MULTITHREADED DIMENSION REDUCTION IN LOCATION THEORY

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Abstract. Due to inevitable nondifferentiabilities, the minisum multifacility location problem is regarded as hard to solve. We propose a new chararacterization of points at which nondifferentiabilities occur. By this chararacterization it is possible to check if several variables can be identified with each other, thereby reducing the dimension of the problem and the number of nondifferentiable terms in the objective function.

Key words. facility location, coincidence conditions, dimension reduction, nondifferentiability detection

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1. Introduction.

In multifacility location one has given some fixed facilities and one is looking for the locations of some new facilities. The quality of a choice of locations is measured by an objective function in which the interactions between the different objects are taken into account. Typically, the sum of all interaction costs is considered. Applications for this problem can be found in limit analysis [4], when solving Steiner tree [58] or minimal surface problems, when computing the static of spring systems [40], in location science [46, 47], etc. The complexity of the problem can be seen by the following observation: if a norm is used to determine the distance between two points in $\mathbb{R}^N$, the objective function will not be differentiable, because two objects may have the same location and norms are not differentiable at the origin. But such a *coincidence*, i. e. two objects sharing the same location in a minimum of the multifacility location problem, is likely to occur rather often. Consequently, after the first formulation of the problem at the end of the fifties in the work of Moses [39], a large number of different numerical methods for solving problems of this type have been proposed (see, e. g., [46, 47] for an overview).

For special cases, special lines of attack lead to success [42, 30, 53, 11]. In a more general case, we may approximate the nondifferentiable objective function by an infinitely often differentiable one (hyperbolic approximation, cp., e. g., [54, 17]). For this approach, convergence results have been derived for popular methods [28]. However, the speed of convergence is usually only linear [27]. Other solution attempts ignore the existence of coincidences and use methods from nondifferentiable optimization [31, 18]. But only linear convergence seems to be achievable here, too [33, 57]. It was in the eighties when Newton-like methods, working with projections onto linear manifolds which correspond exactly to the nondifferentiabilities, and achieving quadratic convergence have been developed [40, 5]. Unfortunately, it is not clear how far the amount of work necessary in each step makes these methods unattractive for large problems. Moreover, some assumptions made in the convergence proofs on the structure of the problem seem to hold for special cases only [12]. Another approach consists of solving a differentiable reformulation of the primal or dual problem [50]. However, this reformulation seems to be possible only for the Euclidean norm case. In recent times, interior point methods have been applied to special cases of the minisum multifacility location problem with Euclidean distances [4, 1] and complexity

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results have been proven [59, 58]. Since some of these algorithms have a polynomial time complexity, the methods seem to be not as sensitive with respect to coincidences than the methods mentioned above. First numerical results seem to confirm this [10, 2, 3, 58]. Moreover, the author has developed polynomial-time algorithms for convex multifacility location problems with mixed nonsymmetric gauges [23]. Unfortunately, up to now no numerical experiences have been gathered. An extension to nonconvex problems seems to be rather questionable.

Only a few authors are concerned with *nonsymmetric* distances instead of symmetric ones. The actual proposal was already made by Witzgall [55], who viewed symmetry as an artifact of modeling. If the locational space is a real vector space, we arrive in a natural way at the concept of *gauges* [38] as the nonsymmetric counterpart of norms.

Another generalization of the minisum multifacility location problem is obtained when distances occurring in the objective function are weighted with negative numbers. With this, we extend the purely economical interpretation of these terms, and objects whose interactions with other objects are measured by negative weighted distances can now be interpreted as *obnoxious*, *semi-obnoxious*, or *repulsive facilities*. Consequently, the applications proposed in the literature often consider waste disposal sites, chemical plants, nuclear reactors, etc. (cp., e. g., [16, 46, 15, 47] for overviews of other objective functions used for these problems). The loss of convexity encountered here means, of course, that we lose many of the tools from convex analysis. Consequently, this type of problem has been attacked rather lately. The computation times documented [8, 7, 52] seem to suggest that only special, low-dimensional problem classes can be solved with the methods proposed. In addition to the problems caused by the nondifferentiability mentioned above we have to take care of the nonconvexity also. Therefore, a special treatment of the nondifferentiable points in the locational space seems to be even more urgent than before.

By considering such a special treatment, the question of detecting and characterizing the nondifferentiabilities turns up immediately. In most models, norms are used to measure distances. Therefore, points of nondifferentiability have to be expected whenever two objects share the same location. Sufficient conditions for such a coincidence of facilities in a solution of a multifacility location problem are called *coincidence conditions*. By using such conditions, we may simultaneously reduce the number of unknowns, i. e. we carry out a *dimension reduction*. In this way it is possible to drastically reduce the amount of work necessary for an optimization method employed to solve a given problem. Witzgall [55, p. 30] considered such conditions for the single-facility case, while Juel and Love [32] considered the minisum multifacility problem with Euclidean norms. The sensitivity of some special cases of the Euclidean problem has been analyzed by Pesamosca and Mazzarella [41]. Plastria [44] generalized Witzgall's result to the case of a nonsymmetric distance functions, while Lefebvre, Michelot, and Plastria [36, 37] considered the multifacility case with an arbitrary metric. Finally, problems with mixed gauges have been considered by Fliege [19, 20, 22]. If not only metrical, but also analytical properties of the functions employed are used, we can derive stronger, but not as general results. This is usually done with tools from convex analysis [34, 35, 19, 29, 9]. Metrical and analytical properties can be used even when mixed distance functions are employed [45, 19]. But in none of the works cited above there have been considered negative weights in the form discussed in the last paragraph. After a trivial case was considered [51], Plastria [43] extended the conditions known to the single-facility case with positive and negative weights in pseudometric spaces. These results have been generalized to multifacility location problems with positive and negative weights in weak metric spaces by Fliege [24]. Frenk [26] generalized these conditions to minisum multifacility location problems with concave costfunctions in weak metric spaces. More general conditions, which contain all the conditions cited above as special cases, can be found in this work.

In this paper we will consider minisum multifacility location problems in which the functions used to measure distances fulfill only the triangle inequality or the inverse triangle inequality. With this concept, we may use, e. g., positive or negative weighted weak metrics. Mixed distance functions may be used in one problem simultaneously. All minisum multifacility location problems attacked in the works cited above belong to this category. We will prove new coincidence conditions for this problem class which contain all metric coincidence conditions cited above as special cases (cp. Theorem 4). Because of the wide variety of applications, these conditions can be used in *any* iterative code for solving minisum multifacility location problems to reduce the problem dimension prior or during the iteration. Furthermore, additional information of the model is gained, which may be important during an economic analysis or an interpretation of the solution set. Moreover, nondifferentiable terms in the objective function can be dropped, which results in a subsequent easier numerical treatment.

The rest of this work is as follows. In § 2, we will present the necessary definitions with respect to distance cost functions and the problem at hand, and we will formulate the location problem as a graph embedding problem. Section 3 contains some auxiliary results needed in § 4. There, the main result of this paper, Theorem 4, is derived and several examples of its application are presented.

Efficient algorithms for checking the conditions proposed here and computational results are presented in a companion paper [25].

2. Notation. In this section, the notation used is introduced and the optimization problem considered is defined.

For a fixed set $X \neq \emptyset$ we define the set Δ by $\Delta := \{(x,x) \in X \times X \mid x \in X\}$, the *diagonal* of $X \times X$. For an arbitrary set Λ and an arbitrary index set I define

$$\Lambda^I := \{(\lambda_i)_{i \in I} \mid \forall\, i \in I : \lambda_i \in \Lambda\},$$

the set of all families $(\lambda_i)_{i \in I}$ with $\lambda_i \in \Lambda$ for all $i \in I$.

2.1. Distance Costs. A function

$$d : X \times X \longrightarrow \mathbb{R}$$

is called *convex*, if for all $x, y, z \in X$ the *triangle inequality*

$$d(x,y) \leq d(x,z) + d(z,y)$$

holds. For example, all metrics on X are convex. The function $f \circ d$ fulfills the triangle inequality, if $f : \mathbb{R} \longrightarrow \mathbb{R}$ is monotone nondecreasing and subadditive and d fulfills the triangle inequality. A function $d : X \times X \longrightarrow \mathbb{R}$ is called *concave* if $-d$ is convex. For $d : X \times X \longrightarrow \mathbb{R}$ we define $\overline{d}$ by $\overline{d}(x,y) := d(y,x)$. The function $\overline{d}$ is convex if and only if d is convex. Moreover, $\overline{d}$ is concave if and only if d is concave.

A function $d : X \times X \longrightarrow \mathbb{R}$ is called *semi-definite* if for all $x \in X$ the equation $d(x,x) = 0$ holds. For a semi-definite function d we define

$$[d] := \begin{cases} d & \text{if } d \text{ is convex,} \\ |\overline{d}| & \text{else.} \end{cases}$$

Note that $[\,\overline{d}\,] = \overline{[d]}$. A semi-definite function $d : X \times X \longrightarrow \mathrm{IR}$ which is either convex or concave is called *distance cost function* on X. Multiplying a distance cost function with an arbitrary real number again yields a distance cost function. A nonnegative convex distance cost function d is also called a *weak metric* [49, 55, 56, 24, 48]. For an overview of the different terminologies used in location science we refer to [48].

As an example, define $X := \mathrm{IR}$ and $d(x, y) := x - y$. Then, the function d is semi-definite, convex and therefore a distance cost function. Moreover, it is concave. By definition, we have $[d] = d$. If d is a convex distance cost function and $f : \mathrm{IR} \longrightarrow \mathrm{IR}$ is monotone nondecreasing and subadditive with $f(0) = 0$, then $f \circ d$ is a convex distance cost function. This has important applications in several economic models, since transportation costs are often modelled as monotone nondecreasing concave functions of distance [6, 48].

In what follows, we write for given functions $c, d : X \times X \longrightarrow \mathrm{IR}$ the inequality $c \leq d$, if this inequality holds for all arguments $(x, y) \in X \times X$. For semi-definite functions c, d the inequality above is called *strict*, if it is strict for all arguments $(x, y) \in (X \times X) \setminus \Delta$.

