A New Adaptive Algorithm for Convex Quadratic Multicriteria Optimization

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Abstract

We present a new adaptive algorithm for convex quadratic multicriteria optimization. The algorithm is able to adaptively refine the approximation to the set of efficient points by way of a warm-start interior-point scalarization approach. Numerical results show that this technique is faster than a standard method used for this problem.

1 Introduction

Multicriteria optimization problems are a class of difficult optimization problems in which several different objective functions have to be taken care of at the same time. It will usually be the case that no single point will minimize all of the several objective functions given at once. Therefore, we are in search for so-called *efficient* points, i. e. feasible points for which there does not exist a different feasible point with the same or smaller objective function values such that there is a strict decrease in at least one objective function value. Since two different efficient points will usually be not only quite different from each other in terms of objective function values, but also incomparable with each other, we have to gain as much information as possible about the solution set of a given problem, preferably by constructing a well-defined approximation to it. This is the subject of this paper.

Applications of multicriteria optimization can be found in various areas, e. g. in engineering design [8, 7, 25], space exploration [29], antenna design [24, 23], management science [9, 2, 18, 27, 31, 1], environmental analysis [26, 10, 11], cancer treatment planning [21], bilevel programming [14] location science [3], statistics [4], etc.

The rest of this paper is as follows. In Section 2, we consider the problem of solving one single-criteria convex-quadratic optimization problem by an interior-point method, namely by an infeasible point method. Section 3, containing the main theoretical results of this paper, considers perturbed optimization problems and a strategy to compute a *warm-start point*, i. e. a point feasible for the new, perturbed problem, computed out of a point feasible for the unperturbed problem. Armed with this technique, we are ready to tackle our main problem. After explaining in short what multicriteria optimization is and where the main difficulties lie (Section 5), we come to the main part of the paper, Section 6. There, we describe a new efficient adaptive interior-point technique for solving convex quadratic multicriteria problems. Numerical results are presented and discussed, too.

2 The Interior-Point Algorithm

2.1 The Problem

Let there be given a primal quadratic optimization problem (PQP) of the form

$$\begin{array}{ll} \min & \frac{1}{2}x^TQx + c^Tx \\ \text{s. t.} & Ax = b, \\ & x \geq 0, \end{array} \tag{1}$$

with $c \in \mathrm{IR}^n$, $b \in \mathrm{IR}^m$, $A \in \mathrm{IR}^{m\times n}$ and a symmetric matrix $Q \in \mathrm{IR}^{n\times n}$. The vector $x \in \mathrm{IR}^n$ represents the primal variables.

A dual problem (DQP) to (PQP) is

$$\begin{array}{ll} \max & -\frac{1}{2}x^TQx + b^T\lambda \\ \text{s. t.} & -Qx + A^T\lambda + s = c, \\ & s \geq 0, \end{array} \tag{2}$$

with the dual variables $s \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}^m$. The set of primal-dual feasible 3-tupel $w = (x, \lambda, s)$ is given by

$$\Omega := \{w = (x, \lambda, s) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^n \mid Ax = b,\ -Qx + A^T\lambda + s = c,\ (x, s) \geq 0\}.$$

For a primal-dual feasible point $w \in \Omega$, the difference between the primal and the dual objective function values is given by

$$s^T x = c^T x - x^T Q x + b^T \lambda = s^T x \geq 0$$

and is called *duality gap*. Denote by

$$\Omega^0 := \{w = (x, \lambda, s) \in \Omega \mid (x, s) > 0\}$$

the set of *strictly feasible points*.

In the rest of the paper, we make the following three assumptions.

1. The set of primal-dual feasible points is nonempty: $\Omega \neq \emptyset$.
2. The constraint matrix A has full row rank.
3. The matrix Q is positive semidefinite.

Clearly, under these assumptions, a primal-dual point $w = (x, \lambda, s)$ is optimal for (PQP) as well as (DQP) (i. e. x is optimal for (PQP) and (λ, s) is optimal for (DQP)) if the KKT-conditions hold:

$$F(x, \lambda, s) := \begin{bmatrix} -Qx + A^T\lambda + s - c \\ Ax - b \\ SXe \end{bmatrix} = 0, \qquad (x, s) \geq 0. \tag{3}$$

Here, as usual, $e := (1, 1, ..., 1)^T \in \mathbb{R}^n$, $S := \operatorname{diag}(s) \in \mathbb{R}^{n \times n}$ and $X := \operatorname{diag}(x) \in \mathbb{R}^{n \times n}$.

2.2 The Algorithm

We now perturb the right hand side of the system of equations $F(x, \lambda, s) = 0$ in the usual way by considering a parameter $\tau > 0$ and the perturbed system

$$F_\tau(x, \lambda, s) := \begin{bmatrix} -Qx + A^T\lambda + s - c \\ Ax - b \\ SXe - \tau e \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \qquad (x, s) > 0. \tag{4}$$

A Newton step for the nonlinear system of equations $F_\tau(x, \lambda, s) = 0$ amounts in solving the linear system

$$\begin{bmatrix} -Q & A^T & I \\ A & 0 & 0 \\ S & 0 & X \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta\lambda \\ \Delta s \end{bmatrix} = \begin{bmatrix} -r_c \\ -r_b \\ -(SXe - \sigma\mu e) \end{bmatrix}. \tag{5}$$

Here, we have used the abbreviations $r_b := Ax - b$ and $r_c := -Qx + A^T\lambda + s - c$. (Note that for these abbreviations, r_b as well as r_c depend on (x, λ, s).) Moreover, we used $\tau = \sigma\mu$ with the duality measure

$$\mu = \frac{x^T s}{n} \tag{6}$$

and the centering parameter $\sigma \in]\,0, 1[$. This parameter weights the competing aims of convergence of μ to zero and closeness to the central path. Denote by $\Delta w = (\Delta x, \Delta\lambda, \Delta s)$ the solution of the linear system (5).

Basically, we want to execute Newton steps for F_τ with the parameter $\tau > 0$ converging to 0. For prespecified parameters $\gamma \in]\,0, 1\,[$ and $\beta \geq 1$ and a starting point (x^0, λ^0, s^0) defining residuals r_b^0 and r_c^0 and a duality measure μ_0, all iterates of the algorithm presented below will lie in the set

$$\begin{aligned} \mathcal{N}_{-\infty}(\gamma, \beta) := \Bigg\{ (x, \lambda, s) \ \Bigg| \ & \|(r_b, r_c)\|_2 \leq \frac{\|({r_b}^0, {r_c}^0)\|_2}{\mu_0}\beta\mu, \ (x, s) > 0, \\ & x_i s_i \geq \gamma\mu \quad \forall\, i = 1, 2, \ldots, n \Bigg\}, \end{aligned} \tag{7}$$

Here, the inequality $x_i s_i \geq \gamma\mu$ serves the purpose of hindering some products $x_i s_i$ to converge faster to zero than other ones. Moreover, due to $\beta \geq 1$, we have $(x^0, \lambda^0, s^0) \in \mathcal{N}_{-\infty}(\gamma, \beta)$.

In order to stay inside the set $\mathcal{N}_{-\infty}(\gamma, \beta)$ with all iterates we will introduce a step length $\alpha > 0$. (Indeed, full Newton steps for F_τ might lead outside of the set.) We will choose α in such a way that with $w^{k+1} = w^k + \alpha\Delta w$ we still have $w^{k+1} \in \mathcal{N}_{-\infty}(\gamma, \beta)$. For this, define

$$(x(\alpha), \lambda(\alpha), s(\alpha)) := (x, \lambda, s) + \alpha(\Delta x, \Delta\lambda, \Delta s)$$

as well as

$$\mu(\alpha) := x(\alpha)^T s(\alpha)/n.$$

Now we are ready to describe the primal-dual infeasible-point long-step algorithm for quadratic problems, see Table 1, p. 5.

Algorithm QIP

(S0) Choose $\varepsilon > 0$, $\beta \geq 1$, $\gamma \in]\,0,1\,[$, $\sigma_{\min}$ and $\sigma_{\max}$ with $0 < \sigma_{\min} < \sigma_{\max} < 1/2$ and $(x^0, \lambda^0, s^0) \in \mathcal{N}_{-\infty}(\gamma, \beta)$. Set $k := 0$.

(S1) If

$$\mu_k := \frac{(x^k)^T s^k}{n} \leq \varepsilon, \tag{8}$$

then stop.

(S2) Choose $\sigma_k \in]\sigma_{\min}, \sigma_{\max}]$ and compute r_c^k, r_b^k as well as r_{xs}^k by

$$\begin{array}{rcl} r_c^k & = & -Qx^k + A^T\lambda^k + s^k - c \\ r_b^k & = & Ax^k - b \\ r_{xs}^k & = & S^k X^k e - \sigma_k \mu_k e. \end{array} \tag{9}$$

Compute $(\Delta x^k, \Delta\lambda^k, \Delta s^k)$ by solving

$$\begin{bmatrix} -Q & A^T & I \\ A & 0 & 0 \\ S^k & 0 & X^k \end{bmatrix} \begin{bmatrix} \Delta x^k \\ \Delta\lambda^k \\ \Delta s^k \end{bmatrix} = \begin{bmatrix} -r_c^k \\ -r_b^k \\ -r_{xs}^k \end{bmatrix}. \tag{10}$$

Choose α_k as the largest $\alpha \in [\,0,1\,]$ such that

$$(x^k(\alpha), \lambda^k(\alpha), s^k(\alpha)) \in \mathcal{N}_{-\infty}(\gamma, \beta) \tag{11}$$

as well as the Armijo condition

$$\mu_k(\alpha) \leq (1 - 0.01\alpha)\mu_k \tag{12}$$

holds.

(S4) Define

$$(x^{k+1}, \lambda^{k+1}, s^{k+1}) := (x^k(\alpha_k), \lambda^k(\alpha_k), s^k(\alpha_k))$$

as well as $k := k+1$. Go to S2.

Table 1: Algorithm QIP for solving a convex-quadratic optimization problem subject to linear equality and inequality constraints in standard form.

Remark 1 In case of $r_b = 0$ as well as $r_c = 0$, i. e. if the starting point fulfills already the primal-dual equality constraints, we have a *feasible point algorithm*. In this case, all iterates generated are strictly feasible. Such an algorithm for the problem considered here is discussed in [13]. However, finding such a starting point turns out to be rather difficult. Indeed, there exist problems with $\Omega \neq \emptyset$ but $\Omega^0 = \emptyset$. In the algorithm above, we do not need such a feasible starting point, just a point with $(x, s) > 0$. We will argue below that the iterates constructed in the algorithm converge to an optimal primal-dual solution, although all iterates can be infeasible. To achieve this, we have to prove that the residuals r_b and r_c converge as fast to zero as the duality measure μ.

