HOMOTOPY EXPONENTS OF SOME HOMOGENEOUS SPACES

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Abstract. Let p be an odd prime. Using homotopy decompositions and spherical fibrations, under certain dimensional restrictions, we obtain upper bounds of the p-primary homotopy exponents of some homogeneous spaces such as generalized complex Stiefel manifolds, generlized complex Grassmann manifolds, $SU(2n)/Sp(n)$, E_6/F_4 and F_4/G_2 (the latter for $p=2$ and $p \geq 5$).

1. Introduction

To determine the homotopy groups of a topological space is one of the central problems in homotopy theory. Although easy to formulate, the explicit calculation of homotopy groups is a notoriously difficult problem, even for a one-cell complex, sphere. Reduced in complexity new problems arise, such as the homotopy exponent problem. Since higher homotopy groups are abelian, for a prime p we can define the *p-primary homotopy exponent* of a space X, written $\exp_p(X)$, as p^t if t is the minimal power of p which annihilates the p-torsion of $\pi_*(X)$.

Throughout this paper, p is an odd prime, and $q = 2(p-1)$. We work in the homotopy category of simply-connected p-local spaces. The mod p homology (cohomology) of a space X is denoted by $H_*(X)$ ($H^*(X)$). Let $\widetilde{H}_*(X)$ ($\widetilde{H}^*(X)$) denote the reduced mod p homology (cohomology). Let $E(A)$ be the exterior algebra generated by a vector space A. A generator x_i of $H_*(X)$ or $H^*(X)$ has degree i. The term homotopy exponent is used for the p-primary homotopy exponent.

We start by reviewing known results on the homotopy exponents of spheres and Moore spaces. Cohen, Moore, and Neisendorfer [5] showed that for an odd prime p, $\exp_p(S^{2n+1}) = p^n$. When $p = 2$, the homotopy exponent problem is much harder and $\exp_2(S^{2n+1})$ has not yet been determined. James [14] showed that $\exp_2(S^{2n+1})$ is bounded by 2^{2n} and later Selick [24] improved this upper bound to $2^{2n-\lfloor n/2 \rfloor}$, where $\lfloor k \rfloor$ denotes the greatest integer which is less or equal to k. Barratt and Mahowald conjectured that

$$\exp_2(S^{2n+1}) = \begin{cases} 2^n, & n \equiv 0,3 \mod 4 \\ 2^{n+1}, & n \equiv 1,2 \mod 4. \end{cases}$$

The Moore space $P^m(p^r)$ is the homotopy cofibre of the degree p^r map on S^{m-1}. When p is an odd prime, $m \geq 3$, and $r \geq 1$, Neisendorfer [23] proved that $\exp_p(P^m(p^r)) = p^{r+1}$. The 2-primary homotopy exponent was studied by Theriault [29] who showed that $\exp_2(P^m(2^r)) = 2^{r+1}$ for $m \geq 4$ and $r \geq 6$. The 2-primary homotopy exponent of $P^m(2^r)$ is conjectured to be

$$\exp_2(P^m(2^r)) = \begin{cases} 2^{r+1}, & 3 \leq m, 2 \leq r \\ 2^{r+2}, & 3 \leq m, r = 1. \end{cases}$$

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The above conjecture is a special case of the long-standing unsolved Barratt conjecture which states that if the degree p^r map on $\Sigma^2 X$ is null homotopic, then $\exp_p(\Sigma^2 X) \le p^{r+1}$.

Except in the case of spheres, Moore spaces, and Lie groups of low rank [26, 27, 28], the homotopy exponents of topological spaces are mostly unknown. In this paper, we are interested in homogeneous spaces. The homotopy theory of homogeneous spaces has a lot of applications in fibre bundle theory. For example, the homotopy groups of a complex Stiefel manifold $W_{m+k,k} = U(m+k)/U(m)$ and a complex Grassmann manifold $G_{m+k,k} = U(m+k)/(U(m) \times U(k))$ are needed in the obstruction theory of orientable fibre bundles. Gilmore [7] computed, up to some range, the homotopy groups $\pi_*(W_{m+k,k})$. James [12] showed that the homotopy groups of a complex Grassmann manifold are the direct sum of the homotopy groups of a complex Stiefel manifold and a unitary group. We gerneralise this result by showing that the splitting holds on the level of spaces. The homotopy groups of $SU(2n)/Sp(n)$ are needed in the study of the existence of almost quaternion structures. Up to some range, $\pi_*(SU(2n)/Sp(n))$ are obtained by Harris [10] and Mimura [16]; while Hirato, Kachi, and Mimura [11] partially calculated the 2-primary part of $\pi_*(E_6/F_4)$ and $\pi_*(F_4/G_2)$. In this paper, under some restricted conditions, we give upper bounds of the homotopy exponents of the homogeneous spaces such as $W_{m+k,k}$, $G_{m+k,k}$, $SU(2n)/Sp(n)$, E_6/F_4, and F_4/G_2. Additionally, the homotopy exponents of the generalized complex Grassmann manifolds and generalized complex Stiefel manifolds are studied.

Homotopy decompositions play an important role in determining the homotopy properties of a topological space. For example, we can decompose a space as a product of indecomposable factors, investigate them and the way they are assembled to understand the original space. In this paper we use homotopy decompositions to look into the homotopy exponents of homogeneous spaces.

When $m + k \le (p-1)^2 + 1$ and $m > 0$, we decompose $\Omega W_{m+k,k}$ as a product of loop spaces. By analysing the homotopy exponent of each factor, we obtain an upper bound of the homotopy exponent of $W_{m+k,k}$.

Theorem 1.1. *Let $m + k \le (p-1)^2 + 1$ and $m > 0$. Then*

(1) *if $k \le p-1$, then $\exp_p(W_{m+k,k}) = p^{m+k-1}$.*

Suppose further that $p \le k$, and let there be $\bar{i}$ such that $1 \le \bar{i} \le p-1$ and $2(m + \bar{i} - 1) + (k_{\bar{i}} - 1)q + 1 = 2(m+k) - 1$. Then

(2) *if $(m+\bar{i}-1)+(t-1)(p-1) \not\equiv 0 \mod p$ for every t such that $1 \le t \le k_{\bar{i}}-1$, then $\exp_p(W_{m+k,k}) \le p^{m+k+k_{\bar{i}}-2}$;*

(3) *if $(m+\bar{i}-1)+(t-1)(p-1) \equiv 0 \mod p$ for some t such that $1 \le t \le k_{\bar{i}}-1$, then $\exp_p(W_{m+k,k}) \le p^{m+k+k_{\bar{i}}-3}$.*

The definition of $k_{\bar{i}}$ is given in Section 3.

In Section 4 we decompose the homogeneous space $SU(2n)/Sp(n)$ when $2n \le (p-1)^2 + 2$ into a product of H-spaces. By studying the homotopy exponent of each factor, we obtain an upper bound of $\exp_p(SU(2n)/Sp(n))$.

Theorem 1.2. *Let $2n \le (p-1)^2 + 2$. Then*

(1) *if $2n \le p+1$, then $\exp_p(SU(2n)/Sp(n)) = p^{2n-2}$.*

Suppose further that $p + 3 \le 2n$, and let there be $\bar{i}$ such that $1 \le \bar{i} \le (p-1)/2$ and $4\bar{i} + (r^{(2n)}_{2\bar{i}} - 1)q + 1 = 4n - 3$. Then

(2) *if* $2\bar{i} + (t-1)(p-1) \not\equiv 0 \mod p$ *for every* t *such that* $1 \le t \le r^{(2n)}_{2\bar{i}} - 1$, *then* $\exp_p(SU(2n)/Sp(n)) \le p^{2n + r^{(2n)}_{2\bar{i}} - 1}$;

(3) *if* $2\bar{i} + (t-1)(p-1) \equiv 0 \mod p$ *for some* t *such that* $1 \le t \le r^{(2n)}_{2\bar{i}} - 1$, *then* $\exp_p(SU(2n)/Sp(n)) \le p^{2n + r^{(2n)}_{2\bar{i}} - 2}$.

The definition of $r^{(2n)}_{2\bar{i}}$ is given in Section 3.

In Section 5 we show that the homotopy groups of a complex Grassmann manifold $G_{m+k,k} = U(m+k)/(U(m) \times U(k))$ are determined by those of $W_{m+k,k}$ and $SU(k)$ when $k \le m$.

Theorem 1.3. *For* $G_{m+k,k} = U(m+k)/(U(k) \times U(m))$ *with* $k \le m$, *there is a homotopy equivalence*

$$\Omega G_{m+k,k} \simeq \Omega W_{m+k,k} \times SU(k) \times S^1.$$

Assume that $m + k \le (p-1)^2 + 1$ *and let there be* $\bar{i}$ *such that* $1 \le \bar{i} \le p-1$ *and* $2(m + \bar{i} - 1) + (k_{\bar{i}} - 1)q + 1 = 2(m+k) - 1$. *Then*

(1) *if* $(m+\bar{i}-1) + (t-1)(p-1) \not\equiv 0 \mod p$ *for every* t *such that* $1 \le t \le k_{\bar{i}} - 1$, *then* $\exp_p(G_{m+k,k}) \le p^{m+k+k_{\bar{i}}-2}$;

(2) *if* $(m+\bar{i}-1) + (t-1)(p-1) \equiv 0 \mod p$ *for some* t *such that* $1 \le t \le k_{\bar{i}} - 1$, *then* $\exp_p(G_{m+k,k}) \le p^{m+k+k_{\bar{i}}-3}$.

Further on, we extend the results of Theorems 1.1, and 1.3 to results on upper bounds of the homotopy exponents of generalized complex Grassmann manifolds and generalized complex Stiefel manifolds.

The homogeneous space E_6/F_4 is spherically resolved when localized at odd primes, while F_4/G_2 is spherically resolved when localized at any prime except 3. In Section 6 by studying the spherical fibrations related to E_6/F_4, and F_4/G_2, we obtain upper bounds of their homotopy exponents.

Theorem 1.4. *For* $p = 3$ *or* 5, *an upper bound of the* p*-primary homotopy exponent of* E_6/F_4 *is given by*

$$\exp_p(E_6/F_4) \le p^9.$$

For $p > 5$, *the* p*-primary homotopy exponent of* E_6/F_4 *is* p^8.

