SOME NEW COINCIDENCE CONDITIONS IN MINISUM MULTIFACILITY LOCATION PROBLEMS WITH MIXED GAUGES

Jörg Fliege
University of Dortmund
Fachbereich Mathematik, 44221 Dortmund
Tel.: 0231 / 755–5415
e-mail: fliege@euler.mathematik.uni-dortmund.de

Abstract

In minisum multifacility location problems the optimal locations of clusters of facilities frequently tend to coincide. Several conditions for coincidence using attraction trees have been published recently. These conditions are extended and improved to more general location problems. Using trees of sets of facilities, new sufficient coincidence conditions for location problems with mixed asymmetric gauges are given. Moreover, sufficient conditions are formulated, which only involve the given weights and certain values from a preprocessing step.

Keywords: Coincidence conditions, multifacility location problem, gauges.

The minisum multifacility location problem could be stated as follows: let A be a set of fixed facilities located at $x_a \in \mathrm{IR}^N$ $(a \in A)$ and V be a set of new facilities, which locations $x_v \in \mathrm{IR}^N$ $(v \in V)$ should be chosen so as to minimize the weighted sum of some of the distances between the facilities. The distances between pairs of points in IR^N are measured with arbitrary gauges, a notion that goes back to Minkowski [5, pp. 131 – 135].

1 Preliminaries

1.1 Gauges

Definition 1.1 (Gauges) *If $B \subset \mathrm{IR}^N$ is a convex, compact set with $0 \in \mathrm{int}(B)$, the* gauge γ of B *is given by*

$$\gamma(x) := \inf\{\lambda > 0 \mid x \in \lambda B\}.$$

The gauge of $-B$ will be denoted by $\overline{\gamma}$.

Obviously, there is $\overline{\gamma}(x) = \gamma(-x)$ for all $x \in \mathrm{IR}^N$. Gauges are nonnegative, positive homogeneous, convex and finite-valued functions. Particularly, the *triangle inequality*,

$$\gamma(x-y) \leq \gamma(x-z) + \gamma(z-y),$$

holds for all $x, y, z \in \mathrm{IR}^N$. Moreover, one has $\gamma(x) = 0$ if and only if $x = 0$. Any function γ with all these features is a gauge of the compact convex unit ball $B := \{x \in \mathrm{IR}^N \mid \gamma(x) \leq 1\}$. Consequently, any positive linear combination of gauges is a gauge. But not every gauge is a norm: this holds only for symmetric unit balls B.

1.1.1 Comparison of Gauges

The "comparison step" should be a significant part of any computer code for testing coincidence conditions of the minisum multifacility location problem with mixed gauges. In this step two or more gauges are examined in order to find the one which is the "strongest", i. e. the one which measures greater distances between pairs of points than the other gauges. This subsection shows an efficient way of transforming this task into a problem which could be solved in a preprocessing step.

Theorem 1.2 *Let γ_1 and γ_2 be gauges. Then there exist two values $\alpha, \beta > 0$ with*

$$\alpha\gamma_1(x) \leq \gamma_2(x) \leq \beta\gamma_1(x) \text{ for all } x \in \mathrm{IR}^N. \tag{1}$$

Proof: Let B_1 and B_2 be the unit balls of γ_1 and γ_2. Choose $\alpha > 0$ with $\alpha B_2 \subseteq B_1$. Then $B_2 \subseteq (1/\alpha)B_1$ and one has for all $x \in \mathrm{IR}^N$, that

$$\begin{aligned}\alpha\gamma_1(x) = \gamma_1(\alpha x) &= \inf\{\lambda > 0 \mid x \in (\lambda/\alpha)B_1\} \\ &\leq \inf\{\lambda > 0 \mid x \in \lambda B_2\} = \gamma_2(x).\end{aligned}$$

For any $\beta' > 0$ with $\beta' B_1 \subseteq B_2$ one has $\beta'\gamma_2(x) \leq \gamma_1(x)$ and with $\beta := 1/\beta'$ the result holds. □

Therefore, it is necessary and sufficient for (1) that

$$\alpha B_2 \subseteq B_1 \subseteq \beta B_2;$$

moreover, the estimate (1) cannot be improved, if

$$\alpha = \sup\{\mu > 0 \mid \mu B_2 \subseteq B_1\} \tag{2}$$

and

$$\beta = \inf\{\mu > 0 \mid B_1 \subseteq \mu B_2\}. \tag{3}$$

Definition 1.3 (Comparison Factors) *If two gauges γ_1 and γ_2 are given, then two scalars $\alpha, \beta > 0$ with (1) are named* lower and upper comparison factors of γ_1 and γ_2*.*

If α satisfies (2), then α is named best lower comparison factor of γ_1 and γ_2*; if β satisfies (3), then β is named* best upper comparison factor of γ_1 and γ_2*.*

Plastria [7] defines a similar concept for distances defined by a *quasimetric*, the asymmetric equivalent of a metric. In the notation used here the *skewness factor* defined in [7] is the best upper comparison factor between a gauge γ and $\overline{\gamma}$.