With respect to convergence, we just note that we can use the convergence proof of Wright [32, p. 110ff] for algorithm IPF (stated there for linear optimization problems) almost verbatim. The only difference of any interest is that for the pair $(\overline{x}, \overline{s})$ defined in Lemma 6.3 of [32] the equality $\overline{x}^T\overline{s} = 0$ does not hold. Instead, we have $\overline{x}^T\overline{s} \geq 0$. This, however, is sufficient for the rest of the reasoning to work. We summarize the main results in the following two theorems.

Theorem 1 (Convergence) *Let $\{(x^k, \lambda^k, s^k)\}$ be a sequence constructed by Algorithm QIP. Then,*

1. *the sequence $\{\mu_k\}$ of duality measures converges Q-linearly to zero, and*
2. *the sequence $\{\|(r_b^k, r_c^k)\|_2\}$ of residuals converges Q-linearly to zero.*

Theorem 2 (Complexity) *Let $\varepsilon > 0$ be given. Suppose that for the starting point (x^0, λ^0, s^0) we have that*

$$(x^0, \lambda^0, s^0) = \zeta(e, 0, e) \quad \text{and} \quad \zeta \geq \|(x^*, s^*)\|_\infty$$

holds for a solution (x^*, λ^*, s^*) of (PQP) and (DQP). Moreover, let constants $C, \kappa > 0$ be chosen such that

$$\zeta^2 \leq \frac{C}{\varepsilon^\kappa}.$$

Then, there exists an index K with

$$K = \mathcal{O}(n^2 |\log \varepsilon\,|),$$

such that for the iterates (x^k, λ^k, s^k) constructed by algorithm QIP *we have that*

$$\mu_k \leq \varepsilon \qquad \textit{for all} \quad k \geq K$$

holds.

3 Warm-start Points

Let there be given the primal and the dual problem (PQP) and (DQP) from Subsection 2.1. Both problems, (PQP) as well as (DQP), can be described in a unique way by the 4-tuple

$$d := (A, b, Q, c). \tag{13}$$

We define the norm of such a data instance d by the maximum of the 2-norm of the components,

$$\|d\|_2 := \max\{\|A\|_2, \|b\|_2, \|Q\|_2, \|c\|_2\}. \tag{14}$$

(Of course, the matrix norms are matrix norms induced by the Euclidean norms in the corresponding vector spaces.)

Let us now consider the data instance $d = (A, b, Q, c)$ as well as the perturbed instance $\tilde{d} := d + \Delta d$ with the perturbation $\Delta d = (0, 0, \Delta Q, \Delta c)$. (We will see later, in Section 5, that we are in exactly such a situation if we want to solve a multicriteria optimization problem). If we solve the problem described by d with algorithm QIP, all iterates will lie in $\mathcal{N}_{-\infty}(\gamma, \beta)$. If we want to solve the problem represented by $d + \Delta d$ with algorithm QIP, too, we have to use a similar set for the corresponding sequence of iterates.

Remark 2 To differentiate between variables and parameters for the original problem with data instance d from variables and parameters for the perturbed problem with data instance $\tilde{d} := d + \Delta d$, we will use a tilde ($\tilde{}$) on all variables and parameters for the latter to signify the perturbation of the data.

Now let $w = (x, \lambda, s)$ be a prima-dual strictly feasible point of the original problem. We want to construct a *warm-start point*

$$\tilde{w} := w + \Delta w = (x + \Delta x, \lambda + \Delta\lambda, s + \Delta s)$$

of the pertubed problem. This warm start point should have the same residuals $(\tilde{r}_b, \tilde{r}_c)$ as those given by w, i. e. (r_b, r_c), the residuals of w for the original problem. For the problem instance $d + \Delta d$, and the residuals $(\tilde{r}_b, \tilde{r}_c)$, we consider the sets

$$
\begin{aligned}
\tilde{\Omega} &:= \{(x, \lambda, s) \mid Ax - b = \tilde{r}_b,\ -(Q + \Delta Q)x + A^T\lambda + s - (c + \Delta c) = \tilde{r}_c,\\
&\qquad (x, s) \geq 0\},\\
\tilde{\Omega}^0 &:= \{(x, \lambda, s) \in \tilde{\Omega} \mid (x, s) > 0\}
\end{aligned}
$$

as well as

$$
\tilde{\mathcal{N}}_{-\infty}(\gamma, \beta) := \left\{ (x, \lambda, s) \in \tilde{\Omega}^0 \;\middle|\; \begin{array}{l} \|(\tilde{r}_b, \tilde{r}_c)\|_2 \leq \dfrac{\|(r_b^0,\ r_c^0)\|_2}{\mu_0}\beta\tilde{\mu}, \\ x_i s_i \geq \gamma\tilde{\mu}, \quad i = 1, 2, \ldots, n \end{array} \right\}. \tag{15}
$$

Note that the residuals r_b^0 and r_c^0 and the duality gap μ_0 stem from the starting point of the original problem represented by d. Now we are in search of a corrector step $\Delta w = (\Delta x, \Delta\lambda, \Delta s)$ such that the warm start point $\tilde{w}$ defined by $\tilde{w} = w + \Delta w$ is in the set $\tilde{\mathcal{N}}_{-\infty}(\gamma, \beta)$.

With the assumptions from above, we can now make the following observations.

Due to $\tilde{r}_b = r_b$, we have

$$
\begin{aligned}
& \tilde{r}_b = A\tilde{x} - b = A(x + \Delta x) - b = \underbrace{Ax - b}_{= r_b} + A\Delta x \\
\Leftrightarrow \qquad & A\Delta x = 0.
\end{aligned} \tag{16}
$$

Moreover, with $\tilde{r}_c = r_c$, it follows that

$$
\begin{aligned}
\tilde{r}_c &= -(Q + \Delta Q)\tilde{x} + A^T\tilde{\lambda} + \tilde{s} - (c + \Delta c) \\
&= -(Q + \Delta Q)(x + \Delta x) + A^T(\lambda + \Delta\lambda) + (s + \Delta s) - (c + \Delta c) \\
&= \underbrace{-Qx + A^T\lambda + s - c}_{= r_c} - (Q + \Delta Q)\Delta x + A^T\Delta\lambda + \Delta s - \Delta c - \Delta Q x \\
\Leftrightarrow \qquad & -(Q + \Delta Q)\Delta x + A^T\Delta\lambda + \Delta s = \Delta c + \Delta Q x
\end{aligned} \tag{17}
$$

In addition, we want that the duality gap of $\tilde{w}$ is at most as large as w. This can be achieved by

$$
S\Delta x + X\Delta s = 0. \tag{18}
$$

We will show in the proof of Theorem 3 (p. 10) that this is indeed sufficient for $\tilde{x}^T\tilde{s} \le x^T s$.

Taking (16), (17), and (18) together, we see that we need to consider the following system of linear equations:

$$\begin{bmatrix} -\tilde{Q} & A^T & I \\ A & 0 & 0 \\ S & 0 & X \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta\lambda \\ \Delta s \end{bmatrix} = \begin{bmatrix} \Delta c + \Delta Q x \\ 0 \\ 0 \end{bmatrix}. \tag{19}$$

Here,

$$\tilde{Q} := Q + \Delta Q.$$

Solving (19) for $\Delta w = (\Delta x, \Delta\lambda, \Delta s)$, we get

$$\begin{aligned} \Delta\lambda &= (AM^{-1}A^T)^{-1}AM^{-1}(\Delta c + \Delta Q x), & (20)\\ \Delta s &= D^{-2}M^{-1}(\Delta c + \Delta Q x - A^T\Delta\lambda), & (21)\\ \Delta x &= -D^2\Delta s. & (22) \end{aligned}$$

Here, we have used the abbreviations

$$D^2 := S^{-1}X$$

and

$$M := \tilde{Q} + D^{-2}.$$

Note, however, that in order for the inclusion

$$\tilde{w} \in \tilde{\mathcal{N}}_{-\infty}(\gamma, \beta) \tag{23}$$

to hold, we need strict positivity of all components of $\tilde{x}$ as well as $\tilde{s}$. Unfortunately, this is not necessarily true for $\tilde{w} = w + \Delta w$. Using a step length $\alpha \neq 1$ like in algorithm QIP does not help, either, since only full steps can assure that the residuals stay the same. We are therefore in search for sufficient conditions for (23). This is the subject of the next section.

4 Warmstart Criteria

We follow the strategy outlined before in Nunez & Freund [30], Yildirim & Wright [33], and Fliege & Heseler [13].

4.1 Necessary and Sufficient Conditions

A necessary and sufficient condition for strict positivity of $\tilde{x}$ and $\tilde{s}$ can be found by taking a closer look at (22). Indeed, written in the actual components, we get

$$s_i \Delta x_i + x_i \Delta s_i = 0, \qquad i = 1, 2, ..., n.$$

Using $(x, s) > 0$, this is equivalent to

$$\frac{\Delta x_i}{x_i} + \frac{\Delta s_i}{s_i} = 0, \qquad i = 1, 2, ..., n. \tag{24}$$

Therefore, $\tilde{x}$ and $\tilde{s}$ are strictly positive componentwise if and only if

$$\frac{\Delta x_i}{x_i} < -1 \text{ and } \frac{\Delta s_i}{s_i} < -1, \qquad i = 1, 2, ..., n. \tag{25}$$

Combining (24) and (25) we arrive at

$$\left| \frac{\Delta x_i}{x_i} \right| = \left| \frac{\Delta s_i}{s_i} \right| < 1, \qquad i = 1, 2, ..., n$$

i. e.

$$\left\| X^{-1} \Delta x \right\|_\infty = \left\| S^{-1} \Delta s \right\|_\infty < 1. \tag{26}$$

This, now, is a criteria that a feasible warm-start point has to fulfill.

4.2 Sufficient Conditions

Unfortunately, before we can check (26), we have to compute $\Delta\lambda$ as well as Δs. While we will present in what follows an algorithm which does exactly this, it might be argued that this is slightly inefficient: in such a scheme, we first compute a warm-start point, then we check for feasibility. The next theorem is a step further on in our search for simple sufficient conditions for feasibility of a warm-start point.

