Introduction
In this Section we consider two important features of complex functions. The Cauchy Riemann equations introduced on page 2 provide a necessary and sufficient condition for a function \( f(z) \) to be analytic in some region of the complex plane; this allows us to find \( f'(z) \) in that region by the rules of the previous Section. A mapping between the \( z \)-plane and the \( w \)-plane is said to be conformal if the angle between two intersecting curves in the \( z \)-plane is equal to the angle between their mappings in the \( w \)-plane. Such a mapping has widespread uses in solving problems in fluid flow and electromagnetics, for example, where the given problem geometry is somewhat complicated.

Prerequisites
Before starting this Section you should...

Learning Outcomes
After completing this Section you should be able to...

✓ use the Cauchy Riemann equations to obtain the derivative of complex functions
✓ appreciate the idea of a conformal mapping
① understand the idea of a complex function and its derivative
1. Cauchy-Riemann equations

Remembering that \( z = x + iy \) and \( w = u + iv \) we note that there is a very useful test to determine whether a function \( w = f(z) \) is analytic at a point. This is provided by the Cauchy-Riemann equations. These state that \( w = f(z) \) is differentiable at a point \( z = z_0 \) if, and only if,

\[
\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}
\]

at that point.

When these equations hold then it can be shown that the complex derivative may be determined by using either \( \frac{df}{dz} = \frac{\partial f}{\partial x} \) or \( \frac{df}{dz} = -i\frac{\partial f}{\partial y} \).

(The use of ‘if, and only if,’ means that if the equations are valid, then the function is differentiable and vice versa).

If we consider \( f(z) = z^2 = x^2 - y^2 + 2ixy \) then \( u = x^2 - y^2 \) and \( v = 2xy \) so that

\[
\frac{\partial u}{\partial x} = 2x, \quad \frac{\partial u}{\partial y} = -2y, \quad \frac{\partial v}{\partial x} = 2y, \quad \frac{\partial v}{\partial y} = 2x.
\]

It should be clear that, for this example, the Cauchy-Riemann equations are always satisfied; therefore, the function is analytic everywhere. We find that

\[
\frac{df}{dz} = \frac{\partial f}{\partial x} = 2x + 2iy = 2z \quad \text{or, equivalently} \quad \frac{df}{dz} = -i\frac{\partial f}{\partial y} = -i(-2y + 2ix) = 2z
\]

This is the result we would expect to get by simply differentiating \( f(z) \) as if it was a real function. For analytic functions this will always be the case.

Example Show that the function \( f(z) = z^3 \) is analytic everywhere and hence obtain its derivative.

Solution

\[
w = f(z) = (x + iy)^3 = x^3 - 3xy^2 + (3x^2y - y^3)i
\]

Hence

\[
u = x^3 - 3xy^2 \quad \text{and} \quad v = 3x^2y - y^3.
\]

Then

\[
\frac{\partial u}{\partial x} = 3x^2 - 3y^2, \quad \frac{\partial u}{\partial y} = -6xy, \quad \frac{\partial v}{\partial x} = 6xy, \quad \frac{\partial v}{\partial y} = 3x^2 - 3y^2.
\]

The Cauchy-Riemann equations are identically true and \( f(z) \) is analytic everywhere. Furthermore

\[
\frac{df}{dz} = \frac{\partial f}{\partial x} = 3x^2 - 3y^2 + (6xy)i = 3(x + iy)^2 = 3z^2 \quad \text{as we would expect.}
\]

We can easily find functions which are not analytic anywhere and others which are only analytic in a restricted region of the complex plane. Consider again the function \( f(z) = \bar{z} = x - iy \).

Here
\[ u = x \] so that \( \frac{\partial u}{\partial x} = 1, \] and \( \frac{\partial u}{\partial y} = 0; \]

\[ v = -y \] so that \( \frac{\partial v}{\partial x} = 0, \) \( \frac{\partial v}{\partial y} = -1. \)

The Cauchy-Riemann equations are never satisfied so that \( \bar{z} \) is not differentiable anywhere and so is not analytic anywhere.