1.2 Graphs

If F is a nonempty set and $E \subseteq F \times F$ is given, then $G := (F, E)$ is named *directed graph with node set F and edge set E.* The notation uv will shortly refer to an edge $(u, v) \in E$.

For two subsets $U_1, U_2 \subseteq F$, the set $E(U_1, U_2) := E \cap (U_1 \times U_2)$ denotes the set of edges that run from a node in U_1 to a node in U_2. Also, let

$$\boldsymbol{E}(U_1, U_2) := \boldsymbol{E}(U_2, U_1) := E(U_1, U_2) \cup E(U_2, U_1).$$

For a node $f \in F$ and a subset $U \subseteq F$ let $E(f, U) := E(\{f\}, U)$ be the set of edges which run from f into U. Similarly, let $E(U, f) := E(U, \{f\})$.

2 Formulation of the Problem

In what follows, let F be a nonempty, finite set of nodes, $\emptyset \neq E \subseteq F \times F$ and $G := (F, E)$ be a directed graph. The node set F represents a set of facilities, while the edge set E corresponds to interactions between these objects. Let the node set F be partitioned into the nonempty sets A and V. For all $a \in A$ let a fixed vector $x_a \in \mathbb{R}^N$ be given. The nodes $v \in V$ represent facilities for which the best location is still unknown. For every edge $e \in E$ let there be given a weight $\omega_e \geq 0$ and a gauge γ_e. The cost of the interaction between two nodes connected by an edge is measured by the corresponding gauge, the weight ω_e serves as a proportionality factor. The set of families $(x_v)_{v \in V}$ with $x_v \in \mathbb{R}^N$ for all $v \in V$ will be denoted by $(\mathbb{R}^N)^V$. This is just the domain of the objective function. Now, let in addition to the vectors $x_a \in \mathbb{R}^N$ $(a \in A)$, a family $(x_v)_{v \in V} \in (\mathbb{R}^N)^V$ be given. For any edge $e = (f, g) \in E$ set $x_e := x_f - x_g$. The length of this vector will be measured with the gauge γ_e. Sometimes, a second family $(\hat{x}_v)_{v \in V}$ is needed. Then the notation $\hat{x}_e := \hat{x}_f - \hat{x}_g$ (where clearly $\hat{x}_a := x_a$ $(a \in A)$) will be used.

Definition 2.1 (Multifacility Location Problem) *The* continuous minisum multifacility location problem with different gauges *is the following optimization problem:*

Find a minimum $(\hat{x}_v)_{v \in V} \in (\mathbb{R}^N)^V$ of the function

$$\begin{aligned} \Psi : (\mathbb{R}^N)^V &\longrightarrow \mathbb{R} \\ (x_v)_{v \in V} &\longmapsto \sum_{e \in E} \omega_e \gamma_e(x_e). \end{aligned} \tag{4}$$

Particularly, there are no restrictions on the $(x_v)_{v \in V}$.

Many assumptions about the structure of the graph G or the parameters could be made without reducing the class of problems. Some of them are found in Plastria [6] and in Fliege [1]. However, only three of them are necessary for the following discussion:

1. The weights are strictly positive, i. e. $\omega_e > 0$ for all $e \in E$.
2. The graph G is simple: $(f, f) \notin E$ for all $f \in F$ (the removal of such an edge does not change Ψ) and if $(f, g) \in E$ then $(g, f) \notin E$ (otherwise, the new gauge $\omega_{fg}\gamma_{fg} + \omega_{gf}\overline{\gamma_{gf}}$ with the new weight 1 could be used on the edge (f, g); the edge (g, f) could be removed).
3. The points x_a $(a \in A)$ are mutually different. If $x_a = x_b$ for two nodes $a, b \in A$ with $a \neq b$, one could reduce the problem by identifying a and b and redefining all involved edges.

The function Ψ is convex, nonnegative and not bounded from above. Therefore, the set of minima of Ψ is nonempty, convex and compact (Francis and Cabot [2], Fliege [1]).

3 Conditions for Coincidence

It is a well observed fact that often several new facilities tend to coincide at an optimal solution of the minisum multifacility location problem. Even if all gauges are as smooth as p-Norms with $1 < p < \infty$, this leads to a nondifferentiability in a minimum of the original problem. But by testing sufficient conditions for coincidence, it is possible to collapse some coinciding facilities into one. This leads to a reduction of the problem's dimension and of the degree of nondifferentiability [4, p. 437].