Theorem 3 *Let d be a problem instance, w be a point for this instance with $(x, s) > 0$ and residuals $(r_b, r_c) \neq 0$. Let a perturbation $\Delta d = (0, 0, \Delta Q, \Delta c)$ be given, let $\Delta w = (\Delta x, \Delta\lambda, \Delta s)$ be a solution to (19) and set $\tilde{w} = (\tilde{x}, \tilde{\lambda}, \tilde{s}) = w + \Delta w$. With*

$$T := I - A^T (AM^{-1}A^T)^{-1} AM^{-1},$$

suppose that

$$\left\| \begin{bmatrix} \Delta c \\ \Delta Q x \end{bmatrix} \right\|_\infty < \frac{1}{\|X^{-1}M^{-1}[T,T]\|_\infty}, \tag{27}$$

holds. Then,

$$\tilde{w} \in \tilde{\Omega}^0$$

as well as

$$\tilde{x}^T \tilde{s} \leq x^T s \tag{28}$$

follow.

Proof: (Cmp. Yildirim & Wright [34], Proposition 5.1, p. 797f.) Using (20) and (21) we arrive at the following chain of equalities and inequalities.

$$\begin{aligned}
& \left\| S^{-1} \Delta s \right\|_\infty \\
= \; & \left\| S^{-1} D^{-2} M^{-1} [\Delta c + \Delta Q x - A^T \Delta \lambda] \right\|_\infty \\
= \; & \left\| X^{-1} M^{-1} \left[\Delta c + \Delta Q x - A^T \left((AM^{-1}A^T)^{-1} AM^{-1} (\Delta c + \Delta Q x) \right) \right] \right\|_\infty \\
= \; & \Big\| X^{-1} M^{-1} \Big[\left(\Delta c - A^T (AM^{-1}A^T)^{-1} AM^{-1} \Delta c \right) \\
& + \left(\Delta Q x - A^T (AM^{-1}A^T)^{-1} AM^{-1} \Delta Q x \right) \Big] \Big\|_\infty \\
\leq \; & \left\| X^{-1} M^{-1} \left(I - A^T (AM^{-1}A^T)^{-1} AM^{-1} \right) [I, I] \right\|_\infty \left\| \begin{bmatrix} \Delta c \\ \Delta Q x \end{bmatrix} \right\|_\infty \\
= \; & \left\| X^{-1} M^{-1} (T, T) \right\|_\infty \left\| \begin{bmatrix} \Delta c \\ \Delta Q x \end{bmatrix} \right\|_\infty .
\end{aligned}$$

Therefore, as long as

$$\left\| \begin{bmatrix} \Delta c \\ \Delta Q x \end{bmatrix} \right\|_\infty < \left\| X^{-1} M^{-1} (T, T) \right\|_\infty^{-1} \tag{29}$$

holds, we have $\|S^{-1}\Delta s\|_\infty < 1$ and therefore $\tilde{w} \in \tilde{\Omega}^0$.

It remains to show (28). We have

$$x^T \Delta s + s^T \Delta x = 0.$$

Due to (24), we know that Δx_i and Δs_i have different signs for all $i = 1, 2, \ldots, n$. This results in $(\Delta x)^T \Delta s \leq 0$, which in turn leads to

$$\tilde{x}^T \tilde{s} = (x + \Delta x)^T (s + \Delta s) = x^T s + x^T \Delta s + s^T \Delta x + (\Delta x)^T \Delta s \leq x^T s.$$

$\square$

Of course, we need that the warm-start point generated by (19) is not only strictly feasible, but also in $\tilde{\mathcal{N}}_{-\infty}(\gamma, \beta)$ for some $\gamma \in]\,0, 1\,[$ and a $\beta \geq 1$. More precisely, we need that

$$\tilde{x}_i \tilde{s}_i \geq \gamma \tilde{\mu} \qquad i = 1, 2, \ldots, n$$

and

$$\|(\tilde{r}_b, \tilde{r}_c)\|_2 \leq \|(r_b^0, r_c^0)\|_2 \frac{\beta \tilde{\mu}}{\mu_0}$$

holds. Before taking a closer look at these inequalities, we consider the following lemma.

Lemma 4 *Under the assumptions of Theorem 3, define*

$$\theta := 1 - \left\| S^{-1} \Delta s \right\|_\infty . \tag{30}$$

Then,

$$\tilde{\mu} \geq \theta \mu. \tag{31}$$

Proof: Due to (26) and (30), we have $\theta > 0$. Moreover, (30) can be written as

$$\tilde{x}_i := x_i + \Delta x_i \geq \theta x_i \quad \text{and} \quad \tilde{s}_i := s_i + \Delta s_i \geq \theta s_i, \quad i = 1, 2, \ldots, n.$$

According to the proof of Theorem 3, we have that

$$\Delta x_i \Delta s_i \leq 0 \quad \text{forall} \quad i = 1, 2, \ldots, n.$$

On the one hand, assuming $\Delta x_i \geq 0$ leads immediately to $\tilde{x}_i \geq x_i$. With (30), we get

$$\tilde{x}_i \tilde{s}_i \geq x_i \tilde{s}_i \geq \theta x_i s_i.$$

On the other hand, assuming $\Delta s_i \geq 0$ we get in an analogous way

$$\tilde{x}_i \tilde{s}_i \geq \tilde{x}_i s_i \geq \theta x_i s_i.$$

Taking both cases together, we arrive at

$$\tilde{\mu} \geq \theta \mu.$$

$\square$

Corollary 5 *Let all assumptions of Theorem 3 hold and let a* $\theta \in]\,0,1\,[$ *be given with*

$$\theta \leq 1 - \left\| S^{-1} \Delta s \right\|_\infty . \tag{32}$$

Suppose now that

$$\left\| \begin{bmatrix} \Delta c \\ \Delta Qx \end{bmatrix} \right\|_\infty \leq \frac{1-\theta}{\|X^{-1}M^{-1}(T,T)\|_\infty}$$

holds. Then,

$$\tilde{w} \in \tilde{\Omega}^0$$

as well as $\tilde{\mu} \leq \mu$. *Moreover, if for some parameters* $\gamma_0 \in]\,0,1\,[$ *and* $\beta_0 \geq 1$ *we have* $w \in \mathcal{N}_{-\infty}(\gamma_0, \beta_0)$, *then*

$$\tilde{w} \in \tilde{\mathcal{N}}_{-\infty}(\theta\gamma_0, \beta_0/\theta).$$

Proof: The descent of the duality gap as well as $\tilde{w} \in \tilde{\Omega}^0$ follows with Theorem 3. Now let $w \in \mathcal{N}_{-\infty}(\gamma_0, \beta_0)$. Using (32) and (28), we get

$$\tilde{x}_i \tilde{s}_i \geq \theta x_i s_i \geq \theta\gamma\mu \geq \theta\gamma\tilde{\mu}$$

Moreover, due to (31), we have

$$\|(\tilde{r}_b, \tilde{r}_c)\|_2 = \|(r_b, r_c)\|_2 \leq \|(r_b^0, r_c^0)\|_2 \frac{\beta\mu}{\mu_0} \leq \|(r_b^0, r_c^0)\|_2 \frac{\frac{\beta_0}{\theta}\tilde{\mu}}{\mu_0}.$$

As a consequence, $\tilde{w} \in \tilde{\mathcal{N}}_{-\infty}(\theta\gamma_0, \beta_0/\theta)$. □

Up to now, we have just found criteria which make it possible to check if, starting with a prespecified perturbation, a given point (i. e. an iterate) can be used to construct a warm-start point. For the complexity analysis still to follow (see Subsection 4.3), we want to couple the size of a possible perturbation with the duality gap of a given point. We prepare the road with some concepts and some preliminary results.

The set of data instances for which there exists a strictly feasible point is denoted by

$$\mathcal{L} = \{(A, b, Q, c) \mid \exists\, x, \lambda, s : (x, s) > 0, Ax = b, -Qx + A^T\lambda + s = c\}.$$

Let the complement of $\mathcal{L}$ be denoted by $\mathcal{L}^{\mathcal{C}}$. The set $\mathcal{L}^{\mathcal{C}}$ contains all those data instances for which either (PQP) or (DQP) or both do not have a strictly feasible point. Denote the boundary between $\mathcal{L}$ and $\mathcal{L}^{\mathcal{C}}$ by

$$\mathcal{B} := \mathrm{cl}(\mathcal{L}) \cap \mathrm{cl}(\mathcal{L}^{\mathcal{C}}).$$

Due to $(0, 0, 0, 0) \in \mathcal{B}$ we have $\mathcal{B} \neq \emptyset$. A data instance $d \in \mathcal{B}$ is called *ill-posed*: an arbitrary small perturbation Δd can result in $d + \Delta d \in \mathcal{L}$ or $d + \Delta d \in \mathcal{L}^{\mathcal{C}}$. The *distance to ill-posedness* is defined by

$$\rho(d) := \inf\{\|\Delta d\|_2 \mid d + \Delta d \in \mathcal{B}\}.$$

At last, the *condition number* of a data instance d is defined by

$$\mathcal{C}(d) := \frac{\|d\|_2}{\rho(d)}$$

(resp. $\mathcal{C}(d) = \infty$ in case of $\rho(d) = 0$).

Remark 3 *Since for $\Delta d = -d$ we have $d + \Delta d \in \mathcal{B}$, we always have $\rho(d) \leq \|d\|_2$ and $\mathcal{C}(d) \geq 1$.*

A sufficient condition for the infeasibility of a convex-quadratic problem is given in the following lemma. There, as usual, all inequalities between vectors are to be understood componentwise.

Lemma 6 *Let there be given matrices $Q \in \mathrm{IR}^{n \times n}$ and $A \in \mathrm{IR}^{m \times n}$ as well as a vector $c \in \mathrm{IR}^n$. Then, the systems*

$$A^T \lambda < c + Qx \tag{33}$$

and

$$\begin{array}{rcl} Ax & = & 0, \\ x & \geq & 0, \\ c^T x + x^T Q x & \leq & 0, \\ x & \neq & 0 \end{array} \tag{34}$$

can not be solved simultaneously by vectors $x \in \mathrm{IR}^n$ and $\lambda \in \mathrm{IR}^m$.