Theorem 1.5. *An upper bound of the* p*-primary homotopy exponent of* F_4/G_2 *is given by*

$$\exp_p(F_4/G_2) \le \begin{cases} 2^{21}, & p = 2 \\ 5^{12}, & p = 5. \end{cases}$$

For $p > 5$, *the* p*-primary homotopy exponent of* F_4/G_2 *is* p^{11}.

This paper is organized as follows. In Section 2, we relate the functor $A^{\min}$, introduced by Selick, and Wu [25], with the functor M, constructed by Cohen, and Neisendorfer [4], by obtaining new fibrations which connect them. We start Section 3 by decomposing, under certain dimensional restrictions, suspended stunted complex projective spaces into a wedge sum. To these summands we apply the functor $\overline{M}$ in order to study the homotopy exponent of newly obtained spaces. Using the results of Section 3, the proofs of Theorems 1.1, 1.2, and 1.3 are given in Sections 4, and 5. Finally the proofs of Theorems 1.4, and 1.5 are given in Section 6.

2. Relating the functors $A^{\min}$ and M

For a p-local path-connected CW-complex X, Selick, and Wu [25] showed that $\Omega\Sigma X$ admits a functorial decomposition

$$\Omega\Sigma X \simeq A^{\min}(X) \times B^{\max}(X)$$

where $A^{\min}(X)$ is the minimal functorial retract whose mod p homology contains $\widetilde{H}_*(X)$. As retracts of $\Omega\Sigma X$, $A^{\min}(X)$ and $B^{\max}(X)$ are both H-spaces. Let λ_X be the composite

$$\lambda_X : X \xrightarrow{E} \Omega\Sigma X \xrightarrow{r_X} A^{\min}(X)$$

where E is the suspension map and r_X is the functorial retraction. Then λ_X induces an injection in homology.

Let X^l be the p-localization of a CW complex $S^{n_1} \bigcup e^{n_2} \bigcup \cdots \bigcup e^{n_l}$ where $n_1 \leq n_2 \leq \cdots \leq n_l$ and each n_i is odd. Following the notation in [4], we identify $A^{\min}(X^l)$ with $M(X^l)$ when $l < p-1$. Cohen, and Neisendorfer [4] showed that the homology of $M(X^l)$ is the exterior algebra $E(\widetilde{H}_*(X^l))$. For $1 < j \leq l$, denote the homotopy cofibre of the inclusion $X^{j-1} \longrightarrow X^l$ by X^l_j. If $l < p-1$, then the cofibration $X^{j-1} \longrightarrow X^l \longrightarrow X^l_j$ induces a fibration

$$M(X^{j-1}) \longrightarrow M(X^l) \longrightarrow M(X^l_j). \tag{1}$$

In general, a suitable H-space $M(X^l)$ fails to exists when $l = p-1$. To approach the problem when $l = p-1$, we study $A^{\min}(X^{p-1})$. For the space X^{p-1}, we define $b = \sum\limits_{i=1}^{p-1} n_i$.

Proposition 2.1. [31] *There is a fibre sequence*

$$\Omega A^{\min}(\Sigma^b X^{p-1}) \xrightarrow{P} E(X^{p-1}) \xrightarrow{E_p} A^{\min}(X^{p-1}) \xrightarrow{H_p} A^{\min}(\Sigma^b X^{p-1}) \tag{2}$$

such that the following properties hold:

1. $H_*(A^{\min}(X^{p-1})) \cong H_*(E(X^{p-1})) \otimes H_*(A^{\min}(\Sigma^b X^{p-1}))$ *as coalgebras;*
2. $H_*(E(X^{p-1})) \cong E(\widetilde{H}_*(X^{p-1}))$ *as coalgebras;*
3. E_p *induces an injection, and* H_p *induces an epimorphism in homology.*

Remark 2.2. $E(X^{p-1})$ is in general not an H-space. Wu [31] gave some criteria under which $E(X^{p-1})$ is an H-space.

Consider fibre sequence (2). Since $\dim(X^{p-1}) = n_{p-1}$ is less than the connectivity of $A^{\min}(\Sigma^b X^{p-1})$, the composite $H_p \circ \lambda_{X^{p-1}}$ is null homotopic. Let $\lambda\colon X^{p-1} \longrightarrow E(X^{p-1})$ be a lift of $\lambda_{X^{p-1}}$, that is, $E_p \circ \lambda \simeq \lambda_{X^{p-1}}$.

Corollary 2.3. *The map* $\lambda\colon X^{p-1} \longrightarrow E(X^{p-1})$ *induces an injection in homology.*

Proof. Since E_p and $\lambda_{X^{p-1}}$ both induce injections in homology, so does λ. □

Remark 2.4. In general λ is not a unique map; however if E_p admits a retraction (in which case $E(X^{p-1})$ is an H-space) then λ is unique.

Corollary 2.5. *Suppose that* $X^l \simeq X_1 \vee X_2$, *where* X_1 *and* X_2 *are not contractible spaces. Then*

1. *when* $l < p-1$, *there is a homotopy equivalence* $M(X^l) \simeq M(X_1) \times M(X_2)$;
2. *when* $l = p-1$, *there is a homotopy equivalence* $E(X^{p-1}) \simeq M(X_1) \times M(X_2)$.

Proof. The proof of (1) is due to Cohen, and Neisendorfer [4]. To prove (2), define the maps

$$f_i \colon A^{\min}(X^{p-1}) \simeq A^{\min}(X_1 \vee X_2) \xrightarrow{A^{\min}(\pi_i)} A^{\min}(X_i) = M(X_i)$$

where π_i is the i-th coordinate projection. As the composite

$$E(X^{p-1}) \xrightarrow{E_p} A^{\min}(X^{p-1}) \xrightarrow{\Delta} A^{\min}(X^{p-1}) \times A^{\min}(X^{p-1}) \xrightarrow{f_1 \times f_2} M(X_1) \times M(X_2)$$

induces an isomorphism in homology, it is a homotopy equivalence. □

In what follows, we give an analogous fibration to (1) when $l = p-1$.

Proposition 2.6. *If $1 < j \leq p-1$, then the cofibration $X^{j-1} \xrightarrow{i} X^{p-1} \xrightarrow{k} X_j^{p-1}$ induces a fibration*

(3) $$M(X^{j-1}) \xrightarrow{f} E(X^{p-1}) \xrightarrow{g} M(X_j^{p-1}).$$

Proof. Applying the functor $A^{\min}$ to the cofibration $X^{j-1} \xrightarrow{i} X^{p-1} \xrightarrow{k} X_j^{p-1}$ and keeping the same names of the maps i and k (the context will make the difference), we have the composite

$$h \colon A^{\min}(X^{j-1}) \xrightarrow{i} A^{\min}(X^{p-1}) \xrightarrow{k} A^{\min}(X_j^{p-1})$$

which is then null homotopic. Since the dimension of $A^{\min}(X^{j-1})$ is less than the connectivity of $A^{\min}(\Sigma^b X^{p-1})$, the composite

$$A^{\min}(X^{j-1}) \xrightarrow{i} A^{\min}(X^{p-1}) \xrightarrow{H_p} A^{\min}(\Sigma^b X^{p-1})$$

is null homotopic. Hence the map $i \colon A^{\min}(X^{j-1}) \longrightarrow A^{\min}(X^{p-1})$ admits a lift $f \colon A^{\min}(X^{j-1}) \longrightarrow E(X^{p-1})$ via fibre sequence (2). Let g be the composite

$$g \colon E(X^{p-1}) \xrightarrow{E_p} A^{\min}(X^{p-1}) \xrightarrow{k} A^{\min}(X_j^{p-1})$$

and let F_g be the homotopy fibre of g. Then $g \circ f \simeq h$ and thus it is null homotopic. Hence f admits a lift $l \colon A^{\min}(X^{j-1}) \longrightarrow F_g$ via the fibration

(4) $$F_g \xrightarrow{\zeta} E(X^{p-1}) \xrightarrow{g} A^{\min}(X_j^{p-1})$$

such that $\zeta \circ l = f$. Thus we obtain the commutative diagram

(5) $$\begin{array}{ccccc}
F_g & \xleftarrow{\ l\ } & A^{\min}(X^{j-1}) & \longrightarrow & \Omega\Sigma X^{j-1} \\
{\scriptstyle \zeta}\downarrow & \swarrow{\scriptstyle f} & \downarrow{\scriptstyle i} & & \downarrow \\
E(X^{p-1}) & \xrightarrow{E_p} & A^{\min}(X^{p-1}) & \longrightarrow & \Omega\Sigma X^{p-1} \\
{\scriptstyle g}\downarrow & & \downarrow{\scriptstyle k} & & \downarrow \\
A^{\min}(X_j^{p-1}) & = & A^{\min}(X_j^{p-1}) & \longrightarrow & \Omega\Sigma X_j^{p-1}
\end{array}$$

where the two rightmost squares commute by the definition of $A^{\min}$. Then in homology we obtain the following commutative diagram

$$\begin{array}{ccccccc}
E(\widetilde{H}_*(X^{j-1})) & = & H_*(A^{\min}(X^{j-1})) & \longrightarrow & T(\widetilde{H}_*(X^{j-1})) & \xrightarrow{\text{ab}} & E(\widetilde{H}_*(X^{j-1})) \\
\downarrow f_* & & \downarrow i_* & & \downarrow & & \downarrow E(i_*) \\
E(\widetilde{H}_*(X^{p-1})) & \xrightarrow{(E_p)_*} & H_*(A^{\min}(X^{p-1})) & \longrightarrow & T(\widetilde{H}_*(X^{p-1})) & \xrightarrow{\text{ab}} & E(\widetilde{H}_*(X^{p-1})) \\
\downarrow g_* & & \downarrow k_* & & \downarrow & & \downarrow E(k_*) \\
E(\widetilde{H}_*(X_j^{p-1})) & = & H_*(A^{\min}(X_j^{p-1})) & \longrightarrow & T(\widetilde{H}_*(X_j^{p-1})) & \xrightarrow{\text{ab}} & E(\widetilde{H}_*(X_j^{p-1}))
\end{array}$$

where the rows are isomorphisms. In the rightmost column, the homomorphism $E(\widetilde{H}_*(X^{j-1})) \xrightarrow{E(i_*)} E(\widetilde{H}_*(X^{p-1}))$ induced by the monomorphism $\widetilde{H}_*(X^{j-1}) \xrightarrow{i_*} \widetilde{H}_*(X^{p-1})$ is a monomorphism, and $E(\widetilde{H}_*(X^{p-1})) \xrightarrow{E(k_*)} E(\widetilde{H}_*(X_j^{p-1}))$ induced by the epimorphism $\widetilde{H}_*(X^{p-1}) \xrightarrow{k_*} \widetilde{H}_*(X_j^{p-1})$ is an epimorphism. It follows that f_* is a monomorphism, g_* is an epimorphism, and thus the Serre spectral sequence for fibration (4) collapses. Hence we have $E(\widetilde{H}_*(X^{p-1})) \cong E(\widetilde{H}_*(X_j^{p-1})) \otimes H_*(F_g)$. Thus $H_*(F_g)$ has the same Euler-Poincare series as $E(\widetilde{H}_*(X^{j-1}))$. Since f_* is a monomorphism and $f_* = \zeta_* \circ l_*$, we conclude that l_* is also a monomorphism. Thus l_* is an isomorphism and hence l is a homotopy equivalence. The stated fibration is thus obtained. □