The following definition is an extension of the definition of *attraction* from Lefebvre, Michelot and Plastria [4].

Definition 3.1 *Let $W_1, W_2 \subseteq F$. The set W_1* attracts *the set W_2, if the following conditions are true:*

- $W_1 \neq \emptyset$.
- $\emptyset \neq W_2 \subseteq V$.
- $W_1 \cap W_2 = \emptyset$.
- *The set W_1 contains at most one $a \in A$.*
- *With $W := W_1 \cup W_2$, the following condition holds for all $w \in W_2$ and all $y \in \mathbb{R}^N$:*

$$\sum_{e \in E(w, F \setminus W)} \omega_e \gamma_e(y) \quad + \sum_{e \in E(F \setminus W, w)} \omega_e \overline{\gamma_e}(y) \quad \leq \sum_{e \in E(W_1, w)} \omega_e \gamma_e(y) \quad + \sum_{e \in E(w, W_1)} \omega_e \overline{\gamma_e}(y). \tag{5}$$

Now let W_1 attract W_2 and let at least one $w \in W_2$ exists such that (5) is strict for all $y \in \mathbb{R}^N$, $y \neq 0$. Then W_2 is called strictly attracted *by W_1 and w is called a* witness of strict attraction.

Remark 3.2 *Let $\emptyset \neq W_1 \subseteq F$, $\emptyset \neq W_2 \subseteq V$, $W_1 \cap W_2 = \emptyset$ and $W := W_1 \cup W_2$. The set W_1 should contain at most one $a \in A$. If*

$$\boldsymbol{E}(W_2, F \setminus W) = \emptyset,$$

then W_1 attracts the set W_2. Moreover, if at least one $w \in W_2$ exists with $\boldsymbol{E}(w, W_1) \neq \emptyset$, then W_2 is strictly attracted by W_1 with witness w.

Proof: For all $w \in W_2$ is $\boldsymbol{E}(w, F \setminus W_2) = \emptyset$, therefore the left side of (5) is 0. If, in addition, at least one $w \in W_2$ exists with $\boldsymbol{E}(w, W_1) \neq \emptyset$, the inequality (5) is strict for all $y \neq 0$. $\square$

Theorem 3.3 *Let $W_1, W_2 \subseteq F$, $W := W_1 \cup W_2$ and let W_1 attract the set W_2. Let a family $(x_v)_{v\in V}$ be given with $x_w = x$ for all $w \in W_1$ and define $(\hat{x}_v)_{v\in V}$ by $\hat{x}_w := x$ for all $w \in W$ and $\hat{x}_v := x_v$ for all $v \in V \setminus W$. Then it follows that*

$$\Psi((\hat{x}_v)_{v\in V}) \leq \Psi((x_v)_{v\in V}).$$

If W_1 strictly attracts W_2 and there exists a witness $w \in W_2$ with $x_w \neq x$, it follows that $\Psi((\hat{x}_v)_{v\in V}) < \Psi((x_v)_{v\in V})$.

If $(x_v)_{v\in V}$ is a minimum of Ψ with $x_f = x_g$ for all $f, g \in W_1$, then it follows for all $(u, v) \in E(W_2, W_2)$ that $x_u = x_v$.

Proof: The difference $\Psi((\hat{x}_v)_{v\in V}) - \Psi((x_v)_{v\in V})$ will be estimated. The following cases cover all $e \in E$:

1. $e \in E(F\setminus W, F\setminus W)$ or $e \in E(W_1, W_1)$:we have $\hat{x}_e = x_e$ and then $\gamma_e(\hat{x}_e) - \gamma_e(x_e) = 0$.

2. $e \in E(W_2, W_2)$: then $\gamma_e(\hat{x}_e) - \gamma_e(x_e) = -\gamma_e(x_e)$.

3. $e \in \boldsymbol{E}(W_1, F \setminus W)$: here one has $\gamma_e(\hat{x}_e) - \gamma_e(x_e) = 0$.

4. $e \in \boldsymbol{E}(W_1, W_2)$: we have $\gamma_e(\hat{x}_e) - \gamma_e(x_e) = -\gamma_e(x_e)$.

5. (a) $e = (w, f) \in E(W_2, F \setminus W)$: here, $\hat{x}_e = x - x_f$ and $x_e = x_w - x_f$. The triangle inequality now shows that $\gamma_e(\hat{x}_e) \leq \gamma_e(x - x_w) + \gamma_e(x_w - x_f) = \gamma_e(x - x_w) + \gamma_e(x_e)$. Consequently, one has $\gamma_e(\hat{x}_e) - \gamma_e(x_e) \leq \gamma_e(x - x_w)$.