By contrast if we consider the function \( f(z) = \frac{1}{z} \) we find that

\[
 u = \frac{x}{x^2 + y^2}; \quad v = \frac{y}{x^2 + y^2}.
\]

As can readily be shown, the Cauchy-Riemann equations are satisfied everywhere except for \( x^2 + y^2 = 0 \), i.e. \( x = y = 0 \) (or, equivalently, \( z = 0 \)). At all other points \( f'(z) = -\frac{1}{z^2} \). This function is analytic everywhere except at the single point \( z = 0 \).

**Analyticity is a very powerful property of a function of a complex variable. Such functions tend to behave like functions of a real variable.**

**Example** Show that if \( f(z) = z\bar{z} \) then \( f'(z) \) exists only at \( z = 0 \).

**Solution**

\[
 f(z) = x^2 + y^2 \quad \text{so that} \quad u = x^2 + y^2, \quad v = 0.
\]

\[
 \frac{\partial u}{\partial x} = 2x, \quad \frac{\partial u}{\partial y} = 2y, \quad \frac{\partial v}{\partial x} = 0, \quad \frac{\partial v}{\partial y} = 0.
\]

Hence the Cauchy-Riemann equations are satisfied only where \( x = 0 \) and \( y = 0 \), i.e. where \( z = 0 \). Therefore this function is not analytic anywhere.

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**Analytic Functions and Harmonic Functions**

Using the Cauchy-Riemann equations in a region of the \( z \)-plane where \( f(z) \) is analytic,

\[
 \frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial x} \left( \frac{\partial u}{\partial y} \right) = \frac{\partial}{\partial x} \left( -\frac{\partial v}{\partial x} \right) = -\frac{\partial^2 v}{\partial x^2}
\]

and

\[
 \frac{\partial^2 u}{\partial y \partial x} = \frac{\partial}{\partial y} \left( \frac{\partial u}{\partial x} \right) = \frac{\partial}{\partial y} \left( \frac{\partial v}{\partial y} \right) = \frac{\partial^2 v}{\partial y^2}.
\]

If these differentiations are possible then \( \frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x} \) so that

\[
 \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \text{(Laplace's equation)}.
\]

In a similar way we find that

\[
 \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0 \quad \text{(can you show this?)}
\]

When \( f(z) \) is analytic the functions \( u \) and \( v \) are called **conjugate harmonic functions**.
Suppose \( u = u(x, y) = xy \) then it is easy to verify that \( u \) satisfies Laplace’s equation (try this).

We now try to find the conjugate harmonic function \( v = v(x, y) \).

First, using the Cauchy-Riemann equations:

\[
\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} = y \quad \text{and} \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = -x.
\]

Integrating the first equation gives \( v = \frac{1}{2}y^2 + \) a function of \( x \). Integrating the second equation gives \( v = -\frac{1}{2}x^2 + \) a function of \( y \). Bearing in mind that an additive constant leaves no trace after differentiation we pool the information above to obtain

\[
v = \frac{1}{2}(y^2 - x^2) + C \quad \text{where} \quad C \text{ is a constant}
\]

Note that \( f(z) = u + iv = xy + \frac{1}{2}(y^2 - x^2)i + D \) where \( D \) is a constant (replacing \( Ci \)).

We can write \( f(z) = -\frac{1}{2}iz^2 + D \) (as you can verify). This function is analytic everywhere.

Show that the function \( u = x^2 - x - y^2 \) is harmonic.

**Your solution**

\[
\frac{\partial u}{\partial x} = 2x - 1, \quad \frac{\partial^2 u}{\partial x^2} = 2, \quad \frac{\partial u}{\partial y} = -2y, \quad \frac{\partial^2 u}{\partial y^2} = -2.
\]

Hence

\[
\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \text{and} \quad u \text{ is harmonic.}
\]

Now find the conjugate harmonic function \( v \).