Corollary 2.7. *There is a map* $\bar{\lambda}\colon X^{j-1} \longrightarrow M(X^{j-1})$ *such that there is a commutative diagram*

$$\begin{array}{ccccc}
X^{j-1} & \longrightarrow & X^{p-1} & \longrightarrow & X_j^{p-1} \\
\downarrow \bar{\lambda} & & \downarrow \lambda & & \downarrow \lambda_{X_j^{p-1}} \\
M(X^{j-1}) & \xrightarrow{f} & E(X^{p-1}) & \xrightarrow{g} & M(X_j^{p-1})
\end{array}$$

where the top row is a cofibration and the bottom row is fibration (3). *The map* $\bar{\lambda}$ *induces an injection in homology.*

Proof. The right square is given by the commutative diagram

$$\begin{array}{ccccc}
 & & X^{p-1} & \longrightarrow & X_j^{p-1} \\
 & \swarrow^{\lambda} & \downarrow \lambda_{X^{p-1}} & & \downarrow \lambda_{X_j^{p-1}} \\
E(X^{p-1}) & \xrightarrow{E_p} & A^{\min}(X^{p-1}) & \xrightarrow{k} & A^{\min}(X_j^{p-1}).
\end{array}$$

Consider fibration (3). Since the composite

$$X^{j-1} \longrightarrow X^{p-1} \xrightarrow{\lambda} E(X^{p-1}) \xrightarrow{g} M(X_j^{p-1})$$

is null homotopic, the composite $X^{j-1} \longrightarrow X^{p-1} \xrightarrow{\lambda} E(X^{p-1})$ admits a lift $\bar{\lambda}\colon X^{j-1} \longrightarrow M(X^{j-1})$ which gives the left square. Since $f \circ \bar{\lambda}$ and f both induce injections in homology, so does $\bar{\lambda}$. □

For $0 \le l \le p-1$, we define $X^0 := *$ and

$$\overline{M}(X^l) := \begin{cases} * & \text{for } l = 0, \\ M(X^l) & \text{for } 1 \le l < p-1, \\ E(X^{p-1}) & \text{for } l = p-1. \end{cases}$$

Combining fibrations (1), and (3) gives the following result.

Theorem 2.8. *For $0 < j \le l \le p-1$, the cofibration $X^{j-1} \longrightarrow X^l \longrightarrow X^l_j$ induces a fibration $\overline{M}(X^{j-1}) \longrightarrow \overline{M}(X^l) \longrightarrow \overline{M}(X^l_j)$.*

3. Preliminary results on homotopy exponents

First recall that the homology of the unitary group $SU(n)$ is

$$H_*(SU(n)) = E(x_3, x_5, \ldots, x_{2n-1})$$

while the reduced homology of the suspended complex projective space $\Sigma\mathbb{C}P^{n-1}$ is

$$\widetilde{H}_*(\Sigma\mathbb{C}P^{n-1}) = \mathbb{Z}/p\{x_3, x_5, \ldots, x_{2n-1}\}.$$

There is a canonical map $\Sigma\mathbb{C}P^{n-1} \longrightarrow SU(n)$ (see for example [20]) which induces an injection in homology. Mimura, Nishida, and Toda [17] showed that there exists a wedge decomposition

$$\Sigma\mathbb{C}P^{n-1} \simeq \bigvee_{i=1}^{p-1} A_i(n) \tag{6}$$

where the reduced homology of $A_i(n)$ is

$$\widetilde{H}_*(A_i(n)) = \mathbb{Z}/p\{x_{2i+1}, x_{2i+q+1}, \ldots, x_{2i+(r_i^{(n)}-1)q+1}\}$$

and $r_i^{(n)} = \lfloor (n-i-1)/(p-1) \rfloor + 1$. If $r_i^{(n)} = 0$, then $\widetilde{H}_*(A_i(n))$ is trivial, in which case $A_i(n) = *$. Let $j_i^{(n)}$ denote the composite $j_i^{(n)}\colon A_i(n) \longrightarrow \Sigma\mathbb{C}P^{n-1} \longrightarrow SU(n)$. Then $j_i^{(n)}$ induces an injection in homology. Since $SU(n)$ is a homotopy associative H-space, then $j_i^{(n)}$ extends to an H-map $\bar{j}_i^{(n)}\colon \Omega\Sigma A_i(n) \longrightarrow SU(n)$. Thus we have the commutative diagram

$$\begin{array}{ccc} A_i(n) & \xrightarrow{\;j_i^{(n)}\;} & SU(n) \\ {\scriptstyle E}\downarrow & \nearrow{\scriptstyle \bar{j}_i^{(n)}} & \\ \Omega\Sigma A_i(n). & & \end{array} \tag{7}$$

Let $m > 0$. From the cofibration $\Sigma\mathbb{C}P^{m-1} \longrightarrow \Sigma\mathbb{C}P^{m+k-1} \longrightarrow \Sigma\mathbb{C}P^{m+k-1}_m$ and wedge decomposition (6) we deduce a wedge decomposition $\Sigma\mathbb{C}P^{m+k-1}_m \simeq \bigvee_{i=1}^{p-1} A^{k_i}_{j_i}$ where $A^{k_i}_{j_i}$ is defined by the cofibration $A_i(m) \longrightarrow A_i(m+k) \longrightarrow A^{k_i}_{j_i}$. Thus the reduced homology of $A^{k_i}_{j_i}$ is

$$\widetilde{H}_*(A^{k_i}_{j_i}) = \mathbb{Z}/p\{x_{2j_i+1}, x_{2j_i+q+1}, \cdots, x_{2j_i+(k_i-1)q+1}\}$$

where $j_i = i + m - 1$ and $k_i = \lfloor (k-i)/(p-1) \rfloor + 1$.

When $m + k \le (p-1)(p-2) + 1$, we have $r_i^{(m)} \le r_i^{(m+k)} < p-1$. When $(p-1)(p-2) + 2 \le m + k \le (p-1)^2 + 1$, we have $r_i^{(m)} \le r_i^{(m+k)} = p-1$ for

$1 \leq i \leq m+k-(p-1)(p-2)-1$, and $r_i^{(m)} \leq r_i^{(m+k)} < p-1$ for $m+k-(p-1)(p-2) \leq i \leq p-1$. Thus from Theorem 2.8, we obtain the following result.

Proposition 3.1. *Let $m+k \leq (p-1)^2+1$ and $1 \leq i \leq p-1$. Then the cofibration $A_i(m) \longrightarrow A_i(m+k) \longrightarrow A_{j_i}^{k_i}$ induces a fibration*

$$\overline{M}(A_i(m)) \longrightarrow \overline{M}(A_i(m+k)) \longrightarrow \overline{M}(A_{j_i}^{k_i}).$$

□

In what follows, we will consider the homotopy exponent of $\overline{M}(A_{j_i}^{k_i})$. First let us review a method for calculating an upper bound of the homotopy exponent of spherically resolved spaces. If there is a fibration $X \longrightarrow Y \longrightarrow Z$ with $\exp_p(X)$ and $\exp_p(Z)$ known, then we have $\exp_p(Y) \leq \exp_p(X) \cdot \exp_p(Z)$. But in most cases this estimate is quite rough. Theriault showed that a better estimate can be obtained for certain spherically resolved spaces.

Lemma 3.2. [28] *Suppose there is a fibration*

$$X \xrightarrow{f} Y \xrightarrow{g} S^{2n+1}$$

and a map $i\colon S^{2n+1} \longrightarrow Y$ such that $g \circ i \simeq p^r$.

(1) *If Y is an H-space, then there is a fibration*

$$\Omega X \times \Omega S^{2n+1} \xrightarrow{\Omega f\cdot(-\Omega i)} \Omega Y \longrightarrow S^{2n+1}\{p^r\}.$$

(2) *If Y is not an H-space, and X, Y are 2-connected, then there is a fibration*

$$\Omega^2 X \times \Omega^2 S^{2n+1} \xrightarrow{\Omega^2 f\cdot(-\Omega^2 i)} \Omega^2 Y \longrightarrow \Omega S^{2n+1}\{p^r\}.$$

Recall that $S^{2n+1}\{p^r\}$ is the homotopy fibre of the p^r power map on S^{2n+1}; its homotopy exponent is p^r (see [22]). Thus from Lemma 3.2, we have

$$\exp_p(Y) \leq \exp_p(S^{2n+1}\{p^r\}) \cdot \max\{\exp_p(X), \exp_p(S^{2n+1})\}$$

that is, $\exp_p(Y) \leq p^r \cdot \max\{\exp_p(X), p^n\}$.