   (b) $e = (f, w) \in E(F \setminus W, W_2)$: in the same way, as before, one has $\gamma_e(\hat{x}_e) - \gamma_e(x_e) \leq \gamma_e(x_w - x)$.

Now all edges in E are examined.

By the above considerations it follows that

$$\Psi((\hat{x}_v)_{v\in V}) - \Psi((x_v)_{v\in V}) = \sum_{e\in E} \omega_e(\gamma_e(\hat{x}_e) - \gamma_e(x_e))$$

$$\leq \sum_{w\in W_2} \left(\sum_{e\in E(w,F\setminus W)} \omega_e\gamma_e(x - x_w) \quad + \sum_{e\in E(F\setminus W,w)} \omega_e\gamma_e(x_w - x) \right.$$

$$
\begin{aligned}
& \qquad\qquad - \sum_{e \in E(w,W_1)} \omega_e \gamma_e(x_e) \quad - \sum_{e \in E(W_1,w)} \omega_e \gamma_e(x_e) \Bigg) \\
& \quad - \sum_{e \in E(W_2,W_2)} \omega_e \gamma_e(x_e) \\
\leq & \; - \sum_{e \in E(W_2,W_2)} \omega_e \gamma_e(x_e) \\
\leq & \; 0
\end{aligned}
$$

because of (5). Therefore, the first statement follows. If (5) is strict for at least one $w \in W_2$ with $x_w \neq x$ and all $y \in \mathrm{IR}^N \setminus \{0\}$, the last but one inequatlity is also strict. The statement $\Psi((\hat{x}_v)_{v \in V}) < \Psi((x_v)_{v \in V})$ follows.

Now let $(x_v)_{v \in V}$ be a minimum of Ψ. Then

$$0 \leq \Psi((\hat{x}_v)_{v \in V}) - \Psi((x_v)_{v \in V}) \leq - \sum_{e \in E(W_2,W_2)} \omega_e \gamma_e(x_e) \leq 0$$

and then $\gamma_e(x_e) = 0$, i. e. $x_e = 0$ for all $e \in E(W_2, W_2)$. □

Consider now the special case when all gauges γ_e are equal to one norm γ. Let (5) hold, let $W_1 = \{g\}$ and let $\boldsymbol{E}(g, w) \neq \emptyset$ hold for all $w \in W_2$. Then W_2 is the leaf-set of an attraction tree (cf. [4]) of height 1 with root g. The suppressed edges are the edges in $E(W_2, W_2)$. If $W_2 = \{f\}$, then (5) simplifies to the attraction condition of [4].

The next example (due to Plastria [8]) uses only one norm instead of several different ones.

Example 3.4 Let $A = \{a, b, c, d\}$, $V = \{t, u, v, w\}$,

$$E = \{(u, a), (u, b), (u, t), (v, a), (v, d), (v, t), (w, a), (w, c), (w, t)\}.$$

All γ_e are equal to a norm γ. The weights are $\omega_{ua} = \omega_{va} = \omega_{wa} = \omega_{ut} = \omega_{wt} = \omega_{vt} = 3$ and $\omega_{ub} = \omega_{vd} = \omega_{wc} = 1$. Taking $W_1 := \{a\}$ and $W_2 := V$ one gets strict attraction between W_1 and W_2. Especially, every element of V except t is a witness of strict attraction. Therefore, it is $x_a = x_t = x_u = x_v = x_w$ for all minima of the constructed minisum multifacility location problem.

It should be noted that in this example no attraction tree as defined in [4] exists. Therefore, none of Lefebvre, Michelot and Plastria's results will yield any coincidences. The same holds for example 2.2 in [1]. This example is smaller than the one presented here, but uses mixed norms instead of a single one.

Since (5) is a functional inequality, it is not clear how to check these coincidence condition in a computer code. However, it is possible to derive sufficient conditions for

(5), which only involve the given weights and certain comparison factors. In a computer code, these conditions can then be used to detect coincidences in location problems with mixed gauges.