Proof: Suppose that both systems can be solved. Then (33) is equivalent to

$$c > A^T \lambda - Qx.$$

Substituting this in the third block of (34), we get

$$0 \geq c^T x + x^T Q x > \lambda^T \underbrace{Ax}_{=0} - x^T Q^T x + x^T Q x = 0,$$

which is a contradiction. Therefore, the conclusion holds. □

Lemma 7 *Let there be given a data instance $d = (A, b, Q, c) \in \mathcal{L}$ and a point $w = (x, \lambda, s)$ with residuals (r_b, r_c). Define the data instance $\hat{d}$ by $\hat{d} := (A, b + r_b, Q, c + r_c)$. Then, it follows that*

$$\|x\|_2 \leq \frac{\max\left\{\|\hat{b}\|_2, |\hat{c}^T x + x^T Q x|, \|P\|_2\right\}}{\rho(\hat{d})} \tag{35}$$

holds for all positive semidefinite matrices $P \in \mathrm{IR}^{n \times n}$,

Proof: We will modify the idea of Nunez & Freund [30, p. 11f]. Let there be given the data instance d and the point $w = (x, \lambda, s)$ with $x \neq 0$. The residuals are given by $r_b = Ax - b$ and $r_c = -Qx + A^T\lambda + s - c$. Define

$$\hat{b} := b + r_b \quad \text{and} \quad \hat{c} := c + r_c.$$

Then, w is strictly feasible for the data instance $\hat{d} = (A, \hat{b}, Q, \hat{c})$. Now consider the perturbation $\Delta d = (\Delta A, 0, \Delta Q, \Delta c)$ defined by

$$\begin{aligned} \Delta A &:= -\hat{b} x^T \frac{1}{\|x\|_2^2}, \\ \Delta c &:= \frac{-|\hat{c}^T x + x^T Q x|}{\|x\|_2^2} x, \\ \Delta Q &:= -\frac{1}{\|x\|_2} P \end{aligned}$$

for some positive semidefinite matrix P. Then,

$$(A + \Delta A)x = Ax - \hat{b}\frac{x^T x}{\|x\|_2^2} = Ax - \hat{b} = 0$$

and

$$\begin{aligned} &(\hat{c} + \Delta c)^T x + x^T (Q + \Delta Q) x \\ &= \hat{c}^T x + \Delta c^T x + x^T Q x + x^T \Delta Q x \\ &= \hat{c}^T x + x^T Q x - |\hat{c}^T x + x^T Q x| - \frac{1}{\|x\|_2} x^T Q^* x \\ &\leq 0. \end{aligned}$$

Using Lemma 6, we see that there does not exist a λ with

$$(A + \Delta A)^T \lambda < \hat{c} + \Delta c + (Q + \Delta Q)x$$

and the dual problem to the data instance $\hat{d} + \Delta d$ is infeasible. Therefore,

$$\begin{aligned} \rho(\hat{d}) \leq \|\Delta d\|_2 &= \max\{\|\Delta A\|_2, \|\Delta c\|_2, \|\Delta Q\|_2\} \\ &= \frac{\max\{\|\hat{b}\|_2, |\hat{c}^T x + x^T Q x|, \|P\|_2\}}{\|x\|_2}, \end{aligned}$$

and the conclusion follows with

$$\|x\|_2 \leq \frac{\max\{\|\hat{b}\|_2, |\hat{c}^T x + x^T Q x|, \|P\|_2\}}{\rho(\hat{d})}. \tag{36}$$

□

Corollary 8 *Let there be given a data instance $d = (A, b, Q, c) \in \mathcal{L}$ and a point $w = (x, \lambda, s)$ with residuals (r_b, r_c). Define the data instance $\hat{d}$ by $\hat{d} := (A, b + r_b, Q, c + r_c)$. Then, it follows that*

$$\|x\|_2 \leq \mathcal{C}(\hat{d}) + \frac{|\hat{c}^T x + x^T Q x|}{\rho(\hat{d})}.$$

Proof: Use $P = 0$ in the last lemma. Then $\Delta Q = 0$. With (36) it follows that

$$\begin{aligned} \|x\|_2 &\leq \frac{\|\hat{b}\|_2 + |\hat{c}^T x + x^T Q x|}{\rho(\hat{d})} \\ &\leq \frac{\|\hat{d}\|_2}{\rho(\hat{d})} + \frac{|\hat{c}^T x + x^T Q x|}{\rho(\hat{d})} \\ &= \mathcal{C}(\hat{d}) + \frac{|\hat{c}^T x + x^T Q x|}{\rho(\hat{d})}. \end{aligned}$$

□

Corollary 9 *Let there be given a data instance $d = (A, b, Q, c) \in \mathcal{L}$ and a point $w = (x, \lambda, s)$ strictly feasible for d. Then,*

$$\|x\|_2 \leq \frac{\max\left\{\|b\|_2, |c^T x + x^T Q x|\right\}}{\rho(d)} \leq \mathcal{C}(d) + \frac{|c^T x + x^T Q x|}{\rho(d)}.$$

Now we are almost ready to couple the size of a given perturbation with a given duality gap. We just need one more technical result.

Lemma 10 *Let there be given symmetric and positive definite matrices* $M_j \in \mathbb{R}^{n\times n}$ $(j = 1, \ldots, p)$*. For* $\alpha \in \mathbb{R}^p$*,* $\alpha \geq 0$*, define* $M(\alpha) := \sum_{j=1}^{p} \alpha_j M_j$*. If* A *has full row rank, then*

$$\chi(A) := \sup_{\substack{\alpha\geq 0:\\ M(\alpha)\, p.\, d.}} \|(A(M(\alpha))^{-1}A^T)^{-1}A(M(\alpha))^{-1}\|_\infty < \infty$$

holds. Here, "p. d." stands for positive definite.

Proof: This follows directly with inequality (5.6) from the proof of Corollary 5.2 in Forsgren and Sporre [15]. Note that this Corollary 5.2 is based directly on Theorem 5.1 in the same paper. □

To use the result above, we assume in what follows that we have given a data instance $d = (A, b, Q, c)$ and a perturbation of the primal objective function, i. e. a data perturbation of the form $\Delta d = (0, 0, \Delta Q, \Delta c)$. Furthermore, let us assume that we have given symmetric positive definite matrices $Q_1, Q_2, \ldots, Q_p$ and a vector $\alpha \in \mathbb{R}^p$, $\alpha \geq 0$ such that

$$Q + \Delta Q = \sum_{i=1}^{p} \alpha_i Q_i.$$

We will see in Section 5 that these additional assumptions fit perfectly into the framework of multicriteria optimization.

Theorem 11 *Let there be given a data instance* $d = (A, b, Q, c) \in \mathcal{L}$ *and a perturbation* $\Delta d = (0, 0, \Delta Q, \Delta c)$*. Let* $w = (x, \lambda, s)$ *be a point with* $(x, s) > 0$ *and residuals* (r_b, r_c) *(with respect to* d*). Define the data instance* $\hat{d}$ *by* $\hat{d} := (A, b + r_b, Q, c + r_c)$*. Let* A *have full row rank and define*

$$\psi(A) := 1 + \|A^T\|_\infty \chi(A).$$

Suppose that $w \in \mathcal{N}_\infty(\gamma, \beta)$ *for some* $\beta, \gamma > 0$*. Then,*

$$\frac{1}{\|X^{-1}M^{-1}(T,T)\|_\infty} \geq \frac{\gamma\mu}{2n^{1/2}\left(\mathcal{C}(\hat{d}) + \frac{|\hat{c}^T x + x^T Q x|}{\rho(\hat{d})}\right)\psi(A)}. \tag{37}$$

Proof: The proof follows exactly the lines of a similar result in Fliege & Heseler [13, p. 14f] and is therefore omitted here. □

We are now ready to state the main result, connecting the size of the duality gap with the size of a perturbation and vice versa.

Define

$$\|\Delta d\|_\infty := \max\{\|\Delta c\|_\infty, \|\Delta Q\|_\infty\} \tag{38}$$

as well as

$$\xi := 1 + \frac{|\hat{c}^T x + x^T Q x|}{\|\tilde{d}\|_2} \geq 1.$$

Then,

$$\|x\|_2 \leq \xi \mathcal{C}(\hat{d}) = \mathcal{C}(\hat{d}) + \mathcal{C}(\hat{d}) \frac{|\hat{c}^T x + x^T Q x|}{\|\hat{d}\|_2} = \mathcal{C}(\hat{d}) + \frac{|\hat{c}^T x + x^T Q x|}{\rho(\hat{d})}. \tag{39}$$

Theorem 12 *Let there be given parameters γ and γ_0 with $0 < \gamma < \gamma_0 < 1$ and define $\theta := \gamma/\gamma_0$. Define ξ as above. Let w and $\tilde{w}$ as well as β be as in Corollary 5. If*

$$\mu \geq \frac{2\|\Delta d\|_\infty}{\gamma_0(1-\theta)} \xi n^{1/2} \mathcal{C}(\hat{d}) \left(\mathcal{C}(\hat{d}) + \frac{|\hat{c}^T x + x^T Q x|}{\rho(\hat{d})} \right) \psi(A).$$

holds, then $\tilde{w} \in \tilde{\mathcal{N}}_{-\infty}(\gamma, \beta)$.

Proof: See Fliege & Heseler [13, p. 15f]. □

4.3 Complexity

Let there be given a problem in form of a data instance d as well as a primal-dual starting point (x^0, λ^0, s^0), possibly infeasible for d. Denote, as usual, the residuals by (r_b^0, r_c^0) and let μ_0 be the duality measure at the starting point. Moreover, let $\varepsilon > 0$ be given. Suppose furthermore that we have already solved that problem with algorithm QIP by computing iterates $w^k = (x^k, \lambda^k, s^k)$, $k = 1, 2, 3, \ldots$. Our complexity analysis has shown that

$$\mu_k \leq \|d\|_2 \varepsilon$$

holds for

$$k \geq K = \mathcal{O}\left(n^2 \log \frac{\mu_0}{\|d\|_2 \varepsilon}\right).$$

Now suppose that we perturb our problem d to $d + \Delta d$ and construct a warm-start point by our warm-start strategy out of the iterate w^j. This warm-start point is then used by algorithm QIP to solve the perturbed problem. Clearly, after

$$k \geq K_{\text{warm}} = \mathcal{O}\left(n^2 \log \frac{\mu_j}{\|d + \Delta d\|_2 \varepsilon}\right)$$

iterations we have that

$$\tilde{\mu}_k \leq \|d + \Delta d\|_2 \varepsilon$$

holds. Here, $\tilde{\mu}_k$ are the duality measures for the perturbed problem, as computed by algorithm QIP. Furthermore, if $\|\Delta d\|_2 \leq \|d\|_2/2$ holds, we can use the estimate

$$\frac{1}{\|d + \Delta d\|_2} \leq \frac{1}{\|d\|_2 - \|\Delta d\|_2} \leq \frac{2}{\|d\|_2}$$

to conclude

$$K_{\text{warm}} = \mathcal{O}\left(n^2 \log \frac{\mu_j}{\|d\|_2 \varepsilon}\right).$$

5 An Application: Multicriteria Optimization

In this section, we give a short introduction to multicriteria optimization. We follow roughly the chain of arguments presented in [13] and repeat the main points for the sake of completeness here.

