1}) = I(\alpha_1)$. Let $\zeta = s_{m+1}$ in the principal diagram (16). Then $\zeta \circ \alpha_1 \neq 0$ and it is the image of the map ξ in the proof of Proposition 7.8 under double suspension. That is, $E\xi = \zeta \circ \alpha_1$. With this composition, consider the two squares on the left side of the diagram (15). The maps β and η in (15) correspond to the present maps α_1 and $\zeta^\sharp$ respectively, where $\zeta^\sharp$ is the adjoint of ζ. Further, with $\beta = \alpha_1$ we have $\Sigma^{2m+2r-3}C = \Sigma^{4m+2p-5}\overline{A}$, where $\overline{A} = S^{2m-1} \cup_{\alpha_1} e^{2m+2p-3}$. So the bottom left square of (15) is

$$\begin{tikzcd} S^{6m-3} \arrow[d, "s_m"] & \Sigma^{4m+2p-5}\overline{A} \arrow[l, "D(\alpha_1)"'] \arrow[d, "k"] \\ S^{2m-1} & S^{6m+4p-8}. \arrow[l, "\zeta^\sharp"] \end{tikzcd}$$

Thus to show that $j \circ s_m \circ D(\alpha_1) \neq 0$ it is equivalent to show that $j \circ \zeta^\sharp \circ k \neq 0$.

Since $\overline{A}$ is the cone on α_1 and j maps into this cone, we can say two things. If $j \circ \zeta^\sharp = 0$ then $\zeta^\sharp = \alpha_1 \circ x$ for some map x. Also, if $\zeta^\sharp \circ k = 0$ then $\zeta^\sharp = \alpha_1 \circ y$ for some map y. So to show that $j \circ \zeta^\sharp \circ k \neq 0$ it is equivalent to show that the equation

$$\zeta^\sharp = \alpha_1 \circ x + y \circ \alpha_1$$

cannot hold. If the equation does hold, then as in the proof of Proposition 7.8, we have $H(\zeta^\sharp) = I(\beta_1)$ and $H(\alpha_1 \circ x) = 0$. Since y lies in $\pi_{6m+1}(S^{2m-1})$, taking Hopf invariants on both sides of the equation above yields $I(\beta_1) = H(y) \circ \alpha_1$. But this equation of Hopf invariants cannot hold since $\beta_1 \in \langle \alpha_1, \alpha_1, \alpha_1 \rangle$. □

Proposition 7.10. *If $m \equiv 0\ (3)$, then $B_m(\alpha_1)$ is a 3-local H-space.*

Proof. In this case we have $s_m \circ \alpha_1 = 0$ and, up to sign, $H(s_{m+1}) = I(\alpha_1)$. Moreover, $s_{m+1} \circ \alpha_1 = 0$ in the principal diagram. Thus $\eta = P(z)$ for some $z \in \pi_{6m+3}(\Omega W_m)$. The only nonzero possibility is $\eta = P \circ I(\alpha_2) = s_m \circ \alpha_2$. But $s_m \circ \alpha_2 = 0$ for $m \equiv 0 \ (3)$ by the computing diagram in [T, p.37]. Hence $\eta = 0$ and $s_m \circ D(\alpha_1) = 0$ for $m \equiv 0 \ (3)$. Therefore $j \circ s_m \circ D(\alpha_1) = 0$ so Theorem 1.3 implies that $B_m(\alpha_1)$ is a 3-local H-space. □

Next, we consider the case when $m = 2$.