**Your solution**
Integrating $\frac{\partial v}{\partial y} = 2x - 1$ gives $v = 2xy - y + \text{function of } x$. Integrating $\frac{\partial v}{\partial x} = + 2y$ gives $v = 2xy + \text{function of } y$. Ignoring the duplication, $v = 2xy - y + C$, where $C$ is a constant.

Finally, find $f(z)$ in terms of $z$.

**Your solution**

$A + z - \bar{z} = (z)f$

Thus $\hat{h} + x = z$ and $\hat{h}x\bar{z} + \bar{h} - \bar{x} = \bar{z}$

Now constant $\bar{A}$, $A + \hat{h} - \hat{h}x\bar{z} + \bar{h} - x - \bar{x} = n + n = (z)f$

**Exercises**

1. Find the singular point of the rational function $f(z) = \frac{z}{z - 2i}$. Find $f'(z)$ at other points and evaluate $f'(-i)$.

2. Show that the function $f(z) = z^2 + z$ is analytic everywhere and hence obtain its derivative.

3. Show that the function $u = x^2 - y^2 - 2y$ is harmonic, find the conjugate harmonic function $v$ and hence find $f(z) = u + iv$ in terms of $z$.

$$1 + z\bar{z} = \frac{\hat{h}\bar{z}}{\bar{a}} + 1 + x\bar{z} = \frac{x\bar{Q}}{\bar{f}Q} = \frac{zP}{fP}$$

Here the Cauchy-Riemann equations are identically true and is analytic everywhere.

$$1 + x\bar{z} = \frac{\bar{h}\bar{z}}{a\bar{Q}} \hbar + x\bar{z} = \frac{x\bar{Q}}{n\bar{Q}}$$

$$I + x\bar{z} = \frac{\bar{h}\bar{z}}{a\bar{Q}} \hbar + x\bar{z} = \frac{x\bar{Q}}{n\bar{Q}}$$

$$\frac{\bar{z}(\bar{z} - z)}{\bar{z} - 1} = \frac{\bar{z}(\bar{z} - z)}{\bar{z} - 1(\bar{z} - z)} = (z)f$$

Here $f(z)$ is singular at $\bar{z}$.
2. Conformal Mapping

In Section 1 we saw that the real and imaginary parts of an analytic function each satisfies Laplace’s equation. We shall show now that the curves
\[ u(x, y) = \text{constant} \quad \text{and} \quad v(x, y) = \text{constant} \]
intersect each other at right angles (we say that they are orthogonal). To see this we note that along the curve \( u(x, y) = \text{constant} \) we have \( du = 0 \). Hence
\[ du = \frac{\partial u}{\partial x} \, dx + \frac{\partial u}{\partial y} \, dy = 0. \]

Thus, on these curves the gradient at a general point is given by
\[ \frac{dy}{dx} = \frac{-\frac{\partial u}{\partial x}}{\frac{\partial u}{\partial y}}. \]

Similarly along the curve \( v(x, y) = \text{constant} \), we have
\[ \frac{dy}{dx} = \frac{-\frac{\partial v}{\partial x}}{\frac{\partial v}{\partial y}}. \]

The product of these gradients is
\[ \left( \frac{\partial u}{\partial x} \right) \left( \frac{\partial v}{\partial x} \right) = - \left( \frac{\partial u}{\partial y} \right) \left( \frac{\partial u}{\partial y} \right) = -1 \]
where we have made use of the Cauchy-Riemann equations. We deduce that the curves are orthogonal.

As an example of the practical application of this work consider two-dimensional electrostatics. If \( u = \text{constant} \) gives the equipotential curves then the curves \( v = \text{constant} \) are the electric lines of force. Figure 1 shows some curves from each set in the case of oppositely-charged particles near
to each other; the dashed curves are the lines of force and the solid curves are the equipotentials.

Figure 1
In ideal fluid flow the curves \( v = \text{constant} \) are the *streamlines* of the flow.
In these situations the function \( w = u + iv \) is the complex potential of the field.