The reduced cohomology of the suspended stunted projective space $\Sigma\mathbb{C}P_m^{m+k-1}$

$$\widetilde{H}^*(\Sigma\mathbb{C}P_m^{m+k-1}) = \mathbb{Z}/p\{x_{2m+1}, x_{2m+3}, \ldots, x_{2(m+k)-1}\}$$

admits an action of the Steenrod algebra given by $\mathcal{P}^j(x_{2r+1}) = r!/(j!(r-j)!) \cdot x_{2r+jq+1}$. In particular, $\mathcal{P}^j(x_{2(m+k)-1}) = 0$ for $j > 0$ and $\mathcal{P}^1(x_{2r+1}) = r \cdot x_{2r+q+1}$ for $m \leq r \leq m+k-1$. As $A_{j_i}^{k_i}$ is a summand of the wedge decomposition $\Sigma\mathbb{C}P_m^{m+k-1} \simeq \bigvee_{i=1}^{p-1} A_{j_i}^{k_i}$, its reduced cohomology

$$\widetilde{H}^*(A_{j_i}^{k_i}) = \mathbb{Z}/p\{x_{2j_i+1}, x_{2j_i+q+1}, \cdots, x_{2j_i+(k_i-1)q+1}\}$$

inherits the action of the Steenrod algebra from $\widetilde{H}^*(\Sigma\mathbb{C}P_m^{m+k-1})$. Specially, for $1 \leq t \leq k_i - 1$, we have $\mathcal{P}^1(x_{2j_i+(t-1)q+1}) = (j_i + (t-1)(p-1)) \cdot x_{2j_i+tq+1}$. When $m+k \leq (p-1)^2+1$, we see that $k_i \leq p-1$ which implies that $1 \leq t \leq p-2$. It follows that $j_i + (t-1)(p-1) \equiv 0 \mod p$ for at most one value of t. Hence in the set $\{x_{2j_i+1}, x_{2j_i+q+1}, \ldots, x_{2j_i+(k_i-2)q+1}\}$ of $\widetilde{H}^*(A_{j_i}^{k_i})$ there is at most one generator on which $\mathcal{P}^1$ acts trivially.

Theorem 3.3. *Let $m+k \leq (p-1)^2+1$. Let $1 \leq i \leq p-1$ and $1 \leq t \leq k_i - 1$. Suppose that $k_i > 1$ for some i. Then*

(1) *if* $j_i+(t-1)(p-1)\not\equiv 0 \mod p$ *for every* t, *then* $\exp_p(\overline{M}(A^{k_i}_{j_i}))\leq p^{j_i+(k_i-1)p}$;
(2) *if* $j_i+(t-1)(p-1)\equiv 0 \mod p$ *for some* t, *then* $\exp_p(\overline{M}(A^{k_i}_{j_i}))\leq p^{j_i+(k_i-1)p-1}$.

Proof. Let $0\leq l_i\leq k_i-1$. Look at $A^{k_i-l_i}_{j_i+l_i(p-1)}$, and $A^{k_i-l_i}_{j_i}$ which are the summands of $\Sigma\mathbb{C}P^{m+k-1}_{j_i+l_i(p-1)}$, and $\Sigma\mathbb{C}P^{j_i+(k_i-l_i-1)(p-1)}_m$, respectively. Their reduced cohomology is

$$\widetilde{H}^*(A^{k_i-l_i}_{j_i+l_i(p-1)})=\mathbb{Z}/p\{x_{2j_i+l_iq+1},x_{2j_i+(l_i+1)q+1},\ldots,x_{2j_i+(k_i-1)q+1}\}$$
$$\widetilde{H}^*(A^{k_i-l_i}_{j_i})=\mathbb{Z}/p\{x_{2j_i+1},x_{2j_i+q+1},\ldots,x_{2j_i+(k_i-l_i-1)q+1}\}.$$

There are also two kinds of cofibrations connecting them

$$S^{2j_i+l_iq+1}\longrightarrow A^{k_i-l_i}_{j_i+l_i(p-1)}\longrightarrow A^{k_i-l_i-1}_{j_i+(l_i+1)(p-1)}$$
$$A^{k_i-l_i-1}_{j_i}\longrightarrow A^{k_i-l_i}_{j_i}\longrightarrow S^{2j_i+(k_i-l_i-1)q+1}.$$

By Theorem 2.8, to these two cofibrations we can associate the following two fibrations

$$S^{2j_i+l_iq+1}\longrightarrow \overline{M}(A^{k_i-l_i}_{j_i+l_i(p-1)})\longrightarrow \overline{M}(A^{k_i-l_i-1}_{j_i+(l_i+1)(p-1)})\tag{8}$$

$$\overline{M}(A^{k_i-l_i-1}_{j_i})\longrightarrow \overline{M}(A^{k_i-l_i}_{j_i})\longrightarrow S^{2j_i+(k_i-l_i-1)q+1}.\tag{9}$$

Proof of part (1). Let $l=k_i-l_i$ for $0\leq l_i\leq k_i-2$. Thus $2\leq l\leq k_i$. When $l=2$, we have

$$\widetilde{H}^*(A^{k_i-l_i}_{j_i+l_i(p-1)})=\widetilde{H}^*(A^{2}_{j_i+(k_i-2)(p-1)})=\mathbb{Z}/p\{x_{2j_i+(k_i-2)q+1},x_{2j_i+(k_i-1)q+1}\}.$$

Since there is a nontrivial Steenrod operation

$$\mathcal{P}^1(x_{2j_i+(k_i-2)q+1})=(j_i+(k_i-2)(p-1))\cdot x_{2j_i+(k_i-1)q+1}$$

the top dimensional cell of $A^2_{j_i+(k_i-2)(p-1)}$ is attached by $\alpha_1\colon S^{2j_i+(k_i-1)q}\longrightarrow S^{2j_i+(k_i-2)q+1}$ which is a stable map detected by the Steenrod operation $\mathcal{P}^1$. Hence we have the cofibre sequence

$$S^{2j_i+(k_i-1)q}\xrightarrow{\alpha_1}S^{2j_i+(k_i-2)q+1}\longrightarrow A^2_{j_i+(k_i-2)(p-1)}\longrightarrow S^{2j_i+(k_i-1)q+1}.$$

Associated to it there is a fibration

$$S^{2j_i+(k_i-2)q+1}\longrightarrow \overline{M}(A^2_{j_i+(k_i-2)(p-1)})\longrightarrow S^{2j_i+(k_i-1)q+1}$$

and by Corollary 2.7 a commutative diagram

$$\begin{array}{ccccc}
S^{2j_i+(k_i-2)q+1} & \longrightarrow & A^2_{j_i+(k_i-2)(p-1)} & \longrightarrow & S^{2j_i+(k_i-1)q+1}\\
\| & & \downarrow & & \|\\
S^{2j_i+(k_i-2)q+1} & \longrightarrow & \overline{M}(A^2_{j_i+(k_i-2)(p-1)}) & \longrightarrow & S^{2j_i+(k_i-1)q+1}.
\end{array}\tag{10}$$

Since $\alpha_1\in\pi_{2j_i+(k_i-1)q}(S^{2j_i+(k_i-2)q+1})=\mathbb{Z}/p$ is of order p, we have the following commutative diagram of cofibre sequences

(11)

$$\begin{array}{ccccccc}
S^{2j_i+(k_i-1)q} & \longrightarrow & * & \longrightarrow & S^{2j_i+(k_i-1)q+1} & = & S^{2j_i+(k_i-1)q+1}\\
\downarrow p & & \downarrow & & \downarrow & & \downarrow p\\
S^{2j_i+(k_i-1)q} & \xrightarrow{\alpha_1} & S^{2j_i+(k_i-2)q+1} & \longrightarrow & A^2_{j_i+(k_i-2)(p-1)} & \longrightarrow & S^{2j_i+(k_i-1)q+1}.
\end{array}$$

Combining the rightmost squares of diagrams (10), and (11) gives a commutative diagram

$$(12)\qquad \begin{array}{ccc} S^{2j_i+(k_i-1)q+1} & \xrightarrow{p} & S^{2j_i+(k_i-1)q+1} \\ \big\downarrow b_2 & & \big\| \\ \overline{M}(A^2_{j_i+(k_i-2)(p-1)}) & \longrightarrow & S^{2j_i+(k_i-1)q+1}. \end{array}$$

Suppose, by induction, that for $2 \leq l < k_i$, there is a commutative diagram

$$(13)\qquad \begin{array}{ccc} S^{2j_i+(k_i-1)q+1} & \xrightarrow{p^{l-1}} & S^{2j_i+(k_i-1)q+1} \\ \big\downarrow b_l & & \big\| \\ \overline{M}(A^l_{j_i+(k_i-l)(p-1)}) & \longrightarrow & S^{2j_i+(k_i-1)q+1} \end{array}$$

where the map $\overline{M}(A^l_{j_i+(k_i-l)(p-1)}) \longrightarrow S^{2j_i+(k_i-1)q+1}$ is the composite

$$\overline{M}(A^l_{j_i+(k_i-l)(p-1)}) \longrightarrow \overline{M}(A^{l-1}_{j_i+(k_i-l+1)(p-1)}) \longrightarrow \cdots$$
$$\cdots \longrightarrow \overline{M}(A^2_{j_i+(k_i-2)(p-1)}) \longrightarrow S^{2j_i+(k_i-1)q+1}.$$

There is a fibration of type (8)

$$(14)\qquad \Omega\overline{M}(A^{l+1}_{j_i+(k_i-l-1)(p-1)}) \longrightarrow \Omega\overline{M}(A^l_{j_i+(k_i-l)(p-1)}) \longrightarrow S^{2j_i+(k_i-l-1)q+1}.$$

Let $\tilde{b}_l\colon S^{2j_i+(k_i-1)q} \longrightarrow \Omega\overline{M}(A^l_{j_i+(k_i-l)(p-1)})$ be the adjoint of b_l in diagram (13). Since $\pi_{2j_i+(k_i-1)q}(S^{2j_i+(k_i-l-1)q+1}) = \mathbb{Z}/p$ by [30, Theorem 13.4], it follows that the composite

$$S^{2j_i+(k_i-1)q} \xrightarrow{\tilde{b}_l} \Omega\overline{M}(A^l_{j_i+(k_i-l)(p-1)}) \longrightarrow S^{2j_i+(k_i-l-1)q+1}$$

is of order p. Hence the composite

$$\tilde{b}_l \circ p\colon S^{2j_i+(k_i-1)q} \longrightarrow S^{2j_i+(k_i-1)q} \longrightarrow \Omega\overline{M}(A^l_{j_i+(k_i-l)(p-1)})$$

admits a lift $\tilde{b}_{l+1}\colon S^{2j_i+(k_i-1)q} \longrightarrow \Omega\overline{M}(A^{l+1}_{j_i+(k_i-l-1)(p-1)})$ via fibration (14). Thus we have the commutative diagram

$$\begin{array}{ccc} S^{2j_i+(k_i-1)q} & \xrightarrow{p} & S^{2j_i+(k_i-1)q} \\ \big\downarrow \tilde{b}_{l+1} & & \big\downarrow \tilde{b}_l \\ \Omega\overline{M}(A^{l+1}_{j_i+(k_i-l-1)(p-1)}) & \longrightarrow & \Omega\overline{M}(A^l_{j_i+(k_i-l)(p-1)}). \end{array}$$