Proposition 7.11. *If $m = 2$ and $\alpha = \alpha_1$ or $\alpha = \alpha_1 \circ \gamma$ where γ and β_1 commute up to sign on S^6, then $B_2(\alpha)$ is a 3-local H-space.*

Proof. Here $\alpha_1^2 \circ D(\alpha_1) = \alpha_1\beta_1 \circ k$, implying that $j \circ s_2 \circ D(\alpha_1) = j \circ \alpha_1^2 \circ D(\alpha_1) = 0$. Therefore Theorem 1.3 implies that $B_2(\alpha_1)$ is a 3-local H-space. As well, by Lemma 7.5 (b), $\alpha_1^2 \circ D(\alpha_1 \circ \gamma) = \alpha_1 \circ \beta_1 \circ \gamma^{(2)} \circ k' = 0$, provided β_1 and γ commute up to sign on S^6. Therefore $j \circ s_2 \circ D(\alpha_1 \circ \gamma) = 0$, so Theorem 1.3 implies that $B_2(\alpha_1 \circ \gamma)$ is a 3-local H-space. □

Theorem 7.1 (f) and the remainder of Theorem 7.1 (e) are consequences of the proofs for α_1.

Proposition 7.12. *If $\alpha = \alpha_1 \circ \gamma$ or $\alpha = \beta_1 \circ \gamma$ and $m \equiv 0 \ (3)$, then $B_m(\alpha)$ is a 3-local H-space.*

Proof. Suppose $\alpha = \alpha_1 \circ \gamma$. The composition property $D(\alpha) = D(\alpha_1) \circ Q(\gamma)$ along with the equation $s_m \circ D(\alpha_1) = 0$ for $m \equiv 0 \ (3)$ in the proof of Proposition 7.10 imply that $s_m \circ D(\alpha) = 0$. Thus Theorem 1.3 implies that $B_m(\alpha)$ is a 3-local H-space.

Suppose $\alpha = \beta_1 \circ \gamma$. The fact that $s_m \circ \beta_1 \equiv 0 \ (3)$ follows from the details of the proof of Proposition 7.8 where β_1 is a birth certificate. By Lemma 7.5 (b), $D(\alpha) = \Sigma^{4m-2}\beta_1 \circ Q(\gamma)$. Thus $s_m \circ D(\alpha) = 0$. Theorem 1.3 therefore implies that $B_m(\alpha)$ is a 3-local H-space. □

The next few paragraphs are a digression from our calculations in order to discuss the best known stable family $\{\widetilde{\alpha}_s\}$ as described in Toda's memoir [T, p.34]. If $\nu_3(s)$ denotes the highest power of 3 that divides s, then $\widetilde{\alpha}_s$ generates a cyclic summand in the $(4s-1)$-stem having order $3^{\nu_3(s)+1}$. For $s \geq 3$, we cannot complete a calculation of $s_m \circ D(\widetilde{\alpha}_s)$ because a direct use of the principal diagram (16) puts us out of range of Toda's calculations, and we have not developed suitable factorizations of $\widetilde{\alpha}_s$ for use in Lemma 7.5. Nevertheless, enough computation can be made to suggest the following conjecture.

Conjecture 7.13. If $\nu_3(m+s) \leq s-1$ and $\nu_3(m+s) + \nu_3(m+2s) \leq 2s-2$, then $s_m \circ D(\widetilde{\alpha}_s) = 0$ for suitable $\widetilde{\alpha}_s$ on S^{2m-1}, where $m \geq \nu_3(s) + 2$.

To explain this formulation, we use information developed by Gray [Gr1]. In particular, for the prime 3, certain elements

$$\chi_{m,s} \in \pi_{6m+4s-1}(S^{2m+1})$$

are constructed provided $\nu_3(m+s) \le s-1$. Except for some anomolous behavior at extreme values (of no concern here), we have

$$H(\chi_{m,s}) = I(\widetilde{\alpha}_s)$$

in the principal diagram (16) (assuming Toda's "assertion A" in [T, p.38]) and hence that $s_m \circ \widetilde{\alpha}_s = 0$.

If $\nu_3(m+s) = 0$, it can be shown that

$$\chi_{m,s} \circ \widetilde{\alpha}_s = s_{m+1} \circ \widetilde{\alpha}_{2s-1}$$

which is 0 if $\nu_3(m+2s) \le 2s-2$. If $\nu_3(m+s) > 0$, it can be shown that

$$\chi_{m,s} \circ \widetilde{\alpha}_s = 0 \qquad \text{if } \nu_3(m+s) \le s-1.$$

The formulation in the conjecture is an interpolation between the extreme values of $\nu_3(m+s)$ such that the first condition is met.