2.2. Graphs. Let $F \neq \emptyset$ be an arbitrary set and $E \subseteq F \times F$. Then $G := (F, E)$ is called *graph* with *node set* F and *edge set* E. A 2-tuple $(f, g) \in F \times F$ will also be denoted by fg. For $f \in F$ and $U \subseteq F$ we define the following subsets of F:

$$\begin{aligned} F_G(f, U) &:= \{g \in U \mid (f, g) \in E\}, \\ F_G(U, f) &:= \{g \in U \mid (g, f) \in E\}, \\ \boldsymbol{F}_G(f, U) = \boldsymbol{F}_G(U, f) &:= F_G(f, U) \cup F_G(U, f). \end{aligned}$$

For $U_1, U_2 \subseteq F$ and $f \in F$ define $E_G(U_1, U_2) := E \cap (U_1 \times U_2)$,

$$\boldsymbol{E}_G(U_1, U_2) = \boldsymbol{E}_G(U_2, U_1) := E_G(U_1, U_2) \cup E_G(U_2, U_1)$$

and $E_G(f, U) := E_G(\{f\}, U)$, $E_G(U, f) := E_G(U, \{f\})$, as well as

$$\boldsymbol{E}_G(f, U) = \boldsymbol{E}_G(U, f) := E_G(f, U) \cup E_G(U, f).$$

If the graph G is fixed or it is clear which given graph is meant, we will omit the index on $E_G, \boldsymbol{E}_G, F_G$ and $\boldsymbol{F}_G$. Now let $G' := (F', E')$ be a subgraph of G, i. e. we have $F' \subseteq F$ and $E' \subseteq E \cap (F' \times F')$. Then we define $F'(f, U) := F_{G'}(f, U)$ and modify the rest of the notation analogously. The same modifications will be used if a graph is denoted by G_0 or some other symbols.

2.3. The Problem. In what follows, let $G = (F, E)$ be a directed simple graph with finite node set F. This node set should be partitioned into two nonempty sets A and V, i. e. we have $A \cup V = F$ and $A \cap V = \emptyset$. For all $a \in A$ there should be given an $x_a \in X$. For all $e \in E$ there should be given a distance cost function $d_e : X \times X \longrightarrow \mathrm{IR}$. Our goal is to embed the graph G into the space X in such a way that the total cost induced by the interactions between nodes in F and measured by the functions d_e is minimized. Since we have already given the locations of the facilities $a \in A$, we are in search for locations for the facilities $v \in V$. Such a location $x_v \in X$ for a node $v \in V$ is called *new location* of the object v. The locations $x_a \in X$ $(a \in A)$ are the *known locations* of the optimization problem defined below. Moreover, let there be given a set $C \subseteq X^V$ of feasible families of new locations.

If, in addition to the fixed locations $x_a \in X$ $(a \in A)$, a family of new locations $(x_v)_{v \in V} \in X^V$ is given, we can define for each $(f, g) \in F \times F$ the 2-tuple $x_{fg} :=$

$(x_f, x_g) \in X \times X$. If $e = (f,g) \in E$ we will write $x_e := x_{fg}$. If, in addition to $(x_v)_{v\in V}$, a second family $(\tilde{x}_v)_{v\in V} \in X^V$ is given, we will use the obvious definition $\tilde{x}_a := x_a$ $(a \in A)$ and write $\tilde{x}_{fg} := (\tilde{x}_f, \tilde{x}_g)$ for $(f,g) \in F \times F$ and $\tilde{x}_e := \tilde{x}_{fg}$ for $e = (f,g) \in E$. This notation will also be used for additional families $(\hat{x}_v)_{v\in V}, (y_v)_{v\in V} \in X^V$ etc. For example, $\hat{x}_{fg} := (\hat{x}_f, \hat{x}_g)$, $y_e := y_{fg} := (y_f, y_g)$ if $e = (f,g) \in E$ etc.

Definition 1 (Minisum Multifacility Location Problem). *The* constrained minisum multifacility location problem with mixed distance cost functions *is the following optimization problem:*

Find a minimum $(\hat{x}_v)_{v\in V} \in X^V$ *of the function*

$$\begin{aligned} \Psi : X^V &\longrightarrow \mathbb{R} \\ (x_v)_{v\in V} &\longmapsto \sum_{e\in E} d_e(x_e) \end{aligned} \tag{2.1}$$

subject to the constraint $(\hat{x}_v)_{v\in V} \in C$.

Since X^V has no natural topology, the minimum $(\hat{x}_v)_{v\in V}$ of the objective function Ψ that we are in search for according to the definition above is a point $(\hat{x}_v)_{v\in V} \in C \subseteq X^V$ such that $\Psi((\hat{x}_v)_{v\in V}) \leq \Psi((x_v)_{v\in V})$ holds for all $(x_v)_{v\in V} \in C$. Of course, this problem does not need to have a solution. A discussion about the assumptions needed to prove the existence of optimal families can be found in [55, p. 32–33] or [14].

If the graph G contains multiple edges we have a slightly more general problem than the one defined above. However, the results stated in the following sections still hold (cp. [21, Bemerkung 2.1.3,Bemerkung 2.2.5]). For the sake of an easier notation, we will omit this generalization at this point.

In what follows, we may assume without loss of generality that all functions d_e $(e \in E)$ are unequal to the constant function 0. Moreover, we will not allow the presence of edges between two facilities with known location: $(a,b) \notin E$ for all $a, b \in A$. Edges of this type only show up as constant terms $d_{ab}(x_a, x_b)$ in Ψ. On the other hand, we may use on each "nonexisting" edge $(f,g) \notin E$ the distance cost function $d_{fg} \equiv 0$ and add (f,g) to the edge set E. This will be done below as soon as it simplifies the notation.

For other assumptions which can be made without loss of generality, the reader is referred to Plastria [45] and Fliege [19].

2.4. Compatible Partitions. Let two subsets $E_{\leq} \subseteq E$ and $E_{\geq} \subseteq E$ of E be given with $E_{\leq} \cap E_{\geq} = \emptyset$. Then, the notation of § 2.2 can be used here immediately. As an example, we set $E_{\leq}(f,U) := E_{\leq} \cap E(f,U)$, $F_{\geq}(U,f) := F_{(F,E_{\geq})}(U,f)$ etc. The obviously arbitrary indices $\leq$ and $\geq$ on the sets $E_{\leq}$ and $E_{\geq}$ will be motivated in § 4. Here, it suffices to say that it will later be meaningful to define $E_{\geq}$ as the set of all edges $e \in E$ such that d_e is concave.

Now let a number $n \in \mathbb{N}$, $n \geq 2$, be given and $\mathcal{W} = \{W_0, \ldots, W_n\}$ be a given *partition* of F, i. e. we have $W_i \cap W_j = \emptyset$ for $i \neq j$, $i,j = 0, \ldots, n$, $W_i \neq \emptyset$ for $i = 0, \ldots, n$, and $\bigcup_{i=0}^{n} W_i = F$. Then $\mathcal{W}$ is called *compatible with* Ψ, if at most one object with known location is in W_0 and all W_i with $1 \leq i \leq n-1$ contain only objects with unknown location: $|W_0 \cap A| \leq 1$ and $\bigcup_{i=1}^{n-1} W_i \subseteq V$. In this case the set $K := W_0$ is called *kernel* of $\mathcal{W}$ and the set $S := W_n$ is called *shell* of $\mathcal{W}$. This naming scheme is motivated by the strategy of dimension reduction described in § 3 and 4. For a given compatible partition we will try to identify the nodes in the sets W_i $(1 \leq i \leq n-1)$ with the nodes in the kernel K, while we will leave the nodes in the shell S as they are. For $i \leq n$ define $U_i := \bigcup_{j=0}^{i} W_j$. For $f \in F$ let $W(f)$ be the (well-defined) set $W_i \in \mathcal{W}$ such that $f \in W_i$. Now let $f \in F \setminus S$ and $W_i = W(f)$. Then the set $\mathrm{pre}(f) := U_{i-1}$ is called the *predecessor* of f and the set $\mathrm{succ}(f) := U_{n-1} \setminus U_i$ is called the *successor* of f. If a partition $\mathcal{W} = \{W_0, \ldots, W_n\}$ is given we may without loss of generality assume that $\boldsymbol{E}(U_i, F \setminus U_i) = E(U_i, F \setminus U_i)$ $(0 \leq i \leq n-1)$ holds. If this is not the case, we turn around each edge e with the wrong direction and replace the distance cost function d_e by $\overline{d_e}$.

Now let $\mathcal{W} = \{W_0, \ldots, W_n\}$ be a partition compatible with Ψ with kernel K and shell S (i. e. $K = W_0$ and $S = W_n$). If another set $E_s \subseteq E(F \setminus S, F \setminus S)$ is given, we define the subgraph $G' = (F, E')$ by $E' := E \setminus E_s$.

3. A Single Threaded Reduction Theorem. In this section and the following one we will derive new sufficient conditions for coincidences in minisum multifacility location problems. If one of these conditions is fulfilled, we have in at least one minimum $(\hat{x}_v)_{v \in V}$ of the objective function Ψ an identity of the form $\hat{x}_f = \hat{x}_g$. In this case, we may identify the nodes f and g in the graph G. Likewise, the number of unknowns and the number of terms in the objective function which are nondifferentiable in at least one minimum are reduced. In case we have a constrained problem, we can deduce which constraints have to be active at a solution of our problem.

The next theorem contains a sufficient condition for a coincidence between exactly two objects in the node set of the graph given. This condition is formulated as an inequality between the distance cost functions which measure the distance to and from the locations of the two objects. This means that the coincidence condition is strictly local with respect to the topology of the graph that has to be embedded. A more general result characterizing coincidences of several objects is contained in the next section.