Adjointing gives

$$(15)\qquad \begin{array}{ccc} S^{2j_i+(k_i-1)q+1} & \xrightarrow{p} & S^{2j_i+(k_i-1)q+1} \\ \big\downarrow b_{l+1} & & \big\downarrow b_l \\ \overline{M}(A^{l+1}_{j_i+(k_i-l-1)(p-1)}) & \longrightarrow & \overline{M}(A^l_{j_i+(k_i-l)(p-1)}). \end{array}$$

Combine diagrams (13), and (15), we get the commutative diagram

$$
\begin{array}{ccc}
S^{2j_i+(k_i-1)q+1} & \xrightarrow{p^l} & S^{2j_i+(k_i-1)q+1} \\
\Big\downarrow{\scriptstyle b_{l+1}} & & \Big\| \\
\overline{M}(A^{l+1}_{j_i+(k_i-l-1)(p-1)}) & \longrightarrow & S^{2j_i+(k_i-1)q+1}.
\end{array}
\tag{16}
$$

This finishes the induction and thus we obtain the commutative diagram

$$
\begin{array}{ccc}
S^{2j_i+(k_i-1)q+1} & \xrightarrow{p^{k_i-1}} & S^{2j_i+(k_i-1)q+1} \\
\Big\downarrow{\scriptstyle b_{k_i}} & & \Big\| \\
\overline{M}(A^{k_i}_{j_i}) & \longrightarrow & S^{2j_i+(k_i-1)q+1}.
\end{array}
\tag{17}
$$

For a fibration of type (9),

$$\overline{M}(A^{k_i-1}_{j_i}) \longrightarrow \overline{M}(A^{k_i}_{j_i}) \longrightarrow S^{2j_i+(k_i-1)q+1}$$

using diagram (17) and Lemma 3.2, we have

$$\exp_p(\overline{M}(A^{k_i}_{j_i})) \leq p^{k_i-1} \cdot \max\{\exp_p(\overline{M}(A^{k_i-1}_{j_i})), p^{j_i+(k_i-1)(p-1)}\}.$$

For every $3 \leq s \leq k_i$, we similarly have

$$\exp_p(\overline{M}(A^{s}_{j_i})) \leq p^{s-1} \cdot \max\{\exp_p(\overline{M}(A^{s-1}_{j_i})), p^{j_i+(s-1)(p-1)}\}. \tag{18}$$

For $\overline{M}(A^2_{j_i})$, by the previous statement for $\overline{M}(A^2_{j_i+(k_j-2)(p-1)})$, we get the commutative diagram

$$
\begin{array}{ccc}
S^{2j_i+q+1} & \xrightarrow{p} & S^{2j_i+q+1} \\
\Big\downarrow{\scriptstyle b_2} & & \Big\| \\
\overline{M}(A^2_{j_i}) & \longrightarrow & S^{2j_i+q+1}.
\end{array}
$$

Applying Lemma 3.2 to the fibration $S^{2j_i+1} \longrightarrow \overline{M}(A^2_{j_i}) \longrightarrow S^{2j_i+q+1}$, we have

$$\exp_p(\overline{M}(A^2_{j_i})) \leq p \cdot \max\{p^{j_i}, p^{j_i+p-1}\} = p \cdot p^{j_i+p-1} = p^{j_i+p}.$$

Suppose, by induction, that for $2 \leq s < k_i$ we know that

$$\exp_p(\overline{M}(A^s_{j_i})) \leq p^{j_i+(s-1)p}.$$

Then by (18), we have

$$
\begin{array}{rcl}
\exp_p(\overline{M}(A^{s+1}_{j_i})) & \leq & p^s \cdot \max\{\exp_p(\overline{M}(A^s_{j_i})), p^{j_i+s(p-1)}\} \\
& \leq & p^s \cdot \max\{p^{j_i+(s-1)p}, p^{j_i+s(p-1)}\} \\
& = & p^s \cdot p^{j_i+s(p-1)} = p^{j_i+sp}.
\end{array}
$$

This finishes the induction and we conclude that

$$\exp_p(\overline{M}(A^{k_i}_{j_i})) \leq p^{j_i+(k_i-1)p}.$$

Proof of part (2). If there exists t such that $j_i + (t-1)(p-1) \equiv 0 \mod p$, then the generators $x_{2j_i+(t-1)q+1}$ and x_{2j_i+tq+1} of

$$\widetilde{H}^*(A^{k_i-t+1}_{j_i+(t-1)(p-1)}) = \mathbb{Z}/p\{x_{2j_i+(t-1)q+1}, x_{2j_i+tq+1}, \ldots, x_{2j_i+(k_i-1)q+1}\}$$

are not connected by a nontrivial Steenrod operation. Since the generator α_1 of $\pi_{2j_i+tq}(S^{2j_i+(t-1)q+1}) = \mathbb{Z}/p$ is detected by $\mathcal{P}^1$, the attaching map for $(2j_i+tq+1)$-dimensional cell in $A^{k_i-t+1}_{j_i+(t-1)(p-1)}$ is trivial. Hence the $2j_i + tq + 1$ skeleton of $A^{k_i-t+1}_{j_i+(t-1)(p-1)}$ is homotopy equivalent to $S^{2j_i+(t-1)q+1} \vee S^{2j_i+tq+1}$ and there is a cofibration

$$S^{2j_i+(t-1)q+1} \vee S^{2j_i+tq+1} \longrightarrow A^{k_i-t+1}_{j_i+(t-1)(p-1)} \longrightarrow A^{k_i-t-1}_{j_i+(t+1)(p-1)}.$$

Since $p \geq 3$, by Corollary 2.5 we have

$$\overline{M}(S^{2j_i+(t-1)q+1} \vee S^{2j_i+tq+1}) \simeq S^{2j_i+(t-1)q+1} \times S^{2j_i+tq+1}.$$

Then by Theorem 2.8, the above cofibration induces a fibration

$$\Omega\overline{M}(A^{k_i-t+1}_{j_i+(t-1)(p-1)}) \longrightarrow \Omega\overline{M}(A^{k_i-t-1}_{j_i+(t+1)(p-1)}) \longrightarrow S^{2j_i+(t-1)q+1} \times S^{2j_i+tq+1}. \tag{19}$$

Since the generators of $\widetilde{H}^*(A^{k_i-t-1}_{j_i+(t+1)(p-1)})$ are all connected by the nontrivial Steenrod operation $\mathcal{P}^1$, by the same method as for $A^{k_i}_{j_i}$ in the proof of part (1), we have a diagram analogues to (17)

$$\begin{array}{ccc} S^{2j_i+(k_i-1)q+1} & \xrightarrow{p^{k_i-t-2}} & S^{2j_i+(k_i-1)q+1} \\ \downarrow{\scriptstyle b_{k_i-t-1}} & & \| \\ \overline{M}(A^{k_i-t-1}_{j_i+(t+1)(p-1)}) & \longrightarrow & S^{2j_i+(k_i-1)q+1}. \end{array} \tag{20}$$

According to [30, Theorem 13.4], we have

$$\pi_{2j_i+(k_i-1)q}(S^{2j_i+(t-1)q+1} \times S^{2j_i+tq+1}) = \begin{cases} \mathbb{Z}/p \oplus \mathbb{Z}/p & 1 \leq t < k_i - 1, \\ \mathbb{Z}/p & t = k_i - 1. \end{cases}$$

Let $\tilde{b}_{k_i-t-1} \colon S^{2j_i+(k_i-1)q} \longrightarrow \Omega\overline{M}(A^{k_i-t-1}_{j_i+(t+1)(p-1)})$ be the adjoint of b_{k_i-t-1}. Hence the composite

$$S^{2j_i+(k_i-1)q} \xrightarrow{\tilde{b}_{k_i-t-1}} \Omega\overline{M}(A^{k_i-t-1}_{j_i+(t+1)(p-1)}) \longrightarrow S^{2j_i+(t-1)q+1} \times S^{2j_i+tq+1}$$

is of order p. Then via fibration (19), the composite

$$\tilde{b}_{k_i-t-1} \circ p \colon S^{2j_i+(k_i-1)q} \longrightarrow S^{2j_i+(k_i-1)q} \longrightarrow \Omega\overline{M}(A^{k_i-t-1}_{j_i+(t+1)(p-1)})$$

admits a lift $\tilde{b}_{k_i-t+1} \colon S^{2j_i+(k_i-1)q} \longrightarrow \Omega\overline{M}(A^{k_i-t+1}_{j_i+(t-1)(p-1)})$ such that the following diagram commutes

$$\begin{array}{ccc} S^{2j_i+(k_i-1)q} & \xrightarrow{p} & S^{2j_i+(k_i-1)q} \\ \downarrow{\scriptstyle \tilde{b}_{k_i-t+1}} & & \downarrow{\scriptstyle \tilde{b}_{k_i-t-1}} \\ \Omega\overline{M}(A^{k_i-t+1}_{j_i+(t-1)(p-1)}) & \longrightarrow & \Omega\overline{M}(A^{k_i-t-1}_{j_i+(t+1)(p-1)}). \end{array}$$

Adjointing gives the commutative diagram

$$\begin{array}{ccc} S^{2j_i+(k_i-1)q+1} & \xrightarrow{p} & S^{2j_i+(k_i-1)q+1} \\ \downarrow{\scriptstyle b_{k_i-t+1}} & & \downarrow{\scriptstyle b_{k_i-t-1}} \\ \overline{M}(A^{k_i-t+1}_{j_i+(t-1)(p-1)}) & \longrightarrow & \overline{M}(A^{k_i-t-1}_{j_i+(t+1)(p-1)}). \end{array} \tag{21}$$

Putting diagram (15) for $k_i - t - 1 \leq l \leq k_i - 1$ together, we get the following diagram