We now turn to the case $s=2$, where Conjecture 7.13 holds.

Proposition 7.14. *If $m \not\equiv 7 \ (9)$ and $m \not\equiv 23 \ (27)$, then $B_m(\alpha_2)$ is a 3-local H-space.*

Proof. The first condition on m shows that $s_m \circ \alpha_2 = 0$. Applying the principal diagram (16) with $\alpha = \alpha_2$, the middle square then shows that $\chi_{m,2}$ exists and can be chosen to satisfy $H(\chi_{m,2}) = I(\widetilde{\alpha}_2)$.
Case 1: $\nu_3(m+2) = 0$. Toda's calculating diagram [T, p.37] yields

$$\chi_{m,2} = \langle s_{m+1}, [3], \alpha_1 \rangle.$$

Thus $\chi_{m,2} \circ \alpha_2 = s_{m+1} \circ \widetilde{\alpha}_3$, up to sign, with 0 indeterminacy. The second condition on m yields the information that the map η in the principal diagram (16) satisfies $\eta = P(z)$ for $z \in \pi_{6m+11}(\Omega W_m)$. The only nonzero possibility is $z = I(\alpha_4)$, but $s_m \circ \alpha_4 = 0$ since $\nu_3(m+4) \le 2$. Thus $\eta = P(z) = PI(\alpha_4) = s_m \circ \alpha_4 = 0$, with the third equality from the middle right square in the principal diagram. But the upper right square in the principal diagram then says that $s_m \circ D(\alpha_2) = k \circ \eta^\sharp = 0$. Hence Theorem 1.3 implies that $B_m(\alpha_2)$ is a 3-local H-space.

Case 2: $\nu_3(m+2) = 1$. Now $\nu_3(m+2) = 1$ implies that $m \equiv 1 \ (3)$. So the double suspension of $\chi_{m,2}$ is s_{m+2}, from Toda's calculating diagram [T, p.40] (of course, all this is in [Gr1] as well). Since $\nu_3(m+4) = 0$ for $m \equiv 1 \ (3)$, we have $s_{m+2} \circ \alpha_2 = 0$. Thus we know two things about $\chi_{m,2} \circ \alpha_2$. It is in the image of P from $\Omega^3 W_{m+1}$, say $P(z) = \chi_{m,2} \circ \alpha_2$, and it has trivial Hopf invariant. We claim that $\chi_{m,2} \circ \alpha_2 = 0$. To see this, consider the exact sequence

$$\pi_{6m+11}(\Omega^5 S^{6m+5}) \xrightarrow{I} \pi_{6m+11}(\Omega^3 W_{m+1}) \xrightarrow{J} \pi_{6m+11}(\Omega^6 S^{6m+7}) \tag{17}$$

with $\widetilde{\alpha}_3$ generating the group on the left and β_1 generating the group on the right. Since $\nu_3(m+4) = 0$, we have $s_{m+1} \circ \widetilde{\alpha}_3 = 0$. Thus $z \ne I(\widetilde{\alpha}_3)$, for if so then $0 \ne P(z) = PI(\widetilde{\alpha}_3) = s_m \circ \widetilde{\alpha}_3 = 0$, a contradiction. On the other hand, the Hopf invariant of $PJ^{-1}(\beta_1)$ satisfies

$$JHPJ^{-1}(\beta_1) = \chi \alpha_1 \beta_1$$

with $\chi \not\equiv 0$ (3) for $m \equiv 1$ (3). Thus $J(z) \neq \beta_1$, for if so then $z = J^{-1}\beta_1$ and $\chi_{m,2} \circ \alpha_2 = P(z) = PJ^{-1}\beta_1$ has nontrivial Hopf invariant, a contradiction. Hence exactness in (17) implies that $\chi_{m,2} \circ \alpha_2 = 0$. The proof is now completed by invoking the condition $m \not\equiv 23$ (27) and arguing as in Case 1. □