THEOREM 2. *Let $f \in V$, $g \in F$ and $(x_v)_{v \in V} \in X^V$ be given. Define $(\hat{x}_v)_{v \in V} \in X^V$ by $\hat{x}_h := x_h$ for $h \neq f$ and $\hat{x}_f := x_g$. If*

$$\sum_{e \in E(f,F)} [d_e] + \sum_{\substack{e \in E(F,f) \\ e \neq gf}} [\overline{d_e}] \leq d_{gf} \tag{3.1}$$

holds, it follows that

$$\Psi((\hat{x}_v)_{v \in V}) \leq \Psi((x_v)_{v \in V}).$$

If (3.1) *is strict and $x_f \neq x_g$, we have that $\Psi((\hat{x}_v)_{v \in V}) < \Psi((x_v)_{v \in V})$.*

Proof. The proof is a simple generalization of the proof of the attraction theorem found, e. g., in [24]. It can also be found in [21]. □

COROLLARY 3. *Let the assumptions of Theorem 2 be fulfilled. From $(x_v)_{v \in V} \in C$ it should always follow that $(\hat{x}_v)_{v \in V} \in C$. If there exists a solution to problem* (2.2), *then there exists at least one solution $(\hat{x}_v)_{v \in V}$ of* (2.2) *such that $\hat{x}_f = \hat{x}_g$. If, additionally,* (3.1) *is strict, then for each solution $(\tilde{x}_v)_{v \in V}$ of* (2.2) *we have that $\tilde{x}_f = \tilde{x}_g$.*

The relation (3.1) is often called *attraction* between f and g [20, 36, 37]. Another terminology uses the notion of *majority*, cp. [43, 44, 48].

Some remarks about further generalizations to Theorem 2 can be found in [21]. Especially, we may associate each edge $e \in E$, $e \neq (g, f)$, of the graph G with a sum $\sum_{i=1}^{m_e} d_e^{(i)}$ $(m_e \in \mathbb{N})$ of distance cost functions instead of a single function d_e. With this, a similar result holds. Moreover, the theorem above still holds if some of the functions d_e with $e \notin \boldsymbol{E}(f, F)$ are neither convex nor concave [21].

Theorem 2 considers an attraction between exactly two objects f and g. But not all coincidences fit into this scheme. We will extend the coincidence condition derived above in the next section.

4. A Multithreaded Reduction Theorem. Let $E_\leq \subseteq E$ and $E_\geq \subseteq E$ be two arbitrary fixed edge sets with $E_\leq \cup E_\geq = E$ and $E_\leq \cap E_\geq = \emptyset$. For example, we may choose $E_\leq := \{e \in E \mid d_e \text{ convex}\}$.

The next theorem defines a complex attraction structure of the following type: if a subset of all distance cost functions used fulfills certain conditions, we may identify the locations of the nodes connected by the corresponding edges without increasing the objective function value. These sufficient conditions are formulated in terms of the existence of certain distance cost functions. Since the identification process can be carried out for all families of new locations without increasing the objective function value, we can reduce the dimension of the problem, i. e. the number of unknowns. Let a partition $\mathcal{W}$ be given which is compatible with Ψ. The idea of the new attraction structure consists of the "splitting" of all nodes $v \in W_i$ $(i = n-1, \ldots, 1)$ into as many nodes as there are edges in $E'_\leq(\mathrm{pre}(v), v)$. Each of these edges (f, v) will then be redefined such that the node f is connected to its corresponding new node. This represents a "thread", starting in the set W_{n-1} and ultimately ending in the kernel K. Further, since the node v has been split up, edges of the kind $(v, s) \in E(v, S)$ will be split up into several different ones. As a consequence, the corresponding distance cost functions d_{vs} have to be split up, too. But this split into several functions, say $d_s^{f,v}$, has to be consistent with the function d_{vs}, which means that the sum of the new functions $d_s^{f,v}$ has to be equal to d_{vs}. This splitting process has to be done for all edges involved. If this is done successively for all nodes $v \in W_i$ and all W_i, $i = n-1, \ldots, 1$, we may apply in each step Theorem 2 on the spliced nodes to get attraction along the edges in $E'_\leq(\mathrm{pre}(v), v)$. Because of this attraction, we may identify pairs of nodes connected by those edges. Likewise, we may identify resulting multiple edges. In the last step, we have to use our consistency conditions to compare the constructed sequence of graphs and corresponding embedding problems with the original graph and the original embedding problem.

THEOREM 4. *Let $\mathcal{W} = \{W_0, \ldots, W_n\}$ be a partition of F which is compatible with Ψ and which has the kernel K and the shell S. Define $U := F \setminus (K \cup S)$. Let $E_s \subseteq E(F \setminus S, F \setminus S)$ be an edge set such that $d_e \geq 0$ holds for all $e \in E_s$ and define $E' := E \setminus E_s$. Assume that $F'_\leq(\mathrm{pre}(v), v) \neq \emptyset$ holds for all $v \in U$.*

For all $v \in U$, $w \in W(v) \setminus \{v\}$, $f \in F'_\leq(\mathrm{pre}(v), v)$, $g \in F'_\leq(\mathrm{pre}(w), w)$, $h \in \mathrm{pre}(v) \setminus F'_\leq(\mathrm{pre}(v), v)$, and $s \in S$ there should exist distance cost functions

$$d_s^{f,v}, \quad d_{v,w}^{f,g}, \quad d_{w,v}^{g,f}, \quad d_h^{f,v}, \quad \text{and} \quad d^{f,v},$$

such that the following conditions hold:

1. *The attraction conditions*

$$\sum_{s \in S} [d_s^{f,v}] + \sum_{\substack{w \in W(v) \setminus \{v\} \\ g \in F'_\leq(\mathrm{pre}(w), w)}} \left([d_{v,w}^{f,g}] + \left[\overline{d_{w,v}^{g,f}} \right] \right) + \sum_{h \in \mathrm{pre}(v) \setminus F'_\leq(\mathrm{pre}(v), v)} \left[\overline{d_h^{f,v}} \right] \leq d^{f,v} \tag{4.1a}$$

for all $v \in U$ and $f \in F'_\leq(\mathrm{pre}(v), v)$,

2. *the consistency conditions*

$$\sum_{f \in F'_\leq(\mathrm{pre}(v), v)} d_s^{f,v} = \sum_{vs \in E} d_{vs} + \sum_{u \in F'_\leq(v, \mathrm{succ}(v))} d_s^{v,u} \tag{4.1b}$$

for all $v \in U$ and $s \in S$,

$$\sum_{\substack{f \in F'_{\leq}(\mathrm{pre}(v),v) \\ g \in F'_{\leq}(\mathrm{pre}(w),w)}} d^{f,g}_{v,w} = \sum_{vw \in E'} d_{vw} + \sum_{\substack{u \in F'_{\leq}(w,\mathrm{succ}(w)) \\ \text{with } vu \notin E'_{\leq}}} d^{w,u}_{v} + \sum_{\substack{t \in F'_{\leq}(v,\mathrm{succ}(v)) \\ u \in F'_{\leq}(w,\mathrm{succ}(w)) \\ W(t)=W(u), t \neq u}} d^{v,w}_{t,u} \tag{4.1c}$$

for all $v, w \in U$ with $w \in W(v) \setminus \{v\}$,

$$\begin{aligned} \sum_{f \in F'_{\leq}(\mathrm{pre}(v),v)} d^{f,v}_{h} = {} & \sum_{hv \in E'_{\geq}} d_{hv} + \sum_{\substack{u \in F'_{\leq}(v,\mathrm{succ}(v)) \\ \text{with } h \notin F'_{\leq}(\mathrm{pre}(u),u)}} d^{v,u}_{h} \\ & + \sum_{\substack{u \in F'_{\leq}(h,\mathrm{succ}(v)) \\ \text{with } v \notin F'_{\leq}(\mathrm{pre}(u),u)}} \overline{d^{h,u}_{v}} + \sum_{\substack{t \in F'_{\leq}(v,\mathrm{succ}(v)) \\ u \in F'_{\leq}(h,\mathrm{succ}(v)) \\ W(t)=W(u), t \neq u}} \left(d^{h,v}_{u,t} + \overline{d^{v,h}_{t,u}} \right) \end{aligned} \tag{4.1d}$$

for all $v \in U$ and $h \in \mathrm{pre}(v) \setminus F'_{\leq}(\mathrm{pre}(v),v)$, and

$$\begin{aligned} d^{f,v} = {} & \sum_{fv \in E'} d_{fv} + \sum_{\substack{u \in F'_{\leq}(v,\mathrm{succ}(v)) \\ \text{with } f \notin F'_{\leq}(\mathrm{pre}(u),u)}} d^{v,u}_{f} \\ & + \sum_{\substack{u \in F'_{\leq}(f,\mathrm{succ}(v)) \\ \text{with } v \notin F'_{\leq}(\mathrm{pre}(u),u)}} \overline{d^{f,u}_{v}} + \sum_{\substack{t \in F'_{\leq}(f,\mathrm{succ}(v)) \\ u \in F'_{\leq}(v,\mathrm{succ}(v)) \\ W(t)=W(u), t \neq u}} \left(d^{f,v}_{t,u} + \overline{d^{v,f}_{u,t}} \right) \end{aligned} \tag{4.1e}$$

for all $v \in U$ and $f \in F'_{\leq}(\mathrm{pre}(v),v)$.
Now, let an $x \in X$ and a family $(x_v)_{v \in V} \in X^V$ be given with $x_k = x$ for all $k \in K$. Define $(\hat{x}_v)_{v \in V} \in X^V$ by $\hat{x}_v := x$ for all $v \in V \setminus S$ and $\hat{x}_s := x_s$ for all $s \in V \cap S$. Then, it follows that

$$\Psi((\hat{x}_v)_{v \in V}) \leq \Psi((x_v)_{v \in V}).$$

This inequality is strict, if (4.1a) is strict for an edge $(f,v) \in E'_{\leq}$ with $x_f \neq x_v$ or if there exists an edge $e \in E_s$ such that $d_e(x_e) \neq 0$ holds.

COROLLARY 5. *Let the assumptions of Theorem 4 hold. Define*

$$\mathcal{K} := \{(x_v)_{v \in V} \in X \mid x_j = x_k \text{ for all } j, k \in K\}.$$

From $(x_v)_{v \in V} \in C \cap \mathcal{K}$ it should always follow that $(\hat{x}_v)_{v \in V} \in C \cap \mathcal{K}$. If there exists a solution $(x_v)_{v \in V}$ of problem (2.2) with $(x_v)_{v \in V} \in \mathcal{K}$, then there exists at least one solution $(\hat{x}_v)_{v \in V}$ of (2.2) with $\hat{x}_u = \hat{x}_k$ for all $u \in U$, $k \in K$. If, additionally, (4.1a) is strict for an edge $(f,v) \in E'_{\leq}$, we have for each solution $(\tilde{x}_v)_{v \in V}$ of (2.2) with $(\tilde{x}_v)_{v \in V} \in \mathcal{K}$ that $\tilde{x}_f = \tilde{x}_v$. Moreover, we have for each solution $(\tilde{x}_v)_{v \in V}$ of (2.2) with $(\tilde{x}_v)_{v \in V} \in \mathcal{K}$ that $d_e(\tilde{x}_e) = 0$ holds for all $e \in E_s$.