(22)

$$\begin{array}{ccccccccc} S^{2j_i+(k_i-1)q+1} & \xrightarrow{p} & S^{2j_i+(k_i-1)q+1} & \xrightarrow{p} \cdots \xrightarrow{p} & S^{2j_i+(k_i-1)q+1} & \xrightarrow{p} & S^{2j_i+(k_i-1)q+1} \\ \downarrow{\scriptstyle b_{k_i}} & & \downarrow{\scriptstyle b_{k_i-1}} & & \downarrow{\scriptstyle b_{k_i-t+2}} & & \downarrow{\scriptstyle b_{k_i-t+1}} \\ \overline{M}(A^{k_i}_{j_i}) & \longrightarrow & \overline{M}(A^{k_i-1}_{j_i+(p-1)}) & \longrightarrow \cdots \longrightarrow & \overline{M}(A^{k_i-t+2}_{j_i+(t-2)(p-1)}) & \longrightarrow & \overline{M}(A^{k_i-t+1}_{j_i+(t-1)(p-1)}). \end{array}$$

Then combining diagrams (20), (21), and (22), we get the commutative diagram

$$\begin{array}{ccc} S^{2j_i+(k_i-1)q+1} & \xrightarrow{p^{k_i-2}} & S^{2j_i+(k_i-1)q+1} \\ \downarrow{\scriptstyle b_{k_i}} & & \| \\ \overline{M}(A^{k_i}_{j_i}) & \longrightarrow & S^{2j_i+(k_i-1)q+1}. \end{array} \tag{23}$$

Hence for the fibration

$$\overline{M}(A^{k_i-1}_{j_i}) \longrightarrow \overline{M}(A^{k_i}_{j_i}) \longrightarrow S^{2j_i+(k_i-1)q+1}$$

by a similar argument as in the proof of part (1), we get

$$\begin{array}{rcl} \exp_p(\overline{M}(A^{k_i}_{j_i})) & \leq & p^{k_i-2} \cdot \max\{\exp_p(\overline{M}(A^{k_i-1}_{j_i})), p^{j_i+(k_i-1)(p-1)}\} \\ & \leq & p^{k_i-2} \cdot \max\{p^{j_i+(k_i-2)p}, p^{j_i+(k_i-1)(p-1)}\} \\ & = & p^{k_i-2} \cdot p^{j_i+(k_i-1)(p-1)} = p^{j_i+(k_i-1)p-1}. \end{array}$$

That finishes the proof of part (2). ☐

4. Homotopy exponents of $W_{m+k,k}$ and $SU(2n)/Sp(n)$

In this section, we use the results of Section 3 to study the homotopy exponents of the Stiefel manifold $W_{m+k,k}$ and $SU(2n)/Sp(n)$. Under certain dimensional restrictions, we decompose $\Omega W_{m+k,k}$ as a product of loop spaces and $SU(2n)/Sp(n)$ as a product of H-spaces.

Proposition 4.1. *Let $m + k \leq (p-1)^2 + 1$ and $m > 0$. Then for the Stiefel manifold $W_{m+k,k}$, there exists a product decomposition*

$$\Omega W_{m+k,k} \simeq \prod_{i=1}^{p-1} \Omega\overline{M}(A^{k_i}_{j_i}).$$

Proof. Since $m + k \leq (p-1)^2 + 1$ and $m > 0$, for $1 \leq i \leq p-1$, we get $0 \leq r_i^{(m)} \leq r_i^{(m+k)} \leq p-1$. There is a commutative diagram

$$\begin{array}{ccc} \overline{M}(A_i(m)) & \longrightarrow & \overline{M}(A_i(m+k)) \\ \downarrow & & \downarrow \\ \Omega\Sigma A_i(m) & \longrightarrow & \Omega\Sigma A_i(m+k) \\ \downarrow{\scriptstyle \bar{j}_i^{(m)}} & & \downarrow{\scriptstyle \bar{j}_i^{(m+k)}} \\ SU(m) & \longrightarrow & SU(m+k) \end{array} \tag{24}$$

where the maps $\bar{j}_i^{(m)}$ and $\bar{j}_i^{(m+k)}$ come from diagram (7) and the bottom square commutes by the universality of the James construction; the top square commutes by the functoriality of $\Omega\Sigma$ and $\overline{M}$. We denote the left column by $\tilde{j}_i^{(m)}\colon \overline{M}(A_i(m)) \longrightarrow SU(m)$, and the right column by $\tilde{j}_i^{(m+k)}\colon \overline{M}(A_i(m+k)) \longrightarrow SU(m+k)$. The composites $\tilde{j}_i^{(m)} \circ \lambda_{A_i(m)}$, and $\tilde{j}_i^{(m+k)} \circ \lambda_{A_i(m+k)}$ induce injections in homology. Thus $\tilde{j}_i^{(m)}$ and $\tilde{j}_i^{(m+k)}$ both induce the injections of the generating sets in homology as $\lambda_{A_i(m)}$ and $\lambda_{A_i(m+k)}$ both induce injections in homology.

Multiplying diagram (24) for $1 \le i \le p-1$ gives the following commutative diagram

$$
\begin{array}{ccccc}
\prod_{i=1}^{p-1} \Omega\overline{M}(A_{j_i}^{k_i}) & \longrightarrow & \prod_{i=1}^{p-1} \overline{M}(A_i(m)) & \longrightarrow & \prod_{i=1}^{p-1} \overline{M}(A_i(m+k)) \\
 & & \big\downarrow{\scriptstyle \Pi\tilde{j}_i^{(m)}} & & \big\downarrow{\scriptstyle \Pi\tilde{j}_i^{(m+k)}} \\
\big\downarrow{\scriptstyle h} & & \prod_{i=1}^{p-1} SU(m) & \longrightarrow & \prod_{i=1}^{p-1} SU(m+k) \\
 & & \big\downarrow{\scriptstyle \mu} & & \big\downarrow{\scriptstyle \mu} \\
\Omega W_{m+k,k} & \longrightarrow & SU(m) & \longrightarrow & SU(m+k).
\end{array}
\tag{25}
$$

The top row is the product of fibrations which were defined in Proposition 3.1. The map μ is the multiplication on $SU(n)$ and h is the induced map of fibrations. The right two columns induce coalgebra homomorphisms in homology which are isomorphisms on the sets of generators. Dualizing to cohomology, these two columns induce algebra homomorphisms which are isomorphisms on the sets of generators. It follows that these two homomorphisms in cohomology are isomorphisms. Thus the right two columns are homotopy equivalences. From the 5-lemma, h is a homotopy equivalence which gives the stated homotopy decomposition. $\square$

Remark 4.2. Beben [1] gave a proof of Proposition 4.1 when $m+k \le (p-1)(p-2)+1$.

By using the above decomposition and Theorem 3.3, we can now obtain an upper bound of the homotopy exponent of Stiefel manifolds.

Proof of Theorem 1.1. (1) If $k \le p-1$, then $k_i = 0$ or 1 for $1 \le i \le p-1$. Thus we have the product decomposition

$$\Omega W_{m+k,k} \simeq \prod_{i=1}^{k} \Omega S^{2(m+i)-1}.$$

It follows that

$$
\begin{aligned}
\exp_p(W_{m+k,k}) &= \max\{\exp_p(S^{2m+1}), \exp_p(S^{2m+3}), \ldots, \exp_p(S^{2(m+k)-1})\} \\
&= \max\{p^m, p^{m+1}, \ldots, p^{m+k-1}\} = p^{m+k-1}.
\end{aligned}
$$

In what follows we assume that $m+p \le m+k \le (p-1)^2+1$ which implies that there exists at least one $k_i > 1$ for $1 \le i \le p-1$. By the product decomposition given in Proposition 4.1, we have

$$\exp_p(W_{m+k,k}) = \max\{\exp_p(\overline{M}(A_{j_i}^{k_i})) \mid 1 \le i \le p-1\}.$$

Since $2(m+\bar{i}-1)+(k_{\bar{i}}-1)q+1 = 2(m+k)-1$, we have $k_{\bar{i}} > 1$. By Theorem 3.3, we obtain

$$\exp_p(\overline{M}(A_{j_i}^{k_i})) \le p^{j_i+(k_i-1)p}$$

for all $k_i > 1$.

(2) If $(m+\bar{i}-1)+(t-1)(p-1) \not\equiv 0 \mod p$ for every t such that $1 \leq t \leq k_{\bar{i}}-1$, then by Theorem 3.3, $\exp_p(\overline{M}(A_{j_{\bar{i}}}^{k_{\bar{i}}})) \leq p^{j_{\bar{i}}+(k_{\bar{i}}-1)p}$. For $i \neq \bar{i}$ such that $k_i > 1$, we have

$$j_{\bar{i}} + (k_{\bar{i}}-1)p - (j_i + (k_i-1)p) = (\bar{i}-i) + (k_{\bar{i}}-k_i)p > 0.$$

It follows that $p^{j_i+(k_i-1)p} < p^{j_{\bar{i}}+(k_{\bar{i}}-1)p}$. If $k_i = 1$, then $k_{\bar{i}} = 2$. Since

$$\exp_p(\overline{M}(A_{j_i}^{k_i})) = \exp_p(S^{2(m+i-1)+1}) = p^{m+i-1}$$

and $j_{\bar{i}} + (k_{\bar{i}}-1)p - (m+i-1) = p+\bar{i}-i > 0$, we also have $p^{m+i-1} < p^{j_{\bar{i}}+(k_{\bar{i}}-1)p}$. Thus we get

$$\exp_p(W_{m+k,k}) \leq p^{j_{\bar{i}}+(k_{\bar{i}}-1)p}$$

or equivalently

$$\exp_p(W_{m+k,k}) \leq p^{m+k+k_{\bar{i}}-2}.$$

(3) If $(m+\bar{i}-1)+(t-1)(p-1) \equiv 0 \mod p$ for some t as in the assumption, then by Theorem 3.3, $\exp_p(\overline{M}(A_{j_{\bar{i}}}^{k_{\bar{i}}})) \leq p^{j_{\bar{i}}+(k_{\bar{i}}-1)p-1}$. From the proof of part (2), it follows that $\exp_p(M(A_{j_i}^{k_i})) \leq p^{j_i+(k_i-1)p} \leq p^{j_{\bar{i}}+(k_{\bar{i}}-1)p-1}$ for $i \neq \bar{i}$. Thus

$$\exp_p(W_{m+k,k}) \leq p^{j_{\bar{i}}+(k_{\bar{i}}-1)p-1}$$

or equivalently

$$\exp_p(W_{m+k,k}) \leq p^{m+k+k_{\bar{i}}-3}.$$

□

Now let us consider the homotopy exponent of $SU(2n)/Sp(n)$. Recall that the homology of $SU(2n)/Sp(n)$ is

$$H_*(SU(2n)/Sp(n)) = E(x_5, x_9, \ldots, x_{4n-3})$$

where the generator $x_{4i+1} (1 \leq i \leq n-1)$ is the projection of the degree $4i+1$ generator of $H_*(SU(2n))$ under the quotient map $SU(2n) \longrightarrow SU(2n)/Sp(n)$.