We consider what happens at the excluded values of m in Proposition 7.14. The condition $m \not\equiv 7$ (9) yields $m = 7, 16, 25, \ldots$. If $m \geq 25$ we are out of the known range. If $m = 7$ then $s_7 \circ \alpha_2 = \alpha_1\beta_1^3 = \alpha_2\beta_2$, up to sign. It is easy to extend β_2 to a suitable x so that $s_2 \circ D(\alpha_2) = \alpha_2 \circ x$. Thus Theorem 1.3 implies that $B_7(\alpha_2)$ is a 3-local H-space. If $m = 16$, $s_{16} \circ \alpha_2 \neq 0$ and no equation $s_{16} \circ \alpha_2 = \alpha_2 \circ x$ is possible because the stable 62-stem is generated by $\beta_1\beta_2^2$ and $\alpha_2\beta_1\beta_2^2 = 0$ on S^{31} by inspection of the 69-stem in Toda's table [T].

The condition $m \not\equiv 23$ (27) yields $m = 23, 50, \ldots$. We are now out of the known range for all values of m. Moreover, for $m = 23$, we have $\chi_{23,2} \circ \alpha_2 = s_{24} \circ \widetilde{\alpha}_3$, the value of which could depend on whether 3 divides s_{27}.

The negative results in Theorem 7.1 (h) and (i) are based on our work for α_1. For part (h) we have the following.

Proposition 7.15. *Suppose $m \equiv 2$ (3), $m \neq 2$, $s_m \circ \alpha_1 \circ \gamma \neq 0$, $I(\beta_1 \circ \gamma) \neq 0$ and only elements from the $\{\widetilde{\alpha}_s\}$ family are in stems $(\text{stem}\,\gamma) + 6, (\text{stem}\,\gamma) + 7$. Then $B_m(\alpha_1 \circ \gamma)$ is not a 3-local H-space.*

Proof. First, the element ξ appearing in the proof of Proposition 7.8 satisfies the equations

$$s_m \circ \Sigma^{4m-2}\alpha_1 = E\xi \qquad \text{and} \qquad H(\xi) = I(\beta_1).$$

Hence

$$s_m \circ \Sigma^{4m-2}(\alpha_1 \circ \gamma) = E(\xi \circ \Sigma^{4m-4}\gamma) \qquad \text{and} \qquad H(\xi \circ \Sigma^{4m-4}\gamma) = I(\beta_1 \circ \gamma).$$

The hypotheses $s_m \circ \alpha_1 \circ \gamma \neq 0$ and $I(\beta_1 \circ \gamma) \neq 0$ therefore imply that $E(\xi \circ \Sigma^{4m-4}\gamma) \neq 0$ and $H(\xi \circ \Sigma^{4m-4}\gamma) \neq 0$. Second, consider the sequence

$$\pi_c(\Omega^2 S^{6m-7}) \xrightarrow{I} \pi_c(W_{m-1}) \xrightarrow{J} \pi_c(\Omega^3 S^{6m-5})$$

where $c = 6m - 2 + \text{stem}\,\gamma$. If $s_m \circ \Sigma^{4m-2}\alpha_1 \circ \gamma = 0$ then $\xi \circ \Sigma^{4m-4}\gamma = P(z)$. By hypothesis, only elements from the $\{\widetilde{\alpha}_s\}$ family are in stems $(\text{stem}\,\gamma)+6$, $(\text{stem}\,\gamma)+7$. So z lies in a group determined only by the $\{\widetilde{\alpha}_s\}$ family, and HP on these elements are also in the $\{\widetilde{\alpha}_s\}$ family. From the first and second observations we can rule out the existence of an equation $s_m \circ \Sigma^{4m-2}(\alpha_1 \circ \gamma) = \alpha_1 \circ \gamma \circ x$ by arguing as in the proof of Proposition 7.8. Therefore Theorem 1.3 implies that $B_m(\alpha_1 \circ \gamma)$ is not a 3-local H-space. □

Examples meeting the hypotheses of Proposition 7.15 are β_1, β_2 and β_2^2.