Proof of Theorem 4. To proof the theorem, we define for each index i, $i = n-1, \ldots, 1$, a new multifacility location problem in which a graph G'_i has to be embedded into the space X by minimizing a function Ψ'_i. Fixed nodes in G'_i will be the nodes in $S \cup A$, free nodes will be the nodes in $U_{i-1} \cap V$. Moreover, we will split each node $v \in W_i$ in as many nodes as there are edges in $E'_{\leq}$ connecting v with U_{i-1}.

By using (4.1a), we will then show attraction of the split up nodes into the set U_{i-1}. The structure of the graph G'_{n-1} is given by the embedding of G induced by $(x_v)_{v\in V}$. The embedding induced by the coincidence result on G'_i will induce the structure of the graph G'_{i-1}. Finally, we will use the consistency conditions (4.1b)–(4.1e) to compare the embedding costs of G'_1 with $\Psi((\hat{x}_v)_{v\in V})$, and we will conclude that these two costs are equal.

In what follows, note that $E(U_i, F\setminus U_i) = \boldsymbol{E}(U_i, F\setminus U_i)$ for $0\le i\le n-1$.

For $i = n-1,\dots,1$ define

$$V_i := \left(V \setminus \bigcup_{j=i}^{n-1} W_j\right) \cup \{v_e \mid e\in E'_{\le}(U_{i-1},W_i)\}$$

and $\tilde{V}_i := V_i\setminus V$. Here, the nodes v_e $(e\in E'_{\le}(U_{i-1},W_i))$ should not be contained in V and for edges $e_1,e_2\in E'_{\le}(U_{i-1},W_i)$ with $e_1\ne e_2$ we should have $v_{e_1}\ne v_{e_2}$.

Now define $F_i := A\cup V_i$ and

$$\begin{aligned}
E'_i := {} & (U_{i-1}\times U_{i-1})\setminus\{(f,f)\mid f\in U_{i-1}\}\\
&\cup \left(\tilde{V}_i\times S\right)\cup (U_{i-1}\times S)\cup E'(S,S)\\
&\cup \{(f,v_e)\mid e=(f,v)\in E'_{\le}(U_{i-1},W_i)\}\\
&\cup \{(h,v_e)\mid e=(f,v)\in E'_{\le}(U_{i-1},W_i),\ h\in U_{i-1}\setminus F'_{\le}(\mathrm{pre}(v),v)\}\\
&\cup \{(v_{e_1},v_{e_2})\mid e_1=(f,v)\in E'_{\le}(U_{i-1},W_i),\\
&\qquad\qquad e_2=(g,w)\in E'_{\le}(U_{i-1},W_i),\ v\ne w\}
\end{aligned}$$

as well as $G'_i := (F_i,E'_i)$. Note that the graph G'_i may contain multiple edges, since for nodes $f,g\in U_{i-1}$ with $f\ne g$ we have $(f,g)\in E'_i$ and $(g,f)\in E'_i$. Likewise, for edges $e_1,e_2\in E'_{\le}(U_{i-1},W_i)$, $e_1=(f,v)$, $e_2=(g,w)$ with $v\ne w$ we have $(v_{e_1},v_{e_2})\in E'_i$ and $(v_{e_2},v_{e_1})\in E'_i$.

Next, we have to provide the graph G'_i with semi-definite functions $d_e^{(i)}$. This is done in the following way:

1. $e=(f,g)\in U_{i-1}\times U_{i-1}$, $f\ne g$: for $i=n-1$ set $d_e^{(n-1)} := \sum_{e\in E'} d_e$, for $1\le i<n-1$ set

$$d_e^{(i)} := d_e^{(i+1)} + \sum_{\substack{u\in F'_{\le}(f,W_{i+1})\\ v\in F'_{\le}(g,W_{i+1})\\ u\ne v}} d_{u,v}^{f,g} + \sum_{\substack{v\in F'_{\le}(g,W_{i+1})\\ \text{with } f\notin F'_{\le}(\mathrm{pre}(v),v)}} d_f^{g,v}.$$

Usually, the functions $d_{fg}^{(i)}$ are neither convex nor concave. However, each such function is a sum of convex or concave semi-definite functions.

2. $e=(v_{\overline{e}},s)\in\tilde{V}_i\times S$ with $\overline{e}=(f,v)\in E'_{\le}(U_{i-1},W_i)$: $d_e^{(i)} := d_s^{f,v}$.
3. $e=(f,s)\in U_{i-1}\times S$: set

$$d_e^{(i)} := \sum_{fs\in E'} d_{fs} + \sum_{j=i+1}^{n-1}\ \sum_{u\in F'_{\le}(f,W_j)} d^{(j)}_{(v_{fu},s)}. \tag{4.2}$$

As in the first case, this function is a sum of convex or concave semi-definite functions.

4. $e\in E'(S,S)$: $d_e^{(i)} := d_e$.

5. $e = (f, v_{\overline{e}})$ with $\overline{e} = (f, v) \in E'_{\leq}(U_{i-1}, W_i)$: $d_e^{(i)} := d^{f,v}$.

6. $e = (h, v_{\overline{e}})$ with $\overline{e} = (f, v) \in E'_{\leq}(U_{i-1}, W_i)$, $h \in U_{i-1}$ but $h \notin F'_{\leq}(\mathrm{pre}(v), v)$: $d_e^{(i)} := d_h^{f,v}$.

7. $e \in (v_{e_1}, v_{e_2})$ with $e_1, e_2 \in E'_{\leq}(U_{i-1}, W_i)$, $e_1 = (f, v)$, $e_2 = (g, w)$, $v \neq w$: $d_e^{(i)} := d_{v,w}^{f,g}$.

Define the families $(x_v^{(i)})_{v \in V_i} \in X^{V_i}$ and $(\tilde{x}_v^{(i)})_{v \in V_i} \in X^{V_i}$ by

$$\begin{aligned}
x_v^{(i)} &:= x_v \text{ for } v \in V \cap V_i = V \setminus \tilde{V}_i = (U_{i-1} \cup S) \cap V, \\
x_{v_e}^{(i)} &:= x_v \text{ for } e = (f, v) \in E'_{\leq}(U_{i-1}, W_u), \text{ i. e. } v_e \in \tilde{V}_i, \\
\tilde{x}_v^{(i)} &:= x_v \text{ for } v \in V \cap V_i, \text{ and} \\
\tilde{x}_{v_e}^{(i)} &:= x_f \text{ for } e = (f, v) \in E'_{\leq}(U_{i-1}, W_i), \text{ i. e. } v_e \in \tilde{V}_i.
\end{aligned}$$

This means that the family $(\tilde{x}_v^{(i)})_{v \in V_i}$ is generated from the family $(x_v^{(i)})_{v \in V_i}$ by coinciding each node $v_{fv} \in \tilde{V}_i$ with the node $f \in U_{i-1}$.

As auxiliary functions, we define

$$\Psi'_i((y_v)_{v \in V_i}) := \sum_{e \in E'_i} d_e^{(i)}(y_e),$$

where $(y_v)_{v \in V_i} \in X^{V_i}$. For an index $i \in \{1, \ldots, n-1\}$, let a node $v_e \in \tilde{V}_i$ with $e = (f, v) \in E'_{\leq}(U_{i-1}, W_i)$ be given. Then we have

$$\begin{aligned}
\boldsymbol{E}'_i(v_e, F_i) = (\{v_e\} \times S) &\cup \{(f, v_e)\} \\
&\cup \{(h, v_e) \mid h \in U_{i-1} \setminus F'_{\leq}(\mathrm{pre}(v), v)\} \\
&\cup \{(v_e, v_{\overline{e}}) \mid \overline{e} = (g, w) \in E'_{\leq}(U_{i-1}, W_i), v \neq w\} \\
&\cup \{(v_{\overline{e}}, v_e) \mid \overline{e} = (g, w) \in E'_{\leq}(U_{i-1}, W_i), v \neq w\},
\end{aligned}$$

and the definitions of the $d_e^{(i)}$ above show us that

$$\sum_{e \in E'_i(v_e, F_i)} d_e^{(i)} = \sum_{s \in S} d_s^{f,v} + \sum_{\substack{gw \in E'_{\leq}(U_{i-1}, W_i) \\ \text{with } w \neq v}} d_{v,w}^{f,g}$$

and

$$\sum_{\substack{e \in E'_i(F_i, v_e) \\ \text{with } e \neq (f, v_e)}} d_e^{(i)} = \sum_{h \in \mathrm{pre}(v) \setminus F'_{\leq}(\mathrm{pre}(v), v)} d_h^{f,v} + \sum_{\substack{gw \in E'_{\leq}(U_{i-1}, W_i) \\ \text{with } w \neq v}} d_{w,v}^{g,f}$$

holds. Moreover, we have $d_{(f, v_e)}^{(i)} = d^{f,v}$. Because of (4.1a), the node v_e is attracted by f in the graph G'_i. Using Theorem 2, we conclude

$$\Psi'_i((\tilde{x}_v^{(i)})_{v \in V_i}) \leq \Psi'_i((x_v^{(i)})_{v \in V_i}). \tag{4.3}$$

This inequality is strict, if an edge $(f, v) \in E'_{\leq}(U_{i-1}, W_i)$ with $x_f \neq x_v$ exists such that (4.1a) is strict for all arguments $(x, y) \in (X \times X) \setminus \Delta$.