Theorem 3.5 *Let $\emptyset \neq W_1 \subset F$, $\emptyset \neq W_2 \subseteq V$ and $W := W_1 \cup W_2$. Also, at most one $a \in A$ should be in W_1. For all $w \in W_2$ let $\gamma_w^{(1)}$ and $\gamma_w^{(2)}$ be two gauges and let $\alpha_{e,w}^{(i)}, \beta_{e,w}^{(i)} > 0$ $(i = 1, 2)$ be comparison factors with*

$$\alpha_{e,w}^{(i)} \gamma_w^{(i)} \leq \gamma_e \leq \beta_{e,w}^{(i)} \gamma_w^{(i)} \qquad (e \in E,\ i = 1, 2).$$

If

$$\sum_{e \in E(w, F \setminus W)} \omega_e \beta_{e,w}^{(1)} \leq \sum_{e \in E(W_1, w)} \omega_e \alpha_{e,w}^{(1)} \tag{6}$$

and

$$\sum_{e \in E(F \setminus W, w)} \omega_e \beta_{e,w}^{(2)} \leq \sum_{e \in E(w, W_1)} \omega_e \alpha_{e,w}^{(2)} \tag{7}$$

hold for all $w \in W_2$, then W_2 is attracted by W_1.

If, additionally, (6) or (7) is strict for at least one $w \in W_2$, then W_2 is strictly attracted by W_1.

Proof: For all $e \in E$ one has

$$\alpha_{e,w}^{(i)} \overline{\gamma_w^{(i)}} \leq \overline{\gamma_e} \leq \beta_{e,w}^{(i)} \overline{\gamma_w^{(i)}} \qquad (i = 1, 2).$$

Let $w \in W_2$. Then

$$\begin{aligned}
&\sum_{e \in E(w, F \setminus W)} \omega_e \gamma_e + \sum_{e \in E(F \setminus W, w)} \omega_e \overline{\gamma_e} \\
\leq\ & \sum_{e \in E(w, F \setminus W)} \omega_e \beta_{e,w}^{(1)} \gamma_w^{(1)} + \sum_{e \in E(F \setminus W, w)} \omega_e \beta_{e,w}^{(2)} \overline{\gamma_w^{(2)}} \\
=\ & \left(\sum_{e \in E(w, F \setminus W)} \omega_e \beta_{e,w}^{(1)} \right) \gamma_w^{(1)} + \left(\sum_{e \in E(F \setminus W, w)} \omega_e \beta_{e,w}^{(2)} \right) \overline{\gamma_w^{(2)}} \\
\leq\ & \left(\sum_{e \in E(W_1, w)} \omega_e \alpha_{e,w}^{(1)} \right) \gamma_w^{(1)} + \left(\sum_{e \in E(w, W_1)} \omega_e \alpha_{e,w}^{(2)} \right) \overline{\gamma_w^{(2)}} \\
=\ & \sum_{e \in E(W_1, w)} \omega_e \alpha_{e,w}^{(1)} \gamma_w^{(1)} + \sum_{e \in E(w, W_1)} \omega_e \alpha_{e,w}^{(2)} \overline{\gamma_w^{(2)}} \\
\leq\ & \sum_{e \in E(W_1, w)} \omega_e \gamma_e + \sum_{e \in E(w, W_1)} \omega_e \overline{\gamma_e}
\end{aligned}$$

and (5) follows. Also, (5) is strict for all $y \in \mathbb{R}^N \setminus \{0\}$, whenever (6) or (7) is strict. □

The next example, however, shows that (6) and (7) are not necessary for (5):

Example 3.6 This example is in dimension $N = 2$. Choose $W_2 = \{w\}$, $\boldsymbol{E}(w, F \setminus W) = E(w, F \setminus W) = \{e_1, e_2\}$, $\boldsymbol{E}(w, W_1) = E(W_1, w) = \{e_3, e_4\}$, and assume that (7) holds for all gauges $\gamma^{(2)}$. Let also $\omega_{e_i} := 1$ $(i = 1, 2, 3, 4)$ and

$$\begin{aligned}\gamma_{e_1}(x_1, x_2) := \gamma_{e_3}(x_1, x_2) &:= (1/2)|x_1| + (1/2)|x_2| \\ \gamma_{e_2}(x_1, x_2) := \gamma_{e_4}(x_1, x_2) &:= \max\{(1/3)|x_1|, |x_2|\}.\end{aligned}$$

Then (5) is true. Now let $\gamma_w^{(1)}$ be an arbitrary gauge. If (6) would hold, one would get $\beta_{e_1,w}^{(1)} + \beta_{e_2,w}^{(1)} \leq \alpha_{e_3,w}^{(1)} + \alpha_{e_4,w}^{(1)}$. With $\alpha_{e_3,w}^{(1)} \leq \beta_{e_1,w}^{(1)}$ and $\alpha_{e_4,w}^{(1)} \leq \beta_{e_2,w}^{(1)}$ it would follow that $\beta_{e_1,w}^{(1)} = \alpha_{e_3,w}^{(1)}$ and $\beta_{e_2,w}^{(1)} = \alpha_{e_4,w}^{(1)}$. But one has $\alpha_{e_3,w}^{(1)}\gamma^{(1)} \leq \gamma_{e_3} = \gamma_{e_1} \leq \beta_{e_1,w}^{(1)}\gamma^{(1)}$ and $\alpha_{e_4,w}^{(1)}\gamma^{(1)} \leq \gamma_{e_4} = \gamma_{e_2} \leq \beta_{e_2,w}^{(1)}\gamma^{(1)}$. Therefore, one would get $\alpha_{e_3,w}^{(1)}\gamma^{(1)} = \gamma_{e_3}$ and $\alpha_{e_4,w}^{(1)}\gamma^{(1)} = \gamma_{e_4}$. It would follow that $\gamma_{e_3} = c\gamma_{e_4}$ with $c > 0$, which leads to a contradiction.