What happens for $\gamma = \beta_1^2$ and $m \equiv 2$ (3)? In the principal diagram (16) we have

$$H(\langle s_{m-1}, [3], \beta_2 \rangle) = \langle \alpha, [3], \beta_2 \rangle = \beta_1^3.$$

In case $m = 5$, $s_5 \circ \alpha_1\beta_1^2 = 0$ and $\alpha_1\beta_1^2$ is the Hopf invariant for Toda's element ϵ_2 on S^{11}. In cases $m = 8, 11, 14$, $s_m \circ \alpha_1\beta_1^2 \neq 0$ from Toda's tables [T] in the $53, 65$ and 77 stems, but the sphere of origin for $s_m \circ \alpha_1\beta_1^2$ is not part of a general pattern. In these cases, an equation of the form $s_m \circ \alpha_1\beta_1^2 = \alpha_1\beta_1^2 \circ x$ can be ruled out. The stems involved are $30, 42, 54$ with possible x values $\beta_1^3, \epsilon_2, 0$. In each case, composition of x with $\alpha_1\beta_1^2$ gives 0. Thus Theorem 1.3 implies that $B_m(\alpha_1\beta_1^3)$ is not a 3-local H-space for $m = 8, 11, 14$, and the issue is unresolved for other cases of $m \equiv 2\ (3)$, $m \neq 2$. It is true that on S^3, $B_2(\alpha_1\beta_1^2)$ is a 3-local H-space.

For $m \equiv 1\ (3)$ the following proposition deals with the relevant statements in Theorem 7.1 (e) and (i).

Proposition 7.16. *If $m \equiv 1\ (3)$, $\alpha_1 \circ \gamma \neq 0$ and $\gamma^{(2)} = 0$, then $B_m(\alpha_1 \circ \gamma)$ is a 3-local H-space. If $\alpha_1 \circ \gamma$ is in the image of the double suspension map, $I(\beta_1 \circ \gamma^{(2)}) \neq 0$ and $(\text{stem}\,\gamma) + 7$ contains only elements in the $\{\widetilde{\alpha}_s\}$ family with $\widetilde{\alpha}_s \circ \alpha_1 = 0$, then $B_m(\alpha_1 \circ \gamma)$ is not a 3-local H-space.*

Proof. Since $m \equiv 1\ (3)$, as in the proof of Proposition 7.9 we have $s_m \circ \alpha_1 = 0$ and

$$s_m \circ D(\alpha_1) = \xi^\sharp \circ k$$

with $H(\xi^\sharp) = I(\beta_1)$. By Lemma 7.5, we have

$$s_m \circ D(\alpha_1 \circ \gamma) = \xi^\sharp \circ \gamma^{(2)} \circ k'.$$

For the positive result, by hypothesis $\gamma^{(2)} = 0$, so $s_m \circ D(\alpha_1 \circ \gamma) = 0$. Theorem 1.3 therefore implies that $B_m(\alpha_1 \circ \gamma)$ is a 3-local H-space.

By Theorem 1.3, the negative result holds if $j \circ s_m \circ D(\alpha_1 \circ \gamma) \neq 0$. By the first paragraph, this is equivalent to showing that $j \circ \xi^\sharp \circ \gamma^{(2)} \circ k' \neq 0$. To see this, as in Proposition 7.9, it is equivalent to show that there are no maps x and y satisfying an equation

$$\xi^\sharp \circ \gamma^{(2)} = \alpha_1 \circ \gamma \circ x + y \circ \alpha_1 \circ \gamma.$$