With

$$\Psi'((y_v)_{v \in V}) := \sum_{e \in E'} d_e(y_e),$$

$(y_v)_{v\in V} \in X^V$, it follows that

$$\Psi'_{n-1}((x_v^{(n-1)})_{v\in V_{n-1}}) = \sum_{e\in E'_{n-1}} d_e^{(n-1)}(x_e^{(n-1)})$$

$$= \sum_{e\in E'(U_{n-2},U_{n-2})} d_e(x_e) + \sum_{(f,v)\in E'_{\leq}(U_{n-2},W_{n-1})} \sum_{s\in S} d_s^{f,v}(x^{(n-1)}_{(v_{fv},s)})$$

$$+ \sum_{(f,v)\in E'_{\leq}(U_{n-2},W_{n-1})} d^{f,v}(x^{(n-1)}_{(f,v_{fv})}) + \sum_{\substack{f\in U_{n-2}\\ s\in S}} \sum_{fs\in E'} d_{fs}(x_{fs})$$

because of (4.2),

$$+ \sum_{e\in E'(S,S)} d_e(x_e) + \sum_{(f,v)\in E'_{\leq}(U_{n-2},W_{n-1})} \sum_{h\in \mathrm{pre}(v)\setminus F'_{\leq}(\mathrm{pre}(v),v)} d_h^{f,v}(x^{(n-1)}_{(h,v_{fv})})$$

$$+ \sum_{\substack{fv\in E'_{\leq}(U_{n-2},W_{n-1})\\ gw\in E'_{\leq}(U_{n-2},W_{n-1})\\ v\neq w}} d_{v,w}^{f,g}(x^{(n-1)}_{(v_{fv},v_{gw})})$$

because of the definitions of the $d_e^{(n-1)}$ and of E'_{n-1},

$$= \sum_{\substack{e\in E'(U_{n-2},U_{n-2})\\ e\in E'(S,S)}} d_e(x_e) + \sum_{e\in E'(U_{n-2},S)} d_e(x_e)$$

$$+ \sum_{v\in W_{n-1}} \sum_{s\in S} d_{vs}(x_{vs}) + \sum_{(f,v)\in E'_{\leq}(U_{n-2},W_{n-1})} d_{fv}(x_{fv})$$

because of (4.1b) and of (4.1e),

$$\sum_{v\in W_{n-1}} \sum_{h\in F'_{\geq}(\mathrm{pre}(v),v)} d_{hv}(x_{hv}) + \sum_{v\in W_{n-1}} \sum_{vw\in E'(v,W_{n-1})} d_{vw}(x_{vw})$$

because of (4.1d) and of (4.1c),

$$= \Psi'((x_v)_{v\in V}).$$

(In the penultimate step we have used $\mathrm{succ}(v) = \emptyset$ for all $v \in W_{n-1}$.) Let $2 \leq i \leq n-1$. We will now show

$$\Psi'_{i-1}((x_v^{(i-1)})_{v\in V_{i-1}}) = \Psi'_i((\tilde{x}_v^{(i)})_{v\in V_i}). \tag{4.4}$$

We have to examine the distance cost functions corresponding to the edges in E'_{i-1} and E'_i, resp. There are seven different cases to consider:

(i) For $e \in E'_i(S,S) = E'_{i-1}(S,S) = E'(S,S)$ we have $\tilde{x}_e^{(i)} = x_e^{(i-1)} = x_e$ and $d_e^{(i-1)} = d_e^{(i)} = d_e$. It follows that $d_e^{(i-1)}(x_e^{(i-1)}) = d_e^{(i)}(\tilde{x}_e^{(i)})$.

(ii) Let $s \in S$ be fixed.

1. Let $f \in U_{i-2}$. Then, $e = (f,s) \in E'_{i-1} \cap E'_i$ and $x_{fs}^{(i-1)} = \tilde{x}_{fs}^{(i)}$. Using (4.2), it follows that

$$d_e^{(i-1)} = \sum_{fs\in E'} d_{fs} + \sum_{u\in F'_{\leq}(f,\mathrm{succ}(f))\setminus U_{i-1}} d_s^{f,u}$$

and, analogously,

$$d_e^{(i)} = \sum_{fs\in E'} d_{fs} + \sum_{u\in F'_{\leq}(f,\mathrm{succ}(f))\setminus U_i} d_s^{f,u}.$$

Now let $v \in W_i = U_i \setminus U_{i-1}$ and $v \in F'_{\leq}(f, \mathrm{succ}(f))$, i. e. $\overline{e} = (f, v) \in E'_{\leq}$. Then there exists exactly one $v_{\overline{e}} \in \tilde{V}_i$ with $(f, v_{\overline{e}}) \in E'_i$, $d^{(i)}_{(v_{\overline{e}}, s)} = d^{f,v}_s$, and $\tilde{x}^{(i)}_{v_{\overline{e}}} = x_f$. This means that

$$d^{(i-1)}_{fs}(x^{(i-1)}_{fs}) = d^{(i)}_{fs}(\tilde{x}^{(i)}_{fs}) + \sum_{v \in F'_{\leq}(f, W_i)} d^{(i)}_{(v_{fv}, s)}(\tilde{x}^{(i)}_{(v_{fv}, s)}). \tag{4.5}$$

2. Let $v \in W_{i-1}$. For each $f \in U_{i-2}$ with $e = (f, v) \in E'_{\leq}$ there exists exactly one $v_e \in \tilde{V}_{i-1}$ with $x^{(i-1)}_{v_e} = x_v$ and $(f, v_e), (v_e, s) \in E'_{i-1}$. Moreover, (4.1b) leads to

$$\sum_{e = fv \in E'_{\leq}(\mathrm{pre}(v), v)} d^{(i-1)}_{(v_e, s)} = \sum_{vs \in E} d_{vs} + \sum_{u \in F'_{\leq}(v, \mathrm{succ}(v))} d^{v,u}_s.$$

But with (4.2) we have

$$d^{(i)}_{vs} = \sum_{vs \in E} d_{vs} + \sum_{u \in F'_{\leq}(v, \mathrm{succ}(v)) \setminus U_i} d^{v,u}_s.$$

For exactly those $u \in W_i$ with $(v, u) \in E'_{\leq}$ there exists a $v_{vu} \in F'_i$ such that $(v, v_{vu}), (v_{vu}, s) \in E'_i$, $\tilde{x}_{v_{vu}} = x_v$, and $d^{(i)}_{(v_{vu}, s)} = d^{v,u}_s$. By using $x^{(i-1)}_{(v_e, s)} = \tilde{x}^{(i)}_{vs} = \tilde{x}^{(i)}_{(v_{vu}, s)}$ we arrive at

$$\sum_{e = fv \in E'_{\leq}(\mathrm{pre}(v), v)} d^{(i-1)}_{(v_e, s)}(x^{(i-1)}_{(v_e, s)}) = d^{(i)}_{vs}(\tilde{x}^{(i)}_{vs}) + \sum_{u \in F'_{\leq}(v, W_i)} d^{(i)}_{(v_{vu}, s)}(\tilde{x}^{(i)}_{(v_{vu}, s)}). \tag{4.6}$$

Using (4.5) and (4.6), we see that

$$\sum_{\substack{e \in E'_{i-1}(\tilde{V}_{i-1}, S) \\ e \in E'_{i-1}(U_{i-2}, S)}} d^{(i-1)}_e(x^{(i-1)}_e) = \sum_{\substack{e \in E'_i(\tilde{V}_i, S) \\ e \in E'_i(U_{i-1}, S)}} d^{(i)}_e(\tilde{x}^{(i)}_e). \tag{4.7}$$

(iii) Let $e = (v_{e_1}, v_{e_2}) \in E'_{i-1}(\tilde{V}_{i-1}, \tilde{V}_{i-1})$, which means that we have $e_1 = (f, v) \in E'_{\leq}(U_{i-2}, W_{i-1})$ and $e_2 = (g, w) \in E'_{\leq}(U_{i-2}, W_{i-1})$ for some $f, g \in U_{i-2}$ and $v, w \in W_{i-1}$. Moreover, it is $v \neq w$. The definition above shows us that $d^{(i-1)}_e = d^{f,g}_{v,w}$. Likewise, the definition of the $d^{(i)}_{vw}$ for $i < n-1$ shows us that we have

$$\begin{aligned} d^{(i)}_{vw} &= d^{(i+1)}_{vw} + \sum_{\substack{t \in F'_{\leq}(v, W_{i+1}) \\ u \in F'_{\leq}(w, W_{i+1}) \\ t \neq u}} d^{v,w}_{t,u} + \sum_{\substack{t \in F'_{\leq}(w, W_{i+1}) \\ \text{with } v \notin F'_{\leq}(\mathrm{pre}(t), t)}} d^{w,t}_v \\ &= \sum_{vw \in E'} d_{vw} + \sum_{\substack{t \in F'_{\leq}(v, \mathrm{succ}(v)) \setminus W_i \\ u \in F'_{\leq}(w, \mathrm{succ}(w)) \setminus W_i \\ t \neq u, W(t) = W(u)}} d^{v,w}_{t,u} + \sum_{\substack{t \in F'_{\leq}(w, \mathrm{succ}(w)) \setminus W_i \\ \text{with } v \notin F'_{\leq}(\mathrm{pre}(t), t)}} d^{w,t}_v. \end{aligned}$$