Instead of (6) and (7), one could also use

$$\sum_{e \in E(w, F\setminus W)} \omega_e \beta_{e,w}^{(1)} \leq \sum_{e \in E(w, W_1)} \omega_e \overline{\alpha}_{e,w}^{(1)} \tag{8}$$

and

$$\sum_{e \in E(F\setminus W, w)} \omega_e \beta_{e,w}^{(2)} \leq \sum_{e \in E(W_1, w)} \omega_e \overline{\alpha}_{e,w}^{(2)} \tag{9}$$

(here we have $\overline{\alpha}_{e,w}^{(i)}\gamma_w^{(i)} \leq \overline{\gamma_e}$, $(w \in W_1,\ e \in E,\ i = 1, 2)$). These two conditions are also sufficient for (5). But the last example could be used to construct a counterexample to show that (5) is not sufficient for (8) and (9).

Now let $\tilde{\gamma}_w = \gamma_w^{(1)} = \gamma_w^{(2)}$ for a $w \in W_2$ and let $\overline{\tilde{\alpha}}_{e,w}$, $\overline{\tilde{\beta}}_{e,w}$ $\tilde{\alpha}_{e,w}$, $\tilde{\beta}_{e,w} > 0$ be comparison factors with $\tilde{\alpha}_{e,w}\tilde{\gamma}_w \leq \gamma_e \leq \tilde{\beta}_{e,w}\tilde{\gamma}_w$ and $\overline{\tilde{\alpha}}_{e,w}\tilde{\gamma}_w \leq \overline{\gamma_e} \leq \overline{\tilde{\beta}}_{e,w}\tilde{\gamma}_w$ $(e \in \boldsymbol{E}(w, F \setminus W_2))$. Instead of (6) and (7), the following condition could be used:

$$\sum_{e \in E(w, F\setminus W)} \omega_e \tilde{\beta}_{e,w} + \sum_{e \in E(F\setminus W, w)} \omega_e \overline{\tilde{\beta}}_{e,w} \leq \sum_{e \in E(W_1, w)} \omega_e \tilde{\alpha}_{e,w} + \sum_{e \in E(w, W_1)} \omega_e \overline{\tilde{\alpha}}_{e,w}. \tag{10}$$

This condition also implies (5). But the existence of two gauges $\gamma^{(1)}$ and $\gamma^{(2)}$ with (6) and (7) is neither necessary nor sufficient for the existence of a gauge $\tilde{\gamma}$ with (10), as the following examples show.

Example 3.7 Let $w \in W_2$, $E(w, F \setminus W) = \{e_1\}$, $E(F \setminus W, w) = \{e_2\}$, $E(W_1, w) = \{e_3\}$ and $E(w, W_1) = \{e_4\}$. The gauges γ_{e_i} and the weights ω_{e_i} $(i = 1, 2, 3, 4)$ are chosen as in example 3.6. With $\gamma^{(1)} = \gamma_{e_1} = \gamma_{e_3}$ and $\gamma^{(2)} = \gamma_{e_2} = \gamma_{e_4}$, (6) and (7) hold. Now let (10) hold for a gauge $\tilde{\gamma}_w$. Then $\tilde{\beta}_{e_1,w} + \overline{\tilde{\beta}}_{e_2,w} \leq \tilde{\alpha}_{e_3,w} + \overline{\tilde{\alpha}}_{e_4,w}$ would hold. But as in example 3.6, a contradiction would follow.

Example 3.8 Let $w \in W_2$, $E(w, F \setminus W) = \{e_1\}$, $E(F \setminus W, w) = \{e_2\}$, $E(W_1, w) = \{e_3\}$ and $E(w, W_1) = \{e_4\}$. Let $\gamma_{e_1} = \gamma_{e_2} = \gamma_{e_3} = \gamma_{e_4} =: \gamma$ be a norm, $\omega_{e_1} = \omega_{e_4} = 2$ and $\omega_{e_2} = \omega_{e_3} = 1$. With $\tilde{\gamma} = \gamma$ one has (10). Now let $\gamma^{(1)}$ be an arbitrary gauge. Then (6) is equivalent to $2\beta^{(1)}_{e_1,w} \leq \alpha^{(1)}_{e_3,w}$. But with $\gamma_{e_1} = \gamma_{e_3}$ one has $\alpha^{(1)}_{e_3,w} \leq \beta^{(1)}_{e_1,w}$, so (6) cannot be true.