**Conformal Mapping**

A function \( w = f(z) \) can be regarded as a mapping, which ‘maps’ a point in the \( z \)-plane to a point in the \( w \)-plane. Curves in the \( z \)-plane will be mapped into curves in the \( w \)-plane.

Consider aerodynamics. The idea is that we are interested in the fluid flow, in a complicated geometry (say flow past an aerofoil). We first find the flow in a simple geometry that can be mapped to the aerofoil shape (the complex plane with a circular hole works here). Most of the calculations necessary to find physical characteristics such as lift and drag on the aerofoil can be performed in the simple geometry - the resulting integrals being much easier to evaluate than in the complicated geometry.

Consider the mapping

\[
 w = z^2.
\]

The point \( z = 2 + i \) maps to \( w = (2 + i)^2 = 3 + 4i \). The point \( z = 2 + i \) lies on the intersection of the two lines \( x = 2 \) and \( y = 1 \). To what curves do these map? To answer this question we note that a point on the line \( y = 1 \) can be written as \( z = x + i \). Then

\[
 w = (x + i)^2 = x^2 - 1 + 2xi
\]

As usual, let \( w = u + iv \), then

\[
 u = x^2 - 1 \quad \text{and} \quad v = 2x
\]

Eliminating \( x \) we obtain:

\[
 4u = 4x^2 - 4 = v^2 = 4 \quad \text{or} \quad v^2 = 4 + 4u.
\]
Example To what curve does the line $x = 2$ map?

Solution
A point on the line is $z = 2 + yi$. Then
$$w = (2 + yi)^2 = 4 - y^2 + 4yi$$
Hence $u = 4 - y^2$ and $v = 4y$ so that, eliminating $y$ we obtain
$$16u = 64 - v^2 \quad \text{or} \quad v^2 = 64 - 16u$$

In Figure 2(a) we sketch the lines $x = 2$ and $y = 1$ and in Figure 2(b) we sketch the curves into which they map. Note these curves intersect at the point $(3,4)$.

![Figure 2](image)

The angle between the original lines was clearly $90^\circ$; what is the angle between the curves at the point of intersection?

The curve $v^2 = 4 + 4u$ has a gradient $\frac{dv}{du}$. Differentiating the equation implicitly we obtain
$$2v \frac{dv}{du} = 4 \quad \text{or} \quad \frac{dv}{du} = \frac{2}{v}$$
At the point $(3,4)$ $\frac{dv}{du} = \frac{1}{2}$.

Find $\frac{dv}{du}$ for the curve $v^2 = 64 - 16u$ and evaluate it at the point $(3,4)$

Your solution
\[ \zeta - = \frac{np}{\pi} \quad \text{and} \quad \varphi = a \, 1 \, \psi \]

Note that the product of the gradients at \( (3,4) \) is \(-1\) and therefore the angle between the curves at their point of intersection is also \(90^0\). Since the angle between the lines and the angle between the curves is the same we say the angle is preserved.

In general, if two curves in the \( z \)-plane intersect at a point \( z_0 \), and their image curves under the mapping \( w = f(z) \) intersect at \( w_0 = f(z_0) \) and the angle between the two original curves at \( z_0 \) equals the angle between the image curves at \( w_0 \) we say that the mapping is **conformal** at \( z_0 \).

An analytic function is conformal everywhere except where \( f'(z) = 0 \).

At which points is \( w = e^z \) not conformal?

**Your solution**

Since this is never zero the mapping is conformal everywhere.