But for $i = n-1$ it is $d^{(i)}_{vw} = \sum_{vw \in E'} d_{vw}$. This means that

$$d^{(i)}_{vw} = \sum_{vw \in E'} d_{vw} + \sum_{\substack{t \in F'_{\leq}(v, \mathrm{succ}(v)) \setminus W_i \\ u \in F'_{\leq}(w, \mathrm{succ}(w)) \setminus W_i \\ t \neq u, W(t) = W(u)}} d^{v,w}_{t,u} + \sum_{\substack{t \in F'_{\leq}(w, \mathrm{succ}(w)) \setminus W_i \\ \text{with } v \notin F'_{\leq}(\mathrm{pre}(t), t)}} d^{w,t}_v \tag{4.8}$$

holds for $2 \le i \le n-1$, since for $i = n-1$ the index sets of the last two sums are empty. Using (4.1c) and (4.8), we see that

$$\sum_{\substack{f \in F'_{\le}(\mathrm{pre}(v),v) \\ g \in F'_{\le}(\mathrm{pre}(w),w)}} d^{f,g}_{v,w} = d^{(i)}_{vw} + \sum_{\substack{t \in W_i \cap F'_{\le}(v,\mathrm{succ}(v)) \\ u \in W_i \cap F'_{\le}(w,\mathrm{succ}(w)) \\ t \ne u}} d^{v,w}_{t,u} + \sum_{\substack{t \in W_i \cap F'_{\le}(w,\mathrm{succ}(w)) \\ \text{with } v \notin F'_{\le}(\mathrm{pre}(t),t)}} d^{w,t}_{v}$$

holds for all nodes $v, w \in W_{i-1}$ with $v \ne w$. For $t \in W_i \cap F'_{\le}(v, \mathrm{succ}(v)) = F'_{\le}(v, W_i)$ and $u \in W_i \cap F'_{\le}(w, \mathrm{succ}(w)) = F'_{\le}(w, W_i)$ with $t \ne u$ there exist $v_{(v,t)}, v_{(w,u)} \in \tilde{V}_i$ with $(v_{(v,t)}, v_{(w,u)}) \in E'_i$ and, using the definitions, $d^{(i)}_{(v_{vt}, v_{wu})} = d^{v,w}_{t,u}$. On the other hand, for all edges $e = (v_{vt}, v_{wu}) \in E'_i(\tilde{V}_i, \tilde{V}_i)$ there exist nodes $t, u \in W_i$ with $t \ne u$ and $(v,t), (w,u) \in E'_{\le}(U_{i-1}, W_i)$. Likewise, for exactly those $t \in F'_{\le}(w, W_i)$ with $v \notin F'_{\le}(\mathrm{pre}(t), t)$ there exists a node $v_{(w,t)} \in \tilde{V}_i$ such that $(v, v_{(w,t)}) \in E'_i$ and $d^{(i)}_{(v, v_{wt})} = d^{w,t}_{v}$ holds. For $e = (v_{vt}, v_{wu}) \in E'_i$ or $e = (v, v_{wt}) \in E'_i$ we have $\tilde{x}^{(i)}_e = x_{vw}$. For $e = (v_{e_1}, v_{e_2}) \in E'_{i-1}$ as above it follows that $x^{(i-1)}_e = x_{vw}$. Therefore, for fixed $v, w \in W_{i-1}$ with $v \ne w$ we have

$$\sum_{\substack{e=(v_{fv}, v_{gw}) \\ \in E'_{i-1}(\tilde{V}_{i-1}, \tilde{V}_{i-1})}} d^{(i-1)}_e(x^{(i-1)}_e) = d^{(i)}_{vw}(\tilde{x}^{(i)}_{vw}) + \sum_{\substack{e=(v, v_{wu}) \in E'_i(v, \tilde{V}_i) \\ \text{with } v \notin F'_{\le}(\mathrm{pre}(u),u)}} d^{(i)}_e(\tilde{x}^{(i)}_e) + \sum_{e=(v_{vt}, v_{wu}) \in E'_i(\tilde{V}_i, \tilde{V}_i)} d^{(i)}_e(\tilde{x}^{(i)}_e). \tag{4.9}$$

Here, the sum on the left-hand side is taken over all edges $e \in E'_{i-1}(\tilde{V}_{i-1}, \tilde{V}_{i-1})$ which can be written as $e = (v_{fv}, v_{gw})$ with some $f \in F'_{\le}(\mathrm{pre}(v), v)$ and $g \in F'_{\le}(\mathrm{pre}(w), w)$.

(iv) Let $e = (f, v_{\overline{e}}) \in E'_i(U_{i-1}, \tilde{V}_i)$ with $\overline{e} = (f, v) \in E'_{\le}(U_{i-1}, W_i)$. Since $\tilde{x}^{(i)}_e = x_{ff}$ we have

$$d^{(i)}_e(\tilde{x}^{(i)}_e) = 0.$$

(v) Let $f, g \in U_{i-2}$ with $f \ne g$. Using the definition of $d^{(i-1)}_{fg}$ and the same way of reasoning as in case (iii), we arrive at

$$d^{(i-1)}_{fg}(x^{(i-1)}_{fg}) = d^{(i)}_{fg}(\tilde{x}^{(i)}_{fg}) + \sum_{\substack{e=(f, v_{gv}) \in E'_i(U_{i-2}, \tilde{V}_i) \\ \text{with } f \notin F'_{\le}(\mathrm{pre}(v),v) \\ gv \in E'_{\le}}} d^{(i)}_e(\tilde{x}^{(i)}_e) + \sum_{e=(v_{fu}, v_{gv}) \in E'_i(\tilde{V}_i, \tilde{V}_i)} d^{(i)}_e(\tilde{x}^{(i)}_e). \tag{4.10}$$

(vi) Let $e = (f, v_{\overline{e}}) \in E'_{i-1}(U_{i-2}, \tilde{V}_{i-1})$ with $\overline{e} = (f, v) \in E'_{\le}(U_{i-2}, W_{i-1})$. Using (4.1e) and the ideas that lead to (4.8), we see that

$$d^{(i-1)}_{(f, v_{\overline{e}})}(x^{(i-1)}_{(f, v_{\overline{e}})}) = d^{(i)}_{fv}(\tilde{x}^{(i)}_{fv}) + d^{(i)}_{vf}(\tilde{x}^{(i)}_{vf}) + \sum_{\substack{e=(v_{ft}, v_{vu}) \in E'_i(\tilde{V}_i, \tilde{V}_i)(t \ne u) \\ e=(v_{vu}, v_{ft}) \in E'_i(\tilde{V}_i, \tilde{V}_i)(t \ne u)}} d^{(i)}_e(\tilde{x}^{(i)}_e) + \sum_{\substack{e=(f, v_{vu}) \in E'_i(f, \tilde{V}_i) \\ \text{with } f \notin F'_{\le}(\mathrm{pre}(u),u)}} d^{(i)}_e(\tilde{x}^{(i)}_e) + \sum_{\substack{e=(v, v_{fu}) \in E'_i(v, \tilde{V}_i) \\ \text{with } v \notin F'_{\le}(\mathrm{pre}(u),u)}} d^{(i)}_e(\tilde{x}^{(i)}_e) \tag{4.11}$$

holds.

(vii) Let $(h, v_{\overline{e}}) \in E'_{i-1}(U_{i-2}, \tilde{V}_{i-1})$ with $\overline{e} = (f, v) \in E'_{\leq}(U_{i-2}, W_{i-1})$, but $(h, v) \notin E'_{\leq}(U_{i-2}, W_{i-1})$. Using (4.1d) and the ideas leading to (4.8) again, we get

$$
\begin{aligned}
&\sum_{\substack{\overline{e}=(f,v)\in E'_{\leq}(U_{i-2},W_{i-1})\\ \text{with } h\notin F'_{\leq}(\mathrm{pre}(v),v)}} d^{(i-1)}_{(h,v_{\overline{e}})}(x^{(i-1)}_{(h,v_{\overline{e}})}) = d^{(i)}_{hv}(\tilde{x}^{(i)}_{hv}) + d^{(i)}_{vh}(\tilde{x}^{(i)}_{vh}) \\
&+ \sum_{\substack{e=(v_{ht},v_{vu})\in E'_i(\tilde{V}_i,\tilde{V}_i)\\ e=(v_{vu},v_{ht})\in E'_i(\tilde{V}_i,\tilde{V}_i)\\ t\neq u}} d^{(i)}_e(\tilde{x}^{(i)}_e) + \sum_{\substack{e=(h,v_{vu})\in E'_i(h,\tilde{V}_i)\\ \text{with } h\notin F'_{\leq}(\mathrm{pre}(u),u)}} d^{(i)}_e(\tilde{x}^{(i)}_e) \\
&+ \sum_{\substack{e=(v,v_{hu})\in E'_i(v,\tilde{V}_i)\\ \text{with } v\notin F'_{\leq}(\mathrm{pre}(u),u)}} d^{(i)}_e(\tilde{x}^{(i)}_e).
\end{aligned} \tag{4.12}
$$

We have examined all edges necessary for the rest of the proof. We see that

$$
\Psi'_{i-1}((x^{(i-1)}_v)_{v\in V_{i-1}}) = \sum_{e\in E'_{i-1}} d^{(i-1)}_e(x^{(i-1)}_e)
$$

$$
= \sum_{e\in E'_i(S,S)} d^{(i)}_e(\tilde{x}^{(i)}_e)
$$

because of the first case of the examination above,

$$
+ \sum_{e\in E'_i(U_{i-1},S)} d^{(i)}_e(\tilde{x}^{(i)}_e) + \sum_{e\in E'_i(\tilde{V}_i,S)} d^{(i)}_e(\tilde{x}^{(i)}_e)
$$

because of (4.7),

$$
+ \sum_{e\in E'_{i-1}(F_{i-1}\setminus S, F_{i-1}\setminus S)} d^{(i-1)}_e(x^{(i-1)}_e)
$$

$$
\begin{aligned}
= &\sum_{e\in E'_i(S,S)} d^{(i)}_e(\tilde{x}^{(i)}_e) + \sum_{e\in E'_i(U_{i-1},S)} d^{(i)}_e(\tilde{x}^{(i)}_e) + \sum_{e\in E'_i(\tilde{V}_i,S)} d^{(i)}_e(\tilde{x}^{(i)}_e) \\
&+ \sum_{e\in E'_i(W_{i-1},W_{i-1})} d^{(i)}_e(\tilde{x}^{(i)}_e) + \sum_{\substack{e=(v,v_{wu})\in E'_i(W_{i-1},\tilde{V}_i)\\ \text{with } v\neq w\in W_{i-1},\, v\notin F'_{\leq}(\mathrm{pre}(u),u)}} d^{(i)}_e(\tilde{x}^{(i)}_e) \\
&+ \sum_{\substack{e=(v_{vt},v_{wu})\in E'_i(\tilde{V}_i,\tilde{V}_i)\\ t,u\in W_i;\, v,w\in W_{i-1},\, v\neq w}} d^{(i)}_e(\tilde{x}^{(i)}_e)
\end{aligned}
$$

because of (4.9),

$$
\begin{aligned}
&+ \sum_{e\in E'_i(U_{i-2},U_{i-2})} d^{(i)}_e(\tilde{x}^{(i)}_e) + \sum_{\substack{e=(f,v_{gw})\in E'_i(U_{i-2},\tilde{V}_i)\\ \text{with } f\notin F'_{\leq}(U_{i-2},w),\, g\in U_{i-2}\setminus\{f\}}} d^{(i)}_e(\tilde{x}^{(i)}_e) \\
&+ \sum_{\substack{e=(v_{fu},v_{gw})\in E'_i(\tilde{V}_i,\tilde{V}_i)\\ \text{with } f,g\in U_{i-2}}} d^{(i)}_e(\tilde{x}^{(i)}_e)
\end{aligned}
$$