5.1 The Problem

Let there be given $p > 1$ convex quadratic objective functions of the form

$$f_i(x) = \frac{1}{2} x^T Q_i x + c_i^T x, \qquad i = 1, \dots, p, \tag{40}$$

with positive semidefinite matrices $Q_i \in \mathbb{R}^{n \times n}$ and vectors $c_i \in \mathbb{R}^n$ for all i. Moreover, let

$$G := \{x \in \mathbb{R}^n \mid Ax = b\,,\ x \geq 0\} \tag{41}$$

be the set of feasible points. We are interested in minimizing simultaneously the functions

$$f_1, \dots, f_p : G \longrightarrow \mathbb{R} \tag{42}$$

on the set G in a sense specified as follows. The element $y^* \in f(G)$ is called *efficient*, if and only if there is no other $y \in f(G)$ with

$$y_i \leq y_i^* \quad \forall\, i \in \{1, 2, \dots, p\}$$

and

$$y_k < y_k^* \quad \text{for at least one } k \in \{1, 2, \ldots, p\}.$$

The set of all efficient points of the set $f(G)$ is called the *efficient set*, $E(f(G))$. Now, define the function $f : G \longrightarrow \mathbb{R}^p$ by $f = (f_1, \ldots, f_p)^T$. With the definition of efficiency as above, it becomes clear that in multicriteria optimization we are in search for the whole set $E(f(G))$ and, obviously, for the corresponding set of optimal decision variables $f^{-1}(E(f(G)))$. For typical examples for this type of problem we refer to, e. g., [13]. Note that two efficient points $f(x^{(1)}), f(x^{(2)}) \in E(f(G))$ $(x^{(1)}, x^{(2)} \in G)$ with $f(x^{(1)}) \neq f(x^{(2)})$ are incomparable to each other with respect to the order defined above. Therefore, just one efficient point can not capture the possible optimal alternatives we face when solving a multicriteria optimization problem. This clearly shows that human decision makers need information about the whole set $E(f(G))$.

5.2 Scalarization

It is well-known that we can find a point close to $E(f(G))$ of the problem specified by (42) by solving the single-objective optimization problem

$$\begin{array}{ll} \min & \omega^T f(x) \\ \text{s. t.} & x \in G, \end{array} \tag{43}$$

with ω an arbitrary weight vector from the set

$$Z := \left\{ \omega \in \mathbb{R}^p \;\middle|\; \sum_{i=1}^{p} \omega_i = 1,\ \omega_i > 0\ \forall\, i \in \{1, 2, \ldots, p\} \right\}. \tag{44}$$

This approach is often called *scalarization*. (For a discussion of this and other scalarization techniques see e. g. [17, 22, 20, 12].) Indeed, defining the set of *properly efficient points* P by

$$P(f(G)) := \left\{ f(x^*) \;\middle|\; \omega \in Z,\ x^* \in G,\ f(x^*) = \min_{x \in G} \omega^T f(x) \right\},$$

it can be shown [16, 28] that

$$P(f(G)) \subseteq E(f(G)) \subseteq \mathrm{cl}(P(f(G))) \tag{45}$$

holds. Here, $\mathrm{cl}(\cdot)$ is the closure operator. In fact, this result holds for arbitrary functions $f : G \longrightarrow \mathbb{R}^p$ as long as $f(G) + \mathbb{R}^p_+$ is closed and convex. Since we can not distinguish numerically between a set and its closure, we can therefore replace E by P in all applications involving convex functions. Turning our attention to (42), (41), and (40), we see that we have to consider several scalar problems of the form

$$\begin{array}{ll} \min & \frac{1}{2}x^T Q x + c^T x \\ \text{s. t.} & Ax = b, \\ & x \geq 0, \end{array} \tag{46}$$

where $Q = \sum_{i=1}^{p} \omega_i Q_i$, $c = \sum_{i=1}^{p} \omega_i c_i$, and $\omega = (z_1, \ldots, \omega_p)^T \in Z$ is a given parameter or weight vector.

As a consequence of the discussion above, we are able to approximate the set $E(f(G))$ as well as $f^{-1}(E(f(G)))$ by solving optimization problems of the form (46). These ersatz problems are defined by choosing different weights ω, see (43).

The basic idea is now as follows. Our aim is to compute a discrete approximation of the set of efficient points. We have to solve a standard scalar optimization problem for each efficient point we want to compute. The different optimization problems we have to consider can be viewed as perturbations of each other, with vectors of weights $\omega \in Z$ serving as parameters defining the perturbations. We propose to use an adaptive discretization technique for the set of weights Z. Basically, we want to use more parameters in those regions of the parameter space where weight vectors which are close together result in efficient points whose images in the image space $\mathbb{R}^p$ are far apart from each other. But in contrast to [13], where efficient points were calculated first and then a new weight was chosen adaptively by using information about the last optimal function values computed, we will now introduce more parameters (i. e. more scalar problems) during the solution process of other scalar problems, before these other problems are completely solved.

Furthermore, to save work when computing the new efficient points (i. e. when solving the new optimization problems), we propose to use a warm-start strategy. With such a strategy, points from the iteration history of scalar problems already solved are used as starting points for the optimization problems currently under consideration.

6 The EffTree-Algorithm

6.1 The Basic Idea

When solving a multicriteria optimization problem, we have to solve many standard scalar problems, each of them a perturbation of each other one. Of course, we can solve one scalarized problem first, use one of the iterates computed to generate a warm-start point for the next scalarized problem considered, etc. This is basically the idea outlined in Fliege & Heseler [13]. However, in the algorithm presented below we want to do warm-starts as early as possible, thereby considering many scalarized problems simultaneously. In this way, we will be able to generate an approximation of the set of efficient points even if none of the scalar problems considered is solved up to a prespecified accuracy, yet. (I. e. even if all duality gaps of all scalar problems considered are still rather large.)

Figure 1 (p. 23) illustrates the idea for bicriteria problems. The basic idea for bicriteria problems is explained below.

Suppose we start with a prespecified scalarization parameter ω, defining a problem with data instance denoted by d_ω. Choose a starting point $(x^0, \lambda^0, s^0)_\omega$ for algorithm QIP. In Figure 1, the image of this point under two real-valued objective functions is shown in the upper right hand corner. Executing one step with algorithm QIP results in the point $(x^1, \lambda^1, s^1)_\omega$. Next, we choose two new scalarization parameters ω_l and ω_r and define the corresponding data instances d_{ω_l} and d_{ω_r}. These two data instances can be seen as perturbations of d_ω simply by noting $d_{\omega_l} = d_\omega + (d_{\omega_l} - d_\omega)$ resp. $d_{\omega_r} = d_\omega + (d_{\omega_r} - d_\omega)$. Accordingly, we can try to construct warm-start points based on the information given by $(x^1, \lambda^1, s^1)_\omega$. One additional step from each of these warm-start points with algorithm QIP leads to $(x^2, \lambda^2, s^2)_{\omega_l}$ and $(x^2, \lambda^2, s^2)_{\omega_r}$, see Figure 1. Moreover, one additional step with algorithm QIP for the problem with parameter ω from $(x^1, \lambda^1, s^1)_\omega$ leads to the point $(x^2, \lambda^2, s^2)_\omega$. This scheme can now be applied recursively for each of the three points $(x^2, \lambda^2, s^2)_{\omega_l}, (x^2, \lambda^2, s^2)_{\omega_l}, (x^2, \lambda^2, s^2)_{\omega_r}$, until the images of neighboring iterates are closer to each other than a prespecified distance. A precise statement of the algorithm can be found in the next subsection.

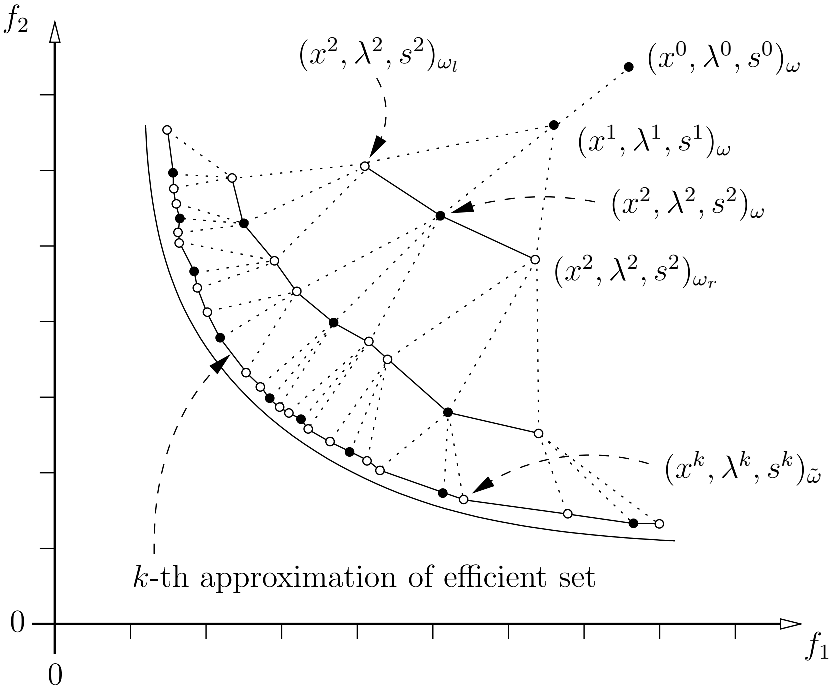


Figure 1: Approximation of the image of the set of efficient points by early warm-starts. An explanation is given in Section 6.1.

6.2 The Algorithm EffTree

Let us introduce two scalar values $\varepsilon, \delta > 0$ as measures of accuracy of an approximation to the the set of solutions of our given bicriteria problem. Both measures of accuracy will be used on points in the image space of the problem. For this problem, we want to calculate a number of points in the image space (as well as their preimages in the decision space) which are "close" to the image of the set of efficient points in the sense that each point is an ε-solution to a scalar problem. (Or, even more precisely, the dual gap of each scalar problem considered should be at most ε.) Moreover, for bicriteria problems, the distance between two image points should be at most δ.