**Inversion**

The mapping

\[ w = f(z) = \frac{1}{z} \]

is called an **inversion**. It maps the interior of the unit circle in the \( z \)-plane to the exterior of the unit circle in the \( w \)-plane, and vice-versa. Note that

\[ w = u + iv = \frac{x}{x^2 + y^2} - \frac{y}{x^2 + y^2}i \quad \text{and similarly} \quad z = x + iy = \frac{u}{u^2 + v^2} - \frac{v}{u^2 + v^2}i \]

so that

\[ u = \frac{x}{x^2 + y^2} \quad \text{and} \quad v = -\frac{y}{x^2 + y^2}. \]

A line through the origin in the \( z \)-plane will be mapped into a line through the origin in the \( w \)-plane. To see this consider the line \( y = mx \), for \( m \) constant. Then

\[ u = \frac{x}{x^2 + m^2x^2} \quad \text{and} \quad v = -\frac{mx}{x^2 + m^2x^2} \]

so that \( v = -mu \), which is a line through the origin in the \( w \)-plane.
Consider the line $ax + by + c = 0$ where $c \neq 0$. This represents a line in the $z$-plane which does not pass through the origin. To what sort of curve does it map in the $w$-plane?

**Your solution**

The mapped curve is

$$au^2 + v^2 - bv^2 + c = 0$$

Hence

$$au - bv + c(u^2 + v^2) = 0.$$ Dividing by $c$ we obtain the equation:

$$u^2 + v^2 + au - bv = 0.$$ The mapped curve is a circle in the $w$-plane which passes through the origin.

Similarly, it can be shown that a circle in the $z$-plane passing through the origin maps to a line in the $w$-plane which does not pass through the origin and a circle in the $z$-plane which does not pass through the origin maps to a circle in the $w$-plane which does not pass through the origin. The inversion mapping is an example of the **bilinear transformation**:

$$w = f(z) = \frac{az + b}{cz + d} \quad \text{where we demand that} \quad ad - bc \neq 0$$

(If $ad - bc = 0$ the mapping reduces to $f(z) = \text{constant}$).

Find the bilinear transformations which map $z = 2$ to $w = 1$.

**Your solution**

$$p + \bar{z} = q + \bar{v} \quad \text{Hence} \quad \frac{p + \bar{z}}{q + \bar{v}} = 1$$
If in addition, $z = -1$ is mapped to $w = 3$ find the class of transformation which is possible.

**Your solution**

$$p\xi + q\eta = r + s \quad \text{where} \quad \frac{p + q}{r + s} = \xi$$

If, further, $z = 0$ is mapped to $w = -5$ then $-5 = \frac{b}{d}$ so that $b = -5d$. Substituting this last relation into the first two obtained we obtain

$$2a - 2c - 6d = 0$$
$$-a + 3c - 8d = 0$$

Solving these two in terms of $d$ we find $2c = 11d$ and $2a = 17d$. Hence the transformation is:

$$w = \frac{17z - 10}{11z + 2} \quad \text{(note that the d’s cancel in the numerator and denominator).}$$

Some other mappings are shown in Figure 3.

As an engineering application we consider the transformation

$$w = z - \frac{\ell^2}{z} \quad \text{where } \ell \text{ is a constant.}$$
It is used to map circles which contain \( z = 1 \) as an interior point and which pass through \( z = -1 \) into shapes resembling aerofoils. Figure 4 shows an example.

\[
\begin{align*}
\frac{1 + z - i}{1 + z} &= p + zp - \overline{p} + \overline{z}p = m \quad \text{hence} \quad p - = q \\
\overline{p} + p &= p + \overline{z} - = p + \overline{z} - = 1 \quad \text{hence} \\
\overline{p} &= q \quad \text{hence} \quad \frac{p}{q} = 1 \quad \text{hence} \quad \frac{q}{p} = 0 = z \\
\frac{p + zq}{q + z\overline{p}} &= m, \quad i
\end{align*}
\]

Figure 4

This creates a cusp at which the associated fluid velocity can be infinite. This can be avoided by adjusting the fluid flow in the \( z \)-plane. Eventually, this can be used to find the lift generated by such an aerofoil in terms of physical characteristics such as aerofoil shape and air density and speed. 

**Exercises**

1. Find a bilinear transformation which maps \( z = 0, -1, -i \) into \( w = i, 0, 1 \) respectively.