because of (4.10),

$$+ \sum_{fv\in E'_{\leq}(U_{i-2},W_{i-1})} \Bigg(d_{fv}^{(i)}(\tilde{x}_{fv}^{(i)}) + d_{vf}^{(i)}(\tilde{x}_{vf}^{(i)}) + \sum_{\substack{e=(v,v_{fu})\in E'_i(v,\tilde{V}_i)\\ \text{with } v\notin F'_{\leq}(\text{pre}(u),u)}} d_e^{(i)}(\tilde{x}_e^{(i)})$$

$$+ \sum_{\substack{e=(f,v_{vu})\in E'_i(f,\tilde{V}_i)\\ \text{with } f\notin F'_{\leq}(\text{pre}(u),u)}} d_e^{(i)}(\tilde{x}_e^{(i)}) + \sum_{\substack{e=(v_{ft},v_{vu})\in E'_i(\tilde{V}_i,\tilde{V}_i)(t\neq u)\\ e=(v_{vu},v_{ft})\in E'_i(\tilde{V}_i,\tilde{V}_i)(t\neq u)}} d_e^{(i)}(\tilde{x}_e^{(i)}) \Bigg)$$

because of (4.11),

$$+ \sum_{\substack{(h,v)\in E'_i(U_{i-2},W_{i-1})\\ \text{with } h\notin F'_{\leq}(\text{pre}(v),v)}} \Bigg(d_{hv}^{(i)}(\tilde{x}_{hv}^{(i)}) + d_{vh}^{(i)}(\tilde{x}_{vh}^{(i)}) + \sum_{\substack{e=(h,v_{vu})\in E'_i(U_{i-2},\tilde{V}_i)\\ \text{with } h\notin F'_{\leq}(\text{pre}(u),u)}} d_e^{(i)}(\tilde{x}_e^{(i)})$$

$$+ \sum_{\substack{e=(v_{ht},v_{vu})\in E'_i(\tilde{V}_i,\tilde{V}_i)\\ e=(v_{vu},v_{ht})\in E'_i(\tilde{V}_i,\tilde{V}_i)\\ t\neq u}} d_e^{(i)}(\tilde{x}_e^{(i)}) + \sum_{\substack{e=(v,v_{hu})\in E'_i(W_{i-1},\tilde{V}_i)\\ \text{with } v\notin F'_{\leq}(\text{pre}(u),u)}} d_e^{(i)}(\tilde{x}_e^{(i)}) \Bigg)$$

because of (4.12),

$$= \sum_{e\in E'_i(S,S)} d_e^{(i)}(\tilde{x}_e^{(i)}) + \sum_{e\in E'_i(U_{i-1},S)} d_e^{(i)}(\tilde{x}_e^{(i)}) + \sum_{e\in E'_i(\tilde{V}_i,S)} d_e^{(i)}(\tilde{x}_e^{(i)})$$
$$+ \sum_{e\in E'_i(U_{i-2},U_{i-2})} d_e^{(i)}(\tilde{x}_e^{(i)}) + \sum_{e\in E'_i(W_{i-1},W_{i-1})} d_e^{(i)}(\tilde{x}_e^{(i)}) + \sum_{e\in E'_i(U_{i-2},W_{i-1})} d_e^{(i)}(\tilde{x}_e^{(i)})$$
$$+ \sum_{e\in E'_i(W_{i-1},\tilde{V}_i)} d_e^{(i)}(\tilde{x}_e^{(i)}) + \sum_{e\in E'_i(\tilde{V}_i,\tilde{V}_i)} d_e^{(i)}(\tilde{x}_e^{(i)}) + \sum_{e\in E'_i(U_{i-2},\tilde{V}_i)} d_e^{(i)}(\tilde{x}_e^{(i)}),$$

since $d_e^{(i)}(\tilde{x}_e^{(i)}) = 0$ for $e = (f, v_{\overline{e}}) \in E'_i(U_{i-1}, \tilde{V}_i)$ with $\overline{e} = (f,v) \in E'_{\leq}(U_{i-1}, W_i)$. This means that

$$\Psi'_{i-1}((x_v^{(i-1)})_{v\in V_{i-1}}) = \Psi'_i((\tilde{x}_v^{(i)})_{v\in V_i}).$$

The next part of the proof consists of showing

$$\Psi'((\hat{x}_v)_{v\in V}) = \Psi'_1((\tilde{x}_v^{(1)})_{v\in V_1}). \tag{4.13}$$

Let $s \in S$ be fixed. For $i = 1, \ldots, n-1$, we have

$$\sum_{fv\in E'_{\leq}(U_{i-1},U_{n-1}\setminus U_{i-1})} d_s^{f,v} = \sum_{j=i}^{n-1} \sum_{v\in W_j} \sum_{f\in F'_{\leq}(U_{i-1},v)} d_s^{f,v}$$
$$= \sum_{v\in W_i} \Bigg(\sum_{vs\in E} d_{vs} + \sum_{u\in F'_{\leq}(v,\text{succ}(v))} d_s^{v,u} \Bigg) + \sum_{j=i+1}^{n-1} \sum_{v\in W_j} \sum_{f\in F'_{\leq}(U_{i-1},v)} d_s^{f,v}$$

because of (4.1b),

$$= \sum_{e\in E(W_i,s)} d_e + \sum_{fv\in E'_{\leq}(U_i,U_{n-1}\setminus U_i)} d_s^{f,v}.$$

Using this, it is possible to show that

$$\sum_{fv\in E'_{\leq}(U_{i-1},U_{n-1}\setminus U_{i-1})} d_s^{f,v} = \sum_{e\in E'_{\leq}(U_{n-1}\setminus U_{i-1},s)} d_e \tag{4.14}$$

holds for $1 \leq i \leq n-1$. This can be done by induction starting at $i = n-1$ and ending at $i = 1$ and is omitted here. Because of (4.2), we see that for all $k \in K$ we have

$$d_{ks}^{(1)} = \sum_{ks\in E'} d_{ks} + \sum_{j=2}^{n-1} \sum_{v\in F'_{\leq}(k,W_j)} d_s^{k,v}.$$

For exactly those $v \in F'_{\leq}(k, W_1)$ with $e := (k, v) \in E'_{\leq}$ there exists a node $v_e \in \tilde{V}_1$ with $(v_e, s) \in E'_1$. For this edge (v_e, s) we have $d^{(1)}_{(v_e,s)} = d_s^{k,v}$. This means that

$$\sum_{e\in E'_1(K\cup\tilde{V},s)} d_e^{(1)} = \sum_{k\in K} \left(d_{ks} + \sum_{j=1}^{n-1} \sum_{v\in F'_{\leq}(k,W_j)} d_s^{k,v} \right).$$

Taking $i = 1$ in (4.14), the equation

$$\sum_{e\in E'_1(K\cup\tilde{V},s)} d_e^{(1)} = \sum_{e\in E(K,s)} d_e + \sum_{j=1}^{n-1} \sum_{e\in E'(W_j,s)} d_e = \sum_{e\in E(F\setminus S,s)} d_e$$

follows. But we have $\tilde{x}_e^{(1)} = (x, x_s)$ for $e \in E'_1(K, s)$ or $e \in E'_1(\tilde{V}_1, s)$ and therefore

$$\sum_{s\in S} \sum_{e\in E'(F\setminus S,s)} d_e(x, x_s) = \sum_{e\in E'_1(K\cup\tilde{V}_1,S)} d_e^{(1)}(\tilde{x}_e^{(1)}). \tag{4.15}$$

The edge set E'_1 can be written as

$$\begin{aligned} E'_1 = {} & (K \times K) \setminus \{(k,k) \mid k \in K\} \cup (K \times S) \cup E'(S,S) \\ & \cup (\tilde{V}_1 \times S) \cup E'_1(K,\tilde{V}_1) \cup E'_1(\tilde{V}_1,\tilde{V}_1). \end{aligned}$$

Because of $\tilde{x}_{v_e}^{(1)} = x_k = x$ for all $v_e \in \tilde{V}_1$ with $e = (k, v) \in E'_{\leq}(K, W_1)$ and $x_k = x$ for all $k \in K$ we have that $d_e^{(1)}(\tilde{x}_e^{(1)}) = 0$ holds for all $e \in E'_1(K, \tilde{V}_1) \cup E'_1(\tilde{V}_1, \tilde{V}_1) \cup (K \times K) \setminus \{(k, k) \mid k \in K\}$. For $e \in E'(S, S)$ it is $d_e^{(1)} = d_e$ and $\tilde{x}_e^{(1)} = x_e$. Therefore, (4.15) leads to

$$\Psi'((\hat{x}_v)_{v\in V}) = \sum_{e=vs\in E'(F\setminus S,S)} d_e(x, x_s) + \sum_{e\in E(S,S)} d_e(x_e).$$

Since $\hat{x}_v = x$ for all $v \in F \setminus S$, (4.13) follows.