Proposition 4.3. *Let $2n \leq (p-1)^2+2$. Then there is a homotopy decomposition*

$$SU(2n)/Sp(n) \simeq \prod_{i=1}^{(p-1)/2} \overline{M}(A_{2i}(2n))$$

such that

$$H_*(\overline{M}(A_{2i}(2n))) = E(\widetilde{H}_*(A_{2i}(2n))) = E(x_{4i+1}, x_{4i+q+1}, \ldots, x_{4i+(r_{2i}^{(2n)}-1)q+1}).$$

Proof. Since $2n \leq (p-1)^2+2$, we get $r_{2i}^{(2n)} \leq p-1$. Define a composite

$$g_{2i} \colon \overline{M}(A_{2i}(2n)) \xrightarrow{h} \Omega\Sigma A_{2i}(2n) \xrightarrow{\bar{j}_{2i}^{(2n)}} SU(2n) \longrightarrow SU(2n)/Sp(n)$$

where the rightmost map is the quotient map and $\bar{j}_{2i}^{(2n)}$ is defined in diagram (7). When $r_{2i}^{(2n)} < p-1$, the map h is the functorial section $M(A_{2i}(2n)) \longrightarrow \Omega\Sigma A_{2i}(2n)$, but when $r_{2i}^{(2n)} = p-1$, h is the composite

$$h \colon E(A_{2i}(2n)) \xrightarrow{E_p} A^{\min}(A_{2i}(2n)) \longrightarrow \Omega\Sigma A_{2i}(2n)$$

where the right map is a functorial section. The map g_{2i} induces an injection of the generating set in homology.

From [8] there is a homotopy equivalence $SU(2n) \simeq Sp(n) \times SU(2n)/Sp(n)$ defining on $SU(2n)/Sp(n)$ an H-structure. By multiplying g_{2i} for $1 \leq i \leq (p-1)/2$, we get a map

$$g\colon \prod_{i=1}^{(p-1)/2} \overline{M}(A_{2i}(2n)) \xrightarrow{\prod g_{2i}} \prod_{i=1}^{(p-1)/2} SU(2n)/Sp(n) \xrightarrow{\mu} SU(2n)/Sp(n)$$

where μ is the multiplication. It follows immediately that g induces an isomorphism of the generating sets in homology. By the same argument as in the last section of the proof of Proposition 4.1, g is a homotopy equivalence which gives the stated homotopy decomposition. $\square$

If in Section 3, we let $m+k=2n$ and $m=1$, then $A_{2i}(2n) = A_{2i}^{r_{2i}^{(2n)}}$. If $2n \leq (p-1)^2+2$, then by Proposition 4.3, we obtain

$$\exp_p(SU(2n)/Sp(n)) = \max\{\exp_p(\overline{M}(A_{2i}^{r_{2i}^{(2n)}})) \mid 1 \leq i \leq (p-1)/2\}.$$

By the same argument as in the proof of Theorem 1.1, we get an upper bound for $\exp_p(SU(2n)/Sp(n))$ as stated in Theorem 1.2.

5. Homotopy exponent of $G_{m+k,k}$

In this section, we will show that a looped complex Grassmann manifold decomposes as the product of a looped complex Stiefel manifold and a unitary group. Thus the homotopy exponent of a complex Grassmann manifold is determined by that of a complex Stiefel manifold and a unitary group. Additionally, the homotopy exponents of generalized complex Grassmann manifolds and generalized complex Stiefel manifolds are also considered. We first recall James' result.

Lemma 5.1. [12] *Let H, K and L be closed subgroups of a Lie group G with K and L subgroups of H. Suppose that H is an extension of K by L which is split. If G/L is connected and K contains a subgroup conjugate to L in G, then H/K is contractible in G/K.*

We first find an upper bound of the homotopy exponent of a Grassmann manifold.

Proof of Theorem 1.3. Given the Stiefel manifold $W_{m+k,k} = U(m+k)/U(m)$ and the Grassmann manifold $G_{m+k,k} = U(m+k)/(U(m) \times U(k))$, there is a fibration

$$\Omega G_{m+k,k} \xrightarrow{\gamma} U(k) \longrightarrow W_{m+k,k} \xrightarrow{\rho} G_{m+k,k}$$

where ρ is the quotient map. Since $k \leq m$, $U(k)$ is a subgroup of $U(m) \subseteq U(m) \times U(k)$ and thus by Lemma 5.1, the inclusion $(U(m) \times U(k))/U(m) = U(k) \longrightarrow W_{m+k,k} = U(m+k)/U(m)$ is null homotopic. It follows that $\gamma\colon \Omega G_{m+k,k} \longrightarrow U(k)$ admits a section $\theta\colon U(k) \longrightarrow \Omega G_{m+k,k}$. Hence there is a homotopy equivalence

$$\Omega W_{m+k,k} \times U(k) \xrightarrow{\Omega\rho\times\theta} \Omega G_{m+k,k} \times \Omega G_{m+k,k} \xrightarrow{\mu} \Omega G_{m+k,k}$$

where μ is the loop multiplication. It follows that

$$\Omega G_{m+k,k} \simeq \Omega W_{m+k,k} \times U(k) \simeq \Omega W_{m+k,k} \times SU(k) \times S^1.$$

Thus we get

$$\exp_p(G_{m+k,k}) = \max\{\exp_p(W_{m+k,k}), \exp_p(SU(k))\}.$$

If $m+k \leq (p-1)^2+1$, then $k \leq (p-1)^2+1$ and according to the last section of the proof of Proposition 4.1, there is a homotopy decomposition

$$SU(k) \simeq \prod_{i=1}^{p-1} \overline{M}(A_i(k))$$

which implies that

$$\exp_p(SU(k)) = \max\{\exp_p(\overline{M}(A_i(k))) \mid 1 \leq i \leq p-1\}.$$

Since $A_i(k)$ is actually $A_i^{r_i^{(k)}}$ in Theorem 3.3, we get $\exp_p(\overline{M}(A_i(k))) \leq p^{i+(r_i^{(k)}-1)p}$. As $r_i^{(k)} = \lfloor (k-i-1)/(p-1) \rfloor + 1$ and $k_{\bar{i}} = \lfloor (k-\bar{i})/(p-1) \rfloor + 1$, then $k_{\bar{i}} - r_i^{(k)} \leq 1$. Since $k \leq m$, we conclude that $m \geq (p-1)^2/2$. It follows that $p^{i+(r_i^{(k)}-1)p} < p^{m+\bar{i}+(k_{\bar{i}}-1)p-2}$. Now using Theorem 1.1, proves the theorem. □

Theorem 1.3 can be generalized to a result about the homotopy exponent of a generalized Grassmann manifold $U(n)/(U(n_1)\times U(n_2)\times\cdots\times U(n_l))$ with $\sum_{i=1}^{l} n_i = n$. First set that $N = \lfloor (n-n_l-\bar{i})/(p-1) \rfloor + 1$ where $1 \leq \bar{i} \leq p-1$ such that $2(n_l+\bar{i}-1)+(N-1)q+1 = 2n-1$.

Theorem 5.2. *Let $G = U(n)/(U(n_1)\times U(n_2)\times\cdots\times U(n_l))$ with $\sum_{i=1}^{l} n_i = n$ and $n_1 \leq n_2 \leq \cdots \leq n_l$. Then there is a homotopy decomposition*

$$\Omega G \simeq \Omega W_{n,n-n_l} \times \prod_{i=1}^{l-1} SU(n_i) \times T^{l-1}$$

where T^{l-1} is the $(l-1)$-dimensional torus. Assume that $n \leq (p-1)^2+1$. Then

(1) *if $(n_l+\bar{i}-1)+(t-1)(p-1) \not\equiv 0 \mod p$ for every t such that $1 \leq t \leq N-1$, then $\exp_p(G) \leq p^{n+N-2}$;*
(2) *if $(n_l+\bar{i}-1)+(t-1)(p-1) \equiv 0 \mod p$ for some t such that $1 \leq t \leq N-1$, then $\exp_p(G) \leq p^{n+N-3}$.*

Proof. Consider the homogeneous spaces $G_i = U(n)/(U(n_i)\times U(n_{i+1})\times\cdots\times U(n_l))$ for $1 \leq i \leq l$. Thus $G_1 = G$ and $G_l = W_{n,n-n_l}$. For $1 \leq i \leq l-1$, there is a fibration

$$\Omega G_{i+1} \longrightarrow \Omega G_i \longrightarrow U(n_i) \longrightarrow G_{i+1} \longrightarrow G_i.$$

Since $n_i \leq n_{i+1}$, $U(n_i)$ is a closed subgroup of $U(n_{i+1})\times\cdots\times U(n_l)$. Then by Lemma 5.1, the inclusion $U(n_i) \longrightarrow G_{i+1}$ is null homotopic and there is a homotopy equivalence

$$\Omega G_i \simeq \Omega G_{i+1} \times U(n_i) \simeq \Omega G_{i+1} \times SU(n_i) \times S^1.$$

By induction, we get

$$\begin{aligned}\Omega G &= \Omega G_1 \simeq \Omega G_2 \times U(n_1) \simeq \Omega G_3 \times U(n_1) \times U(n_2)\\ &\simeq \Omega G_l \times \prod_{i=1}^{l-1} U(n_i) \simeq \Omega W_{n,n-n_l} \times \prod_{i=1}^{l-1} SU(n_i) \times T^{l-1}\end{aligned}$$

and thus

$$\exp_p(G) = \max\{\exp_p(W_{n,n-n_l}), \exp_p(SU(n_1)), \ldots, \exp_p(SU(n_{l-1}))\}. \tag{26}$$

If $n \leq (p-1)^2+1$, we compute an upper bound of $\exp_p(W_{n,n-n_l})$ by Theorem 1.1. As $n_1 \leq \cdots \leq n_{l-1} \leq (p-1)^2/2$, we use the same argument as in the proof of Theorem 1.3 to compare upper bounds of homotopy exponents in (26) and thus produce statements (1) and (2) of the theorem. □

In what follows, we consider the homotopy exponent of generalized complex Stiefel manifold. A generalized complex Stiefel manifold $M(m,n;k)$ is a space of $m\times n$ complex matrices of a fixed rank $k\leq\min\{m,n\}$. If $k=\min\{m,n\}$, then it is a complex Stiefel manifold. Milnor [15] introduced a generalized complex Stiefel manifold to study immersions of complex manifolds. Its homotopy groups are important for determining the existence of a subimmersion between certain complex manifolds.