Each scalar problem considered is defined by a scalarization vector ω. We denote by d_ω the corresponding data instance, by $(x^k, \lambda^k, s^k)_\omega$ the kth iterate of our algorithm for this data instance, and by $r^k_{b,\omega}, r^k_{c,\omega}$ the correspondig residuals. For a given weight vector ω and the corresponding data instance d_ω, a perturbation $\Delta\omega$ of this weight vector induces the data instance $d_{\omega+\Delta\omega}$. The corresponding perturbations of the objective function has the form

$$\Delta Q := \Delta\omega(Q_1 - Q_2) \quad \text{and} \quad \Delta c := \Delta\omega(c_1 - c_2)\,.$$

To simplify notation, we will identify the data instance d_ω with the weight vector ω.

To measure distances between points in the image space, we will make use of the function

$$\begin{array}{rccl} \varphi_k: & [\,0,1\,] & \longrightarrow & \mathsf{IR}^2 \\ & \omega & \longmapsto & f(x^k_\omega)\,, \end{array}$$

where x^k_ω is the primal variable of an iterate $(x^k, \lambda^k, s^k)_\omega$.

Throughout the algorithm, we use the Euclidean norm to measure distances between image points in the IR^2. In what follows, the set $\mathcal{W}$ will contain all those data instances (i. e. weights) which have already been considered up to now. This set depends on the iteration index k, which will be suppressed in what follows for ease of notation. The set $\widehat{\mathcal{W}}$ will contain those data instances which have already been checked as possible data instances to generate warm-start points. The data instances neighboring to a given one, say $\omega \in \mathcal{W}$, will be denoted by

$$\begin{array}{rcl} \omega_l & := & \max\{\omega_l \in \mathcal{W} \cup \{0;1\} \mid \omega_l < \omega\}\,, \\ \omega_r & := & \min\{\omega_r \in \mathcal{W} \cup \{0;1\} \mid \omega_r > \omega\}\,. \end{array}$$

Moreover, we will make use of the notation $\mathcal{W}_\omega := \{\omega_l, \omega_r\} \subseteq \mathcal{W}$.

After the generation of a warm-start point, algorithm QIP will be called to compute i_k ordinary interior-point steps. Only after that is k increased (and i_k is updated to i_{k+1}). As a consequence, the iteration counter k of algorithm EffTree is *not* equal to the current iteration depth of a point $(x^k, \lambda^k, s^k)_\omega$, i. e. it is *not* the case that one has arrived at $(x^k, \lambda^k, s^k)_\omega$ after solving k linear systems of equalities by making either a warm-start step or by making a single step with algorithm QIP. Instead, the total number of steps done by algorithm QIP to arrive at $(x^k, \lambda^k, s^k)_\omega$ is given by $\sum_{j=0}^{k} i_k$. In actual examples, we will use a rather simple sequence of i_k's. Indeed, we will simply choose i_0 fixed as well as i_k constant for $k > 0$. More details on this can be found in Section 7.

Table 2 (p. 25) gives an overview of the parameters and variables used in the algorithm. Note that usually only ϵ_μ, $\varepsilon_{\text{infeas}}$, and δ have to be specified by a user. We are now able to state the algorithm more formally, and this is done in Table 3, p. 26.

ε_μ	max. allowed dual gap
$\varepsilon_{\text{infeas}}$	max. allowed infeasibility
δ	max. dist. between points in images space
k	outer iteration counter
i_k	number if inner iterations in step k
ω	parameter (weight vector) for the data instance d_ω
$\Delta\omega$	perturbation of ω
$\Delta\omega_{\text{mult}}$	backtracking parameter for computing $\Delta\omega$
$\mathcal{W}$	set of all data instances considered
$\mathcal{W}_\omega$	set of all data instances neighboring ω
β_ω	parameter for the set $\mathcal{N}_{-\infty}(\gamma, \beta)$ used for problem ω
γ_ω	parameter for the set $\mathcal{N}_{-\infty}(\gamma, \beta)$ used for problem ω
θ	multiplicator for β_ω and γ_ω
σ_ω	centering parameter used for problem ω
ζ	parameter for starting point
e	$= (1, \dots, 1)^T \in \mathrm{IR}^n$

Table 2: Variables and Parameters used in Algorithm EffTree

The algorithm stops as soon as we have calculated weights $0 = \omega_1 < \omega_2 < \cdots < \omega_N = 1$ and corresponding primal-dual points $(x_{\omega_i}, \lambda_{\omega_i}, s_{\omega_i})$ $(i = 0, \dots, N)$ with duality measure less than ε and infeasibility less than ε, such that in the image space IR^2 of our bicriteria problem, two consecutive solutions $x_{\omega_i}, x_{\omega_{i+1}}$ have a distance from each other of less than δ $(i = 0, \dots, N-1)$.

As it can be seen, one of the crucial parts of EffTree is the subroutine for computing warm-start points, CompWarmStart (cmp. Step S2 of EffTree). This routine is depicted in Table 4, p. 27. Note that the system of linear equations (47) in subroutine CompWarmStart corresponds to the system (19) from Section 3.

We still have to proof that the algorithm is well defined and works correctly. This is done as follows. First, observe that the set $\mathcal{W}$ always contains a finite number of data instances (weight vectors). Therefore, step S3 is well defined. After each execution of one loop step of the while loop in step S2, an

Algorithm EffTree

(S0) // Initialization //

Choose ε_μ, $\varepsilon_{\text{infeas}}$, $\delta > 0$, $\omega_{\text{init}} \in [\,0,1\,]$, $i_k \in \mathbb{N}$ $(k = 0,1,2,\ldots)$,
$\zeta > 0$, $\beta_{\text{init}} \geq 1$, $\gamma_{\text{init}} \in]0,1[$, $\sigma_{\text{init}} \in]\,0,1/2\,[$.

Set $k := 0$, $(x^0, \lambda^0, s^0)_{\omega_{\text{init}}} := \zeta(e,0,e)$, $\mathcal{W} := \{\omega_{\text{init}}\}$.

(S1) // Iterations //

For all $\omega \in \mathcal{W}$:

- $\triangleright$ Compute $(x^{k+1}, \lambda^{k+1}, s^{k+1})_\omega$ by executing i_k iterations with algorithm QIP, using the parameters β_ω, γ_ω, and σ_ω, starting from the point $(x^k, \lambda^k, s^k)_\omega$

end (for all)

(S2) // Compute warm-start points //

Set $\widehat{\mathcal{W}} := \emptyset$.

While $\mathcal{W} \setminus \widehat{\mathcal{W}} \neq \emptyset$:

Choose $\omega \in \mathcal{W} \setminus \widehat{\mathcal{W}}$,
$\omega_l := \max\{\omega_l \in \mathcal{W} \cup \{0,1\} \mid \omega_l < \omega\}$,
$\omega_r := \min\{\omega_r \in \mathcal{W} \cup \{0,1\} \mid \omega_r > \omega\}$,
for all $\hat{\omega} \in \mathcal{W}_\omega := \{\omega_l, \omega_r\}$:
if $\|\varphi_k(\omega) - \varphi_k(\hat{\omega})\|_2 > \delta$

- $\triangleright$ Call routine CompWarmStart to calculate $\tilde{\omega}$, $\gamma_{\tilde{\omega}}$, $\beta_{\tilde{\omega}}$ as well as $(x^{k+1}, \lambda^{k+1}, s^{k+1})_{\tilde{\omega}}$.
- $\triangleright$ Set $\mathcal{W} := \mathcal{W} \cup \{\tilde{\omega}\}$, $\widehat{\mathcal{W}} := \widehat{\mathcal{W}} \cup \{\tilde{\omega}\}$.

end (if)
end (for all)
Set $\widehat{\mathcal{W}} := \widehat{\mathcal{W}} \cup \{\omega\}$.

end (while)

(S3) // Check stopping criterion //

if $\max\{\mu_\omega \mid \omega \in \mathcal{W}\} < \varepsilon_\mu \;\wedge\; \max\{\|r^k_{b,\omega}\|_2, \|r^k_{c,\omega}\|_2 \mid \omega \in \mathcal{W}\} < \varepsilon_{\text{infeas}}$
$\wedge\; \max\{\|\varphi_{k+1}(\omega) - \varphi_{k+1}(\hat{\omega})\|_2 \mid \omega \in \mathcal{W}, \hat{\omega} \in \mathcal{W}_\omega\} < \delta$:
$\triangleright$ STOP.

else:
$\triangleright$ set $k := k+1$, GOTO (S1). $\diamond$

Table 3: Algorithm EffTree for approximating the solution set of a bicriteria convex quadratic optimization problem subject to standard linear equality and inequality constraints.

Subroutine CompWarmStart

(S0) Choose $\Delta\omega_{\text{mult}} \in]\,0,1\,[$, $\theta \in]\,0,1\,[$, set $\Delta\omega := \hat{\omega} - \omega$ and `found` := false.

(S1) While not `found` and $\Delta\omega > \varepsilon$:

Solve

$$\begin{bmatrix} -(Q+\Delta Q) & A^T & I \\ A & 0 & 0 \\ S_\omega^k & 0 & X_\omega^k \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta\lambda \\ \Delta s \end{bmatrix} = \begin{bmatrix} \Delta c + \Delta Q x \\ 0 \\ 0 \end{bmatrix} \tag{47}$$

and define

$$(x^{k+1}, \lambda^{k+1}, s^{k+1})_{\tilde{\omega}} := (x^k, \lambda^k, s^k)_\omega + (\Delta x, \Delta\lambda, \Delta s).$$

If $\|S^k \Delta s\|_\infty < 1$:

▷ define $\tilde{\omega} := \omega + \Delta\omega$, $\gamma_{\tilde{\omega}} := \theta\gamma_\omega$, and $\beta_{\tilde{\omega}} := \beta_\omega/\theta$

if $(x^{k+1}, \lambda^{k+1}, s^{k+1})_{\tilde{\omega}} \in \tilde{\mathcal{N}}_{-\infty}(\gamma_{\tilde{\omega}}, \beta_{\tilde{\omega}})$

▷ `found` := true

end (if)

else

▷ $\Delta\omega := \Delta\omega_{\text{mult}}\Delta\omega$

end (if)

(S4) if not `found`

set $\tilde{\omega} := \omega + |\omega - \hat{\omega}|/2$ and compute $(x^{k+1}, \lambda^{k+1}, s^{k+1})_{\tilde{\omega}}$ by making a *cold-start*:

use the starting point $(x^0, \lambda^0, s^0)_{\tilde{\omega}} := \zeta(e, 0, e)$ as well as the parameters $\gamma_{\text{init}}, \beta_{\text{init}}, \sigma_{\text{init}}$ to execute $\sum_{j=0}^k i_j$ iterations with algorithm QIP.

end (if)

Table 4: Subroutine CompWarmStart, used in Step (S2) of Algorithm EffTree.

element $\omega \in \mathcal{W} \setminus \widehat{\mathcal{W}}$ is put into $\widehat{\mathcal{W}}$. Moreover, additional problems $\tilde{\omega}$ generated by a warm-start are put into $\mathcal{W}$ as well as $\widehat{\mathcal{W}}$. Therefore, the while-loop finishes after at most $|\mathcal{W}|$ loops. Moreover, warm-start points generated by (47) in Step S2 do not increase the duality gap (cmp. Section 3). Therefore, although in each step of the main loop of the algorithm, more problems might be are added to the set $\mathcal{W}$, the maximum duality gap is decreased by performing at least i_k steps of algorithm QIP for each of the problems considered. As a consequence, $\max\{\mu_\omega \mid \omega \in \mathcal{W}\} \longrightarrow 0$ for $k \longrightarrow \infty$ and the first part of the stopping criterion is fulfilled after finitely many steps. (Actually, the maximum of the duality gaps converges Q-linearly to 0, see Theorem 1.) The second stopping criterion is fulfilled after finitely many steps as long as the optimal value function

$$\omega \mapsto \inf \left\{ \sum_{i=1}^{p} \omega_i c_i^T x + \frac{1}{2} x^T \sum_{i=1}^{p} \omega_i Q_i x \;\middle|\; Ax = b,\ x \geq 0 \right\}$$

is continuous. This holds, e. g., as soon as all Q_i are positive definite.