By (4.3) and (4.4) we now have the chain of inequalities

$$\begin{aligned} \Psi'((\hat{x}_v)_{v\in V}) &= \Psi'_1((\tilde{x}_v^{(1)})_{v\in V_1}) \\ &\leq \Psi'_1((x_v^{(1)})_{v\in V_1}) \leq \dots \\ &\leq \Psi'_{n-1}((x_v^{(n-1)})_{v\in V_{n-1}}) \\ &= \Psi'((x_v)_{v\in V}) \\ &\leq \Psi((x_v)_{v\in V}), \end{aligned}$$

since $d_e(x_e) \geq 0$ for all $e \in E_s$. Because of $d_e(\hat{x}_e) = 0$ for all $e \in E_s$ it follows that

$$\Psi'((\hat{x}_v)_{v\in V}) = \Psi((\hat{x}_v)_{v\in V})$$

holds. Therefore,

$$\Psi((\hat{x}_v)_{v\in V}) \leq \Psi((x_v)_{v\in V}).$$

This inequality is strict if there exists a node $v \in U$ and a node $f \in F'_{\leq}(\mathrm{pre}(v), v)$ with $x_f \neq x_v$ such that (4.1a) is strict for all arguments $(x, y) \in (X \times X) \setminus \Delta$. Because of

$$\begin{aligned}\Psi((\hat{x}_v)_{v\in V}) &= \Psi'((\hat{x}_v)_{v\in V})\\ &\leq \Psi'((x_v)_{v\in V})\\ &\leq \Psi'((x_v)_{v\in V}) + \sum_{e\in E_s} d_e(x_e)\\ &= \Psi((x_v)_{v\in V}),\end{aligned}$$

it follows that $\Psi((\hat{x}_v)_{v\in V}) < \Psi((x_v)_{v\in V})$ whenever an edge $e \in E_s$ exists such that $d_e(x_e) \neq 0$ holds. □

Some remarks about further generalizations to Theorem 2 can be found in [21]. That $F'_{\leq}(\mathrm{pre}(v), v) \neq \emptyset$ has to hold for each $v \in U$ means that each node not in the shell S is connected with the kernel K by a path of edges in $E'_{\leq}$. This is a rather weak assumption, because we can always add an edge to this edge set and use the function $d \equiv 0$ on it. It may then be the case that the graph G is not simple any more, but this assumption can be dropped completely. However, this generalization leads to a much more tedious notation and is therefore left out here.

4.1. Some Examples. We will take a look at several rather simple examples to show how the criterion found in the theorem above can be applied to a given multifacility location problem.

EXAMPLE 1. We use $X = \mathbb{R}^N$, $A = \{a, b, c\}$, $V = \{u, v, w\}$,

$$E = \{(a, u), (a, v), (u, b), (u, w), (v, c), (v, w)\},$$

$d_{au}(x, y) = \|x - y\|_1$, $d_{av}(x, y) = \|x - y\|_2$, $d_{ub}(x, y) = \omega_{ub}\|x - y\|_2$, and $d_{vc}(x, y) = \omega_{vc}\|x - y\|_\infty$. Here, let $0 \leq \omega_{ub}, \omega_{vc} \leq 1$ and d_{uw} as well as d_{vw} be arbitrary metrics. Define $E_{\geq} = \emptyset$, $n = 2$, $K = \{a\}$, $S = \{b, c\}$, $W_1 = V$ and $E_s = \{(u, w), (v, w)\}$. Because of our definition of E_s, the metrics d_{uw} and d_{vw} play no role whatsoever. By adding the edge (a, w) to the edge set, we see that we have $F'_{\leq}(\mathrm{pre}(w), w) \neq \emptyset$. Of course, we use the distance cost function $d_{aw}(x, y) = 0$ on this edge, which does not change our optimization problem. Because of $d_{ub} \leq d_{au}$ and $d_{vc} \leq d_{av}$, condition (4.1a) holds for u and v. Moreover, this condition is trivially true for w, as it is the case for (4.1b)–(4.1e). (For (4.1c), we simply choose $\omega^{a;a}_{u,w} = 0$ etc.) We conclude that there is an optimal solution $(\hat{x}_v)_{v\in V}$ with $\hat{x}_u = \hat{x}_v = \hat{x}_w = x_a$. Moreover, in each minimum $(\hat{x}_v)_{v\in V}$ of Ψ we have $\hat{x}_u = \hat{x}_v = \hat{x}_w$, and if $\omega_{ub} < 1$ we have $\hat{x}_u = x_a$ in every solution. Likewise, if $\omega_{vc} < 1$ we have $\hat{x}_v = x_a$ in every solution. In this way, a sensitivity analysis on the weights is also possible.

EXAMPLE 2. Let $A = \{a, b, c, d\}$, $V = \{u, v, w\}$,

$$E = \{(u, a), (u, b), (u, w), (v, a), (v, c), (v, w), (w, d)\},$$

$\omega_{ua} = \omega_{va} = 7$, $\omega_{ub} = \omega_{vc} = 2$, $\omega_{uw} = \omega_{vw} = 6$, and $\omega_{wd} = 8$. Furthermore, let $\Psi((x_u, x_v, x_w)) = \sum_{e\in E} \omega_e d(x_e)$ with a symmetric weak metric d.

Define $E_{\geq} = E_s = \emptyset$, $n = 4$, $K = \{a\}$, $S = \{b, c, d\}$, $W_1 = \{u\}$, $W_2 = \{v\}$, and $W_3 = \{w\}$. Two threads will be used to proof the coincidence of every node in V with the node a. Both of them start at w and end at a, but while one of them passes through u, the node v is used by the other one. Since we are using two threads starting at w, we have to split up this node into two new ones. Because of the symmetry of d we may turn around edges as we wish, and we will do so by using (a, u) and (a, v) instead of (u, a) and (v, a). Since K contains only one element, we have trivially $x_k = x$ for all $k \in K$ and all families $(x_v)_{v \in V}$. As ansatz functions we use

$$d_s^{f,v} = \omega_s^{f,v} d, \quad d_{v,w}^{f,g} = \omega_{v,w}^{f,g} d, \quad d_h^{f,v} = \omega_h^{f,v} d, \quad \text{and} \quad d^{f,v} = \omega^{f,v} d$$

for the corresponding nodes $v, w, f, g, h, s \in F$. We are now in search for weights

$$\omega_s^{f,v}, \omega_{v,w}^{f,g}, \omega_h^{f,v}, \omega^{f,v} \in \mathrm{IR}$$

such that (4.1a)–(4.1e) holds. In order to fulfill (4.1b) with a weight $\omega_d^{u,w}$ and a weight $\omega_d^{v,w}$, we have to choose these weights in such a way that the equality

$$\omega_d^{u,w} + \omega_d^{v,w} = \omega_{wd} = 8$$

holds. Likewise, because (4.1a) has to hold, we have to choose them in such a way that additionally

$$\omega_d^{u,w} \leq \omega_{uw} = 6$$

and

$$\omega_d^{v,w} \leq \omega_{vw} = 6$$

holds. We take care of (4.1c) by defining $\omega_a^{u,w} := \omega_a^{v,w} = 0$. (Instead, we may consider (4.1a) for $f := a$. Because of $\omega_{aw} = 0$ we then have to choose $\omega_d^{a,w} := 0$.) As an example, we choose $\omega_d^{u,w} = 4$ and $\omega_d^{v,w} = 4$. With this and $\omega_b^{u,w} = \omega_c^{u,w} = \omega_b^{v,w} = \omega_c^{v,w} = 0$, (4.1a) and (4.1b) hold. If we follow the construction from the proof of Theorem 4, we see that $G'_3 = (F_3, E'_3)$ with $F_3 = A \cup \{u, v, v_{uw}, v_{vw}\}$ and

$$\begin{aligned} E'_3 = {} & \{(a,u),(u,a),(a,v),(v,a),(u,v),(v,u)\} \uplus (\{a,v,u\} \times \{v_{vw}, v_{uw}\}) \\ & \uplus (\{a,u,v,v_{vw},v_{uw}\} \times \{b,c,d\}). \end{aligned}$$

Of course, we have $d_{ab}^{(3)} \equiv 0$, $d_{au}^{(3)} = \omega_{au} d = 7d$, $d_{ua}^{(3)} \equiv 0$, etc. Likewise, it is $G'_2 = (F_2, E'_2)$ with $F_2 = A \cup \{u, v_{av}\}$ and

$$E'_2 = \{(a,u),(u,a),(a,v_{av}),(u,v_{av})\} \uplus (\{a,u,v_{av}\} \times \{b,c,d\}),$$

and, finally, $G'_1 = (F_1, E'_1)$ with $F_1 = A \cup \{v_{au}\}$ and

$$E'_1 = \{(a, v_{au})\} \uplus (\{a, v_{au}\} \times \{b,c,d\}).$$

In G'_2 we have strict attraction from the node v_{av} to the node a. Likewise, in G'_1 we have strict attraction between the nodes v_{au} and a. This is so because we have chosen $\omega_d^{u,w} = \omega_d^{v,w} = 4$, a choice which leads subsequently to the equations $\omega_d^{a,u} = 4$ and $\omega_d^{a,v} = 4$. It follows that

$$\Psi((\hat{x}_v)_{v \in V}) \leq \Psi((x_v)_{v \in V})$$

for all $(x_v)_{v \in V} \in X^V$. This inequality is strict if $x_w \neq x_u$ or $x_w \neq x_v$ or $x_v \neq x_a$ or $x_u \neq x_a$. An analogous derivation can be achieved by setting $n = 3$, $W_1 := \{u, v\}$ and $W_2 := \{w\}$.

EXAMPLE 3. We use the location problem defined in Example 2 but add the edge (u, v) with the weight $\omega_{uv} = -1/2$ to the graph G. In this case we have to use $E_{\geq} := \{(u, v)\}$, $\omega_u^{a,v} = -1/2$, and $\omega^{a,u} = 13/2$.

EXAMPLE 4. Consider the case in which we add not only the edge (u, v) with $\omega_{uv} = -1/2$ to the graph in Example 2, but also the edge (a, w) with the weight $\omega_{aw} = -1/2$. With this, we define $E_{\geq} := \{(a, w), (u, v)\}$. Using the same partition as above, we choose $\omega_d^{u,w} = 4$ and $\omega_d^{v,w} = 4$. But because of (4.1d), we have to choose weights $\omega_a^{u,w}$ and $\omega_a^{v,w}$ with

$$\omega_a^{u,w} + \omega_a^{v,w} = \omega_{aw} = -1/2,$$

such that with the weights $\omega_d^{u,w}$ and $\omega_d^{v,w}$ as above we still have (4.1a) and (4.1e). This can be done, for example, by choosing $\omega_a^{u,w} := \omega_a^{v,w} := -1/4$. With this, the collapse as above is allowed. The equations $\omega^{a,v} = 27/4$ and $\omega^{a,u} = 25/4$ follow immediately. We still have strict attraction in G_1' together with the final result.

The approach chosen in Example 2–4 is used in a companion paper [25] to develop a polynomial time algorithm for checking the criterion presented in Theorem 4. There, we present numerical results which show that by using this algorithm in conjunction with a nonlinear optimization code it is possible to reduce the computation time of the code while simultaneously enhancing the quality of the solution found.

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