Likewise, (8) and (9) is neither necessary nor sufficient for (10).

Obviously, the new coincidence conditions could be used recursively, once at least one coincidence has been established. In this way, one gets theorems similar to the theorem 3 from Juel and Love [3] (this result could only be used if all gauges are equal to one norm). In multilocation problems with different gauges, one could use different comparison gauges and therefore different comparison values in each recursion step.

In the special case of the Fermat-Weber-problem ($V = \{v\}$) with one quasimetric as a distance function, Plastria [7] proves a majority theorem which plays the same role as the conditions (6)–(10) for the Fermat-Weber-problem. If a gauge γ is used to define a quasimetric, the conditions (6)–(10) collapse into the asymmetric majority theorem from [7], as long as γ is also used as a comparison gauge. But by using different comparison gauges or condition (5), one obtains greater flexibility in checking coincidences.

Still other new conditions could be constructed by using the idea of attraction trees on disjoint sets $W_1, \ldots, W_n$.

Definition 3.9 (Set-Trees) *Let $\mathcal{W}$ be a partition of F and let $\mathcal{O} \in \mathcal{W}$. A tree $T_{\mathcal{W}} = (F_{\mathcal{W}}, E_{\mathcal{W}})$ with node set $F_{\mathcal{W}} := \mathcal{W} \setminus \{\mathcal{O}\}$ is called* set-tree *on F. The set $\mathcal{O}$ is called* outer part *of $T_{\mathcal{W}}$.*

Now let $T_{\mathcal{W}}$ be rooted with root $R \in F_{\mathcal{W}}$. The tree $T_{\mathcal{W}}$ is called compatible with Ψ, *if at most one $a \in A$ is in R and all other sets $W \in F_{\mathcal{W}}$ are subsets of V, i. e. $\bigcup_{W \in F_{\mathcal{W}}} (W \setminus R) \subseteq V$. For $W \in F_{\mathcal{W}}$, $W \neq R$ denote with* $\mathrm{pre}(W)$ *the* father *of W, the first node on the path from W to the root R.*

Definition 3.10 (Cleaned Up Graphs) *Let $\mathcal{W}$ be a partition of F and let $T_{\mathcal{W}} = (F_{\mathcal{W}}, E_{\mathcal{W}})$ be a set-tree on F with root R and outer part $\mathcal{O}$, which is compatible with Ψ. The graph*

$$G'(T_{\mathcal{W}}) := (F, E'(T_{\mathcal{W}}))$$

with

$$E^{'}(T_{\mathcal{W}}) := \boldsymbol{E}(\mathcal{O}, F) \cup \bigcup_{\substack{W \in F_{\mathcal{W}} \\ W \neq R}} \boldsymbol{E}(W, \mathrm{pre}(W))$$

is called cleaned up graph. *The set*

$$E_s^{'}(T_{\mathcal{W}}) := E \setminus E^{'}(T_{\mathcal{W}})$$

is called set of suppressed edges.

Definition 3.11 (Attraction Set-Trees) *Let $\mathcal{W}$ be a partition of F and let $T_{\mathcal{W}} = (F_{\mathcal{W}}, E_{\mathcal{W}})$ be a set-tree on F with root R and outer part $\mathcal{O}$, which is compatible with Ψ. The tree $T_{\mathcal{W}}$ is named* attraction set-tree, *if every $W \in F_{\mathcal{W}}$, $W \neq R$, is attracted by its father* $\mathrm{pre}(W)$ *relative to $G^{'}(T_{\mathcal{W}})$ (an attraction disregarding all suppressed edges).*

Theorem 3.12 *Let $\mathcal{W}$ be a partition of F and $T_{\mathcal{W}} = (F_{\mathcal{W}}, E_{\mathcal{W}})$ be a set-tree on F with root R and outer part $\mathcal{O}$, which is compatible with Ψ and which is an attraction set-tree. Let a family $(x_v)_{v \in V}$ be given with $x_r = x$ for all $r \in R$ and define $(\hat{x}_v)_{v \in V}$ by $\hat{x}_p := x_p$ for all $p \in \mathcal{O}$ and $\hat{x}_f := x$ for all $f \in F \setminus \mathcal{O}$. Then it follows that*

$$\Psi((\hat{x}_v)_{v \in V}) \leq \Psi((x_v)_{v \in V}).$$

If at least one $W \in F_{\mathcal{W}}$ is strictly attracted by its father $\mathrm{pre}(W)$ *relative to $G^{'}(T_{\mathcal{W}})$ and at least one witness $w \in W$ of this exists with $x_w \neq x$, it follows that*

$$\Psi((\hat{x}_v)_{v \in V}) < \Psi((x_v)_{v \in V}).$$

Let additionally $(x_v)_{v \in V}$ be a minimum of Ψ. Then it is $x_f = x_g$ for all $(f, g) \in E_s^{'}(T_{\mathcal{W}})$.