Remark 4 If a user does not want to choose $\delta > 0$ a priori, he/she might be able to specify the number of points to be used in the approximation of the set of efficient points, instead. If such a number M is given, an appropriate albeit rough initialization for δ would be

$$\delta := \frac{\sqrt{2}\|\varphi(0) - \varphi(1)\|_2}{M}. \tag{48}$$

Necessary for this initialization are, of course, optimal function values of the two "extremal" problems with objective functions f_1 and f_2. These can be easily calculated by algorithm QIP.

Algorithm EffTree has been implemented in MATLAB Version 6.5 Release 13. It now forms the core of EfflinGUI, a complete decision support system for multicriteria optimization problems, see Figure 2 (p. 29) and 3 (p. 30).

7 Numerical Results

7.1 Data & Parameters

The algorithm was tested on various problems from power plant optimization. Details on this problems can be found in [13]. Overall, 1440 different

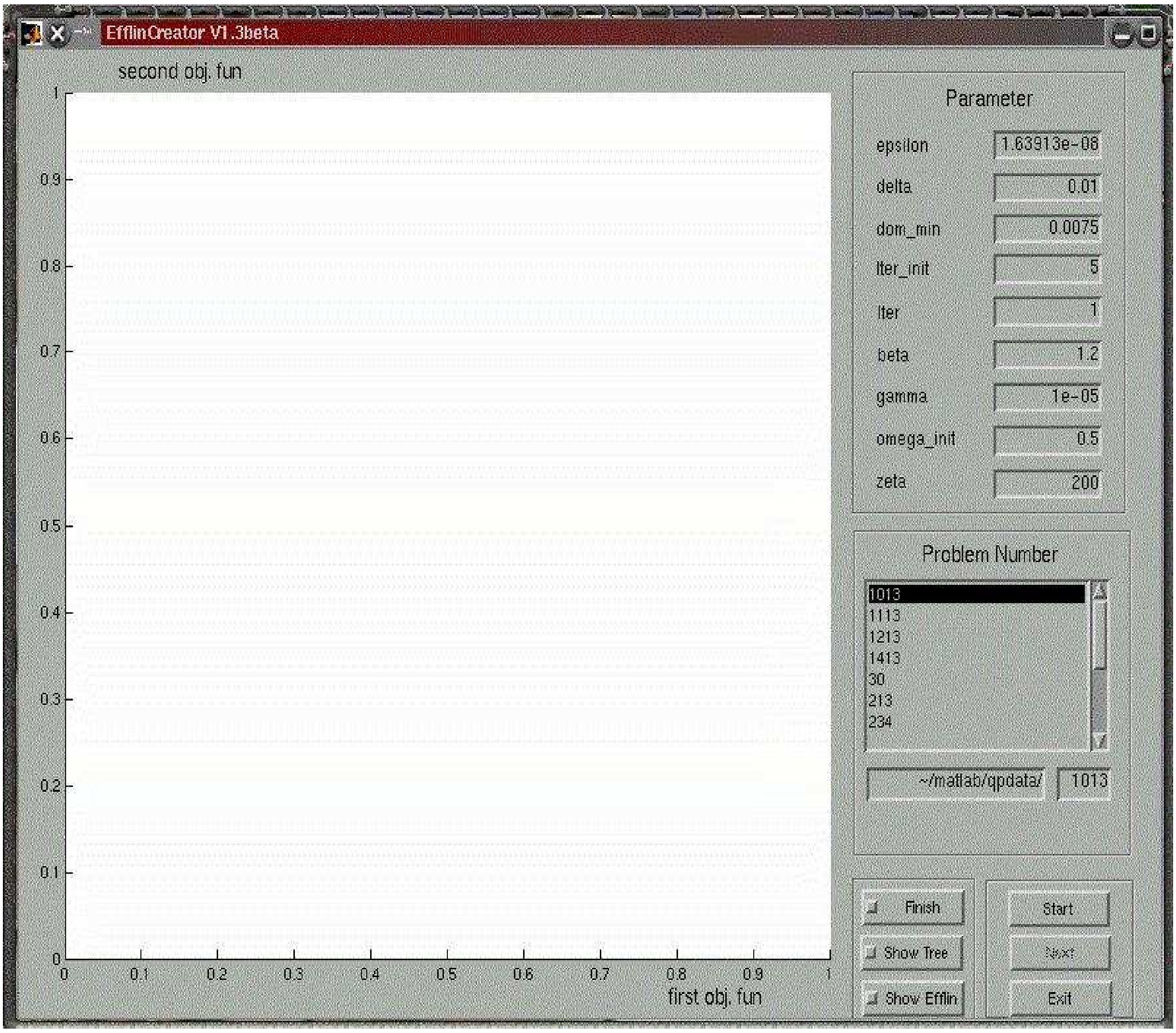


Figure 2: The start screen of EfflinGUI, a decision support system for solving multicriteria optimization problems, based on the algorithm presented in this paper.

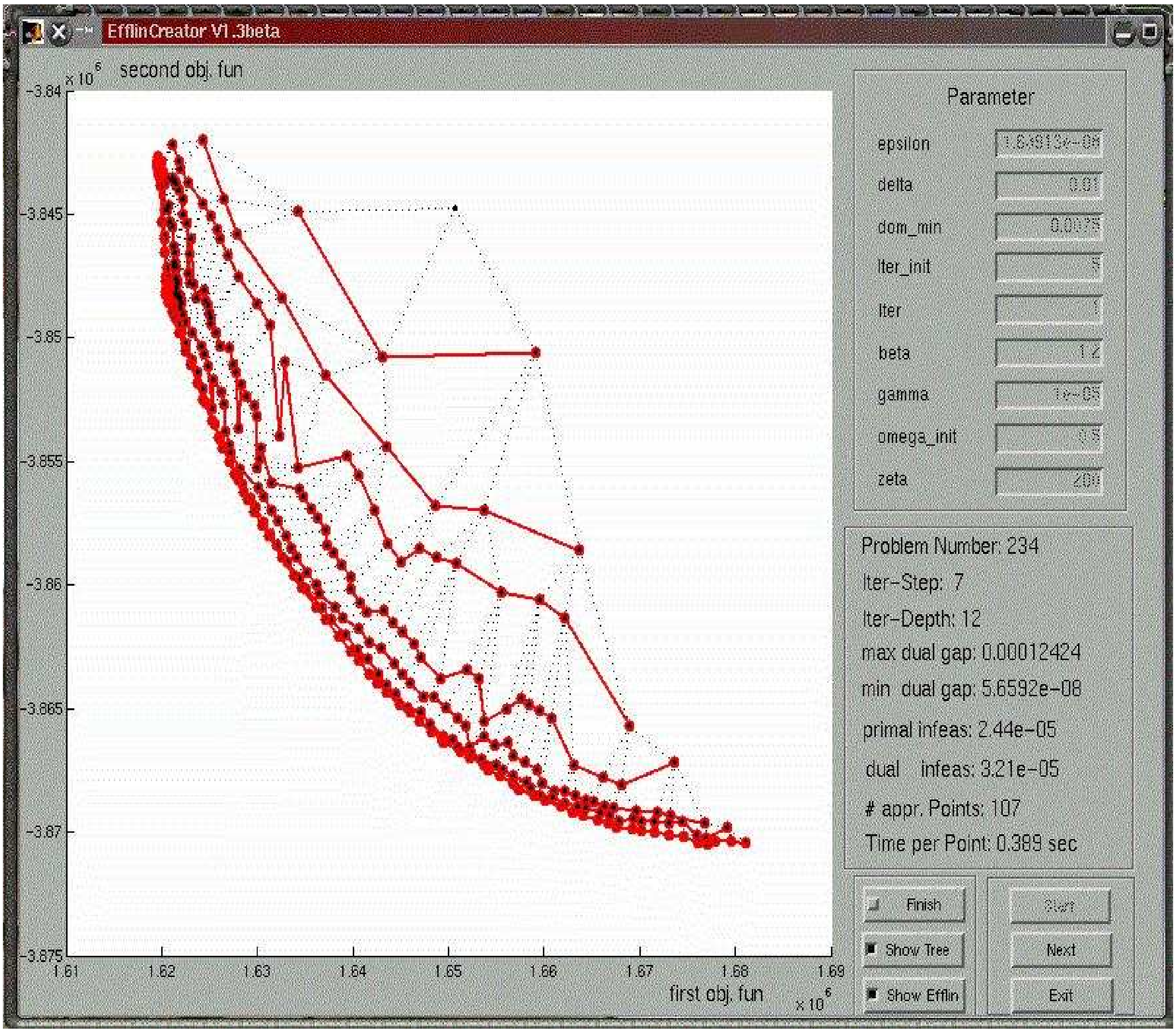


Figure 3: With EfflinGUI it is possible to display the data structure as well as the various approximations of the efficient set found while solving the multicriteria problem.

bicriteria optimization problems (one for each minute of a day) are specified in our test suite.