If this equation were to hold, then taking Hopf invariant of both sides, we obtain $I(\beta_1 \circ \gamma^{(2)}) = H(y) \circ \alpha_1 \circ \gamma$. By hypothesis, the left side of this equation is nonzero, so the right side is also nonzero. In particular, $H(y) \neq 0$. Since $\text{stem}\, H(y) = \text{stem}\,\gamma + 7$, by hypothesis on $\text{stem}\,\gamma + 7$, the nonzero element $H(y)$ must be a nontrivial member of the $\{\widetilde{\alpha}_s\}$ family. But we assume that such elements compose trivially with α_1. Hence the equation of Hopf invariants is impossible. □

A case where the condition is met for a positive result in Proposition 7.16 is $\gamma = \beta_1^2\beta_2$. For $\alpha_1\beta_1^2\beta_2$ is stably nontrivial, while $(\beta_1^2\beta_2)^{(2)} = 0$. To see the latter equality, note that $\beta_1^3\beta_2$ is in the 56-stem which is stably trivial. One suspension of $\xi^\sharp \circ \beta_1^3\beta_2$ in $\pi_{6m+60}(S^{2m-1})$ puts $\beta_1^3\beta_2$ in the stable range if $m \geq 4$.

Cases where the conditions are met for a negative result in Proposition 7.16 are $\gamma = \beta_1, \beta_1^2, \beta_2$. It is interesting to observe that in the case $\gamma = \beta_1^2$ we have $\xi^\sharp \circ \beta_1^4 = s_m \circ \beta_1\epsilon'$, so $0 \in \langle s_m, \alpha_1\beta_1^2, \alpha_1\beta_1^2\rangle$.

As in the discussion of the principal diagram, the map $\eta^\sharp$ in that diagram is also a representative of the bracket $\langle s_m, \alpha_1\beta_1^2, \alpha_1\beta_1^2\rangle$, and the diagram states that $\eta^\sharp \circ k = s_m \circ D(\alpha)$. So if $\eta^\sharp = 0$ then $s_m \circ D(\alpha) = 0$, implying by Theorem 1.3 that $B_m(\alpha) = B_m(\alpha_1\beta_1^2)$ is a 3-local H-space. Our negative result for $B_m(\alpha_1\beta_1^2)$ implies that $\eta^\sharp \neq 0$. That is, the choice of null homotopy used to construct η in the principal diagram sometimes matters.

The cases for Theorem 7.1 (j) are covered by the following.

Proposition 7.17. *If $m \not\equiv 0$ (3), α on S^{2m-1} is in the image of the double suspension map, and $I(\alpha_1 \circ \Sigma^{4m-5}\alpha) \neq 0$ in the $m-1$ version of the principal diagram (16), then $B_m(\alpha)$ is not a 3-local H-space.*

Proof. If $m \not\equiv 0$ (3), then $H(s_m) = I(\alpha_1) \neq 0$. The naturality of H and I therefore implies that $H(s_m \circ \Sigma^{4m-2}\alpha) \neq 0$ since, by hypothesis, $I(\alpha_1 \circ \Sigma^{4m-5}\alpha) \neq 0$. Consequently, $s_m \circ \Sigma^{4m-2}\alpha \neq 0$ because it has a nontrivial Hopf invariant.

On the other hand, since α is in the image of the double suspension, its Hopf invariant is trivial. Thus the equation $s_m \circ \Sigma^{4m-2}\alpha = \alpha \circ x$ cannot hold since the left side has a nontrivial Hopf invariant while the right side has a trivial Hopf invariant. By Remark 7.4, the nonexistence of such an equation implies that $B_m(\alpha)$ is not a 3-local H-space. $\square$

Cases where the hypotheses hold in Proposition 7.17 are $\epsilon', \beta_1\epsilon'$ and μ.

Remark 7.18. If α is born on S^3 and $\alpha_1^2 \circ \Sigma^6\alpha \neq 0$, then $B_2(\alpha)$ is not a 3-local H-space. No examples of this phenomenon are known if α is in an odd stem.

Finally, observe that Propositions 7.7 through 7.17 collectively cover all the cases in Theorem 7.1.

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