Proof: The proof runs the same way as the proof of the attraction theorem in [4] or [1]. Set $E^{'} := E^{'}(T_{\mathcal{W}})$, $E_s^{'} := E_s^{'}(T_{\mathcal{W}})$ and $G^{'} := G^{'}(T_{\mathcal{W}})$. For $(y_v)_{v \in V} \in (\mathrm{IR}^N)^V$ set

$$\Psi^{'}((y_v)_{v \in V}) := \sum_{e \in E^{'}} \omega_e \gamma_e(y_e).$$

By repeated application of theorem 3.3 on $\Psi^{'}$ and the nodes $W \neq R$ and $\mathrm{pre}(W)$ of $T_{\mathcal{W}}$, one obtains $\Psi^{'}((\hat{x}_v)_{v \in V}) \leq \Psi^{'}((x_v)_{v \in V})$. This inequality is strict, if at least one $W \in F_{\mathcal{W}}$ is strictly attracted by its father $\mathrm{pre}(W)$ relative to $G^{'}$ and at least one witness $w \in W$ of this exists with $x_w \neq x$. Moreover, it is $\hat{x}_e = 0$ for all suppressed edges $e \in E_s^{'}$. Therefore,

$$\begin{aligned} \Psi((\hat{x}_v)_{v \in V}) &= \Psi^{'}((\hat{x}_v)_{v \in V}) + \sum_{e \in E_s^{'}} \omega_e \gamma_e(\hat{x}_e) \\ &= \Psi^{'}((\hat{x}_v)_{v \in V}) \leq \Psi^{'}((x_v)_{v \in V}) \ \leq \ \Psi((x_v)_{v \in V}), \end{aligned}$$

and the first and second part of the theorem is proven. Now let additionally $(x_v)_{v\in V}$ be a minimum of Ψ. It follows that $\Psi((\hat{x}_v)_{v\in V}) = \Psi((x_v)_{v\in V})$. Thus

$$\begin{aligned}\Psi((x_v)_{v\in V}) &= \Psi'((x_v)_{v\in V}) + \sum_{e\in E_s'} \omega_e\gamma_e(x_e) \geq \Psi'((x_v)_{v\in V}) \\ &\geq \Psi'((\hat{x}_v)_{v\in V}) = \Psi((\hat{x}_v)_{v\in V}) \;=\; \Psi((x_v)_{v\in V}).\end{aligned}$$

Therefore, $\sum_{e\in E_s'} \omega_e\gamma_e(x_e) = 0$ and the last part follows. □

References

[1] Jörg Fliege. Nichtdiskrete Multilokationsprobleme: Koinzidenzbedingungen und effiziente Punkte. Diplomarbeit, Universität Dortmund, FB Mathematik, Universität Dortmund, 44221 Dortmund, February 1993.

[2] R. L. Francis and A. V. Cabot. Properties of a Multifacility Location Problem Involving Euclidean Distances. *Naval Research Logistics Quarterly*, 19:335–353, 1972.

[3] Henrik Juel and Robert F. Love. Sufficient conditions for Optimal Facility Locations to Coincide. *Transportation Science*, 14(2):125–129, May 1980.

[4] O[dile] Lefebvre, C[hristian] Michelot, and F[rank] Plastria. Sufficient conditions for coincidence in minisum multifacility location problems with a general metric. *Operation Research*, 39(3):437–442, May–June 1991.

[5] Hermann Minkowski. *Gesammelte Abhandlungen*, volume 2. Chelsea Publishing Company, New York, 1967. Reprint.

[6] F[rank] Plastria. When facilities coincide: exact optimality conditions in multifacility location. Report BEIF/6 (Version 2), Centre for Industrial Location and Development, Centre for Management Informatics, Vrije Universiteit Brussel, March 1991.

[7] Frank Plastria. A majority theorem for Fermat-Weber problems in quasimetric spaces with applications to semidirected networks. In *Proceedings of the VIth meeting of the EURO working group on locational analysis*, Puerto de La Cruz (Tenerife), Spain, April 1992.

[8] Frank Plastria. Private communication, May 7, 1993.