In what follows, we used the set of parameters

$$
\begin{array}{rcl|rcl}
\varepsilon_\mu = \varepsilon_{\text{infeas}} & = & \sqrt{\texttt{eps}} & \omega_{\text{init}} & = & 0.5 \\
M & = & 1000 & \zeta & = & 400 \\
\Delta\omega_{\text{mult}} & = & 0.8 & \beta_{\text{init}} & = & 1.2 \\
i_0 & = & 5 & \gamma_{\text{init}} & = & 10^{-4} \\
i_k & = & 1 & \sigma_{\text{init}} & = & 10^{-1}.
\end{array}
$$

and δ was defined for each problem according to (48). Here, `eps` is the machine precision of the hardware platform. The centering parameter σ_k was updated using the well-known rule

$$\sigma_k := \left(\frac{\mu_k}{\mu_{k-1}}\right)^3,$$

cmp. [6].

All tests have been made on an Intel Pentium 4, 2.66GHz and 1024MB using Linux SuSE 9.0Pro as the operating system.

7.2 Results

Several experiments, documented in full by Heermann [19], showed that the combination of parameters $(i_0, i_k) = (5, 1)$ seems to be the most promising. The total performance of the algorithm is relatively stable with respect to $\Delta\omega_{\text{mult}}$. We found that, as long as $\Delta\omega_{\text{mult}} \in [\,0.5, 0.9\,]$, the warm-start search performs rather robust.

Table 5 (p. 32) shows the iteration log of algorithm QIP with respect to the most important values for a typical single-criteria problem encountered during our test. More precisely, the single-criteria problem considered is the multicriteria problem no. 890 with weighting parameters $\omega_1 = 1/2 = 1-\omega_1 = \omega_2$. On this problem, the algorithm stops after 25 iterations. As it can be seen, the maximal duality gap converges superlinearly to 0.

Figure 4 (p. 34) shows the results achieved on the multicriteria problems considered. On average, 1388 points per problem were generated and on average 176.94 seconds were used to solve one multicriteria problem to the accuracy specified above, which means that on average 0.1275 seconds were needed to solve one scalarized problem. This compares favorably with the

k	μ_k	$\|r_b^k\|_2$	$\|r_c^k\|_2$
0	1.600000e+05	7.407255e+03	5.971372e+03
1	8.579048e+04	3.719395e+03	2.998397e+03
2	4.129213e+04	1.514895e+03	1.221235e+03
3	2.016191e+04	6.170106e+02	4.974042e+02
4	8.265110e+03	1.816479e+02	1.464358e+02
5	2.273101e+03	2.906367e+01	2.342973e+01
6	4.783928e+02	4.650187e+00	3.748756e+00
7	1.475387e+02	1.369015e+00	1.103634e+00
8	4.675247e+01	4.030380e-01	3.249098e-01
9	1.490842e+01	1.186544e-01	9.565345e-02
10	2.862102e+00	1.898470e-02	1.530455e-02
11	5.231097e-01	3.037550e-03	2.448730e-03
12	9.540502e-02	4.860103e-04	3.918021e-04
13	1.662533e-02	7.776913e-05	6.269642e-05
14	2.768100e-03	1.244372e-05	1.003240e-05
15	4.530403e-04	1.990987e-06	1.610700e-06
16	7.387028e-05	3.258166e-07	2.620483e-07
17	1.204321e-05	5.099140e-08	4.109352e-08
18	1.961090e-06	1.372712e-08	1.008087e-08
19	3.274416e-07	7.795232e-09	2.087891e-09
20	5.109085e-08	4.216811e-09	1.416855e-09
21	1.078810e-08	2.864808e-09	1.406613e-09

Table 5: Duality gap and primal and dual infeasibility during the course of algorithm QIP for the single-criteria problem constructed out of the bicriteria problem # 890 by choosing $\omega_1 = 1/2$.

standard strategy of using just the same starting point over and over again, until the prespecified accuracy is met. With this strategy, on average 327 seconds were needed to solve one multicriteria optimization problem. Moreover, only in four cases out of 1440 was a cold start necessary at all. In three of them, just one cold-start needed to be made, in the remaining case, two of them were necessary.

The average number of iteration steps executed in algorithm QIP (called within EffTree) per computed efficient point was 9.12, while the minimum resp. maximum number of iterations was 5.71 resp. 11.27. Note that these numbers do not include the warm-start steps made, which tend to decrease the duality gap even more.

7.3 The Perturbation Size during the Algorithm

To measure the size of the maximal possible perturbation allowed during the course of the algorithm, we conducted the following experiment. For fixed values of $\Delta\omega_{\text{mult}}$, we executed i_0 steps of algorithm QIP from the usual starting point ($\zeta = 400$) and with the usual initial weight parameter ($\omega_{\text{init}} = 0.5$). After that, subroutine CompWarmStart was used to calculate a feasible $\Delta\omega_{\text{mult}} > 0$ as well as a corresponding $\Delta\omega_{\text{mult}} < 0$. The results of these experiments for a sample of 200 optimization problems can be seen in Figure 5, (p. 35) . Clearly, the steep decline of the maximal perturbation computed for the case $\Delta\omega_{\text{mult}} = 0.1$ is an artifact of the rather small backtracking parameter. Note that for a value of $\Delta\omega_{\text{mult}} = 0.9$, the values of $|\Delta\omega|$ computed do not seem to converge to zero. According to this numerical evidence, Theorem 12 indeed states a sufficient condition, but not a necessary one. Taking a closer look again at Theorem 12, we see that the theorem can, for a fixed optimization problem, be interpreted roughly as "if $\mu \geq C\|\Delta d\|_\infty = \tilde{C}|\Delta\omega|$, then $w + \Delta w$ is feasible". However, μ converges to 0 and based on this result, we should choose $\Delta\omega$ converging to 0, too. But our numerical evidence shows that this is not necessary.

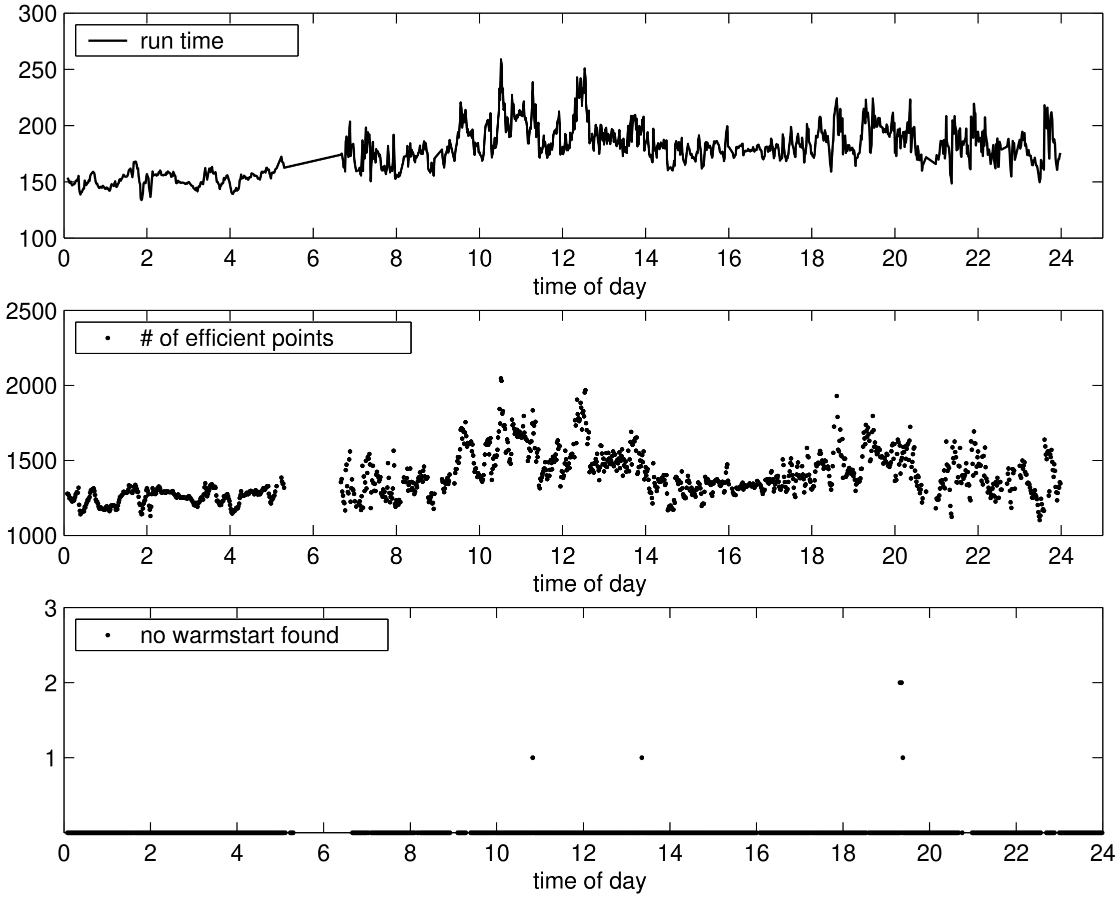
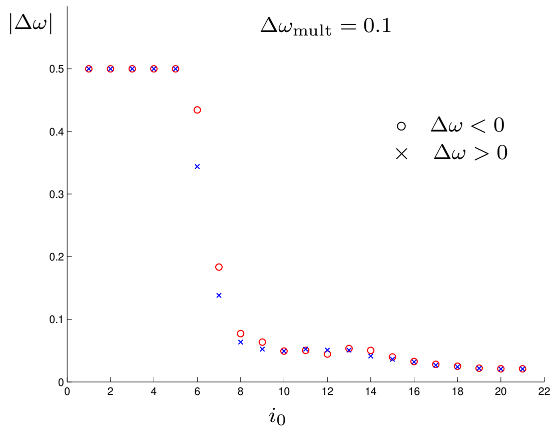


Figure 4: Run time and number of efficient points for the 1440 bicriteria problems considered. These problems are indexed according to the time of the day. Note that the problems at around 6:00 a.m. are infeasible.



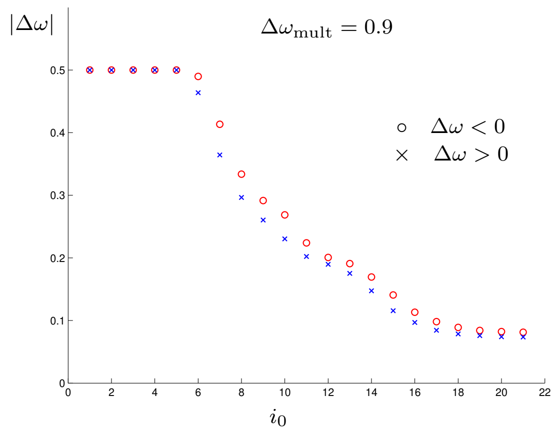


Figure 5: Maximal perturbation size during the execution of QIP.

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