Theorem 5.3. *Let $n\leq m+k$ and $k\leq n$. There is a homotopy equivalence*

$$\Omega M(m+k,n;k)\simeq\Omega W_{m+k,k}\times\Omega G_{n,k}.$$

Assume that $m+k\leq(p-1)^2+1$ and $m>0$. Then

(1) *if $(m+\bar{i}-1)+(t-1)(p-1)\not\equiv 0 \mod p$ for every t such that $1\leq t\leq k_{\bar{i}}-1$, then $\exp_p(M(m+k,n;k))\leq p^{m+k+k_{\bar{i}}-2}$;*

(2) *if $(m+\bar{i}-1)+(t-1)(p-1)\equiv 0 \mod p$ for some t such that $1\leq t\leq k_{\bar{i}}-1$, then $\exp_p(M(m+k,n;k))\leq p^{m+k+k_{\bar{i}}-3}$.*

Proof. Mukherjee [21] showed that since the row space of an $(m+k)\times n$ matrix of rank k spans a k-dimensional complex space, it induces a map $\psi\colon M(m+k,n;k)\longrightarrow G_{n,k}$ with the fibre $W_{m+k,k}$ and the map ψ admits a section s when $n\leq m+k$. Thus there is a split fibration

$$W_{m+k,k}\xrightarrow{j}M(m+k,n;k)\xrightarrow{\psi}G_{n,k}$$

which induces a homotopy equivalence

$$\Omega W_{m+k,k}\times\Omega G_{n,k}\xrightarrow{\Omega j\times\Omega s}\Omega M(m+k,n;k)\times\Omega M(m+k,n;k)\xrightarrow{\mu}\Omega M(m+k,n;k)$$

where μ is the loop multiplication. It follows that

$$\exp_p(M(m+k,n;k))=\max\{\exp_p(W_{m+k,k}),\exp_p(G_{n,k})\}.$$

When $m+k\leq(p-1)^2+1$, the statements (1), and (2) follow from Theorems 1.1, and 1.3. □

6. Homotopy exponents of E_6/F_4 and F_4/G_2

According to [18] and [19], the p-local exceptional Lie groups G_2, F_4 and E_6 admit the following decompositions

$$G_2\simeq\begin{cases}\overline{B}_2(3,11) & p=3\\ \overline{B}(3,11) & p=5\\ S^3\times S^{11} & p>5\end{cases}$$

$$F_4\simeq\begin{cases}\overline{B}(3,11)\times\overline{B}(15,23) & p=5\\ \overline{B}(3,15)\times\overline{B}(11,23) & p=7\\ \overline{B}(3,23)\times S^{11}\times S^{15} & p=11\\ S^3\times S^{11}\times S^{15}\times S^{23} & p>11\end{cases}$$

$$E_6\simeq\begin{cases}F_4\times\overline{B}(9,17) & p=5\\ F_4\times S^9\times S^{17} & p>5\end{cases}$$

where $H^*(\overline{B}_2(3,11),\mathbb{Z}/3)\cong E(x_3,x_{11})$ with the secondary operation $\Phi(x_3)=x_{11}$. The cohomology of $\overline{B}(2r+1,2r+q+1)$ is given by

$$H^*(\overline{B}(2r+1,2r+q+1),\mathbb{Z}/p)\cong E(x_{2r+1},x_{2r+q+1})$$

with the Steenrod operation $\mathcal{P}^1(x_{2r+1}) = x_{2r+q+1}$.

Proof of Theorem 1.4. For $p = 3$ or 5, the mod-p cohomology of E_6/F_4 is given by

$$H^*(E_6/F_4, \mathbb{Z}/p) \cong E(x_9, x_{17})$$

with the secondary operation $\Phi(x_9) = x_{17}$ when $p = 3$ and the Steenrod operation $\mathcal{P}^1(x_9) = x_{16}$ when $p = 5$. When $p = 3$, Bendersky, and Davis [2] showed that E_6/F_4 is spherically resolved. When $p = 5$, there is a homotopy equivalence $E_6 \simeq F_4 \times E_6/F_4$ given in [9]. Then from the decomposition $E_6 \simeq F_4 \times \overline{B}(9, 17)$ we get $E_6/F_4 \simeq \overline{B}(9, 17)$. By [18, Theorem 5.5], $\overline{B}(9, 17)$ is spherically resolved and so is E_6/F_4. Thus for $p = 3$ or 5, there is a fibration

$$S^9 \longrightarrow E_6/F_4 \longrightarrow S^{17}. \tag{27}$$

Let A be the 25-skeleton of E_6/F_4. Then $\widetilde{H}^*(A, \mathbb{Z}/p) = \mathbb{Z}/p\{x_9, x_{17}\}$ with $\Phi(x_9) = x_{17}$ when $p = 3$ and $\mathcal{P}^1(x_9) = x_{16}$ when $p = 5$. Thus there is a cofibration

$$S^{16} \xrightarrow{\alpha} S^9 \longrightarrow A \longrightarrow S^{17} \tag{28}$$

where $\alpha \in \pi_{16}(S^9) = \mathbb{Z}/p$ is the stable element detected by Φ when $p = 3$ and when $p = 5$. Considering fibration (27) and cofibration (28), we get a diagram analogous to diagram (12)

$$\begin{array}{ccc} S^{17} & = & S^{17} \\ \downarrow & & \downarrow{\scriptstyle p} \\ E_6/F_4 & \longrightarrow & S^{17}. \end{array}$$

Then by Lemma 3.2, we obtain

$$\exp_p(E_6/F_4) \leq p \cdot \max\{p^4, p^8\} = p^9.$$

For E_6 and F_4 localized at $p > 5$, combining $E_6 \simeq F_4 \times E_6/F_4$ (see [9]), and $E_6 \simeq F_4 \times S^9 \times S^{17}$, we conclude that E_6/F_4 splits as $S^9 \times S^{17}$. Thus

$$\exp_p(E_6/F_4) = \max\{\exp_p(S^9), \exp_p(S^{17})\} = p^8.$$

□

Proof of Theorem 1.5. Davis and Mahowald [6] showed that, when localized at $p \neq 3$, F_4/G_2 is spherically resolved with respect to a fibration

$$S^{15} \longrightarrow F_4/G_2 \longrightarrow S^{23}.$$

When $p = 2$, Borel [3] showed that $H^*(F_4/G_2, \mathbb{Z}/2) \cong E(x_{15}, x_{23})$ with the Steenrod operation $Sq^8(x_{15}) = x_{23}$. Let C be the 37-skeleton of F_4/G_2. Then C fits into the cofibration

$$S^{22} \xrightarrow{\alpha} S^{15} \longrightarrow C \longrightarrow S^{23}$$

where $\alpha \in \pi_{22}(S^{15}) = \mathbb{Z}/2^4$ by [30, Proposition 5.15]. It follows that α is of order 2^4. Then by the same argument as for E_6/F_4 when $p = 3$ or 5, we get

$$\exp_2(F_4/G_2) \leq \exp_2(S^{23}\{2^4\}) \cdot \max\{\exp_2(S^{15}), \exp_2(S^{23})\}.$$

By [22, Proposition 5.2], we know that $\exp_2(S^{23}\{2^4\}) \leq 2^4$, and Selick [24] proved that $\exp_2(S^{2n+1}) \leq 2^{2n - \lfloor n/2 \rfloor}$. Thus we get

$$\exp_2(F_4/G_2) \leq 2^4 \cdot 2^{17} = 2^{21}.$$

For $p \geq 5$, Borel [3] proved that $H^*(F_4/G_2, \mathbb{Z}/p) \cong E(x_{15}, x_{23})$. When $p = 5$, by [19] there is a Steenrod operation $\mathcal{P}^1(x_{15}) = x_{23}$. Let C' be the 37-skeleton of F_4/G_2. Then C' lies in the cofibration

$$S^{22} \xrightarrow{\alpha_1} S^{15} \longrightarrow C' \longrightarrow S^{23}$$

where $\alpha_1 \in \pi_{22}(S^{15}) = \mathbb{Z}/5$ is a stable element of order 5 detected by $\mathcal{P}^1$. Again by the same argument as for E_6/F_4 when $p = 3$ or 5, we get

$$\exp_5(F_4/G_2) \leq \exp_5(S^{23}\{5\}) \cdot \max\{\exp_5(S^{15}), \exp_5(S^{23})\} = 5^{12}.$$

When $p > 5$, by Toda's [30] calculations $\pi_{22}(S^{15}) \cong 0$, therefore the 37-skeleton of F_4/G_2 is homotopy equivalent to $S^{15} \vee S^{23}$. Define $i_1 \colon S^{15} \longrightarrow F_4/G_2$ and $i_2 \colon S^{23} \longrightarrow F_4/G_2$ to be the inclusions. Recall that the cohomology of $\Omega(F_4/G_2)$ is given by $H^*(\Omega(F_4/G_2)) \cong \mathbb{Z}/p[x_{14}, x_{22}]$. It follows readily that the composite

$$\Omega S^{15} \times \Omega S^{23} \xrightarrow{\Omega i_1 \times \Omega i_2} \Omega(F_4/G_2) \times \Omega(F_4/G_2) \xrightarrow{\mu} \Omega(F_4/G_2)$$

is a homotopy equivalence. Thus

$$\exp_p(F_4/G_2) = \max\{\exp_p(S^{15}), \exp_p(S^{23})\} = p^{11}.$$

□

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